Banach J. Math. Anal. (2020) 14:1305–1325
https://doi.org/10.1007/s43037-020-00061-2

Tusi Mathematical Research Group

ORIGINAL PAPER

Real-valued non compactness measures in topological vector spaces and a pplications

Nabil Machrafi[1] · **Lahbib Oubbi**[2]

Received: 9 December 2019 / Accepted: 3 March 2020 / Published online: 18 March 2020

Abstract

A non compactness measure with values in the lattice of extended non negative real numbers $[0, +\infty]$ is introduced in the general setting of a Hausdorff topological vector space E. This generalizes the classical Kuratowski and Hausdorff non compactness measures. In order to achieve this, we introduce the notions of basic and sufficient collections of zero neighborhoods. We then show that our measure satisfies most of the properties of the classical non compactness measures. We particularly show that if E is locally p-convex for some $0 < p \leq 1$, our measure is stable by the transition to the closed p-convex hull. This allows us to obtain, as applications, generalizations of the well-known three fixed point theorems, namely Schauder, Darbo, and Sadovskii's ones in the setting of locally p-convex spaces. As another application, we establish a quantification of Ascoli theorem in the space $C(X, E)$ of vector-valued continuous functions on a Hausdorff completely regular space with values in a topological vector space E, giving an alternative of Ambrosetti theorem initially stated in the metric spaces setting.

Keywords Topological vector space · Non compactness measure · Precompact set · Completely regular space · Compact open topology · Ascoli theorem · Fixed point

Mathematics Subject Classification 46A03 · 46A17

Communicated by Jose Bonet.

✉ Lahbib Oubbi
oubbi@daad-alumni.de

Nabil Machrafi
nmachrafi@gmail.com

[1] Faculty of Sciences, Department of Mathematics, Team GrAAF, Laboratory LMSA, Center CeReMAR, Mohammed V University in Rabat, 4, Avenue Ibn Battouta, PoBox 1014 RP, 10000 Rabat, Morocco

[2] Ecole Normale Supérieure Takaddoum, Team GrAAF, Laboratory LMSA, Center CeReMAR, Mohammed V University in Rabat, PoBox 5118 Suissi, 10105 Rabat, Morocco

1 Introduction

One of the main tools when seeking for fixed points of a self-map of a metric space X, in which no compactness is assumed, is the notion of non compactness measure. The first such a measure was introduced by Kuratowski in [12] for a bounded subset A of X. This can be thought of as a measurement of the lack of compactness of A and it is defined as follows:

$$\alpha(A) = \inf\left\{r > 0, \exists n \in \mathbb{N}, \exists A_i \subset X \text{ with } \delta(A_i) \le r \text{ and } A \subset \cup_{i=1}^n A_i\right\}, \quad (1)$$

where $\delta(A)$ stands for the diameter of A.

A similar non compactness measure is the so-called Hausdorff (or ball) non compactness measure β of A defined by:

$$\beta(A) = \inf\{r > 0 : \exists n \in \mathbb{N}, \exists x_1, \dots x_n \in X : A \subset \cup_{i=1}^n B(x_i, r)\}, \quad (2)$$

where $B(x, r)$ is the ball centered at x with radius r. These two non compactness measures are equivalent in the sense that, for every bounded subset A of X,

$$\beta(A) \le \alpha(A) \le 2\beta(A).$$

In the setting of a locally convex space E, whose topology is given by a separating family $\mathbb{P}$ of semi-norms, Sadovskii [19] introduced extensions of α and β taking their values in the set of non negative functions on $\mathbb{P}$, rather than non negative real numbers, as in the normed spaces case. These extensions, again denoted by α and β are the functions assigning to a bounded set $A \subset E$ the mappings $\alpha(A) : P \mapsto \alpha(A)(P)$ and $P \mapsto \beta(A)(P)$ respectively, where $\alpha(A)(P)$ (resp. $\beta(A)(P)$) is defined by (1) (resp. (2)), the diameters (resp. the balls) being taken with respect to the semi-norm P.

Later, Kaniok [7] extended the definition of Sadovskii in the more general setting of a topological vector space E, using zero neighborhoods in E instead of semi-norms. Recall that, for a given such neighborhood U, a set $A \subset E$ is said to be U-small (or small of order U [16]), if $A - A \subset U$. For every $A \subset E$, the U-measures of non compactness $\alpha_U(A)$ and $\beta_U(A)$ of A are defined as:

$$\alpha_U(A) = \inf\left\{r > 0 : A \text{ is covered by a finite number of } rU\text{-small sets } A_i\right\}, \quad (3)$$

$$\beta_U(A) = \inf\left\{r > 0 : \exists x_1, \dots, x_n \in E \text{ such that } A \subset \cup_{i=1}^n \left(x_i + rU\right)\right\}, \quad (4)$$

with $\inf \emptyset = +\infty$. In case E is a normed space and U is the closed unit ball of E, α_U and β_U are nothing but the Kuratowski and Hausdorff measures of non compactness, respectively. Thus, if $\mathscr{U}$ denotes a fundamental system of balanced and closed zero neighborhoods in E and $\mathscr{F}_{\mathscr{U}}$ is the space of all functions $\varphi : \mathscr{U} \to \mathbb{R}$, endowed with the pointwise ordering, then, after Kaniok, the Kuratowski (resp. the Hausdorff) measure $\alpha(A)$ (resp. $\beta(A)$) of a subset A of E is the function defined from $\mathscr{U}$ into $[0, +\infty]$ by $\alpha(A)(U) = \alpha_U(A)$ (resp. $\beta(A)(U) = \beta_U(A)$).

The invariance under the transition to the closed convex hull $\overline{co}(A)$ of the subset $A \subset E$ is one of the main properties of the Kuratowski and the Hausdorff measures α

and β. This property remains valid in the setting of locally convex spaces as shown by Sadovskii in [19, Theorem 1.2.3], and is the key of the proof of the classical Darbo and Sadovskii fixed point theorems.

In this paper, we introduce new non compactness measures, again denoted by α and β, generalizing to the setting of a topological vector space E the Kuratowski and the Hausdorff ones. As in the normed case, our measures take their values in $[0, +\infty]$. We then establish some of their properties. In particular, we show that, if the space E is locally p-convex, $0 < p \leq 1$, then these measures are invariant under the transition to the closed s-convex hull, $0 < s \leq p$. As applications, we first extend the Schauder, the Darbo, and the Sadovskii fixed point theorems to the locally p-convex setting. Up to our knowledge, particular attention has been given recently to the study of fixed point results for multi-valued maps defined on p-convex sets (see [5] and the references therein). The usual hypothesis that a multi-valued map has non empty p-convex values does not permit to derive directly the single-valued counterpart for these results when $0 < p < 1$, since in this case the singletons fail to be p-convex. This is the case, for instance, for the p-convex version of Kakutani fixed point theorem given in [5, Corollary 2.13]. Hence, an investigation of p-convex versions of the three above-mentioned fixed point theorems for single-valued maps remains of interest. A complete study of these versions has been recently achieved in [20, 21] in the particular setting of p-normed spaces. Here, we extend this study to the general setting of locally p-convex spaces for the so-called Yanyan continuous maps (Sect. 3, Theorems 4, 5 and Corollary 1). However, whether the three theorems are still valid for continuous maps remains an open question.

A further application consists of a quantification of Ascoli theorem in the space of vector-valued continuous functions on a completely regular space, via an alternative version of Ambrosetti theorem.

After investigating in Sect. 1 non compactness measures with respect to a given zero neighborhood, we define, in Sect. 2, the non compactness measures α and β in a topological vector space E. In order to maintain the scalar character of these measures and to overcome some difficulties occurring when involving the whole set of zero neighborhoods in E, we introduce the notions of basic and sufficient collections of zero neighborhoods (Definition 1). Next, we show that several properties of non compactness measures extend naturally to ours.

In Sect. 3, using our measures, we obtain extensions of Schauder, Darbo and Sadovskii fixed point theorems in the context of locally p-convex spaces. Our results generalize recent fixed point theorems given by Xiao and Lu [20] and Xiao and Zhu [21] in complete p-normed spaces.

Section 4 is devoted to the quantification of Ascoli theorem using non compactness measures in $C(X, E)$, the space of all continuous functions from a Hausdorff completely regular space X into a topological vector space E. Such a quantification was first given by Ambrosetti [1], by showing that, if X and E are two metric spaces with X compact, and D is a bounded and equicontinuous subset of $C(X, E)$, then

$$\widehat{\alpha}(D) = \sup_{x \in X} \alpha(D(x)), \tag{5}$$

where $D(x) := \{f(x) : f \in D\}$, and α and $\widehat{\alpha}$ stand for the standard Kuratowski measures of non compactness on the metric spaces E and $C(X, E)$ respectively. Ascoli theorem is then the particular case where one of the sides of the equality (5) is zero. A general version of Ascoli theorem was given by Kelley [8, p. 234] taking X to be a Hausdorff or a regular k-space and E to be a Hausdorff uniform space. It states that a subset D of $C(X, E)$ is compact in the topology of uniform convergence on compacta if and only if D is closed, $D(x)$ is relatively compact for each $x \in X$ and D is equicontinuous on every compact subset of X. Here, we present a quantified version of Ascoli theorem in a different setting than Ambrosetti's. More precisely, using our non compactness measure α, we give an extension of Ambrosetti theorem, letting X be a Hausdorff completely regular topological space and the range space E a Hausdorff topological vector space (see Theorem 7).

2 Non compactness measures with respect to a zero neighborhood

Throughout this paper, and unless otherwise stated, E will denote a Hausdorff topological vector space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and p will be a real number with $0 < p \leq 1$. The set of all balanced zero neighborhoods in E is denoted by $\mathscr{V}_0$. Recall that $U \in \mathscr{V}_0$ is said to be shrinkable, if it is absorbing, balanced, and $r\overline{U} \subset \mathring{U}$, for every $0 < r < 1$; here $\overline{U}$ stands for the closure of U and $\mathring{U}$ for its interior. Any topological vector space admits a local base at zero consisting of shrinkable sets (see [10] or [6] for details).

Recall that a subset A of E is said to be p-convex if it satisfies $\lambda A + \mu A \subset A$ for all $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$. The case $p = 1$ is the usual case of a convex set. Note that if A is p-convex and contains 0, then it is s-convex for every positive $s \leq p$. In particular, an absolutely p-convex set (i.e., a balanced and p-convex set) is absolutely s-convex for every $0 < s \leq p$. Such a result fails if $0 \notin A$. Actually, a singleton $\{x\}$, $x \in E \setminus \{0\}$, is convex but not s-convex, for any $0 < s < 1$.

We will frequently use the fact that if $A \subset E$ is p-convex, then

$$\lambda A + \mu A \subset \sqrt[p]{\lambda^p + \mu^p}\, A, \qquad \lambda, \mu \geq 0.$$

The p-convex hull of $A \subset E$, denoted by $co_p(A)$ is the smallest p-convex subset of E containing A. This is, equivalently, the set

$$co_p(A) = \bigcup_{n \geq 1} \left\{ \sum_{i=1}^{n} \mu_i x_i, \quad 0 \leq \mu_i, \ \sum_{i=1}^{n} \mu_i^p = 1, \quad x_i \in A \right\}.$$

The set $co_1(A)$ is simply denoted by $co(A)$.

The topological vector space E is said to be a locally p-convex space, if E has a local base at zero consisting of p-convex sets. The topology of a locally p-convex space is always given by an upward directed family $\mathbb{P}$ of p-semi-norms, where a p-semi-norm on E is any non negative real-valued and subadditive functional $\|\ \|_p$ on E, such that $\|\lambda x\|_p = |\lambda|^p \|x\|_p$ for every $x \in E$, $\lambda \in \mathbb{K}$. If E is Hausdorff, then for every $x \neq 0$, there is some $P \in \mathbb{P}$ such that $P(x) \neq 0$. Whenever the family $\mathbb{P}$ is reduced to a singleton, one says that $(E, \| \|_p)$ is a p-semi-normed space. A p-normed space is a Hausdorff p-semi-normed space.

Notice that the case $p = 1$ is the usual locally convex case. Furthermore, a p-normed space is a metric vector space with the translation invariant metric $d_p(x, y) = \|x - y\|_p$, $x, y \in E$. The classical Lebesgue spaces $L_p(\mu)$ defined on a complete measure space $(\Omega, \mathscr{M}, \mu)$, μ being a positive measure on $\mathscr{M}$, are examples of p-normed spaces, where the p-norm is given by

$$\|f\|_p = \int_\Omega |f(x)|^p d\mu, \quad f \in L_p(\mu).$$

If P is a continuous p-semi-norm on E, then the ball $B_P(0, s) := \{x \in E : P(x) < s\}$ is shrinkable, for every $s > 0$. Indeed, if $r < 1$ and $x \in \overline{rB_P(0, s)}$, then there exists a net $(x_i)_i \subset B_P(0, s)$ such that rx_i converges to x. By continuity of P, we get $P(x) \leq r^p s < s$, saying that $\overline{rB_P(0, s)} \subset B_P(0, s)$. More generally, it can be shown that every p-convex $U \in \mathscr{V}_0$ is shrinkable. From now on, denote by $\mathscr{N}_0$ the set of all shrinkable zero neighborhoods in E.

Throughout all the sequel, the results are presented for the Kuratowski type non compactness measure α. Similar arguments can be used to show their analogous for the Hausdorff type one β. Therefore, if $W \in \mathscr{V}_0$, we will denote by α_W the Kuratowski type W-measure of non compactness as given by (3).

Notice that the properties 1–3 in the proposition below are mentioned in [7] without proof.

Proposition 1 *Let $A, B \subset E, U, V \in \mathscr{V}_0$. Then the following assertions hold:*

1. α_U *is semi-additive, i.e.,* $\alpha_U(A \cup B) = \max(\alpha_U(A), \alpha_U(B))$.

2. $\alpha_{rU}(sA) = \frac{|s|}{|r|}\alpha_U(A)$, *for every scalars* $r, s \neq 0$. *In particular, if* $rU \subset V \subset sU$, *then* $\alpha_U(A) = 0$ *if and only if* $\alpha_V(A) = 0$.

3. *If* $V + V \subset U$, *then* $\alpha_U(A + B) \leq \max(\alpha_V(A), \alpha_V(B))$.

4. *If* $V \subset U$, *then* $\alpha_U \leq \alpha_V$.

If U is shrinkable, then

5. $\alpha_U = \alpha_{\overline{U}}$.

6. $\alpha_U(A) = \alpha_U(\overline{A})$.

If U is p-convex for some $0 < p \leq 1$, *then*

7. $\alpha_U(A + B) \leq \sqrt[p]{\alpha_U(A)^p + \alpha_U(B)^p}$. *In particular, if U is convex, i.e.,* $p = 1$, *then* α_U *is algebraically semi-additive, i.e.* $\alpha_U(A + B) \leq \alpha_U(A) + \alpha_U(B)$.

8. *α_U is uniformly continuous, in the sense that, for every $\varepsilon > 0$, there exists $W \in \mathscr{V}_0$ such that, whenever A and B are W-close, we have either $\alpha_U(A) = \alpha_U(B) = +\infty$, or $|\alpha_U(A) - \alpha_U(B)| < \varepsilon$, where A and B are said to be W-close, if $A \subset B + W$ and $B \subset A + W$.*

Proof 1. It is clear that $\alpha_U(A \cup B) \leq \max(\alpha_U(A), \alpha_U(B))$. For the converse, assume that $\alpha_U(A \cup B) < \max(\alpha_U(A), \alpha_U(B))$ and choose a scalar r so that $\alpha_U(A \cup B) < r < \max(\alpha_U(A), \alpha_U(B))$. Then there is a finite covering of both A and B by rU-small sets $C_1, \dots, C_n$. This contradicts $r < \max(\alpha_U(A), \alpha_U(B))$.

2. The homogeneity with respect to A follows easily. The second part derives from the fact that $\mu \in S_A(U)$, if and only if, $\frac{\mu}{r} \in S_A(rU)$, where $S_A(U)$ denotes the set of all $r > 0$ such that A is covered by finitely many rU-small sets. Now, if $rU \subset V \subset sU$, then $\frac{1}{s}\alpha_U \leq \alpha_V \leq \frac{1}{r}\alpha_U$. Therefore, $\alpha_U(A) = 0$ if and only if $\alpha_V(A) = 0$.

3. Indeed, if $\alpha_V(A) = +\infty$ or $\alpha_V(B) = +\infty$, the inequality is trivial. Now, if $r > \alpha_V(A)$ and $s > \alpha_V(B)$, then there are rV-small sets $(A_i)_{i=1,\dots,n}$ and sV-small ones $(B_j)_{j=1,\dots,m}$, such that $A \subset \cup_{i=1}^n A_i$ and $B \subset \cup_{J=1}^m B_j$. Therefore, $A + B \subset \cup_{\substack{i=1,\dots,n \\ j=1,\dots,m}} (A_i + B_j)$. But

$$\begin{aligned}(A_i + B_j) - (A_i + B_j) &= (A_i - A_i) + (B_j - B_j) \\ &\subset rV + sV \\ &\subset \max(r, s)(V + V) \\ &\subset \max(r, s)U.\end{aligned}$$

Passing to the infimum on r and s, we get

$$\alpha_U(A + B) \leq \max(\alpha_V(A), \alpha_V(B)) \leq \alpha_V(A) + \alpha_V(B).$$

4. Indeed, one has $S_A(V) \subset S_A(U)$. Therefore $\inf S_A(U) \leq \inf S_A(V)$.

5. The inequality $\alpha_U \geq \alpha_{\overline{U}}$ is due to *4.* above. Assume, that for some $C \subset E$, $\alpha_U(C) > \alpha_{\overline{U}}(C)$ and choose r and s so that $\alpha_U(C) > s > r > \alpha_{\overline{U}}(C)$. Then there exist $r\overline{U}$-small sets $C_1, \dots, C_n$ covering C. But $C_i - C_i \subset r\overline{U} \subset sU$. Then the C_i's are also sU-small. Hence $\alpha_U(C) \leq s$, whereby $\alpha_U(C) = \alpha_{\overline{U}}(C)$.

6. Indeed, by (1), $\alpha_U(A) \leq \alpha_U(\overline{A})$. Assume that $\alpha_U(A) < \alpha_U(\overline{A})$ and choose $r > 0$, with $\alpha_U(A) < r < \alpha_U(\overline{A})$. Then there exist rU-small subsets $A_1, \dots, A_m$ of E, such that $A \subset \cup_{i=1}^m A_i$. Since $A_i - A_i \subset rU$, we get $\overline{A_i} - A_i \subset r\overline{U}$. Similarly, $\overline{A_i} - \overline{A_i} \subset r\overline{U}$. But $\overline{A} \subset \cup_{i=1}^m \overline{A_i}$. Then $\alpha_U(\overline{A}) \leq r$, a contradiction.

7. Again, if $\alpha_U(A) = +\infty$ or $\alpha_U(B) = +\infty$, the inequality is trivial. Now, assume $r > \alpha_U(A)$ and $s > \alpha_U(B)$. Then there are rU-small sets $(A_i)_{i=1,\dots,n}$ and sU-small ones $(B_j)_{j=1,\dots,m}$, such that $A \subset \cup_{i=1}^n A_i$ and $B \subset \cup_{J=1}^m B_j$. Then

$$\begin{aligned}(A_i + B_j) - (A_i + B_j) &= (A_i - A_i) + (B_j - B_j) \\ &\subset rU + sU \\ &\subset (r^p + s^p)^{\frac{1}{p}} U.\end{aligned}$$

It follows that : $\alpha_U(A + B) \leq \sqrt[p]{\alpha_U(A)^p + \alpha_U(B)^p}$.

8. Fix $\varepsilon > 0$, and assume first that $\alpha_U(B) = +\infty$, while $\alpha_U(A) < +\infty$. Choose then $W \in \mathcal{V}_0$, $r > \alpha_U(A)$, and $\varepsilon' > 0$ small enough so that $W + W \subset U$, $\sqrt[p]{r^p + \varepsilon'^p} < r + \varepsilon$, and A and B are $\varepsilon' W$-close. Then A is covered by finitely many rU-small sets $A_1, \dots, A_n$. For $i = 1, \dots, n$, set $B_i := B \cap (A_i + \varepsilon' W)$. Then the sets B_i, $i = 1 \dots, n$, constitute a covering of B. Moreover, for every $b_1, b_2 \in B_i$, there are $a_1, a_2 \in A_i$ such that $b_1 - a_1 \in \varepsilon' W$ and $b_2 - a_2 \in \varepsilon' W$. But then

$$\begin{aligned} b_1 - b_2 &= (b_1 - a_1) + (a_1 - a_2) + (a_2 - b_2) \\ &\in \varepsilon' W + \varepsilon' W + rU \\ &\in \varepsilon' U + rU \\ &\subset \sqrt[p]{\varepsilon'^p + r^p}\, U \\ &\subset (r + \varepsilon) U. \end{aligned}$$

It follows that B_i is $(r+\varepsilon)U$-small. Hence $\alpha_U(B) \le r + \varepsilon$, contradicting our assertion on B. Therefore $\alpha_U(B)$ and $\alpha_U(A)$ are simultaneously finite or infinite. Now, by the foregoing proof, whenever $\alpha_U(A) < r$, we get $\alpha_U(B) \le r + \varepsilon$. Therefore $\alpha_U(B) \le \alpha_U(A) + \varepsilon$. Similarly, $\alpha_U(A) \le \alpha_U(B) + \varepsilon$. Thus $|\alpha_U(A) - \alpha_U(B)| \le \varepsilon$ and α_U is uniformly continuous. □

Note that in the context of non locally convex spaces, there is no hope for non compactness measures to be invariant under the transition to the convex hull, at least for classical concrete non compactness measures. For example if for $0 < p < 1$ one defines the non compactness measure of a subset A of ℓ^p as in (1), the diameter $\delta(A)$ being relative to the distance defined by the p-norm of ℓ^p, then $\alpha(B_p) \le \sqrt[p]{2}$, while $\alpha(co(B_p)) = +\infty$, where B_p is the closed unit ball of ℓ^p. For $co(B_p)$ is the whole space ℓ^p.

Our following main result establishes the analogous of such a property in the general setting of locally p-convex spaces. As a first step, we show the property for the non compactness measure α_U, when U is taken to be p-convex.

Theorem 1 *If $U \in \mathcal{V}_0$ is p-convex for some $0 < p \le 1$, then $\alpha_U(co_s(A)) = \alpha_U(A)$ for every $A \subset E$ and every $0 < s \le p$.*

Proof Let $A \subset E$ and s be such that $0 < s \le p$. Since $A \subset co_s(A)$, we clearly have $\alpha_U(co_s(A)) \ge \alpha_U(A)$. Assume $\alpha_U(A) < \alpha_U(co_s(A))$ and choose $r > 0$, so that $\alpha_U(co_s(A)) > r > \alpha_U(A)$. Then there exist rU-small sets $A_1, \dots, A_n$, $n \ge 1$, such that $A \subset \cup_{i=1}^n A_i$. Then, for each i, $A_i \subset A_i + rU \subset co_s(A_i) + rU$. Since $co_s(A_i) + rU$ is s-convex, it follows that $co_s(A_i) \subset co_s(A_i) + rU$. Hence $co_s(A_i) - co_s(A_i) \subset rU$. We may then (and we will do) assume each A_i s-convex. Now, choose from each A_i some a_i. Then $A_i \subset a_i + rU$ and, since U is a neighborhood of 0, there exists $M > 0$, such that $a_i \in MU$ for every $i = 1, \dots, n$. Therefore

$$\cup_{i=1}^n A_i \subset MU + rU \subset \sqrt[s]{M^s + r^s}\, U.$$

If x belongs to $co_s(A)$, then there are $n \in \mathbb{N}$ and, for every $1 \le i \le n$, $\mu_i > 0$ and $x_i \in A_i$, such that $\sum_i \mu_i^s = 1$ and $x = \sum_{i=1}^n \mu_i x_i$. For arbitrary $\delta > 0$, since the set $P := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n, \lambda_i > 0 \text{ and } \sum_i \lambda_i^s = 1\}$ is precompact, there exist $m \in \mathbb{N}$ and, for every $j = 1, \dots, m$, $\mu^j := (\mu_1^j, \dots, \mu_n^j) \in P$, such that $P \subset \cup_{j=1}^m (\mu^j + \Delta(\delta))$, with $\Delta(\delta) := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \max_{i=1}^n |\lambda_i| < \delta\}$. Set, for every $j \in \{1, \dots, m\}$, $S^j := \mu_1^j A_1 + \mu_2^j A_2 + \dots + \mu_n^j A_n$. Then clearly $S^j - S^j \subset rU$, and there exists $j \in \{1, \dots, m\}$, such that $\max_{i=1}^n |\mu_i - \mu_i^j| < \delta$. Therefore, for $y := \sum_{i=1}^n \mu_i^j x_i \in S^j$, we have

$$\begin{aligned} x - y &= \sum_{i=1}^n \mu_i x_i - \sum_{i=1}^n \mu_i^j x_i \\ &= \sum_{i=1}^n (\mu_i - \mu_i^j) x_i \\ &\in \delta \sqrt[s]{n(M^s + r^s)}\, U. \end{aligned}$$

Choosing δ small enough, so that $\delta \sqrt[s]{n(M^s + r^s)} \le \varepsilon$, we conclude that $co_s(A)$ is covered by the sets $S^j + \varepsilon U$. But for every $a, b \in S^j$ and $y, z \in U$, we have $(a + \varepsilon y) - (b + \varepsilon z) = (a - b) + \varepsilon(x - y) \in rU + \varepsilon \sqrt[s]{2} U \subset r\sqrt[s]{1 + 2\frac{\varepsilon^s}{r^s}}\, U$. Since ε was arbitrary, $S^j + \varepsilon U$ is $r\overline{U}$-small. But $\alpha_U = \alpha_{\overline{U}}$, then $\alpha_U(co_s(A)) \le r$. This contradicts our assumption on r. □

3 Basic and sufficient collections of zero neighborhoods and relative non compactness measures

In order to define a new non compactness measure in E, we introduce the notions of basic and sufficient collections of zero neighborhoods in a topological vector space. To do this, let us introduce an equivalence relation on $\mathscr{V}_0$ by saying that U is related to V, written $U\mathscr{R}V$, if and only if there exist $r, s > 0$ such that $rU \subset V \subset sU$.

Definition 1 We say that $\mathscr{C} \subset \mathscr{V}_0$ is a basic collection of zero neighborhoods (BCZN in short), if it contains at most one representative member from each equivalence class with respect to $\mathscr{R}$. It will be said to be sufficient (SCZN in short), if it is basic and, for every $V \in \mathscr{V}_0$, there exists some $U \in \mathscr{C}$ and some $r > 0$ such that $rU \subset V$.

In the normed case, if f is a continuous functional on $E, U := \{x \in E : |f(x)| < 1\}$, and V is the open unit ball of E, then $\{U\}$ is basic but not sufficient, but $\{V\}$ is sufficient.

If (E, τ) is a locally convex space, whose topology is given by an upward directed family $\mathbb{P}$ of semi-norms, so that no two of them are equivalent, the collection $(B_P)_{P\in\mathbb{P}}$ is a SCZN, where B_P is the open unit ball of P.

Further, if $\mathscr{W}$ is a fundamental system of zero neighborhoods in a topological vector space, then there exists an SCZN $\mathscr{C}$ consisting of $\mathscr{W}$ members.

Following an idea of [14], a subset A of E is called uniformly bounded with respect to a sufficient collection $\mathscr{C}$ of zero neighborhoods, if there exists $r > 0$ such that $A \subset rV$, for every $V \in \mathscr{C}$. In the locally convex space $C_c(X) := C_c(X, \mathbb{K})$, the set $B_\infty := \{f \in C(X) : \|f\|_\infty \leq 1\}$ is uniformly bounded with respect to the SCZN $\{B_K, K \in \mathscr{K}\}$, where B_K is the (closed or) open unit ball of the semi-norm P_K.

Lemma 1 *If A is an arbitrary bounded set in E, then there exists a SCZN $\mathscr{C}$ with respect to which A is uniformly bounded.*

Proof Fix a SCZN $\mathscr{C}_0$. Since A is bounded, for every $V \in \mathscr{C}_0$, there exists $r_V > 0$ such that $A \subset r_V V$. Now, take $\mathscr{C} := \{r_V V, V \in \mathscr{C}_0\}$. Then $\mathscr{C}$ fulfills the required condition with $r = 1$. □

We are now in a position to introduce a non compactness measure in the topological vector space E.

Definition 2 Let $\mathscr{C}$ be a SCZN in E. For every $A \subset E$, we define the non compactness measure of A with respect to $\mathscr{C}$ as:

$$\alpha_{\mathscr{C}}(A) = \sup_{U \in \mathscr{C}} \alpha_U(A).$$

The semi-additivity of $\alpha_{\mathscr{C}}$, i.e. $\alpha_{\mathscr{C}}(A \cup B) = \max(\alpha_{\mathscr{C}}(A), \alpha_{\mathscr{C}}(B))$ (hence, $\alpha_{\mathscr{C}}$ is monotone in the sense that $A \subset B$ implies $\alpha_{\mathscr{C}}(A) \leq \alpha_{\mathscr{C}}(B)$) is readily derived from Proposition 1 (1). Let us show that $\alpha_{\mathscr{C}}$ is regular, i.e. $\alpha_{\mathscr{C}}(A) = 0$ if and only if A is a precompact subset of E. If $\alpha_{\mathscr{C}}(A) = 0$ and $U \in \mathscr{V}_0$ is arbitrary, since $\mathscr{C}$ is a SCZN, there exists $V \in \mathscr{C}$ and $r > 0$ such that $V \subset rU$. Therefore $\alpha_U(A) \leq r\alpha_V(A) = 0$. Hence A is covered by a finite number of U-small sets. Since U is arbitrary, A is precompact. Conversely, if A is precompact, then $\alpha_U(A) = 0$, for every $U \in \mathscr{V}_0$. In particular $\alpha_U(A) = 0$ for every $U \in \mathscr{C}$. Hence $\alpha_{\mathscr{C}}(A) = 0$.

Some properties are shared by all the $\alpha_{\mathscr{C}}$'s. Indeed, if $\mathscr{C}$ and $\mathscr{C}'$ are two SCZN, then, for every subset A of E, we have:

$$\alpha_{\mathscr{C}}(A) = 0 \text{ if and only if } \alpha'_{\mathscr{C}}(A) = 0.$$

This derives from the definition of a SCZN and Proposition 1 (2). However, there may exist a bounded set $A \subset E$ such that $\alpha_{\mathscr{C}}(A) < \alpha'_{\mathscr{C}}(A) = +\infty$, as the following example shows.

Example 1 Take in $E = C_c(\mathbb{R})$, the space $C(\mathbb{R})$ of all continuous functions from $\mathbb{R}$ into $\mathbb{R}$ endowed with the topology of uniform convergence on compact sets of $\mathbb{R}$, the collections $\mathscr{C} := (B_n)_n$ and $\mathscr{C}' := (\frac{1}{n}B_n)_n$, where $B_n := \{f \in C(\mathbb{R}), \|f\|_n := \sup_{|x| \leq n} |f(x)| \leq 1\}$, $n \in \mathbb{N}$. Then, for $A := \{f \in C(\mathbb{R}), |f(x)| \leq 1,\ x \in \mathbb{R}\}$, we have $\alpha_{\mathscr{C}}(A) = 2$, while $\alpha_{\mathscr{C}'}(A) = +\infty$.

Indeed, if $\alpha_{B_n}(A) < 2$ holds for some n, then also $\alpha_{B_n}(A') < 2$, where $A' := \{f_k : k \in \mathbb{N}\}$ and f_k being defined on $\mathbb{R}$ by:

$$f_k(x) = \begin{cases} -1 & : \ x \leq \frac{1}{k+1} \\ 2k(k+1)x - (2k+1) & : \ \frac{1}{k+1} < x \leq \frac{1}{k} \\ 1 & : \ \text{otherwise.} \end{cases}$$

Choose a real number r so that $\alpha_{B_n}(A') < r < 2$. Then there exist rB_n-small subsets $S_1, \dots, S_m$ of E covering A'. At least one of these sets, say S_i, contains infinitely many elements of A'. But, whenever $f_h, f_k \in S_i$ are such that $h < k$, one has

$$r \geq \left| f_k\left(\frac{1}{h+1}\right) - f_h\left(\frac{1}{h+1}\right) \right| = 2.$$

This contradicts $r < 2$. Hence $\alpha_{B_n}(A) \geq 2$ holds for every n. Since $A \subset B_n$ and B_n is convex, we get $\alpha_{B_n}(A) = 2$ for every n. This shows that $\alpha_{\mathscr{C}}(A) = 2$. But $\alpha_{\frac{1}{n}B_n} = n\alpha_{B_n}(A)$. We conclude that $\alpha_{\mathscr{C}'}(A) = +\infty$.

Similar properties as in Proposition 1 are obtained directly for the non compactness measure $\alpha_{\mathscr{C}}$:

Proposition 2 *Let $\mathscr{C}$ be a SCZN in E consisting of shrinkable sets and $A, B \subset E$. Then, the following hold:*

1. $\alpha_{\mathscr{C}}(A) = \alpha_{\mathscr{C}}(\overline{A})$.
2. $\alpha_{\mathscr{C}}(sA) = |s|\alpha_{\mathscr{C}}(A)$, for every scalar $s \neq 0$.
   Moreover, if $\mathscr{C}$ consists of p-convex sets for some $0 < p \leq 1$, then
3. $\alpha_{\mathscr{C}}(A + B) \leq \sqrt[p]{\alpha_{\mathscr{C}}(A)^p + \alpha_{\mathscr{C}}(B)^p}$.

Some properties of subsets of E can be characterized through the non compactness measure.

Proposition 3 1. *For every SCZN $\mathscr{C}$ in E, a subset A of E is bounded if and only if $\alpha_U(A) < +\infty$ for every $U \in \mathscr{C}$.*
2. *If a SCZN $\mathscr{C}$ in E consists of p-convex sets for some $0 < p \leq 1$ and if A is $\mathscr{C}$-uniformly bounded, then $\alpha_{\mathscr{C}}(A) < +\infty$.*
3. *If E is locally p-convex, then A is bounded if and only if there exists a SCZN $\mathscr{C}$ with $\alpha_{\mathscr{C}}(A) < +\infty$.*

Proof 1. If A is bounded, then its balanced hull B and also $C := B - B$ are bounded. Therefore, for every $U \in \mathscr{C}$ there is $r > 0$ such that $C \subset rU$. But $A \subset B$ and $B - B \subset rU$. Therefore $\alpha_U(A) \leq r < +\infty$. Conversely, for every $U \in \mathscr{V}_0$, there exists $V \in \mathscr{C}$ and $\rho > 0$ satisfying $V + V \subset \rho U$. But $\alpha_V(A) < +\infty$. Then, there exists $r > 0$ and rV-small subsets $A_1, \dots, A_m$ of E such that $A \subset \cup_{i=1}^m A_i$. Fix $a_i \in A_i$ arbitrarily. Then $A \subset \{a_i, i = 1, \dots m\} + rV$. But $\{a_i, i = 1, \dots m\}$ is bounded. Then there exists

$s > r$ so that $\{a_i, i = 1, \ldots n\} \subset sV$. It follows that $A \subset s(V + V) \subset s\rho U$. Hence A is bounded in E.

2. If A is $\mathscr{C}$-uniformly bounded, there exists $r > 0$ such that $A \subset rU$, $U \in \mathscr{C}$. Since $rU - rU \subset r\sqrt[p]{2}U$ for every $U \in \mathscr{C}$, $\alpha_U(A) \le r\sqrt[p]{2}$ for every $U \in \mathscr{C}$. Hence $\alpha_{\mathscr{C}}(A) \le r\sqrt[p]{2} < +\infty$.

3. Assume that E is locally p-convex and that A is bounded. Then by Lemma 1, there exists a SCZN $\mathscr{C}$ consisting of p-convex sets such that A is $\mathscr{C}$-uniformly bounded. Hence by (2), $\alpha_{\mathscr{C}}(A) < +\infty$. For the converse, if $\alpha_{\mathscr{C}}(A) < r$, for some $r > 0$, then by (1), A is bounded. □

We formulate our next main theorem stating the invariance of $\alpha_{\mathscr{C}}$ under the closed s-convex hull in a Hausdorff locally p-convex space E, $0 < s \le p$. This is a consequence of Theorem 1 and Proposition 1 (6). This shows that in the case $s = p = 1$, $\alpha_{\mathscr{C}}$ is a non compactness measure in the sense of Sadovskii [19] and Park [15]. This is

Theorem 2 *Let $\mathscr{C} \subset \mathscr{V}_0$ be a SCZN consisting of p-convex subsets in a Hausdorff topological vector space E. Then, $\alpha_{\mathscr{C}}(\overline{co}_s A) = \alpha_{\mathscr{C}}(A)$ for every $A \subset E$ and every $0 < s \le p$.*

We conclude this section by the following proposition which is an extension of the well-known Cantor type intersection property of non compactness measures in the setting of metric spaces. The proof in this particular setting was given by Kuratowski [12].

Proposition 4 *Let E be a Hausdorff topological vector space, $\mathscr{C}$ a SCZN in E consisting of shrinkable sets, and $(A_n)_{n\ge 0}$ a decreasing sequence of non empty sets in E such that A_{n_0} is complete for some n_0, and $\lim_{n\to\infty} \alpha_U(A_n) = 0$ for every $U \in \mathscr{C}$. Then $\cap_{n=0}^{\infty} \overline{A_n}$ is non empty and compact.*

Proof With no loss of generality, we may assume that A_0 is complete. Let $\varepsilon > 0$ and $U \in \mathscr{C}$. Since $\alpha_U(A_n) \to 0$, there exists $N \in \mathbb{N}$ such that $\alpha_U(A_n) < \varepsilon$ for every $n \ge N$. Choose a sequence $(x_n) \subset E$ such that $x_n \in A_n$ for each n. Since $\alpha_U(F) = 0$ for every finite $F \subset E$, it follows, for every $n \ge N$:

$$\begin{aligned} \alpha_U\big((x_n)\big) &= \alpha_U(\{x_n : n \ge N\}) \\ &\le \alpha_U(A_N) < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ and $U \in \mathscr{C}$ were arbitrary, $\alpha_{\mathscr{C}}\big(\{x_n, n \ge 0\}\big) = 0$, that is, $\{x_n, n \ge 0\}$ is precompact, then relatively compact. Therefore $(x_n)_n$ admits a cluster point $x \in E$. But A_k contains all the x_n's but a finite number. Then x belongs to the closure of A_k for each k. Therefore $\cap_{n=0}^{\infty} \overline{A_n}$ is non empty. Now, the compactness of $\cap_{n=0}^{\infty} \overline{A_n}$ follows easily from the monotonicity, the closure invariance of $\alpha_{\mathscr{C}}$, and the fact that $\alpha_U\big(A_n\big) \to 0$ for every $U \in \mathscr{C}$. □

4 Application: Schauder, Darbo and Sadovskii-type fixed point results in locally *p*-convex spaces

In this section, we will use our results on the introduced non compactness measure to establish the versions of the well known Darbo and Sadovskii fixed point theorems for the so-called Yanyan continuous maps in the setting of locally p-convex spaces. Similarly as with the normed case, the key idea is to come back to a compact set and to apply Schauder fixed point theorem. So, we need for this to establish a p-convex version of Schauder fixed point theorem in our setting of locally p-convex spaces (Theorem 4). This will present an extension of the following recent version established in the p-normed spaces setting.

Throughout this section, C will be a non empty subset of a Hausdorff topological vector space E.

Theorem 3 ([21, Theorem 2.13]) *Let $(E, \|\ \|_p)$ be a complete p-normed space and C be a compact s-convex subset of E, where $0 < s \leq p$. Then, every continuous map $T : C \to C$ has a fixed point.*

Let us say that, in a Hausdorff locally p-convex space (E, τ), a family $\mathbb{P}$ of continuous p-semi-norms on E is sufficient, if the collection $\{U_P, P \in \mathbb{P}\}$ is a SCZN, where U_P denotes the open unit ball of P.

A map $T : C \to C$ will be said to be lipschitzian, if there exists a sufficient family $\mathbb{P}$ of p-semi-norms on E such that:

$$\forall P \in \mathbb{P}, \quad \exists L_P > 0, \quad P(Tx - Ty) \leq L_P P(x - y), \quad x, y \in C.$$

If $L_P < 1$ for every $P \in \mathbb{P}$, then T is called a contraction on C. If there exists a sufficient family $\mathbb{P}$ of p-semi-norms on E, such that, for every $P \in \mathbb{P}$, T is continuous from (C, τ_P) into (C, τ_P), where τ_P is the topology induced on C by the single p-semi-norm P, we say that T is Yanyan-continuous. It is clear that a contraction is lipschitzian, a lipschitzian map is Yanyan-continuous, and that a Yanyan-continuous map is continuous. Moreover, in a normed space, every continuous map is Yanyan-continuous, but need not be lipschitzian. Now, if we consider again the topological linear space $C_c(\mathbb{R})$ of Example 1, then the mapping T defined for every $f \in C(\mathbb{R})$ by $T(f) := f^2$ is Yanyan-continuous, but not lipschitzian. Indeed, the family $(\|\ \|_n)_n$ of semi-norms defined in Example 1 is sufficient and satisfies

$$\|T(f) - T(g))\|_n \leq \|f + g\|_n \|f - g\|_n, \quad f, g \in C(\mathbb{R}).$$

Restricting ourselves to the open ball $B_n(g, r) := \{f \in C(\mathbb{R}) : \|f - g\|_n < r\}$, for any $r > 0$, we get:

$$\|T(f) - T(g)\|_n \leq (\|g\|_n + r)\|f - g\|_n,$$

whence the Yanyan-continuity of T at g and then on $C(\mathbb{R})$ since g is arbitrary. Now, if T were lipschitzian with respect to some sufficient family $\mathbb{P}$ of semi-norms defining the topology of $C(\mathbb{R})$, then for every $P \in \mathbb{P}$, there would exist $M_P > 0$, such that

$$P(T(f) - T(g)) \le M_P P(f - g), \quad f, g \in C(\mathbb{R}).$$

But for every $n \ge 1$, there exist $P_n \in \mathbb{P}$ and a positive number $R(n)$ such that

$$\|f\|_n \le R(n) P_n(f), \quad f \in C(\mathbb{R}).$$

Choose $g,f \in C(\mathbb{R})$, $n \ge 1$ with $\|g\|_n \ne 0$, and $f = \lambda g$, $\lambda \ne 1$. Then, we have

$$|\lambda^2 - 1| \|g^2\|_n = \|T(f) - T(g)\|_n \le R(n) P_n(T(f) - T(g)) \le R(n) M_{P_n} |\lambda - 1| P_n(g).$$

As $\|g^2\|_n = \|g\|_n^2 \ne 0$, this means that

$$|\lambda + 1| \le \frac{R(n) M_{P_n}}{\|g\|_n^2} P_n(g).$$

Since $\lambda \ne 1$ is arbitrary, this is impossible.

Notice at this point that, for every p-semi-norm P on a topological vector space E, the set $\ker P := \{x \in E : P(x) = 0\}$ is a (closed if P is continuous) vector subspace of E.

Our next main result is a Schauder-type fixed point theorem for Yanyan-continuous maps in locally p-convex spaces.

Theorem 4 *Let (E, τ) be a Hausdorff locally p-convex space and $C \subset E$ be a compact and s-convex subset of E, with $0 < s \le p$. Then, every Yanyan-continuous map $T : C \to C$ has a fixed point.*

For the proof, we need the following lemma.

Lemma 2 *Let P be a continuous p-semi-norm on a Hausdorff topological vector space (E, τ), C a compact s-convex subset of E with $0 < s \le p$, and $T : C \to C$ a mapping. If T is P-continuous (i.e., $\forall \varepsilon > 0$, $\exists \eta > 0$, such that $P(Tx - Ty) \le \varepsilon$ for every $x, y \in C$ with $P(x - y) < \eta$), then there exists $x_P \in C$, such that $x_P - Tx_P \in \ker P$.*

Proof It is easily shown that the quotient space $E/\ker P$, endowed with the p-norm $\|\pi(x)\|_p := P(x)$, is a p-normed space, where $\pi : E \to E/\ker P$ stands for the canonical surjection. Thinking of $\pi(C)$ as a subset of the completion $\widehat{E/\ker P}$ of $E/\ker P$, we may assume that $E/\ker P$ is a complete p-normed space. For every $x \in C$, put $\overline{T}(\pi(x)) = \pi(Tx)$. It follows from the P-continuity of T that $\overline{T}$ is a continuous self map on $\pi(C)$. By continuity of P, π is continuous. Therefore $\pi(C)$ is compact. But $\pi(C)$ is also s-convex. Hence, by Theorem 3, there exists $x_P \in C$ such that $\overline{T}(\pi(x_P)) = \pi(x_P)$, that is, $x_P - Tx_P \in \ker P$ as desired. □

Proof of Theorem 4 Let $\mathbb{P}$ be a sufficient family of p-semi-norms on E with respect to which T is Yanyan-continuous on C. By Lemma 2, there exists a family $(x_P)_{P \in \mathbb{P}} \subset C$ with $x_P - Tx_P \in \ker P$ for every $P \in \mathbb{P}$. Denote by A_P, for $P \in \mathbb{P}$, the closure of the set

$$C_P := \{x_Q, Q \in \mathbb{P}, \text{ and } P \leq cQ \text{ for some } c > 0\}.$$

Since $\mathbb{P}$ is sufficient, the collection $(A_P)_{P\in\mathbb{P}}$ satisfies the finite intersection property. By compactness of C, the set $A := \cap\{A_P, P \in \mathbb{P}\}$ contains at least an element $x \in C$. We claim that $Tx = x$. Otherwise, there would exist $P \in \mathbb{P}$ such that $P(Tx - x) > 0$. Choose η so that, $0 < \eta < \frac{P(Tx-x)}{2}$ and whenever $x, y \in C$ with $P(x - y) \leq \eta$, we have $P(Tx - Ty) < \frac{P(Tx-x)}{2}$. Now, choose $Q \in \mathbb{P}$, so that $P \leq cQ$, for some $c > 0$ and $P(x - x_Q) < \eta$. Then

$$\begin{aligned} P(Tx - x) &\leq P(Tx - Tx_Q) + P(Tx_Q - x_Q) + P(x_Q - x) \\ &< \frac{P(x - Tx)}{2} + cQ(Tx_Q - x_Q) + \frac{P(x - Tx)}{2} \\ &= P(Tx - x). \end{aligned}$$

This is absurd. □

Darbo [4] and Sadovskii [18] introduced the notions of set-contractions and condensing maps. They established their famous fixed point theorems in the setting of Banach spaces. Following them, if E is a Hausdorff locally p-convex space, we will say that a map $T : C \to C$ is a set-contraction (resp. condensing), if there is some SCZN $\mathscr{C}$ in E consisting of p-convex sets, such that:

$$\forall U \in \mathscr{C}, \exists 0 < k_U < 1, \quad \alpha_U(T(A)) \leq k_U \alpha_U(A), \quad A \subset C,$$

(resp. $\forall U \in \mathscr{C}, \quad \alpha_U(T(A)) < \alpha_U(A), \quad A \subset C \quad \text{with} \quad \alpha_U(A) > 0$).

It is clear that a contraction on C is a set-contraction and a set-contraction on C is condensing.

Now, our next result is a Sadovskii-type fixed point theorem in the setting of locally p-convex spaces.

Theorem 5 *Let $C \subset E$ be a complete s-convex subset of a Hausdorff locally p-convex space E, with $0 < s \leq p$. If $T : C \to C$ is Yanyan-continuous and condensing, then T has a fixed point.*

Proof Let $\mathscr{C}$ be a sufficient collection of p-convex zero neighborhoods in E with respect to which T is condensing and fix $U \in \mathscr{C}$. Choose some $x_0 \in C$ and let $\mathscr{F}$ be the family of all closed s-convex subsets A of C with $x_0 \in A$ and $T(A) \subset A$. Note that $\mathscr{F}$ is not empty since $C \in \mathscr{F}$. Let $A_0 = \cap_{A\in\mathscr{F}} A$. Then A_0 is a non empty closed s-convex subset of C, such that $T(A_0) \subset A_0$. We shall show that A_0 is compact. Let $A_1 = \overline{co}_s(T(A_0) \cup \{x_0\})$. Since $T(A_0) \subset A_0$ and A_0 is closed and s-convex, $A_1 \subset A_0$. Hence, $T(A_1) \subset T(A_0) \subset A_1$. It follows that $A_1 \in \mathscr{F}$ and therefore $A_1 = A_0$. By Proposition 1 and Theorem 1, we get $\alpha_U(T(A_0)) = \alpha_U(A_0)$. Our assumption on T shows that $\alpha_U(A_0) = 0$. Since U was arbitrary, A_0 is compact as desired. Now, the conclusion follows from Theorem 4 applied to $T : A_0 \to A_0$. □

Corollary 1 (*Darbo-type fixed point theorem*) *Let $C \subset E$ be a complete s-convex subset of a Hausdorff locally p-convex space E, with $0 < s \leq p$. If $T : C \to C$ is Yanyan continuous and a set-contraction, then T has a fixed point.*

Notice that, if T is a contraction on C then it is both Yanyan continuous and a set-contraction. In this case, the preceding corollary can be sharpened, with a standard proof as in Theorem 2.2 of [3], in the following way:

Theorem 6 (The contraction principle) *Let C be a sequentially complete subset of a Hausdorff locally p-convex space E. If $T : C \to C$ is a contraction on C , then T has a unique fixed point x^* in C, and the iterative sequence $(T^n x)$ converges to x^* for every $x \in C$.*

The following two recent results are now particular cases of Theorem 5 and Corollary 1. The cases $s = p = 1$ are the standard Darbo and Sadovskii fixed point theorems. Note that, for a p-normed space E (which is a metric space), α stands for the standard Kuratowski non compactness measure in E.

Corollary 2 (Sadovskii-type, [20, Theorem 4.3]) *Let $(E, \|\cdot\|_p)$ be a complete p-normed space and C be a bounded, closed and s-convex subset of E, where $0 < s \leq p$. Then, every continuous and α-condensing map $T : C \to C$ has a fixed point.*

Proof Take in Theorem 5 $\mathscr{C}\{B_p(0,1)\}$, where $B_p(0,1)$ stands for the closed unit ball of E, and note that it can be easily shown that $\alpha(A) = \left(\alpha_{\mathscr{C}}(A)\right)^p$ for every $A \subset E$, and that T satisfies the conditions of Theorem 5. □

Corollary 3 (*Darbo-type*, [20, *Theorem* 4.1]) *Let $(E, \|\cdot\|_p)$ be a complete p-normed space and C be a bounded, closed and s-convex subset of E, where $0 < s \leq p$. Then, every map $T : C \to C$ which is continuous and a set-contraction has a fixed point.*

We conclude this section by the following open question to which it is alluded in the introduction, on the p-convex versions of the well-known Schauder, Darbo and Sadovskii fixed point theorems in the setting of locally p-convex spaces.

Question : Can the Yanyan-continuity condition in Theorems 4, 5 and Corollary 1 be weakened to continuity ?

5 Noncompactness measure in *C*(*X*, *E*)

In this section, X will be a Hausdorff completely regular topological space, $\mathscr{B}$ will be the von Neumann bornology of E, i.e., the family of all bounded subsets of E, and $\mathscr{K}$ an upward directed collection of compact subsets of X covering the whole X. The smallest such a family is $\mathscr{S} := \{F \subset X \text{ finite}\}$ and the largest is $\mathscr{G} := \{K \subset X \text{ compact}\}$. The linear space of all continuous functions from X into E will be denoted by $C(X, E)$. It will be endowed with the topology $\tau_{\mathscr{K}}$ of uniform convergence on the elements of $\mathscr{K}$. A fundamental system of zero neighborhoods for $\tau_{\mathscr{K}}$ is given by the sets:

$$N(K,U) := \{f \in C(X,E) : f(K) \subset U\}, \quad K \in \mathscr{K}, \quad U \in \mathscr{N}_0.$$

The topological vector space obtained by endowing $C(X, E)$ with the topology $\tau_{\mathscr{K}}$ will be denoted by $C_{\mathscr{K}}(X,E)$. In case $\mathscr{K} = \mathscr{S}$ (resp. $\mathscr{K} = \mathscr{G}$), we will rather write $C_s(X,E)$ (resp. $C_c(X,E)$).

Recall that the topology of X is generated by the uniformity induced on X by $C(X, [0, 1])$, the space of all continuous functions from X into $[0, 1]$. In other terms, the topology of X is nothing but the initial topology associated to $C(X, [0, 1])$.

Now, let us associate to any collection $\mathscr{C}$ of zero neighborhoods in E the collection

$$\widehat{\mathscr{C}} := \{N(K,U),\ K \in \mathscr{K},\ U \in \mathscr{C}\}$$

of zero neighborhoods in $C_{\mathscr{K}}(X,E)$.

Lemma 3 *If $\mathscr{C}$ is a basic (resp. sufficient) collection of zero neighborhoods in E, then so is also the collection $\widehat{\mathscr{C}}$ in $C_{\mathscr{K}}(X,E)$. Moreover, if U is a closed shrinkable zero neighborhood, then $N(K, U)$ is also closed and shrinkable in $C_{\mathscr{K}}(X,E)$.*

Proof Suppose that, for some $K, K' \in \mathscr{K}$ and some zero neighborhoods U and U', there exist $r, s > 0$ such that $rN(K,U) \subset N(K',U') \subset sN(K,U)$. Then $K = K'$ and $U\mathscr{R}U'$, therefore, since $\mathscr{C}$ is basic $U = U'$. Indeed, if some $x \in K$ exists with $x \notin K'$, then there is a continuous function g from X into $[0, 1]$ such that $g(x) = 1$ and g is identically 0 on K'. Let $a \in E$, with $a \notin sU$. Then the function $f := g \otimes a$ defined by $f(y) = g(y)a$ belongs to $N(K',U')$, but $f(x) = a \notin sU$. Hence $K = K'$. Now, assume that there exists $a \in U$, with $ra \notin U'$ and denote by g the constant function $g(y) = a$. Then $g \in N(K,U)$, but $rg(K') = ra \notin U'$. This contradicts our assumption. Similarly, we show that $U' \subset sU$. Hence $U = U'$, since $\mathscr{C}$ is basic.

Now, assume $\mathscr{C}$ is sufficient. Then, by the foregoing proof, $\widehat{\mathscr{C}}$ is basic. Moreover, let W be a zero neighborhood in $C_{\mathscr{K}}(X,E)$. Then there exist a compact set $K \in \mathscr{K}$ and a zero neighborhood V in E such that $N(K,V) \subset W$. But there exist $r > 0$ and $U \in \mathscr{C}$, such that $rU \subset V$. This leads to $rN(K,U) \subset W$.

Assume now that $U \in \mathscr{V}_0$ is closed and shrinkable, and $K \in \mathscr{K}$. Since the evaluation $\delta_x : f \mapsto f(x)$ is continuous from $C_{\mathscr{K}}(X,E)$ into E, $N(K,U) = \cap\{\delta_x^{-1}(U), \quad x \in K\}$ is a closed zero neighborhood in $C_{\mathscr{K}}(X,E)$. Furthermore, for arbitrary $1 > r > 0$, we have $r\overline{U} = rU \subset \mathring{U}$. Therefore, $r\overline{N(K,U)} = rN(K,U) = N(K,rU) \subset N(K,\mathring{U}) \subset N(K,\mathring{U})$. Hence $N(K, U)$ is shrinkable. □

In the following, we let $\mathscr{C}$ denote a SCZN in E. With no loss of generality, since every topological vector space admits a fundamental system of zero neighborhoods consisting of closed shrinkable and balanced sets, we may assume that $\mathscr{C}$ consists also of such sets. Therefore, by Lemma 3, the associated collection $\widehat{\mathscr{C}}$ consists also of closed shrinkable and balanced sets.

Let $\widehat{\alpha}_{\mathscr{C}}$ be the non compactness measure in the space $C_{\mathscr{K}}(X,E)$ defined by the SCZN $\widehat{\mathscr{C}}$. This is

$$\widehat{\alpha}_{\mathscr{C}}(D) = \sup_{\substack{K \in \mathscr{K} \\ U \in \mathscr{C}}} \widehat{\alpha}_{N(K,U)}(D),$$

where $\widehat{\alpha}_{N(K,U)}$ stands for the $N(K,U)$-measure of non compactness in the space $C_{\mathscr{K}}(X,E)$.

Next, we address the question whether the equality

$$\widehat{\alpha}_{\mathscr{C}}(D) = \sup_{K\in\mathscr{K}} \sup_{x\in K} \alpha_{\mathscr{C}}(D(x))$$

holds for a subset D of $C(X, E)$. Here $D(x) := \{f(x) : f \in D\}$.

Definition 3 A subset $D \subset C(X,E)$ is said to be equicontinuous on a subset Y of X, if it is equicontinuous at each point of Y. It is said to be uniformly equicontinuous on Y, if for every $U \in \mathscr{N}_0$, there exist $\mu > 0$ and finitely many functions $g_1, ..., g_n \in C(X, [0, 1])$ such that, whenever $\max_{i=1}^{n} |g_i(x) - g_i(y)| < \mu$, with $x, y \in Y$, we have $f(x) - f(y) \in U$ for every $f \in D$.

Proposition 5 *If $D \subset C(X,E)$ is equicontinuous on a compact $K \subset X$, then D is uniformly equicontinuous on K.*

Proof Let V be an arbitrary zero neighborhood in E and choose $U \in \mathscr{N}_0$ such that $U + U \subset V$. Since D is equicontinuous at every point of K, for every $t \in K$, there exists an open neighborhood Ω_t of t in X such that:

$$f(t) - f(s) \in U, \quad f \in D, \quad s \in \Omega_t. \tag{6}$$

By the complete regularity of X, there exists a function $g_t \in C(X, [0, 1])$ such that $g_t(t) = 1$, and g_t vanishes identically outside of Ω_t. By compactness of K, there are $n \in \mathbb{N}$ and $t_1, t_2, \dots, t_n \in K$, so that $K \subset \cup_{i=1}^{n} \{g_{t_i} > \frac{3}{4}\}$. Put $\eta = \frac{1}{4}$ and $g_i = g_{t_i}$. If $x, y \in K$ satisfy $\max(|g_i(x) - g_i(y)|,\ i = 1, \dots, n) < \eta$, then there is $i_0 \in \{1, \dots, n\}$, such that $g_{i_0}(x) > \frac{3}{4}$. Since $|g_{i_0}(x) - g_{i_0}(y)| < \frac{1}{4}$, it follows $g_{i_0}(y) > \frac{1}{2}$, so that both x and y belong to $\Omega_{t_{i_0}}$. Therefore, using (6), for every $f \in D$ we get

$$f(x) - f(y) = f(x) - f(t_{i_0}) + f(t_{i_0}) - f(y) \in U + U \subset V.$$

Hence, since x, y were arbitrary in K, D is uniformly equicontinuous on K. □

The following corollary is an immediate consequence of Proposition 5.

Corollary 4 *If a subset D of $C(X, E)$ is equicontinuous on X, then it is uniformly equicontinuous on every compact $K \subset X$.*

Now, we are in a position to establish an alternative of Ambrosetti theorem in the general setting of topological vector spaces. This is our next main result:

Theorem 7 *Let D be a subset of C(X, E). If D is equicontinuous on X, then the following equality holds*:

$$\widehat{\alpha}_{\mathscr{C}}(D) = \sup_{K\in\mathscr{K}} \sup_{x\in K} \alpha_{\mathscr{C}}(D(x)).$$

In order to prove this result, we first give the following lemma:

Lemma 4 *Let K be a compact subset of X, U a closed zero neighborhood in E, and D a subset of C(X, E). If D is equicontinuous on K, then the following equality holds*:

$$\widehat{\alpha}_{N(K,U)}(D) = \sup\{\alpha_U(D(x)), \quad x \in K\}.$$

Proof If $\widehat{\alpha}_{N(K,U)}(D) = +\infty$, then obviously $\widehat{\alpha}_{N(K,U)}(D) \geq \sup\{\alpha_U(D(x)),\ x \in K\}$. Now, assume $\widehat{\alpha}_{N(K,U)}(D) < +\infty$ and consider an arbitrary $\eta > \widehat{\alpha}_{N(K,U)}(D)$. Then, there exist $\eta N(K, U)$-small subsets $G_1, \ldots, G_m$ of $C(X, E)$ such that $D \subset \cup_{i=1}^m G_i$. Therefore, for $x \in K$, $D(x) \subset \cup_{i=1}^m G_i(x)$ and $G_i(x) - G_i(x) \subset \eta U$. Hence $\alpha_U(D(x)) \leq \eta$. It follows that $\alpha_U(D(x)) \leq \widehat{\alpha}_{N(K,U)}(D)$, for every $x \in K$.

For the converse, it is clear that if $\sup\{\alpha_U(D(x)), x \in K\} = +\infty$, then $\widehat{\alpha}_{N(K,U)}(D) = \sup\{\alpha_U((D(x)), x \in K\}$. Assume now that $\sup\{\alpha_U(D(x)), x \in K\} < +\infty$ and consider arbitrary $\eta > \sup\{\alpha_U(D(x)),\ \ x \in K\}$ and zero neighborhoods $V, W \in \mathscr{N}_0$, such that $V + V \subset W$. By Proposition 5, D is uniformly equicontinuous on K. Therefore there exist $\mu > 0$ and a finite subset F of $C(X, [0, 1])$, such that:

$$(\forall x, y \in K), (\max_{g\in F} |g(x) - g(y)| < \mu) \Longrightarrow (f(x) - f(y) \in V, f \in D). \tag{7}$$

Now, for arbitrary $x \in K$, put $K_x := \{t \in X : |g(x) - g(t)| < \mu, g \in F\}$. This is an open subset of X containing x. Then the collection $(K_x)_{x\in K}$ is an open covering of K. By compactness, there are $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in K$, such that $K \subset \cup_{i=1}^n K_i$, with $K_i := K_{x_i}$. By the very definition of K_i and (7), we have

$$\big(x \in K_i\big) \Longrightarrow \big(f(x) - f(x_i) \in V, \quad f \in D\big). \tag{8}$$

Now, for every $x \in K$, there is some $i \in \{1, \ldots, n\}$ such that $x \in K_i$. Therefore, due to (8), $D(x) \subset D(x_i) + V$. It follows that $D(K) \subset \cup_{i=1}^n D(x_i) + V$. As $\eta > \alpha_U(D(x_i))$, $i = 1 \ldots, n$, there exist, $m \in \mathbb{N}$ and ηU-small subsets $E_1, \ldots, E_m$ of E, such that $\cup_{i=1}^n D(x_i) \subset \cup_{j=1}^m E_j$. Let M be the set of all mappings from the set $\{1, 2, \ldots, n\}$ into the set $\{1, 2, \ldots, m\}$. Then M is a finite set and, clearly, $D \subset \cup_{\mu\in M} D_\mu$, where $D_\mu := \{f \in D : f(x_i) \in E_{\mu(i)}, i = 1, \ldots, n\}$. Moreover, the sets D_μ are $\eta N(K, U)$-small. Indeed, if $\mu \in M$, $f, g \in D_\mu$ and $x \in K$ are given, then there exists $i \in \{1, \ldots, n\}$ such that $x \in K_i$. By (8), $f(x) - f(x_i) \in V$ and $g(x) - g(x_i) \in V$. Since $E_{\mu(i)} - E_{\mu(i)} \subset \eta U$, we get $f(x_i) - g(x_i) \in \eta U$. It follows that:

$$f(x) - g(x) = f(x) - f(x_i) + f(x_i) - g(x_i) + g(x_i) \\ - g(x) \in V + \eta U + V \subset W + \eta U.$$

Since this holds for $x \in K$ and every $W \in \mathcal{N}_0$, we get $f(x) - g(x) \in \overline{\eta U} = \eta \overline{U} = \eta U$, for U is closed. Hence $D_{\mu(i)} - D_{\mu(i)} \subset N(K, \eta U) = \eta N(K, U)$, saying that D_μ is $\eta N(K, U)$-small. It follows that $\eta \geq \widehat{\alpha}_{N(K,U)}(D)$, achieving the proof. □

Proof of Theorem 7 By definition, $\widehat{\alpha}_{\mathscr{C}}(D) = \sup_{\substack{K \in \mathscr{K} \\ U \in \mathscr{C}}} \widehat{\alpha}_{N(K,U)}(D)$. Since D is equicontinuous on X, it is uniformly equicontinuous on each compact $K \subset X$. Then, Lemma 4 yields

$$\widehat{\alpha}_{N(K,U)}(D) = \sup\{\alpha_U(D(x)), \quad x \in K\}.$$

Then

$$\widehat{\alpha}_{\mathscr{C}}(D) = \sup_{\substack{K \in \mathscr{K} \\ U \in \mathscr{C}}} \sup_{x \in K} \alpha_U(D(x)) \\ = \sup_{K \in \mathscr{K}} \sup_{x \in K} \alpha_{\mathscr{C}}(D(x)),$$

as claimed. □

Since, in any topological vector space, a subset is precompact if and only its non compactness measure is zero, a first immediate corollary of Theorem 7 is the following:

Corollary 5 *An equicontinuous subset D of $C(X, E)$ is precompact in $C_{\mathscr{K}}(X, E)$ if and only if $D(x)$ is precompact in E, for every $x \in X$.*

Notice here that, according to Corollary 5, an equicontinuous subset D of $C(X, E)$ is precompact in $C_s(X, E)$ if and only if it is so in $C_c(X, E)$.

Actually, Corollary 5 can be improved, whenever X is a $k_{\mathscr{K}}$-space and E is a topological vector space. At this point, recall that the completely regular space X is said to be a $k_{\mathscr{K}}$-space, if a function f defined on X, with values in $\mathbb{R}$ (or equivalently in any completely regular space), is continuous, provided its restriction to any $K \in \mathscr{K}$ is relatively continuous. Whenever $\mathscr{K} = \mathscr{G}$, we get a so-called $k_{\mathbb{R}}$-space (see [11] for further details on such spaces).

Let us denote by $\mathscr{L}_{pc}(C_{\mathscr{K}}(X, E), E)$ the topological vector space $\mathscr{L}(C_{\mathscr{K}}(X, E), E)$ of all linear continuous mappings defined on $C(X, E)$ with values in E, endowed with the topology τ_{pc} of uniform convergence on the precompact subsets of $C_{\mathscr{K}}(X, E)$. A fundamental system of zero neighborhoods for τ_{pc} is given by the sets of the form

$$N(D, U) := \{T \in \mathscr{L}(C_{\mathscr{K}}(X, E), E) \,:\, T(D) \subset U\},$$

D running over the set of precompact subsets of $C_{\mathscr{K}}(X,E)$ and U over $\mathscr{N}_0$. By Δ we will mean the evaluation map defined from X into $\mathscr{L}(C_{\mathscr{K}}(X,E),E)$ by $\Delta(x)=\delta_x$.

The following lemma can be deduced from Proposition 16 of [13] (see also Lemma 2.1 of [9], and [17] in the case E is locally convex). For the sake of completeness, we include a proof of it.

Lemma 5 *Let X be a Hausdorff completely regular space and E a Hausdorff topological vector space. Then the evaluation map $\Delta : X \to \mathscr{L}_{pc}(C_{\mathscr{K}}(X,E),E)$ is continuous if and only if every precompact subset of $C_{\mathscr{K}}(X,E)$ is equicontinuous.*

In particular, if X is a $k_{\mathscr{K}}$-space, every precompact subset of $C_{\mathscr{K}}(X,E)$ is equicontinuous on X.

Proof Necessity: Let D be a $\tau_{\mathscr{K}}$-precompact subset of $C(X,E)$, x_0 an element of X, and U a zero neighborhood in E. Since Δ is continuous at x_0, there exists a neighborhood Ω_0 of x_0, such that $\Delta(x)-\Delta(x_0)\in N(D,U)$, $x\in\Omega_0$. This means that $f(x)-f(x_0)\in U$ for every $f\in D$ and $x\in\Omega_0$, showing that D is equicontinuous at x_0. Since x_0 is arbitrary, D is equicontinuous on X.

Sufficiency: Choose arbitrarily $x_0\in X$, $U\in\mathscr{N}_0$, and a $\tau_{\mathscr{K}}$-precompact set $D\subset C(X,E)$. By assumption D is equicontinuous. Therefore there exists a neighborhood Ω_0 of x_0, such that $f(x)-f(x_0)\in U$, for every $f\in D$ and $x\in\Omega_0$. This is $\Delta(x)-\Delta(x_0)\in N(D,U)$, showing that Δ is continuous at x_0.

Now, choose arbitrary $K\in\mathscr{K}$, $x_0\in K$, and consider $N(D,U)$, a zero neighborhood in $\mathscr{L}_{pc}(C_{\mathscr{K}}(X,E),E)$, where D is a precompact set in $C_{\mathscr{K}}(X,E)$ and $U\in\mathscr{V}_0$. Consider $V\in\mathscr{V}_0$, such that $V+V+V\subset U$. Then, for the zero neighborhood $N(K,V)$ in $C_{\mathscr{K}}(X,E)$, there exist, $m\in\mathbb{N}$ and $f_1,\dots,f_m\in D$, such that $D\subset\bigcup_{1\le j\le m}(f_j+N(K,V))$. This means:

$$\forall f\in D,\ \exists j\in\{1,\dots,m\}:f(x)-f_j(x)\in V,\quad x\in K. \tag{9}$$

Since $\{f_1,\dots,f_m\}$ is equicontinuous, there exists a neighborhood Ω_0 of x_0, such that:

$$f_j(x)-f_j(x_0)\in V,\quad j=1,\dots,m,\quad x\in\Omega_0. \tag{10}$$

It derives from (9) and (10) that, for every $f\in D$ there is some $j\in\{1,\dots,m\}$ such that for every $x\in\Omega_0\cap K$, we have:

$$\begin{aligned}\Delta(x)(f)-\Delta(x_0)(f) &= f(x)-f(x_0)\\ &= f(x)-f_j(x)+f_j(x)-f_j(x_0)+f_j(x_0)-f(x_0)\\ &\in V+V+V\subset U.\end{aligned}$$

This shows that the restriction to K of Δ is continuous at x_0 then on K, since x_0 was arbitrary. As X is a $k_{\mathscr{K}}$-space, Δ is continuous on X. We conclude by the first part of the lemma. □

Corollary 6 *Let X be a Hausdorff completely regular $k_{\mathscr{K}}$-space, E a Hausdorff topological vector space, and D a subset of $C(X, E)$. Then D is precompact in $C_{\mathscr{K}}(X, E)$ if and only if it is equicontinuous on X and $D(x)$ is precompact in E for every $x \in X$.*

Proof Since D is precompact in $C_{\mathscr{K}}(X, E)$, it is equicontinuous on X by Lemma 5. Hence, by Theorem 7, $\alpha_{\mathscr{C}}(D(x)) = 0$, for every $x \in X$. Therefore $D(x)$ is precompact for every $x \in X$, whence the necessity. The sufficiency derives immediately again from Theorem 7. □

References

1. Ambrosetti, A.: Un teorema di esistenza per le equazioni differenziali negli spazi di Banach. Rend. Sem. Mat. Univ. Padova. **39**, 349–361 (1967)
2. Appell, J.: Measures of Noncompactness, Condensing Operators and Fixed Points: an Application-Oriented Survey. Fixed Point Theory. **6**(2), 157–229 (2005)
3. Cain Jr., G.L., Zaki Nashed, M.Z.: Fixed Points and Stability for a Sum of two Operators in Locally Convex Spaces. Pacific J. Math. **39**(3), 581–592 (1971)
4. Darbo, G.: Punti uniti in trasformazioni a condominio non compatto. Rend. Sem. Mat. Univ. Padova. **24**, 84–92 (1955)
5. Gholizadeh, L., Karapınar, E., Roohi, M.: Some fixed point theorems in locally *p*-convex spaces. Fixed Point Theory Appl. **2013**, 312 (2013)
6. Jarchow, H.: Locally Convex Spaces. Mathematische Leitfäden. Teubner, Stuttgart (1981)
7. Kaniok, L.: On measures of non compactness in general topological vector spaces. Comment. Math. Univ. Carol. **31**(3), 479–487 (1990)
8. Kelly, J.L.: General Topology. D. Van Nostrand Company Inc, New York (1955)
9. Khan, L.A., Oubbi, L.: The Arzéla-Ascoli theorem for non-locally convex weighted spaces. Rev. Real Acad. Cien. Zaragoza. **60**, 107–115 (2005)
10. Klee, V.: Shrinkable neighborhoods in Hausdorff linear spaces. Math. Ann. **141**, 281–285 (1960)
11. Klilou, M., Oubbi, L.: Weighted composition operators in Nachbin spaces with operator-valued weights. Commun. Korean Math. Soc. **33**(4), 1125–1140 (2018)
12. Kuratowski, K.: Sur les espaces complets. Fund. Math. **15**, 301–309 (1930)
13. Oubbi, L.: Weighted composition operators on non-locally convex weighted spaces. Rocky Mt J. Math. **35**(6), 2065–2087 (2005)
14. Oubbi, L.: Algebras of Gelfand-continuous functions into Arens-Michael algebras. Commun. Korean Math. Soc. **34**(2), 585–602 (2019)
15. Park, S.: Fixed points of approximable or Kakutani maps in generalized convex spaces. J. Nonlinear Convex Anal. **7**(1), 1–17 (2011)
16. Robertson, L.B.: Topological vector spaces. Publ. Inst. Math. **12**(26), 19–21 (1971)
17. Ruess, W.M., Summers, W.H.: Compactness in spaces of vector-valued continuous functions and asymptotic almost periodicity. Math. Nachr. **135**, 7–33 (1988)
18. Sadovskii, B.N.: On a fixed point principle [in Russian]. Funkts. Analiz Prilozh. **1**(2), 74–76 (1967)
19. Sadovskii, B.N.: Limit-Compact and Condensing Operators. Russian Math. Surv. **27**(1), 86–155 (1972)
20. Xiao, J., Lu, Y.: Some fixed point theorems for *s*-convex subsets in *p*-normed spaces based on measures of non compactness. J. Fixed Point Theory Appl. (2018). https://doi.org/10.1007/s11784-018-0562-1
21. Xiao, J., Zhu, X.: Some fixed point theorems for s-convex subsets in p-normed spaces. Nonlinear Anal. **74**(5), 1738–1748 (2011)