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Primarity of direct sums of Orlicz spaces and Marcinkiewicz spaces

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Abstract

Let $\mathbb{Y}$ be either an Orlicz sequence space or a Marcinkiewicz sequence space. We take advantage of the recent advances in the theory of factorization of the identity carried on by Lechner (Stud Math 248(3):295–319, 2019) to provide conditions on $\mathbb{Y}$ that ensure that, for any $1 \leq p \leq \infty$, the infinite direct sum of $\mathbb{Y}$ in the sense of ℓ_p is a primary Banach space. This way, we enlarge the list of Banach spaces that are known to be primary.

Keywords Subsymmetric basis · Primary Banach space · Factorization of the identity · Marcinkiewicz space · Lorentz space · Orlicz space · Sequence space

Mathematics Subject Classification 46B25 · 46B26 · 46B15 · 46B45

1 Introduction

Within his study of operators through which the identity map factors, Lechner [12] introduced the following condition on the coordinate functionals of an unconditional basis of a Banach space.

Definition 1 Let $(\mathbf{x}_j)_{j=1}^{\infty}$ be an unconditional basis for a Banach space $\mathbb{X}$. We say that its sequence $(\mathbf{x}_j^*)_{j=1}^{\infty}$ of coordinate functionals is a *non-ℓ_1-splicing weak* basis* if for every $A \subseteq \mathbb{N}$ infinite and for every $\theta > 0$ there is a sequence $(A_n)_{n=1}^{\infty}$ consisting of pairwise disjoint infinite subsets of A such that for every $(f_n^*)_{n=1}^{\infty}$ in $B_{\mathbb{X}^*}$ there is a sequence of scalars $(a_n)_{n=1}^{\infty} \in S_{\ell_1}$ satisfying

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$$\left\| \sum_{n=1}^{\infty} a_n \, P_{A_n}^*(f_n^*) \right\| \leq \theta.$$

We say that $(\mathbf{x}_j^*)_{j=1}^{\infty}$ is ℓ_1-splicing if it fails to be non-ℓ_1-splicing.

Here, and throughout this note, $B_{\mathbb{X}}$ (respectively $S_{\mathbb{X}}$) denotes the closed unit ball (resp. unit sphere) of a Banach space $\mathbb{X}$. The symbol P_A denotes the *coordinate projection* on a set $A \subseteq \mathbb{N}$ with respect to an unconditional basis $\mathcal{B} = (\mathbf{x}_j)_{j=1}^{\infty}$ of $\mathbb{X}$, i.e., if $\mathcal{B}^* = (\mathbf{x}_j^*)_{j=1}^{\infty}$ is the sequence of coordinate functionals associated to the basis $\mathcal{B}$, also called the *dual basic sequence* of $\mathcal{B}$, then $P_A \colon \mathbb{X} \to \mathbb{X}$ is defined by

$$P_A(f) = \sum_{j \in A} \mathbf{x}_j^*(f) \, \mathbf{x}_j, \quad f \in \mathbb{X}. \tag{1}$$

Note that the *dual coordinate projection* $P_A^* \colon \mathbb{X}^* \to \mathbb{X}^*$ of P_A is given by

$$P_A^*(f^*) = \text{w*-} \sum_{j \in A} f^*(\mathbf{x}_j) \, \mathbf{x}_j^*, \quad f^* \in \mathbb{X}^*.$$

Since the basis $\mathcal{B}$ is, up to equivalence, univocally determined by the basic sequence $\mathcal{B}^*$ (see [4, Corollary 3.2.4]) it is natural to consider being non-ℓ_1-splicing as a condition on $\mathcal{B}^*$ instead of as a condition on $\mathcal{B}$.

In the aforementioned paper, Lechner achieved the following contribution to the theory of primary Banach spaces and the factorization of the identity. Recall that a Banach space $\mathbb{X}$ is said to be *primary* if whenever $\mathbb{Y}$ and $\mathbb{Z}$ are Banach spaces such that $\mathbb{Y} \oplus \mathbb{Z} \approx \mathbb{X}$, then either $\mathbb{Y} \approx \mathbb{X}$ or $\mathbb{Z} \approx \mathbb{X}$. A basis is said to be *subsymmetric* if it is unconditional and equivalent to all its subsequences. The infinite direct sum of a Banach space $\mathbb{X}$ in the sense of ℓ_p (respectively c_0) will be denoted by $\ell_p(\mathbb{X})$ (resp. $c_0(\mathbb{X})$). $\mathcal{L}(\mathbb{X})$ will denote the Banach algebra of automorphisms of a Banach space $\mathbb{X}$. We say that the identity map on $\mathbb{X}$ factors through an operator $R \in \mathcal{L}(\mathbb{X})$ if there are operators S and $T \in \mathcal{L}(\mathbb{X})$ such that $T \circ R \circ S = \text{Id}_{\mathbb{X}}$.

Theorem 1 (see [12, Theorems 1.1 and 1.2]) *Suppose that $\mathbb{X}$ is a Banach space equipped with a subsymmetric basis whose dual basic sequence is non-ℓ_1-splicing. Let $1 \leq p \leq \infty$ and let $\mathbb{Y}$ be either $\mathbb{X}^*$ or $\ell_p(\mathbb{X}^*)$. Then, given $T \in \mathcal{L}(\mathbb{Y})$, the identity map on $\mathbb{Y}$ factors through either T or $\text{Id}_{\mathbb{Y}} - T$. Consequently, $\ell_p(\mathbb{X}^*)$ is a primary Banach space.*

Before undertaking the task of using Theorem 1 for obtaining new primary Banach spaces, we must go over the state-of-the-art on this topic. Casazza et al. [8] proved that if $\mathbb{X}$ has a *symmetric* basis, i.e., a basis which is equivalent to all its permutations, then the Banach spaces $c_0(\mathbb{X})$ and, in the case when $1 < p < \infty$ and $\mathbb{X}$ is not isomorphic to ℓ_1, $\ell_p(\mathbb{X})$ are primary. Shortly later, Samuel [17] proved that $\ell_p(\ell_r)$, $c_0(\ell_r)$ and $\ell_r(c_0)$ are, for $1 \leq p, r < \infty$, primary Banach spaces. Subsequently, Capon [6] completed the study by proving that $\ell_1(\mathbb{X})$ and $\ell_\infty(\mathbb{X})$ are primary Banach spaces whenever $\mathbb{X}$ possesses a symmetric basis. Symmetric bases are subsymmetric [11,19],

and, in practice, the only information that one needs about symmetric bases in many situations is its subsymmetry. So, it is natural to wonder if the proofs carried on in [6,8] still work when dealing with subsymmetric bases. A careful look at these papers reveals that it is the case. Summarizing, we have the following result.

Theorem 2 (see [6,8,17]) *Let $\mathbb{X}$ be a Banach space endowed with a subsymmetric basis. Then $c_0(\mathbb{X})$ and $\ell_p(\mathbb{X})$, $1 \leq p \leq \infty$, are primary Banach spaces.*

At this point, we must mention that, as Pełczyński decomposition method is a pivotal tool for facing the study of primary Banach spaces, the task of proving that $\ell_p(\mathbb{X})$ is primary is, in some sense, easier than that of proving that $\mathbb{X}$ is. In fact, as far as we know, ℓ_p, $1 \leq p < \infty$, and c_0 are the only known primary Banach spaces endowed with a subsymmetric basis.

In light of Theorem 2, applying Theorem 1 to a Banach space $\mathbb{X}$ equipped with a shrinking (subsymmetric) basis does not add a new space to the list of primary Banach spaces. So, taking into account [4, Theorem 3.3.1], within the goal of using Theorem 1 for finding new primary spaces, we must apply it to Banach spaces $\mathbb{X}$ containing a complemented copy of ℓ_1. Among them, $\mathbb{X} = \ell_1$ seems to be the first space we have to consider. It is timely to bring up the following result.

Theorem 3 (see [8]) *Let $1 \leq p \leq \infty$. Then $\ell_p(\ell_\infty)$ is primary.*

It is known [12, Proposition 6.2] that the unit vector system of ℓ_∞, which is, under the natural pairing, the dual basic sequence of the unit vector system of ℓ_1, is a non-ℓ_1-splicing weak* basis. This result combined with Theorem 1 provides an alternative proof to Theorem 3. From an opposite perspective, in order to take advantage of Theorem 3 for obtaining new primary Banach spaces, we need to find weak* bases, other than the unit vector system of ℓ_∞, that are non-ℓ_1-splicing and are not boundedly complete. Within this area of research, Lechner [12] exhibited that the unit vector system of some Orlicz spaces and the dual basis of the unit vector system of some Lorentz sequence spaces fulfil these requirements.

In this manuscript, we go on with the search of non-ℓ_1-splicing weak* bases and, hence, of new primary Banach spaces, initiated in [12]. In Sect. 4, we generalize [12, Theorem 6.4] by characterizing, in terms of the convex Orlicz M, when the unit vector system of the Orlicz sequence space ℓ_M is non-ℓ_1-splicing. In Sect. 3 we override [12, Theorem 6.5] by describing those weights **s** for which the unit vector system of the Marcinkiewicz space $m(\mathbf{s})$ is non-ℓ_1-splicing. Previously to these sections, in Sect. 2, we carry on a detailed analysis of the concept of non-ℓ_1-splicing weak* basis introduced by Lechner.

Throughout this article we follow standard Banach space terminology and notation as can be found in [4]. We single out the notation that is more commonly employed. We will denote by $\mathbb{F}$ the real or complex field. By a *sign* we mean a scalar of modulus one. We denote by $(\mathbf{e}_k)_{k=1}^{\infty}$ the unit vector system of $\mathbb{F}^{\mathbb{N}}$, i.e., $\mathbf{e}_k = (\delta_{k,n})_{n=1}^{\infty}$, were $\delta_{k,n} = 1$ if $n = k$ and $\delta_{k,n} = 0$ otherwise. The linear span of the unit vector system will be denoted by c_{00}.

Given families of non-negative real numbers $(\alpha_i)_{i \in I}$ and $(\beta_i)_{i \in I}$ and a constant $C < \infty$, the symbol $\alpha_i \lesssim_C \beta_i$ for $i \in I$ means that $\alpha_i \leq C\beta_i$ for every $i \in I$,

while $\alpha_i \approx_C \beta_i$ for $i \in I$ means that $\alpha_i \lesssim_C \beta_i$ and $\beta_i \lesssim_C \alpha_i$ for $i \in I$. A *basis* will be a Schauder basis. Suppose $(\mathbf{x}_j)_{j=1}^{\infty}$ and $(\mathbf{y}_j)_{j=1}^{\infty}$ are bases. We say that $(\mathbf{y}_j)_{j=1}^{\infty}$ *C-dominates* $(\mathbf{x}_j)_{j=1}^{\infty}$ (respectively $(\mathbf{y}_j)_{j=1}^{\infty}$ is *C-equivalent* to $(\mathbf{x}_j)_{j=1}^{\infty}$), and write $(\mathbf{x}_j)_{j=1}^{\infty} \lesssim_C (\mathbf{y}_j)_{j=1}^{\infty}$ (resp. $(\mathbf{x}_j)_{j=1}^{\infty} \approx_C (\mathbf{y}_j)_{j=1}^{\infty}$) if

$$\left\| \sum_{j=1}^{\infty} a_j \mathbf{x}_j \right\| \lesssim_C \left\| \sum_{j=1}^{\infty} a_j \mathbf{y}_j \right\| \left(\text{resp. } \left\| \sum_{j=1}^{\infty} a_j \mathbf{x}_j \right\| \approx_C \left\| \sum_{j=1}^{\infty} a_j \mathbf{y}_j \right\| \right)$$

for $(a_j)_{j=1}^{\infty} \in c_{00}$. In all the above cases, when the value of the constant C is irrelevant, we simply drop it from the notation. A basis is said to be *unconditional* if all its permutations are basic sequences. I turn, we say that a basis $(\mathbf{x}_j)_{j=1}^{\infty}$ is *C-unconditional* if $(\mathbf{x}_j)_{n=1}^{\infty} \approx_C (\epsilon_j \mathbf{x}_j)_{j=1}^{\infty}$ for any choice of signs $(\epsilon_j)_{j=1}^{\infty}$. If $(\mathbf{x}_j)_{j=1}^{\infty}$ is a C-unconditional basis of a Banach space $\mathbb{X}$ and $A \subseteq \mathbb{N}$, then the operator P_A defined as in (1) is well-defined and satisfies $\|P_A\| \leq C$. It is well-known (see e.g. [4, Proposition 3.1.3]) that a basis $\mathcal{B}$ is unconditional if and only if there exists a constant $C \geq 1$ such that $\mathcal{B}$ is C-unconditional.

We say that a sequence in a Banach space is a *basic sequence* if it is a basis of its closed linear span. If $\mathcal{B}$ is a basis of a Banach space $\mathbb{X}$, then its coordinate functionals constitute a basic sequence in $\mathbb{X}^*$. Reciprocally, if $\mathcal{B}$ is a basis of a Banach space $\mathbb{Y}$ and $\mathbb{X} := \mathbb{X}[\mathcal{B}]$ is the closed linear span of $\mathcal{B}^*$ in $\mathbb{Y}^*$, then there is a natural isomorphic embedding of $\mathbb{Y}$ into $\mathbb{X}^* := \mathbb{Y}[\mathcal{B}]$ and, via this embedding, $\mathcal{B}$ is the dual basic sequence of $\mathcal{B}^*$ (see [4, Proposition 3.2.3 and Corollary 3.2.4]). Consequently, any basic sequence is the dual basic sequence of some basis. So, it makes sense to wonder if a given unconditional basic sequence $\mathcal{B}$ of a Banach space (regarded as a sequence in the Banach space $\mathbb{Y}[\mathcal{B}]$ constructed as above) is non-ℓ_1-splicing.

A basis $\mathcal{B} = (\mathbf{x}_j)_{j=1}^{\infty}$ of $\mathbb{X}$ is said to be *boundedly complete* if whenever $(a_j)_{j=1}^{\infty} \in \mathbb{F}^{\mathbb{N}}$ satisfies $\sup_n \|\sum_{j=1}^{n} a_j \mathbf{x}_j\| < \infty$ there is $f \in \mathbb{X}$ such that $\mathbf{x}_j^*(f) = a_j$ for every $j \in \mathbb{N}$. The basis $\mathcal{B}$ is said to be *shrinking* if $\mathcal{B}^*$ is a basis of the whole space $\mathbb{X}^*$. It is known [10] that a basis $\mathcal{B}$ is boundedly complete if and only if $\mathcal{B}^*$ is shrinking.

The symbol $f = \text{w*-}\sum_{n=1}^{\infty} f_n$ means that the series $\sum_{n=1}^{\infty} f_n$ in $\mathbb{X}^*$ converges to $f \in \mathbb{X}^*$ in the weak* topology of the dual space $\mathbb{X}^*$. Recall that if $\mathcal{B} = (\mathbf{x}_j)_{j=1}^{\infty}$ is a basis of a Banach space $\mathbb{X}$ and $\mathcal{B}^*(\mathbf{x}_j^*)_{j=1}^{\infty}$ is its sequence of coordinate functionals, then, for every $f^* \in \mathbb{X}^*$, $(f^*(\mathbf{x}_j))_{j=1}^{\infty}$ is the unique sequence $(a_j)_{j=1}^{\infty} \in \mathbb{F}^{\mathbb{N}}$ such that $f^* = \text{w*-}\sum_{j=1}^{\infty} a_j \mathbf{x}_j^*$. So, $\mathcal{B}^*$ is a weak* basis of $\mathbb{X}^*$.

The *support* of a vector $f \in \mathbb{X}$ with respect to the basis $\mathcal{B}$ is the set

$$\text{supp}(f) = \{j \in \mathbb{N}: \mathbf{x}_j^*(f) \neq 0\},$$

and the *support* of a functional $f^* \in \mathbb{X}^*$ with respect to the basis $\mathcal{B}$ is the set

$$\text{supp}(f^*) = \{j \in \mathbb{N}: f^*(\mathbf{x}_j) \neq 0\}.$$

A sequence $(f_n)_{n=1}^{\infty}$ in either $\mathbb{X}$ or $\mathbb{X}^*$ is said to be *disjointly supported* if $(\text{supp}(f_n))_{n=1}^{\infty}$ is a sequence of pairwise disjoint subsets of $\mathbb{N}$. A *block basic sequence* is a sequence

$(f_n)_{n=1}^\infty$ for which there is an increasing sequence $(k_n)_{n=1}^\infty$ of positive integers such that, with the convention $n_0 = 0$, $\text{supp}(f_n) \subseteq [1 + k_{n-1}, k_n]$ for every $n \in \mathbb{N}$. Block basic sequences are a particular case of disjointly supported sequences. Since any block basic sequence is a basic sequence, our terminology is consistent. Note that any disjointly supported sequence (in either $\mathbb{X}$ or $\mathbb{X}^*$) with respect to an unconditional basis of a Banach space $\mathbb{X}$ is an unconditional basic sequence.

Let $\mathbb{X} \subseteq \mathbb{F}^{\mathbb{N}}$ be a Banach space for which the unit vector system is a basis. We say that a Banach space $\mathbb{Y} \subseteq \mathbb{F}^{\mathbb{N}}$ is the *dual space of $\mathbb{X}$ under the natural pairing* if there is an isomorphism $T : \mathbb{X}^* \to \mathbb{Y}$ such that $T(f)(g) = \sum_{j=1}^\infty a_j b_j$ for every $f = (a_j)_{j=1}^\infty \in \mathbb{Y}$ and every $g = (b_j)_{j=1}^\infty \in c_{00}$. Observe that if $\mathbb{Y}$ is, under the natural pairing, the dual space of $\mathbb{X}$ and $f = (a_j)_{j=1}^\infty$ belongs to either $\mathbb{X}$ or $\mathbb{Y}$, then the support of f with respect to the unit vector system is the set $\text{supp}(f) = \{j \in \mathbb{N} : a_j \neq 0\}$.

A sequence $= (f_j)_{j=1}^\infty$ is a Banach space is said to be *semi-normalized* if $\inf_j \|f_j\| > 0$ and $\sup_j \|f_j\| < \infty$. Note that subsymmetric bases are semi-normalized.

Other more specific notation will be specified in context when needed.

2 Non-ℓ_1-splicing weak* bases

The main goal of the study carried on in this section is to show that, if the basis is subsymmetric, we can describe more simply non-ℓ_1-splicing bases. In order to prove our results, it will be convenient to introduce some additional terminology.

If $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ is a subsymmetric basis of a Banach space $\mathbb{X}$ then [5, Theorem 3.7] there is a renorming of $\mathbb{X}$ with respect to which it is 1-subsymmetric, i.e., $\mathcal{B}$ is 1-unconditional and for every increasing map $\phi : \mathbb{N} \to \mathbb{N}$ the linear operator

$$V_\phi : \mathbb{X} \to \mathbb{X}, \quad \sum_{j=1}^\infty a_j \mathbf{x}_j \mapsto \sum_{j=1}^\infty a_j \mathbf{x}_{\phi(j)} \tag{2}$$

is an isometric embedding. If the basis is 1-subsymmetric then the linear operator

$$U_\phi : \mathbb{X} \to \mathbb{X}, \quad \sum_{j=1}^\infty a_j \mathbf{x}_j \mapsto \sum_{j=1}^\infty a_{\phi(j)} \mathbf{x}_j \tag{3}$$

is norm-one for every increasing map $\phi : \mathbb{N} \to \mathbb{N}$ (see e.g. [5, Lemma 3.3]). The dual operators of V_ϕ and U_ϕ are given by

$$V_\phi^* : \mathbb{X}^* \to \mathbb{X}^*, \quad \text{w*-}\sum_{j=1}^\infty a_j \mathbf{x}_j^* \mapsto \text{w*-}\sum_{j=1}^\infty a_{\phi(j)} \mathbf{x}_j^*,$$

$$U_\phi^* : \mathbb{X}^* \to \mathbb{X}^*, \quad \text{w*-}\sum_{j=1}^\infty a_j \mathbf{x}_j^* \mapsto \text{w*-}\sum_{j=1}^\infty a_j \mathbf{x}_{\phi(j)}^*.$$

Since $U_\phi \circ V_\phi = \mathrm{Id}_{\mathbb{X}}$, we have $V_\phi^* \circ U_\phi^* = \mathrm{Id}_{\mathbb{X}^*}$. Consequently, U_ϕ^* is an isomorphic embedding (isometric embedding if $\mathcal{B}$ is 1-subsymmetric).

With this background in our hands, we are almost ready to prove the aforementioned characterization of non-ℓ_1-splicing weak* subsymmetric bases. Before doing so, we bring up a result that is implicit in [12].

Lemma 1 (cf. [12, Proposition 6.1]) *Let $\mathcal{B}$ be an unconditional basis of a Banach space. Assume that $\mathcal{B}^*$ is ℓ_1-splicing. Then there is a sequence of disjointly supported functionals in $\mathbb{X}^*$ equivalent to the unit vector system of ℓ_1.*

Proof Our hypothesis says that there are $\theta > 0$ and $A \subseteq \mathbb{N}$ infinite such that, for every sequence $(A_n)_{n=1}^\infty$ consisting of pairwise disjoint infinite subsets of A, there is $(f_n^*)_{n=1}^\infty$ in $B_{\mathbb{X}^*}$ with $\theta < \| \sum_{n=1}^\infty a_n P_{A_n}^*(f_n^*) \|$ for every $(a_n)_{n=1}^\infty \in S_{\ell_1}$.

Pick out an arbitrary sequence $(A_n)_{n=1}^\infty$ consisting of pairwise disjoint infinite subsets of A and let $(f_n^*)_{n=1}^\infty$ be as above. If $g_n^* = P_{A_n}^*(f_n^*)$, we have $\mathrm{supp}(g_n^*) \subseteq A_n$ and $\sup_n \|g_n^*\| < \infty$ for every $n \in \mathbb{N}$. We infer that the disjointly supported sequence $(g_n^*)_{n=1}^\infty$ is equivalent to the unit vector basis of ℓ_1. $\quad\square$

For reference, we write down the following elementary lemma, which we will use in the proof of the subsequent theorem.

Lemma 2 *Let $(B_n)_{n=1}^\infty$ be a sequence of disjointly supported subsets of $\mathbb{N}$ and let $(A_n)_{n=1}^\infty$ be a sequence of disjointly supported infinite subsets of $\mathbb{N}$. Then, there exists an increasing map $\phi \colon \mathbb{N} \to \mathbb{N}$ such that $\phi(B_n) \subseteq A_n$ for every $n \in \mathbb{N}$.*

Proof Clearly, it suffices to prove the result in the case when $(B_n)_{n=1}^\infty$ is a partition of $\mathbb{N}$. Define $\nu \colon \mathbb{N} \to \mathbb{N}$ by $\nu(k) = n$ if $k \in B_n$. By hypothesis,

$$D_{n,m} := \{ j \in A_n \colon j > m \}$$

is non-empty for every $n \in \mathbb{N}$ and every $m \in \mathbb{N} \cup \{0\}$. With the convention $\phi(0) = 0$, we define $\phi \colon \mathbb{N} \to \mathbb{N}$ by means of the recursive formula

$$\phi(k) = \min D_{\nu(k), \phi(k-1)}, \quad k \in \mathbb{N}.$$

It is clear that ϕ satisfies the desired properties. $\quad\square$

Theorem 4 *Assume that $\mathcal{B}$ is a subsymmetric basis of a Banach space $\mathbb{X}$. Then its dual basic sequence $\mathcal{B}^*$ is ℓ_1-splicing if and only there is a sequence of disjointly supported functionals in $\mathbb{X}^*$ equivalent to the unit vector system of ℓ_1.*

Proof The "only if" part follows from Lemma 1. Assume that there is a disjointly supported sequence $(f_n^*)_{n=1}^\infty$ in $\mathbb{X}^*$ that is equivalent to the unit vector system of ℓ_1. By dilation, we can assume that $\|f_n^*\| \leq 1$ for every $n \in \mathbb{N}$. Let $c > 0$ be such that

$$c \sum_{n=1}^\infty |a_n| \leq \left\| \sum_{n=1}^\infty a_n f_n \right\|, \quad (a_n)_{n=1}^\infty \in \ell_1.$$

We also assume, without loss of generality, that $\mathcal{B}$ is 1-subsymmetric. Choose $0 < \theta < c$ and $A = \mathbb{N}$. Pick a sequence $(A_n)_{n=1}^{\infty}$ consisting of pairwise disjoint infinite subsets of $\mathbb{N}$. By Lemma 2, there is an increasing map $\phi \colon \mathbb{N} \to \mathbb{N}$ such that $\phi(\operatorname{supp}(f_n^*)) \subseteq A_n$. Put $g_n^* = U_\phi^*(f_n^*)$ for $n \in \mathbb{N}$. Then, taking into account that U_ϕ^* is an isometric embedding, we have $P_{A_n}^*(g_n^*) = g_n^* \in B_{\mathbb{X}^*}$ for every $n \in \mathbb{N}$, and

$$\theta < c \leq \left\| \sum_{n=1}^{\infty} a_n \, g_n^* \right\|$$

for every $(a_n)_{n=1}^{\infty} \in S_{\ell_1}$. Consequently, $\mathcal{B}^*$ is ℓ_1-splicing. $\qquad \square$

Note that, if $\mathbb{X}$ has an unconditional basis and $\mathbb{X}^*$ is non-separable, then ℓ_1 is a subspace of $\mathbb{X}^*$ (see [4, Theorems 2.5.7 and 3.3.1]). So, Theorems 1 and 4 reveal that the position in which ℓ_1 is (and is not) placed inside $\mathbb{X}^*$ has significative structural consequences.

Next, we give some consequences of Theorem 4. First of them was previously achieved in [12].

Lemma 3 *Let $\mathcal{B} = (f_n)_{n=1}^{\infty}$ be a semi-normalized disjointly supported sequence in ℓ_∞. Then $\mathcal{B}$ is equivalent to the unit vector system of ℓ_∞.*

Proof Denote $c = \inf_n \|f_n\|$ and $C = \sup_n \|f_n\|$. It is clear that $c\|g\|_\infty \leq \|\sum_{n=1}^{\infty} a_n f_n\| \leq C\|g\|_\infty$ for every $g = (a_n)_{n=1}^{\infty} \in c_{00}$. $\qquad \square$

Proposition 1 (see [12, Proposition 6.2]) *The unit vector system of ℓ_∞ is non-ℓ_1-splicing.*

Proof The unit vector system of ℓ_∞, denoted by $\mathcal{B}_\infty$ in this proof, is, under the natural pairing, the dual basic sequence of the unit vector basis of ℓ_1, denoted by $\mathcal{B}_1$. Suppose by contradiction that there is a disjointly supported sequence $\mathcal{B}$ in ℓ_∞ that is equivalent to $\mathcal{B}_1$. Then, in particular, $\mathcal{B}$ is semi-normalized. Invoking Lemma 3 we obtain $\mathcal{B}_1 \approx \mathcal{B} \approx \mathcal{B}_\infty$. This absurdity, combined with Theorem 4, proves that $\mathcal{B}_\infty$ f is non-ℓ_1-splicing. $\qquad \square$

Proposition 2 *Let $\mathcal{B}$ be a subsymmetric basis of a Banach space $\mathbb{X}$ whose dual basic sequence $\mathcal{B}^*$ is non-ℓ_1-splicing. Then $\mathcal{B}$ is boundedly complete and $\mathcal{B}^*$ is shrinking.*

Proof If $\mathcal{B}^*$ fails to be shrinking, then, by [4, Theorem 3.3.1], there is a block basic sequence with respect to $\mathcal{B}^*$ equivalent to the unit vector system of ℓ_1. Consequently, by Theorem 4, $\mathcal{B}^*$ is ℓ_1-splicing. We complete the proof by appealing to [4, Theorem 3.2.15]. $\qquad \square$

Corollary 1 *Let $\mathbb{X}$ be a Banach space endowed with a subsymmetric basis $\mathcal{B}$. If $\mathcal{B}^*$ is non-ℓ_1-splicing, then $\mathbb{X}$ is a dual space (and $\mathbb{X}^*$ is a bidual space).*

Proof It is immediate from combining Proposition 2 with [4, Theorem 3.2.15]. $\qquad \square$

Corollary 2 *Let $\mathbb{X}$ be a Banach space endowed with a subsymmetric shrinking basis $\mathcal{B}$. If $\mathcal{B}^*$ is non-ℓ_1-splicing, then $\mathbb{X}$ is reflexive.*

Proof Just combine Proposition 2 with [4, Theorem 3.2.19]. □

Proposition 3 *Let $\mathbb{X}$ be a reflexive Banach space endowed with a subsymmetric basis $\mathcal{B}$. Then $\mathcal{B}$ and $\mathcal{B}^*$ are non-ℓ_1-splicing.*

Proof By [4, Theorems 3.2.15, 3.2.19 and 3.3.1], neither $\mathbb{X}$ nor $\mathbb{X}^*$ contain a subspace isomorphic to ℓ_1. Then, by Theorem 4, $\mathcal{B}$ and $\mathcal{B}^*$ are non-ℓ_1-splicing. □

3 Marcinkiewicz spaces

A weight will be a sequence of positive scalars. Given a weight $\mathbf{s} = (s_j)_{j=1}^{\infty}$ the *Marcinkiewicz space* $m(\mathbf{s})$ is the set consisting of all sequences $f = (a_j)_{j=1}^{\infty} \in \mathbb{F}^{\mathbb{N}}$ such that

$$\|f\|_{m(\mathbf{s})} := \sup \left\{ \frac{1}{s_n} \sum_{j \in A} |a_j| : n \in \mathbb{N}, |A| = n \right\} < \infty.$$

It is clear, and well-known, that $(m(\mathbf{s}), \| \cdot \|_{m(\mathbf{s})})$ is a Banach space and that the unit vector system is a symmetric basic sequence in $m(\mathbf{s})$. If $f \in c_0$ and $(a_n^*)_{n=1}^{\infty}$ denotes its non-increasing rearrangement then

$$\|f\|_{m(\mathbf{s})} = \sup_n \frac{1}{s_n} \sum_{k=1}^{n} a_k^*. \tag{4}$$

It is not hard to prove that if $\lim_n s_n/n = 0$ then $m(\mathbf{s}) \subseteq c_0$ continuously. Otherwise, we have $m(\mathbf{s}) = \ell_{\infty}$ (up to an equivalent norm).

The next proposition gathers some results that relate Marcinkiewicz spaces to Lorentz spaces. Prior to enunciate it, let us fix some terminology. A weight $(w_j)_{j=1}^{\infty}$ is said to be *regular* if it satisfies the Dini condition

$$\sup_n \frac{1}{n w_n} \sum_{j=1}^{n} w_j < \infty.$$

If two weights $\mathbf{w} = (w_j)_{j=1}^{\infty}$ and $\mathbf{s} = (s_n)_{n=1}^{\infty}$ are related by the formula

$$s_n = \sum_{j=1}^{n} w_j, \quad n \in \mathbb{N},$$

we say that $\mathbf{s}$ is the *primitive weight* of $\mathbf{w}$ and that $\mathbf{w}$ is the *discrete derivative* of $\mathbf{s}$. Given a weight $\mathbf{w} = (w_j)_{j=1}^{\infty}$ with primitive weight $\mathbf{s} = (s_j)_{j=1}^{\infty}$, the Lorentz space $d(\mathbf{w}, 1)$ (respectively weak Lorentz space $d(\mathbf{w}, \infty)$) is the set consisting of all sequences $f \in c_0$ whose non-increasing rearrangement $(a_j^*)_{j=1}^{\infty}$ fulfils

$$\|f\|_{d(\mathbf{w},1)} := \sum_{j=1}^{\infty} a_j^* w_j < \infty \left(\text{resp. } \|f\|_{d(\mathbf{w},\infty)} := \sup_{j \in \mathbb{N}} a_j^* s_j < \infty \right).$$

If $(s_n/n)_{n=1}^{\infty}$ is non-increasing, then $d(\mathbf{w}, 1)$ is a Banach space (see [7, Theorem 2.5.10]). In turn, if $\mathbf{s}$ is doubling, then $d(\mathbf{w}, \infty)$ is a quasi-Banach space (see [7, Theorem 2.2.16]). It is not hard to prove that c_{00} is a dense subspace of $d(\mathbf{w}, 1)$. Then, the unit vector system is a symmetric basis of $d(\mathbf{w}, 1)$.

Proposition 4 (See [7, Theorems 2.4.14 and 2.5.10 and Corollary 2.4.26]; see also [2, Section 6]) *Let* $\mathbf{w} = (w_j)_{j=1}^{\infty}$ *be a decreasing weight with* $\lim_j w_j = 0$*, let* $\mathbf{s}$ *denote its primitive weight, and let* $\mathbf{v}$ *denote the discrete derivative of the inverse weight* $\mathbf{w}^{-1} = (1/w_j)_{j=1}^{\infty}$ *of* $\mathbf{w}$*.*

 (i) *$m(\mathbf{s})$ is, under the natural pairing, the dual space of $d(\mathbf{w}, 1)$.*
(ii) *If $\mathbf{w}$ is a regular weight, then $d(\mathbf{v}, \infty) = m(\mathbf{s})$ (up to an equivalent quasi-norm).*

In route to state the main result of the section, we introduce some additional conditions on weights, and we bring up a result involving them. We say that a weight $(s_n)_{n=1}^{\infty}$ is *essentially decreasing* (respectively *essentially increasing*) if

$$\inf_{m \le n} \frac{s_m}{s_n} > 0 \left(\text{resp. } \sup_{m \le n} \frac{s_m}{s_n} < \infty \right).$$

Note that $(s_n)_{n=1}^{\infty}$ is essentially decreasing (resp. essentially increasing) if and only if it is equivalent to a non-increasing (resp. non-decreasing) weight. We say that a weight $(s_n)_{n=1}^{\infty}$ has the *lower regularity property* (LRP for short) if there is a constant $C > 1$ and an integer $r \ge 2$ such that

$$s_{rn} \ge C s_n, \quad n \in \mathbb{N}.$$

Lemma 4 (see [1, Lemma 2.12]) *Let* $\mathbf{s} = (s_n)_{n=1}^{\infty}$ *be a essentially increasing weight such that* $\mathbf{w} = (s_n/n)_{n=1}^{\infty}$ *is essentially decreasing. Then,* $\mathbf{s}$ *has the LRP if and only if* $\mathbf{w}$ *is a regular weight.*

The following result is rather straightforward, and old-timers will surely be aware of it and could produce its proof on the spot. Nonetheless, for later reference and expository ease, we record it.

Lemma 5 *Let* $\mathcal{B} = (\mathbf{x}_j)_{j=1}^{\infty}$ *be a subsymmetric basis of a Banach space* $\mathbb{X}$ *such that* $n \lesssim \|\sum_{j=1}^{n} \mathbf{x}_j\|$ *for* $n \in \mathbb{N}$*. Then* $\mathcal{B}$ *is equivalent to the unit vector basis of* ℓ_1*.*

Proof By [15, Proposition 3.a.4], for $f = \sum_{j=1}^{\infty} a_j \mathbf{x}_j \in \mathbb{X}$ and $n \in \mathbb{N}$ we have

$$\left| \sum_{j=1}^{n} a_j \right| \lesssim \left| \frac{1}{n} \sum_{j=1}^{n} a_j \right| \left\| \sum_{j=1}^{n} \mathbf{x}_j \right\| \lesssim \|f\|.$$

Combining this inequality (which using the terminology introduced by Singer [18] says that $\mathcal{B}$ is a basis of type P^*) with unconditionality yields the desired result. □

We are ready to state and prove the main theorem of the present section.

Theorem 5 *Let* $\mathbf{s} = (s_n)_{n=1}^{\infty}$ *be an increasing weight whose discrete derivative is essentially decreasing. Then, the unit vector system of* $m(\mathbf{s})$ *is non-*ℓ_1*-splicing if and only if*

$$S := \inf_{n \in \mathbb{N}} \sup_{k \in \mathbb{N}} \frac{s_k}{s_{kn}} = 0. \tag{5}$$

Proof We infer from our assumptions on $\mathbf{s}$ that there is a non-increasing weight $\mathbf{w} = (w_n)_{n=1}^{\infty}$ whose primitive weight $\mathbf{t} = (t_n)_{n=1}^{\infty}$ is equivalent to $\mathbf{s}$. So, $m(\mathbf{t}) = m(\mathbf{s})$. Moreover, by Lemma 4, if $\mathbf{s}$ had the LRP, we could choose $\mathbf{w}$ to be regular.

If $\lim_j w_j > 0$ we would have $s_n \approx n$ for $n \in \mathbb{N}$ and, then, $S = 0$. We would also have $m(\mathbf{s}) = \ell_{\infty}$. Therefore, by Proposition 1, $m(\mathbf{s})$ would be non-ℓ_1-splicing. So, we assume from now on that $\lim_j w_j = 0$. Then, by Proposition 4 (i), the unit vector system of $m(\mathbf{t})$ is the dual basic sequence of the unit vector basis of $d(\mathbf{w}, 1)$.

Given a bijection $\pi \colon \mathbb{N}^2 \to \mathbb{N}$ we define a disjointly supported sequence $\mathcal{B}_{\pi} = (f_n)_{n=1}^{\infty}$ in $\mathbb{F}^{\mathbb{N}}$ by

$$f_n = (a_{j,n})_{j=1}^{\infty}, \quad a_{j,n} = \begin{cases} w_i & \text{if } j = \pi(i, n), \\ 0 & \text{otherwise.} \end{cases}$$

The non-increasing rearrangement of each sequence f_n is the sequence $(w_i)_{i=1}^{\infty}$. Then, by (4), $\|f_n\|_{m(\mathbf{t})} = 1$. We infer that $\mathcal{B}_{\pi}$ is a symmetric basic sequence in $m(\mathbf{t})$. Given $m \in \mathbb{N}$ the non-increasing rearrangement of $\sum_{n=1}^{m} f_n$ is the sequence

$$(\underbrace{w_1, \dots, w_1}_{m}, \dots, \underbrace{w_k, \dots, w_k}_{m}, \dots).$$

Therefore, applying (4) and taking into account that $\mathbf{w}$ is non-increasing,

$$\left\| \sum_{n=1}^{m} f_n \right\|_{m(\mathbf{t})} = \sup_{\substack{k \ge 1 \\ 1 \le r \le m}} \frac{r w_k + m \sum_{i=1}^{k-1} w_i}{\sum_{i=1}^{m(k-1)+r} w_i}$$

$$\le \sup_{\substack{k \ge 1 \\ 1 \le r \le n}} a_{k,r}^{(m)},$$

where

$$a_{k,r}^{(m)} = \frac{r w_k + m \sum_{i=1}^{k-1} w_i}{\frac{r}{m} \sum_{i=1+m(k-1)}^{mk} w_i + \sum_{i=1}^{m(k-1)} w_i}.$$

Since, for any $a, b, c, d \in (0, \infty)$, the mapping $t \mapsto (a + bt)/(c + dt)$ is monotone in $(0, \infty)$ we have $a_{k,r}^{(m)} \le \max\{a_{k,0}^{(m)}, a_{k,m}^{(m)}\}$ whenever $1 \le r \le m$ and $k \ge 2$. Put

$$b_k^{(m)} = \frac{\sum_{i=1}^{k} w_i}{\sum_{i=1}^{mk} w_i}, \ k, m \in \mathbb{N}, \quad B^{(m)} = \sup_{k \in \mathbb{N}} b_k^{(m)}, \ m \in \mathbb{N}, \quad B = \inf_{m \in \mathbb{N}} B_m.$$

Note that $a_{k,m}^{(m)} = a_{k+1,0}^{(m)} = mb_k^{(m)}$ for every $k \in \mathbb{N}$, and that $a_{1,r}^{(m)} = mb_1^{(m)}$ for every $r \in \mathbb{N}$. Consequently,

$$\left\| \sum_{n=1}^{m} f_n \right\|_{m(\mathbf{t})} = mB^{(m)}, \quad m \in \mathbb{N}. \tag{6}$$

Taking into account Theorem 4, we have to prove that $B > 0$ if and only if $m(\mathbf{t})$ contains a disjointly supported sequence equivalent to the unit vector basis of ℓ_1.

Assume that $B > 0$. Pick a bijection π from $\mathbb{N}^2$ onto $\mathbb{N}$ and let $\mathcal{B}_\pi = (f_n)_{n=1}^{\infty}$. By (6), $Bm \leq \| \sum_{n=1}^{m} f_n \|_{m(\mathbf{t})}$ for every $m \in \mathbb{N}$. Then, by Lemma 5, $\mathcal{B}_\pi$, regarded as a sequence in $m(\mathbf{t})$, is equivalent to the unit vector system of ℓ_1.

Reciprocally, assume that $B = 0$. In particular, there is $n \in \mathbb{N}$ such that $s_k/s_{nk} \leq 1/2$ for every $k \in \mathbb{N}$. Then, $\mathbf{s}$ has the LRP and, consequently, we can, and we do, assume that the weight $\mathbf{w}$ above chosen is regular. Therefore, by Part (ii) of Proposition 4,

$$m(\mathbf{s}) = m(\mathbf{t}) = d(\mathbf{v}, \infty),$$

where $\mathbf{v}$ is the discrete derivative of. $\mathbf{w}^{-1}$. Let $(g_n)_{n=1}^{\infty}$ be a disjointly supported sequence in $\mathbb{F}^{\mathbb{N}}$ with $\sup_n \|g_n\|_{d(\mathbf{v}, \infty)} < \infty$. By the very definition of the quasi-norm in $d(\mathbf{v}, \infty)$, there is a bijection $\pi : \mathbb{N}^2 \to \mathbb{N}$ such that $(g_n)_{n=1}^{\infty} \lesssim \mathcal{B}_\pi$. Consequently, by (6),

$$\left\| \sum_{n=1}^{m} g_n \right\|_{d(\mathbf{v},\infty)} \lesssim mB^{(m)}, \quad m \in \mathbb{N}.$$

We infer that $\inf_m m^{-1} \| \sum_{n=1}^{m} g_n \|_{d(\mathbf{v},\infty)} = 0$. Then, $(g_n)_{n=1}^{\infty}$, regarded as a sequence in $d(\mathbf{v}, \infty)$, is not equivalent to the unit vector system of ℓ_1. $\qquad \square$

Remark 1 Suppose that $\mathbf{w} = (w_n)_{n=1}^{\infty}$ in $c_0 \setminus \ell_1$ is non-increasing and fulfils

$$\limsup_n \frac{w_k}{nw_{kn}} = 0. \tag{7}$$

Lechner [12] proved that the dual basis of the unit vector system of the sequence Lorentz space $d(\mathbf{w}, 1)$ is non-ℓ_1-splicing. Consequently, in light of Proposition 4 (i) and Theorem 5, the primitive weight $(s_n)_{n=1}^{\infty}$ of $\mathbf{w}$ satisfies (5). Let us write down a direct proof for this fact. Pick $\varepsilon > 0$. There is $n \in \mathbb{N}$ such that $w_i \leq \varepsilon n w_{in}$ for every $i \in \mathbb{N}$. Then, if

$$t_i = \sum_{j=in-n+1}^{in} w_j,$$

we have $w_i \leq \varepsilon t_i$ for every $i \in \mathbb{N}$. For all $k \in \mathbb{N}$ we obtain

$$\frac{s_k}{s_{kn}} = \frac{\sum_{i=1}^{k} w_i}{\sum_{i=1}^{k} t_i} \leq \frac{\sum_{i=1}^{k} \varepsilon t_i}{\sum_{i=1}^{k} t_i} = \varepsilon.$$

We emphasize that the converse "almost" holds. Indeed, if $\mathbf{w} \in c_0$ is essentially decreasing and its primitive weight $\mathbf{s} = (s_n)_{n=1}^\infty$ fulfils (5), the proof of Theorem 5 gives a non-increasing regular weight $\mathbf{w}'$ whose primitive weight is equivalent to $\mathbf{s}$. Then $d(\mathbf{w}, 1) = d(\mathbf{w}', 1)$, and $\mathbf{w}'$ satisfies (7).

To give relevance to Theorem 5 we make the effort of telling apart Marcinkiewicz spaces from ℓ_∞.

Proposition 5 *Let $\mathbf{s} = (s_n)_{n=1}^\infty$ be an increasing weight whose discrete derivative is essentially decreasing. If $\lim_n s_n/n = 0$ then $m(\mathbf{s})$ is not an $\mathcal{L}_\infty$-space. In particular, $m(\mathbf{s})$ is not isomorphic to ℓ_∞.*

Proof Pick $\mathbf{w} \in \mathcal{W}$ whose primitive weight is equivalent to $\mathbf{s}$. Suppose that $m(\mathbf{s})$ is an $\mathcal{L}_\infty$-space. Then, by Proposition 4 (i) and [14, Theorem III], $d(\mathbf{w}, 1)$ is an $\mathcal{L}_1$-space. Since the unit vector system is an unconditional basis of $d(\mathbf{w}, 1)$, taking into account [13, Theorem 6.1], we reach the absurdity $d(\mathbf{w}, 1) = \ell_1$. □

We close this section by writing down the result that arises from combining Theorem 5 with Theorem 1.

Corollary 3 *Let $p \in [1, \infty]$ and let $\mathbf{s} = (s_n)_{n=1}^\infty$ be an increasing weight whose discrete derivative is essentially decreasing and assume that (5) holds. Let $\mathbb{Y}$ be either $m(\mathbf{s})$ or $\ell_p(m(\mathbf{s}))$. Then, if $T \in \mathcal{L}(\mathbb{Y})$, the identity map on $\mathbb{Y}$ factors through either T or $\mathrm{Id}_{\mathbb{Y}} - T$. Consequently, $\ell_p(m(\mathbf{s}))$ is a primary Banach space.*

4 Orlicz sequence spaces

Throughout this section we follow the terminology on Orlicz spaces and Musielak-Orlicz spaces used in the handbooks [15,16]. A *normalized convex Orlicz function* is a convex function $M: [0, \infty) \to [0, \infty)$ such that $M(0) = 0$ and $M(1) = 1$. If M vanishes on a neighborhood of the origin, M is said to be *degenerate*. Given a sequence $\mathbf{M} = (M_n)_{n=1}^\infty$ of normalized convex Orlicz functions, the Musielak–Orlicz norm $\| \cdot \|_{\ell_\mathbf{M}}$ is the Luxemburg norm built from the modular

$$m_\mathbf{M}: \mathbb{F}^\mathbb{N} \to [0, \infty], \quad (a_n)_{n=1}^\infty \mapsto \sum_{n=1}^\infty M_n(|a_n|),$$

that is, $\|f\|_{\ell_\mathbf{M}} = \inf\{t > 0: m_\mathbf{M}(f/t) \leq 1\}$ for all $f \in \mathbb{F}^\mathbb{N}$. The Musielak–Orlicz space $\ell_\mathbf{M}$ is the Banach space consisting of all sequences f for which $\|f\|_{\ell_\mathbf{M}} < \infty$. Orlicz sequence spaces can be obtained as a particular case of Musielak–Orlicz sequence spaces. Namely, if M is a normalized convex Orlicz functions, we put $\ell_M = \ell_\mathbf{M}$, where, if $\mathbf{M} = (M_n)_{n=1}^\infty$, $M_n = M$ for every $n \in \mathbb{N}$. We will denote by h_M the closed linear span of the unit vector system of ℓ_M. It is known (see [15, Proposition 4.a.2]) that

$$h_M = \{f \in \mathbb{F}^\mathbb{N}: \forall s < \infty, \, m_M(sf) < \infty\}. \tag{8}$$

It is clear that the unit vector system is a symmetric basic sequence of ℓ_M for every normalized Orlicz function M. If M^* is the Orlicz function complementary to M we have

$$(h_{M^*})^* = \ell_M \tag{9}$$

under the natural pairing (see [15, Proposition 4.b.1]). So, the unit vector system is a basis of h_M and a weak* basis of ℓ_M.

Let us bring up the following result that we will need.

Theorem 6 (see [16, Theorem 8.11]) *Let $\mathbf{M} = (M_n)_{n=1}^\infty$ and $\mathbf{N} = (N_n)_{n=1}^\infty$ be sequences of normalized convex Orlicz functions. Then $\ell_{\mathbf{N}} \subseteq \ell_{\mathbf{M}}$ if and only if there are a positive sequence $(a_n)_{n=1}^\infty \in \ell_1$, $\delta > 0$, C and $D \in (0, \infty)$ such that*

$$N_n(t) < \delta \implies M_n(t) \leq C N_n(Dt) + a_n.$$

Remark 2 Theorem 6 gives, in particular, that $\ell_M = \ell_N$ if and only if there are $a, b > 0$ such that $M(t) \approx N(bt)$ for $0 \leq t \leq a$ (see also [15, Proposition 4.a.5]).

If we denote, for $b \in (0, \infty)$,

$$M_b(t) = \frac{M(bt)}{M(b)}, \quad t \geq 0,$$

the indexes α_M and β_M of the non-degenerate normalized convex Orlicz function M are defined by $\alpha_M = \sup A_M$ and, with the convention $\inf \emptyset = \infty$, $\beta_M = \inf B_M$, where

$$A_M = \left\{ q \in [1, \infty) \colon \sup_{0 \leq b, t \leq 1} t^{-q} M_b(t) < \infty \right\},$$

$$B_M = \left\{ q \in [1, \infty) \colon \inf_{0 \leq b, t \leq 1} t^{-q} M_b(t) > 0 \right\}.$$

Note that, by convexity, $M(bt) \leq t M(b)$ for every $t \in [0, 1]$ and $b \in [0, \infty)$. Consequently, $1 \in A_M$ and, hence, α_M is well-defined. Note also that, if $q \in A_M \cap B_M$, then $q = \max A_M = \min B_M$. We infer that $1 \leq \alpha_M \leq \beta_M \leq \infty$.

Our characterization of Orlicz sequence spaces whose unit vector system is non-ℓ_1-splicing will be a consequence of the following result.

Theorem 7 (cf. [15, Theorem 4.a.9]) *Let M be a non-degenerate normalized convex Orlicz function and $1 \leq p \leq \infty$. The following are equivalent.*

  (i) *$p \in [\alpha_M, \beta_M]$.*
 (ii) *There is a disjointly supported sequence with respect to the unit vector system of ℓ_M which is equivalent to the unit vector system of ℓ_p.*
(iii) *There is a block basic sequence with respect to the unit vector system of ℓ_M which is equivalent to the unit vector system of ℓ_p.*

We emphasize that the equivalence between items (i) and (iii) can be easily obtained from [15, Theorem 4.a.9]. Indeed, it follows from combining [3, Proposition 2.14], [4, Theorem 3.3.1] and Bessaga–Pelczyński Selection Principle that, if a Banach space equipped with an unconditional basis $\mathcal{U}$ contains a subsymmetric basic sequence $\mathcal{B}$, then there is a block basic sequence with respect to $\mathcal{U}$ that is equivalent to $\mathcal{B}$. As it is obvious that (iii) implies (ii), our contribution to the theory of sequence Orlicz spaces consists in proving that (ii) implies (i). Nonetheless, for expository ease, we will put in order all the arguments that come into play in the proof of Theorem 7. We start by writing down some terminology and claims from [15].

Given a non-degenerate normalized convex Orlicz function, the set $C_{M,1} \subseteq \mathcal{C}([0, 1/2])$ is the smallest closed convex set containing $\{M_b : 0 < b \leq 1\}$. Note that every function in $C_{M,1}$ extends to a normalized convex Orlicz function. So, we can safely define ℓ_F for $F \in C_{M,1}$.

Theorem 8 (cf. [15, Theorem 4.a.8]) *Let M and F be normalized convex Orlicz functions. Assume that M is non-degenerate and that $F \in C_{M,1}$. Then, there is a block basic sequence of the unit vector system of ℓ_M that is equivalent to the unit vector system of ℓ_F.*

Proof Lindenstrauss–Tzafriri's proof of the "if" part of [15, Theorem 4.a.8] gives exactly this result. □

Given $p \in [1, \infty)$, F_p will denote the potential function given by $F_p(t) = t^p$, $t \geq 0$. We denote by F_∞ the degenerate Orlicz function defined by $M(t) = 0$ if $0 \leq t \leq 1/2$ and $M(t) = 2t - 1$ if $t > 1/2$. Of course, $\ell_{F_p} = h_{F_p} = \ell_p$ for $1 \leq p < \infty$, $\ell_{F_\infty} = \ell_\infty$, and $h_{F_\infty} = c_0$.

Theorem 9 (see [15, Comments below Theorem 4.a.9]) *Let M be a non-degenerate normalized convex Orlicz function and let $p \in [\alpha_M, \beta_M]$. Then $F_p \in C_{M,1}$.*

Proof Lindenstrauss–Tzafriri's proof of the "if" part of [15, Theorem 4.a.9] contains a proof of this result. □

We say that a function $M: [0, \infty) \to [0, \infty)$ satisfies the Δ_2-condition at zero if there is $a > 0$ such that $M(2t) \lesssim M(t)$ for $0 \leq t \leq a$. Note that a non-degenerate normalized convex Orlicz function M satisfies the Δ_2-condition at zero if and only if $M(t) \lesssim M(t/2)$ for $0 \leq t \leq 1$.

Theorem 10 (cf. [15, Proof of Theorem 4.a.9]) *Let M be a non-degenerate normalized convex Orlicz function. The following are equivalent.*

  (i) $\beta_M < \infty$.
 (ii) *M satisfies the Δ_2-condition at zero.*
(iii) $\ell_M = h_M$.
(iv) $F_\infty \notin C_{M,1}$.

Proof First, we prove (i) $\Longrightarrow$ (ii). Assume that $\beta_M < \infty$. Then there are $1 \leq q, C < \infty$ such that $M(b) \leq Ct^{-q}M(bt)$ for every $(b, t) \in (0, 1]^2$. In particular, $M(b) \leq C2^q M(b/2)$ for every $0 < b \leq 1$.

(ii) $\Longrightarrow$ (iii) is a part of [15, Proposition 4.a.4].

Let us we prove (iii) $\Longrightarrow$ (iv). If $\ell_M = h_M$ then, by [15, Proposition 4.a.4], the unit vector system is a boundedly complete basis of ℓ_M. Then, by [4, Theorem 3.3.2], no basic sequence of the unit vector system of ℓ_M is equivalent to the unit vector system of $\ell_\infty = \ell_{F_\infty}$. Therefore, by Theorem 8, $F_\infty \notin C_{M,1}$.

We go on by proving (iv) $\Longrightarrow$ (ii). Assume that $F_\infty \notin C_{M,1}$. Then there is constant $c > 0$ such that

$$\sup\{M_b(t) \colon 0 \le t \le 1/2\} = M_b(1/2) \ge c$$

for every $b \in (0, 1]$. In other words, $M(b) \le c^{-1} M(b/2)$ for every $0 \le b \le 1$.

Finally, we prove (ii) $\Longrightarrow$ (i). Let $C \ge 2$ be such that $M(b) \le CM(b/2)$ for every $b \in (0, 1]$. Choose $q = \log_2(C)$. Given $0 < t \le 1$, pick $n \in \mathbb{N}$ such that $2^{-n} < t \le 2^{-n+1}$. We have

$$M(b) \le C^n M(2^{-n}b) = C2^{(n-1)q} M(2^{-n}b) \le Ct^{-q} M(tb).$$

Therefore,

$$\inf\{t^{-q} M_b(t) \colon 0 < b \le 1\} \ge C^{-1} > 0.$$

Consequently, $\beta_M \le q < \infty$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

For tackling the proof of Theorem 7 we need to study functions constructed from sequences belonging to Orlicz spaces. Given a normalized convex Orlicz function M and $f = (b_j)_{j=1}^\infty \in \mathbb{F}^\mathbb{N}$ we define

$$M_f \colon [0, \infty) \to [0, \infty], \quad s \mapsto \sum_{j=1}^\infty M(|b_j|s).$$

Lemma 6 *Let M be a normalized convex Orlicz function and let $f \in \mathbb{F}^\mathbb{N}$ with $0 < R := \|f\|_{\ell_M} < \infty$. We have the following.*

(i) $\{s \in [0, \infty] \colon M_f(s) \le 1\} = [0, 1/R]$.
(ii) *If there is $s > 1/R$ such that $M_f(s) < \infty$, then $M_f(1/R) = 1$.*
(iii) M_f *is convex in $[0, \infty]$.*

Proof By definition, $M_f(s) \le 1$ if $s < 1/R$, and $M_f(s) > 1$ if $s > 1/R$. By the Monotone Convergence Theorem, $M_f(1/R) \le 1$. Consequently, Ad (i) holds. By the Dominated Convergence Theorem, M_f is continuous on the interval $\{s \in [0, \infty) \colon M_f(s) < \infty\}$. Therefore, Ad (ii) also holds. The proof of Ad (iii) is straightforward. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

Lemma 6 alerts us that, even if $f \in S_{\ell_M}$, $M_f(1)$ may be different from 1. This drawback, caused by dealing with ℓ_M instead of its separable part h_M, motivates the following definition.

Definition 2 Let M be a normalized Orlicz function and let $f \in S_{\ell_M}$. We define $N_f \colon [0, \infty) \to [0, \infty)$ by

$$N_f(t) = \begin{cases} M_f(t) & \text{if } 0 \leq t \leq 1/2, \\ (1 - M_f(1/2))(2t - 1) + M_f(1/2) & \text{if } 1/2 \leq t < \infty. \end{cases}$$

Lemma 7 *Let M be a normalized Orlicz function and let $f \in S_{\ell_M}$. Then N_f is a normalized Orlicz function with $M_f(t) \leq N_f(t)$ for every $0 \leq t \leq 1$ and $M_f(t) = N_f(t)$ for every $0 \leq t \leq 1/2$.*

Proof By definition, $N_f(1) = 1$, and N_f is linear in $[1/2, \infty)$. The function M_f is convex in $[0, 1]$, and $M_f(1) \leq 1$. Consequently, for $1/2 \leq t \leq 1$,

$$\frac{M_f(t) - M_f(1/2)}{t - 1/2} \leq \frac{M_f(1) - M_f(1/2)}{1 - 1/2} \leq \frac{1 - M_f(1/2)}{1 - 1/2} = \frac{N_f(t) - N_f(1/2)}{t - 1/2}$$

and, hence, $M_f(t) \leq N_f(t)$. It is clear that $N_f(0) = 0$, that N_f is continuous, and that N_f is convex in $[0, 1/2]$. If $0 \leq s \leq 1/2 \leq t$,

$$\frac{N_f(1/2) - N_f(s)}{1/2 - s} \leq \frac{M_f(1) - M_f(1/2)}{1 - 1/2} \leq \frac{N_f(t) - N_f(1/2)}{t - 1/2}.$$

We infer that N_f is convex. $\qquad\square$

Lemma 8 *Let M be a normalized convex Orlicz function and $\mathcal{F} = (f_n)_{n=1}^{\infty}$ be a disjointly supported sequence in ℓ_M. Then, the basic sequence $\mathcal{F}$ is isometrically equivalent to the unit vector basis of the sequence space defined from the modular*

$$m_{\mathcal{F}}(f) = \sum_{n=1}^{\infty} M_{f_n}(|a_n|), \quad f = (a_n)_{n=1}^{\infty} \in \mathbb{F}^{\mathbb{N}}. \tag{10}$$

Proof Choose $\nu \colon \mathbb{N} \to \mathbb{N}$ such that $j \in \operatorname{supp}(f_{\nu(j)})$ and $j \notin \operatorname{supp}(f_n)$ if $n \neq \nu(j)$. Put $f_n = (b_{j,n})_{j=1}^{\infty}$ for every $n \in \mathbb{N}$. Given $f = (a_n)_{n=1}^{\infty} \in \mathbb{F}^{\mathbb{N}}$ we have $\sum_{n=1}^{\infty} a_n f_n = (a_{\nu(j)} b_{j,\nu(j)})_{j=1}^{\infty}$. Consequently,

$$m_M \left(\sum_{n=1}^{\infty} a_n f_n \right) = \sum_{j=1}^{\infty} M(|a_{\nu(j)} b_{j,\nu(j)}|) = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} M(|a_n b_{j,n}|) = m_{\mathcal{F}}(f).$$

This equality between modulars yields the desired inequality between norms. $\qquad\square$

Lemma 9 *Let M be a normalized convex Orlicz function and $\mathcal{F} = (f_n)_{n=1}^{\infty}$ be a disjointly supported sequence. Suppose that $\|f_n\|_{\ell_M} = 1$ and that $\lambda = \inf_n M_{f_n}(1/2) > 0$. Then, if $\mathbf{N} = (N_{f_n})_{n=1}^{\infty}$,*

$$\left\| \sum_{n=1}^{\infty} a_n f_n \right\|_{\ell_M} \approx \left\| (a_n)_{n=1}^{\infty} \right\|_{\ell_{\mathbf{N}}}$$

for $(a_n)_{n=1}^{\infty} \in \mathbb{F}^{\mathbb{N}}$.

Proof Notice that $M_{f_n}(1/2) \leq M_{f_n}(1) \leq 1$ for every $n \in \mathbb{N}$ and, hence, $\lambda \leq 1$. In light of Lemma 8, we need to prove that, if $\| \cdot \|_{\mathcal{F}}$ denotes the norm induced by $m_{\mathcal{F}}$, the norms $\| \cdot \|_{\ell_{\mathbf{N}}}$ and $\| \cdot \|_{\mathcal{F}}$ are equivalent.

By Lemma 7, $m_{\mathcal{F}}(f) \leq m_{\mathbf{N}}(f)$ whenever $m_{\mathbf{N}}(f) \leq 1$. Consequently, $\|f\|_{\mathcal{F}} \leq \|f\|_{\ell_{\mathbf{N}}}$ for every $f \in \mathbb{F}^{\mathbb{N}}$.

Conversely, suppose that $\|f\|_{\mathcal{F}} < \lambda$. Then, $m_{\mathcal{F}}(f/\lambda) \leq 1$. The convexity of M_{f_n} yields, if $f = (a_n)_{n=1}^{\infty}$,

$$M_{f_n}(|a_n|) \leq \lambda M_{f_n}\left(\frac{|a_n|}{\lambda}\right) \leq \lambda \leq M_{f_n}\left(\frac{1}{2}\right), \quad n \in \mathbb{N}.$$

Consequently, $|a_n| \leq 1/2$ for every $n \in \mathbb{N}$. Therefore,

$$m_{\mathbf{N}}(f) = \sum_{n=1}^{\infty} M_{f_n}(|a_n|) \leq \sum_{n=1}^{\infty} M_{f_n}\left(\frac{|a_n|}{\lambda}\right) \leq 1.$$

Hence, $\|f\|_{\ell_{\mathbf{N}}} \leq 1$. We infer that $\|f\|_{\ell_{\mathbf{N}}} \leq \lambda^{-1}\|f\|_{\mathcal{F}}$ for every $f \in \mathbb{F}^{\mathbb{N}}$. □

Proposition 6 *Let M be a non-degenerate normalized convex Orlicz function. Then $M_f \in C_{M,1}$ for every $f \in S_{\ell_M}$.*

Proof Assume, without loss of generality, that $f = (b_j)_{j=1}^{\infty} \in [0, \infty)^{\mathbb{N}}$. Denote $\lambda_j = M(b_j)$ for $j \in B := \mathrm{supp}(f)$. By Part (i) of Lemma 6, $\sum_{j \in B} \lambda_j \leq 1$. Denote $\lambda_\infty := 1 - \sum_{j \in B} \lambda_j$. In the case when $\lambda_\infty = 0$ we have

$$\sum_{j \in B} \lambda_j = 1 \quad \text{and} \quad M_f = \sum_{j \in B} \lambda_j M_{b_j}.$$

In the case when $\lambda_\infty > 0$, by Part (ii) of Lemma 6 and the identity (8), $f \in \ell_M \setminus h_M$. Therefore, by Theorem 10, $F_\infty \in C_{M,1}$. The identities

$$\lambda_\infty + \sum_{j \in B} \lambda_j = 1 \quad \text{and} \quad M_f(t) = \lambda_\infty F_\infty(t) + \sum_{j \in B} \lambda_j M_{b_j}(t)$$

for every $t \in [0, 1/2]$ yield that, in both cases, M_f is a (possibly infinite) convex combination of functions in $C_{M,1}$. Consequently, $M_f \in C_{M,1}$. □

We are now in a position to complete the proof of the main theorem of this section.

Proof of Theorem 7 (i) $\Longrightarrow$ (iii) follows from combining Theorem 9 with Theorem 8, and (iii) $\Longrightarrow$ (ii) is obvious. In order to prove that (ii) implies (i), we pick a disjointly supported sequence $\mathcal{F} = (f_n)_{n=1}^{\infty}$ in ℓ_M equivalent to the unit vector basis of ℓ_p. By unconditionality, we can assume, without loss of generality, that $\|f_n\|_{\ell_M} = 1$ for every $n \in \mathbb{N}$. Then, by Proposition 6, $M_{f_n} \in C_{M,1}$ for every $n \in \mathbb{N}$.

We consider two opposite situations according to the behavior of the numbers $\lambda_n = M_{f_n}(1/2) \in [0, 1]$ for $n \in \mathbb{N}$.

Assume that $\inf_n \lambda_n = 0$. Then, passing to a suitable subsequence, we can suppose that $\sum_{n=1}^{\infty} \lambda_n \leq 1$. If $m_{\mathcal{F}}$ is as in (10) and $\| \cdot \|_{\mathcal{F}}$ is its associated norm, we have

$$m_{\mathcal{F}}\left(\frac{f}{2\|f\|_{\infty}}\right) \leq 1$$

and, hence, $\|f\|_{\mathcal{F}} \leq 2\|f\|_{\infty}$ for every $f \in \mathbb{F}^{\mathbb{N}}$. Consequently, by Lemma 8, the unit vector system of ℓ_{∞} dominates the basic sequence $\mathcal{F}$. We infer that $p = \infty$. Moreover $\lim_n M_{f_n} = 0$ uniformly in $[0, 1/2]$ and, hence, applying Proposition 6, we obtain $F_{\infty} \in C_{M,1}$.

Assume that $\inf_n \lambda_n > 0$. Taking into account that $C_{M,1}$ is compact by [15, Lemma 4.a.6], passing to a suitable subsequence, we can suppose that there is $F \in C_{M,1}$ such that

$$\sup_{0 \leq t \leq 1/2} |M_{f_n}(t) - F(t)| \leq 2^{-n} \tag{11}$$

for every $n \in \mathbb{N}$. Therefore, if $\mathbf{N} = (N_{f_n})_{n=1}^{\infty}$, applying Theorem 6 yields $\ell_{\mathbf{N}} = \ell_F$. Consequently, by Lemma 9, $\ell_p = \ell_F$.

In both cases, there is $F \in C_{M,1}$ such that $\ell_F = \ell_p$. By Remark 2, for such a function F, there is $a > 0$ such that $F(t) \approx F_p(t) = t^p$ for $0 \leq t \leq a$.

Let $r < \alpha_M$. There is a constant $C_1 < \infty$ such that $M_b(t) \leq C_1 t^r$ for every $0 < b \leq 1$ and every $0 \leq t \leq 1$. By convexity and continuity, $N(t) \leq C_1 t^r$ for every $N \in C_{M,1}$ and every $0 \leq t \leq 1/2$. Consequently, there is $C_2 < \infty$ such that $F_p(t) \leq C_2 t^r$ for every $0 \leq t \leq a$. We infer that $r \leq p$. Letting r tend to α_M we obtain $\alpha_M \leq p$. We prove that $p \leq \beta_M$ in a similar way. □

Theorem 11 *Let M be a non-degenerate normalized convex Orlicz function. Then the unit vector system of ℓ_M is non-ℓ_1-splicing if and only if $1 < \alpha_M$.*

Proof Taking into account the identity (9), the result follows from combining Theorem 4 with Theorem 7. □

Remark 3 Define, for an Orlicz function M,

$$\Omega_n = \inf\left\{\rho > 0 : \sup_{0 < t \leq 1} \frac{M(t/\rho)}{M(t)} \leq \frac{1}{n}\right\}, \quad n \in \mathbb{N}.$$

Lechner [12, Theorem 6.3] proved that if $\lim_n \Omega_n/n = 0$ then the unit vector system of ℓ_M is non-ℓ_1-splicing. Let us give a proof based on Theorem 11 for this result. Assume that $\lim_n \Omega_n/n = 0$. In particular, there is $n \geq 3$ such that $\Omega_n/n < 1/2$. Consequently, there is $\rho \leq R := n/2$ with

$$\frac{M(t/\rho)}{M(t)} \leq \frac{1}{n}, \quad 0 < t \leq 1.$$

Therefore, $M(R^{-1}t) \leq (2R)^{-1}M(t)$ for every $t \in (0, 1]$. We deduce by induction that $M(R^{-k+1}t) \leq (2R)^{-k+1}M(t)$ for every $t \in (0, 1]$ and $k \in \mathbb{N}$. Since $R > 1$,

$q := 1 + \log_R 2 > 1$. Let $u \in (0, 1]$ and pick $k \in \mathbb{N}$ with $R^{-k} < u \leq R^{-k+1}$. We have

$$M(ut) \leq M(R^{-k+1}t) \leq (2R)^{-k+1}M(t) = 2RR^{-qk}M(t) \leq 2Ru^q M(t).$$

Consequently, $q \in A_M$ and, hence, $\alpha_M > 1$.

As in Sect. 3, we close by telling apart Orlicz sequence spaces from ℓ_∞ and writing down the straightforward consequence of combining Theorem 1 with Theorem 11. We emphasize that, in light of Theorem 2 and the results achieved in [9], Theorem 12 is a novelty only in the case when ℓ_M is non-separable, i.e., when $\beta_M = \infty$.

Proposition 7 *Let M be a normalized convex Orlicz function. Then ℓ_M is a $\mathcal{L}_\infty$-space if and only if M is degenerate.*

Proof Let M^* be the complementary Orlicz function of M. If ℓ_M were a $\mathcal{L}_\infty$-space, then h_{M^*} would be a $\mathcal{L}_1$-space. Since the unit vector basis of h_{M^*} is unconditional, we would obtain $h_{M^*} = \ell_1$. Therefore, $\ell_M = \ell_\infty = \ell_{F_\infty}$. Consequently, there would be $0 < a \leq 1/2$ such that $M(t) = F_\infty(t) = 0$ for every $0 \leq t \leq a$. □

Theorem 12 *Let $p \in [1, \infty]$ and M be a non-degenerate normalized convex Orlicz function with $\alpha_M > 1$. Let $\mathbb{Y} = \ell_M$ or $\mathbb{Y} = \ell_p(\ell_M)$. Then, if $T \in \mathcal{L}(\mathbb{Y})$, the identity map on $\mathbb{Y}$ factors through either T or $\mathrm{Id}_{\mathbb{Y}} - T$. Consequently, $\ell_p(\ell_M)$ is a primary Banach space.*

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