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ORIGINAL PAPER

On the classes of functions of generalized bounded variation

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Abstract
The properties of the class of functions of generalized bounded variation are studied. The "anomaly" feature of this class is revealed. There is the notation of absolute continuity with respect to $((p_n), \phi)$ and it's connection with the ordinary absolute continuity is investigated. The problems of approximation by Steklov's functions and singular integrals are studied.

Keywords Bounded variation · Absolute continuity · Convergence in variation

Mathematics Subject Classification 26A45 · 26A46 · 42A24

1 Introduction

The notion of bounded variation was based by Jordan [4]. Wiener [9] considered the class of functions BV_p. Love [6] studied functional properties of this class. Young [10] intoduced the notion of Φ-variation. Musielak and Orlicz [7] studied properties of this class. Waterman [8] studied class of functions of bounded Λ-variation. Chanturia [3] defined notion of modulus of variation. Kita and Yoneda [5] introduced new class of functions of bounded variation. Akhobadze [1,2] generalized the last class and studied properties of it. This bibliography can be continued (see e.g. [11]).

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Definition 1.1 Let $f(t)$ be a function defined on a finite closed interval $[a, b]$. Suppose p_n and $\phi(n)$ be a sequences such that $p_1 \geq 1$, $p_n \uparrow \infty$, $n \to \infty$ and $\phi(1) \geq 1$, $\phi(n) \uparrow \infty$, $n \to \infty$. We say that $f \in BV(p_n \uparrow \infty, \phi, [a, b])$ if

$$V(f, p_n \uparrow \infty, \phi, [a,b]) = \sup_n \sup_\Delta \left(\sum_{i=1}^{m} |f(t_i) - f(t_{i-1})|^{p_n} : \rho(\Delta) \geq \frac{1}{\phi(n)} \right)^{1/p_n} < +\infty,$$

where Δ is $a = t_0 < t_1 < \cdots < t_m = b$ partition of the interval $[a, b]$ and $\rho(\Delta) = \min_i(t_i - t_{i-1})$.

In the case, $\phi(n) = 2^n$, class $BV(p_n \uparrow \infty, \phi, [a, b])$ is considered by Kita and Yoneda [5]. Sometimes for the simplicity we use notation $V(f, p_n \uparrow \infty, \phi)$ in place of $V(f, p_n \uparrow \infty, \phi, [a, b])$.

2 Some properties of functions of generalized bounded variation

It is easy to verify that $BV(p_n \uparrow \infty, \phi, [a, b])$ is a normed space, with the norme

$$||f|| = |f(a)| + V(f, p_n \uparrow \infty, \phi, [a, b]).$$

Proposition 2.1 (a) $BV(p_n \uparrow \infty, \phi)$ *is a linear space and for each* α *and* β *we have*

$$V(\alpha f + \beta g, p_n \uparrow \infty, \phi) \leq |\alpha| V(f, p_n \uparrow \infty, \phi) + |\beta| V(g, p_n \uparrow \infty, \phi).$$

(b) $BV(p_n \uparrow \infty, \phi, [a, b])$ *is a complete space.*
(c) $BV(p_n \uparrow \infty, \phi, [a, b])$ *is not separable.*
(d) *If at each point* t *of* $[a, b]$ *interval* $\lim_{k\to\infty} f_k(t) = f(t)$*, then*

$$V(f, p_n \uparrow \infty, \phi) \leq \liminf_{k\to\infty} V(f_k, p_n \uparrow \infty, \phi).$$

Proof (a) It is clear.
(b) Let (f_s) be a fundamental sequence in $BV(p_n \uparrow \infty, \phi, [a, b])$. Then for every $\varepsilon > 0$ there exists a positive integer $N(\varepsilon)$, such that for each natural numbers $i, r > N(\varepsilon)$ we have

$$||f_i - f_r|| = |(f_i - f_r)(a)| + V(f_i - f_r, p_n \uparrow \infty, \phi, [a, b]) < \varepsilon. \tag{1}$$

By definition of this variation for every $t \in [a, b]$ we have

$$\begin{aligned} |(f_i - f_r)(t)| - |(f_i - f_r)(a)| &\leq |(f_i - f_r)(t) - (f_i - f_r)(a)| \\ &\leq V(f_i - f_r, p_n \uparrow \infty, \phi, [a, b]). \end{aligned}$$

Thus (1) implies that

$$|(f_i - f_r)(t)| \le |(f_i - f_r)(a)| + V(f_i - f_r, p_n \uparrow \infty, \phi, [a,b]) < \varepsilon.$$

This means uniformly convergence of the sequence (f_s). Let $f_r \to f$ uniformly and consider an arbitrary partition of $[a,b]$ such that $\rho(\Delta) \ge \frac{1}{\phi(n)}$. For each $i, r > N(\varepsilon)$ we have

$$\left(\sum_{k=1}^{m} |(f_i - f_r)(t_k) - (f_i - f_r)(t_{k-1})|^{p_n}\right)^{1/p_n} < \varepsilon.$$

Considering the limit $r \to +\infty$ in the last inequality, we get

$$\left(\sum_{k=1}^{m} |(f_i - f)(t_k) - (f_i - f)(t_{k-1})|^{p_n}\right)^{1/p_n} \le \varepsilon.$$

Therefore,

$$V(f_i - f, p_n \uparrow \infty, \phi, [a,b]) \to 0, i \to \infty.$$

Now, by property (a), for each fixed i $(i > N(\varepsilon))$ we obtain

$$\begin{aligned} |f(a)| + V(f, p_n \uparrow \infty, \phi, [a,b]) &\le |f(a)| + V(f - f_i, p_n \uparrow \infty, \phi, [a,b]) \\ +V(f_i, p_n \uparrow \infty, \phi, [a,b]) &\le |f(a)| + \varepsilon + V(f_i, p_n \uparrow \infty, \phi, [a,b]). \end{aligned}$$

(c) Let $a < x_0 < b$ and

$$f_{x_0}(t) = \begin{cases} 0, \ if & a \le t \le x_0, \\ 1, \ if & x_0 < t \le b. \end{cases}$$

It is easy to see that $f \in BV(p_n \uparrow \infty, \phi, [a,b])$ and if f_{x_0} and f_{x_1} are two functions corresponding to distinct points $x_0 < x_1$ from (a,b), then we have

$$||f_{x_0} - f_{x_1}|| = V(f_{x_0} - f_{x_1}, p_n \uparrow \infty, \phi) \ge \left|(f_{x_0} - f_{x_1})(x_1) - (f_{x_0} - f_{x_1})(0)\right| = 1.$$

The set of f_{x_0} functions is uncountable and distance between to two different functions is greater then 1. Thus $BV(p_n \uparrow \infty, \phi, [a,b])$ is not separable.

(d) Let

$$A := \liminf_{k \to \infty} V(f_k, p_n \uparrow \infty, \phi),$$

then there exists such a subsequence f_{k_r} that

$$\lim_{r \to \infty} V(f_{k_r}, p_n \uparrow \infty, \phi) = A.$$

For every $\varepsilon > 0$ there exists a constant $N(\varepsilon)$, such that

$$V(f_{k_r}, p_n \uparrow \infty, \phi) < A + \varepsilon, \quad r > N(\varepsilon).$$

Let $a = t_0 < t_1 < \cdots < t_m = b$ be an arbitrary partition of interval $[a, b]$ and $|t_i - t_{i-1}| \geq \frac{1}{\phi(n)}$, $i = 1, 2, ..., m$, then

$$\left(\sum_{i=1}^{m} \left| f_{k_r}(t_i) - f_{k_r}(t_{i-1}) \right|^{p_n}\right)^{1/p_n} \leq V(f_{k_r}, p_n \uparrow \infty, \phi) < A + \varepsilon.$$

Hence

$$\left(\sum_{i=1}^{m} | f(t_i) - f(t_{i-1})|^{p_n}\right)^{1/p_n} \leq A + \varepsilon.$$

This implies that

$$V(f, p_n \uparrow \infty, \phi) \leq A.$$

Definition 2.2 A sequence of f_n functions will be termed convergent in variation to f if $V(f_n - f, p_n \uparrow \infty, \phi) \to 0$ for $n \to \infty$.

Convergence in variation implies uniformly convergence, in general. If $\phi(n)^{\frac{1}{p_n}}$ is bounded then it is easy to see that they are equivalent. If $\phi(n)^{\frac{1}{p_n}}$ is not bounded then there exists uniformly convergent sequence of functions which is not convergent in variation. Indeed, there exists a subsequence $\phi(n_k)^{\frac{1}{p_{n_k}}} \to \infty$ and let

$$f_k(t) = \frac{1}{\phi(n_k)^{1/p_{n_k}}} \sin\left(2\pi[\phi(n_k)/4]t\right), \quad t \in [0, 1].$$

Here and in the sequel $[a]$ denotes the integer part of a number a. It is clear that $f_k \to 0$ uniformly on $[0, 1]$. Let us consider points $t_i^k = \frac{i}{4[\phi(n_k)/4]}$, $i = 0, 1, \ldots, 4[\phi(n_k)/4]$.

It is obvious that $t_i^k - t_{i-1}^k = \frac{1}{4[\phi(n_k)/4]} \geq \frac{1}{\phi(n_k)}$. We get

$$\left(\sum_{i=1}^{4[\phi(n_k)/4]} \left| f_k(t_i^k) - f_k(t_{i-1}^k) \right|^{p_{n_k}}\right)^{1/p_{n_k}} = (4[\phi(n_k)/4])^{1/p_{n_k}} \cdot \frac{1}{\phi(n_k)^{1/p_{n_k}}} \geq \frac{1}{2},$$

when $\phi(n_k) \geq 8$. This means that $V(f_k, p_n \uparrow \infty, \phi, [0, 1]) \geq \frac{1}{2}$ for every sufficiently big k.

Lemma 2.3 *Let f be a function defined on $[a, b]$ and $t_0 < t_1 < \cdots < t_m$ be an arbitrary set of points in $[a, b]$ such that $|t_i - t_{i-1}| \geq \frac{1}{\phi(n)}$, $i = 1, 2, \ldots, m$. Then*

$$\left(\sum_{i=1}^{m} |f(t_i) - f(t_{i-1})|^{p_n}\right)^{1/p_n} \leq 3V(f, p_n \uparrow \infty, \phi, [a, b]).$$

Proof It is obvious that

$$\left(\sum_{i=1}^{m} |f(t_i) - f(t_{i-1})|^{p_n}\right)^{1/p_n}$$
$$\leq \left(\sum_{i=1}^{m} |f(t_i) - f(t_{i-1})|^{p_n} + |f(t_1) - f(a)|^{p_n} + |f(b) - f(t_{m-1})|^{p_n}\right)^{1/p_n}$$
$$\leq \left(\sum_{i=2}^{m-1} |f(t_i) - f(t_{i-1})|^{p_n} + |f(t_1) - f(a)|^{p_n} + |f(b) - f(t_{m-1})|^{p_n}\right)^{1/p_n}$$
$$+|f(t_1) - f(t_0)| + |f(t_m) - f(t_{m-1})| \leq 3V(f, p_n \uparrow \infty, \phi).$$

□

3 On "anomalous" property of the class of function of generalized bounded variation

Proposition 3.1 *Let $p_1 \geq 1$, $p_n \uparrow \infty$ and $\phi(1) \geq 1$, $\phi(n) \uparrow \infty$. Then for each point $x \in (a, b)$ there exists $y \in (x, b)$, and a function f defined on $[a, b]$ such that*

$$V(f, p_n \uparrow \infty, \phi, [a, y]) < V(f, p_n \uparrow \infty, \phi, [a, x]).$$

Proof (i) Let r be the least positive integer such that : $x - a \geq \frac{2}{\phi(r)}$;

(ii) $c := x - \frac{1}{\phi(r)}$;

(iii) choose a point $y \in (x, b)$ such that,

$$x < y < x + \frac{1}{\phi(r)},$$

and

$$x < y < c + \frac{1}{\phi(r-1)};$$

(iv) choose a number $\xi \in (0, 1)$ such that

$$0 < \xi < (2^{p_{r+1}/p_r} - 2)^{\frac{1}{p_{r+1}}}.$$

Therefore,

$$(2+\xi^{p_{r+1}})^{\frac{1}{p_{r+1}}} < 2^{\frac{1}{p_r}}.$$

Suppose

$$f(t)=\begin{cases}1, if & t=c,\\ \xi, if & t=y,\\ 0, if & t\in[a,b], t\neq c, t\neq y.\end{cases} \tag{2}$$

We get

$$V(f, p_n\uparrow\infty, \phi, [a,x]) = 2^{p_r}.$$

Indeed, let $\Delta=\{a,c,x\}$. (i) and (ii) implies that $\rho(\Delta)=\frac{1}{\phi(r)}$. It is clear that

$$2^{\frac{1}{p_r}} = \left(|f(c)-f(a)|^{p_r}+|f(x)-f(c)|^{p_r}\right)^{\frac{1}{p_r}} \leq V(f, p_n\uparrow\infty, \phi, [a,x]). \tag{3}$$

Let $a=t_0<t_1<\cdots<t_m=x$ be an arbitrary Δ partition of the interval $[a,x]$, then we have two cases:

(a) if c is not a point of the partition Δ, then

$$\left(\sum_{i=1}^{m}|f(t_i)-f(t_{i-1})|^{p_n}\right)^{1/p_n} = 0;$$

(b) if c is a point of the partition Δ, then $\rho(\Delta)=\min_i\{t_i-t_{i-1}\}\leq x-c=\frac{1}{\phi(r)}$. Thus if $\rho(\Delta)\geq\frac{1}{\phi(k)}$ then for each partition which contains c, implies that $\frac{1}{\phi(r)}\geq\frac{1}{\phi(k)}$, hence $k\geq r$. Since p_n is strictly increasing we have

$$\left(\sum_{i=1}^{m}|f(t_i)-f(t_{i-1})|^{p_k}\right)^{1/p_k} = 2^{\frac{1}{p_k}} \leq 2^{\frac{1}{p_r}}.$$

Therefore, from these two cases we conclude that for arbitrary partition $a=t_0<t_1<\cdots<t_m=x$, for which $\rho(\Delta)\geq\frac{1}{\phi(n)}$, we obtain

$$\left(\sum_{i=1}^{m}|f(t_i)-f(t_{i-1})|^{p_n}\right)^{1/p_n} \leq 2^{\frac{1}{p_r}}.$$

Thus, from (3) we conclude that

$$V(f, p_n\uparrow\infty, \phi, [a,x]) = 2^{\frac{1}{p_r}}.$$

Now we have to show that

$$V(f, p_n \uparrow \infty, \phi, [a, y]) < 2^{\frac{1}{p_r}} = V(f, p_n \uparrow \infty, \phi, [a, x]).$$

Let $a = t_0 < t_1 < \cdots < t_m = y$ be an arbitrary partition of the interval $[a, y]$. Then we have three cases:

Case 1 c is not in Δ. Then

$$\left(\sum_{i=1}^{m} |f(t_i) - f(t_{i-1})|^{p_n}\right)^{1/p_n} = |f(y) - f(t_{m-1})| = \xi.$$

Case 2 c is in Δ, but no point from (c, y) is in Δ, i.e $t_m = y$ and $t_{m-1} = c$. Thus, (iii) implies

$$\rho(\Delta) \le y - c < \frac{1}{\phi(r-1)}.$$

Therefore, if $\rho(\Delta) \ge \frac{1}{\phi(k)}$ then $k > r - 1$ and $k \ge r$.

Since for every fixed a $(0 < a < 1)$ function $(1 + a^x)^{\frac{1}{x}}$ is decreasing with respect to x $(x \ge 1)$, by (2) we have

$$\left(\sum_{i=1}^{m} |f(t_i) - f(t_{i-1})|^{p_k}\right)^{1/p_k} = (1 + (1-\xi)^{p_k})^{\frac{1}{p_k}} \le (1 + (1-\xi)^{p_r})^{\frac{1}{p_r}} < 2^{\frac{1}{p_r}}.$$

Case 3 c is in Δ and there is a point in (c, y) which is contained in Δ. From (ii) and (iii) we get

$$y - c = y - x + x - c < \frac{1}{\phi(r)} + \frac{1}{\phi(r)} = \frac{2}{\phi(r)}.$$

In this case we obtain $\rho(\Delta) < \frac{1}{\phi(r)}$. Besides, if $\rho(\Delta) \ge \frac{1}{\phi(k)}$ then $k \ge r + 1$. Hence

$$\left(\sum_{i=1}^{m} |f(t_i) - f(t_{i-1})|^{p_k}\right)^{1/p_k} = (2 + \xi^{p_k})^{\frac{1}{p_k}} \le (2 + \xi^{p_{r+1}})^{\frac{1}{p_{r+1}}}.$$

Therefore, in each three cases, when $\rho(\Delta) \ge \frac{1}{\phi(k)}$ we get

$$\left(\sum_{i=1}^{m} |f(t_i) - f(t_{i-1})|^{p_k}\right)^{1/p_k} \le \max\left\{\xi,\ (1 + (1-\xi)^{p_r})^{\frac{1}{p_r}},\ (2 + \xi^{p_{r+1}})^{\frac{1}{p_{r+1}}}\right\}.$$

Definition 1.1 and (iv) imply that

$$V(f, p_n \uparrow \infty, \phi, [a, y]) \leq \max\left\{\xi,\ (1 + (1-\xi)^{p_r})^{\frac{1}{p_r}},\ (2 + \xi^{p_{r+1}})^{\frac{1}{p_{r+1}}}\right\} < 2^{\frac{1}{p_r}}$$
$$= V(f, p_n \uparrow \infty, \phi, [a, x]).$$

□

Remark 3.2 Let f be defined on $[a, b]$ and $[a_1, b_1] \subset [a, b]$. Lemma 2.3 implies that

$$V(f, p_n \uparrow \infty, \phi, [a_1, b_1]) \leq 3 \cdot V(f, p_n \uparrow \infty, \phi, [a, b]).$$

Remark 3.3 Let $c \in (a, b)$ and $a = t_0 < t_1 < \cdots < t_m = b$ be an arbitrary partition of $[a, b]$ such that $|t_i - t_{i-1}| \geq \frac{1}{\phi(n)}$, $i = 1, 2, \ldots, m$ and $t_{k-1} < c \leq t_k$. Since $\frac{1}{p_n} \leq 1$ we have

$$\left(\sum_{i=1}^{m} |f(t_i) - f(t_{i-1})|^{p_n}\right)^{1/p_n}$$
$$\leq \left(\sum_{i=1}^{k-1} |f(t_i) - f(t_{i-1})|^{p_n}\right)^{1/p_n}$$
$$+|f(c) - f(t_{k-1})| + |f(t_k) - f(c)| + \left(\sum_{i=k+1}^{m} |f(t_i) - f(t_{i-1})|^{p_n}\right)^{1/p_n}.$$

The last inequality and Lemma 2.3 imply

$$V(f, p_n \uparrow \infty, \phi, [a, b]) \leq 4 \cdot V(f, p_n \uparrow \infty, \phi, [a, c]) + 4 \cdot V(f, p_n \uparrow \infty, \phi, [c, b]).$$

4 A generalization of absolute continuity

Definition 4.1 A function f defined on a closed interval $[a, b]$, will be termed $((p_n), \phi)$-absolute continuous if the following condition is satisfied: for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$\left(\sum_{i=1}^{m} |f(\beta_i) - f(\alpha_i)|^{p_n}\right)^{1/p_n} < \varepsilon,$$

for all finite sets of non-overlapping intervals $(\alpha_i, \beta_i) \subset [a, b]$, $i = 1, 2, \ldots, m$, for which $\beta_i - \alpha_i \geq \frac{1}{\phi(n)}$, $i = 1, 2, \ldots, m$, and

$$\left(\sum_{i=1}^{m} (\beta_i - \alpha_i)^{p_n}\right)^{1/p_n} < \delta.$$

We denote this class by $AC(p_n \uparrow \infty, \phi, [a, b])$. Sometimes for the simplicity we use notation $AC(p_n \uparrow \infty, \phi)$. It is clear that if f is $((p_n), \phi)$-absolute continuous then f is continuous.

Lemma 4.2 *Let f be a function on $[a, b]$ and let $(\alpha_i, \beta_i) \subset [a, b]$, $i = 1, 2, \dots, m$, be a finite set of non-overlapping intervals such that $min_i(\beta_i - \alpha_i) \geq \frac{1}{\phi(n)}$. Then*

$$\left(\sum_{i=1}^{m} |f(\beta_i) - f(\alpha_i)|^{p_n}\right)^{1/p_n} \leq 6V(f, p_n \uparrow \infty, \phi, [a, b]).$$

Proof This statement follows from Lemma 2.3. □

Proposition 4.3 *A necessary and sufficient condition for f to be in $AC(p_n \uparrow \infty, \phi, [a, b])$ is that for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$V(f, p_n \uparrow \infty, \phi, [t_1, t_2]) < \varepsilon,$$

for each $[t_1, t_2] \subset [a, b]$ when $t_2 - t_1 < \delta$.

Proof Necessity is obvious. Now we have to show sufficiency of the condition. Suppose $\varepsilon > 0$ is given, then there exists $\eta > 0$ such that

$$V(f, p_n \uparrow \infty, \phi, [t_1, t_2]) < \frac{\varepsilon}{16},$$

for each $[t_1, t_2] \subset [a, b]$ when $t_2 - t_1 < \eta$.

Let $a = x_0 < x_1 < \cdots < x_m = b$ be a fixed partition of $[a, b]$ such that $x_i - x_{i-1} = \eta_1$, $i = 1, 2, \dots, m$, where $\eta_1 < \eta$. Then

$$V(f, p_n \uparrow \infty, \phi, [x_{i-1}, x_i]) < \frac{\varepsilon}{16}, \quad i = 1, 2, \dots, m. \tag{4}$$

Suppose r be a positive integer such that $m^{\frac{1}{p_r}} < 2$, and $\delta = min\{\eta_1, \frac{1}{\phi(r)}\}$. Let $(\alpha_i, \beta_i) \subset [a, b]$, $i = 1, 2, \dots, s$, be a finite set of non-overlapping intervals such that

$$\left(\sum_{i=1}^{s} (\beta_i - \alpha_i)^{p_n}\right)^{1/p_n} < \delta,$$

and

$$\beta_i - \alpha_i \geq \frac{1}{\phi(n)}, \quad i = 1, 2, \dots, s.$$

It is sufficient to show that

$$\left(\sum_{i=1}^{s} |f(\beta_i) - f(\alpha_i)|^{p_n}\right)^{1/p_n} < \varepsilon.$$

Let

$$A_k := \{i : (\alpha_i, \beta_i) \subset [x_{k-1}, x_k]\}, \quad k = 1, 2, \dots, m.$$

By (4) and Lemma 4.2

$$\sum_{i \in A_k} |f(\beta_i) - f(\alpha_i)|^{p_n} \le (6V(f, p_n \uparrow \infty, \phi, [x_{k-1}, x_k]))^{p_n} < \left(\frac{6\varepsilon}{16}\right)^{p_n}.$$

Suppose

$$B_k := \{i : \alpha_i < x_k < \beta_i\}, \quad k = 1, 2, \dots, m.$$

Note that B_k consists at most of one element and if $i \in B_k$ then $\alpha_i \in [x_{k-1}, x_k]$, $\beta_i \in [x_k, x_{k+1}]$. We have

$$\begin{aligned}
&|f(\beta_i) - f(\alpha_i)|^{p_n} \le (|f(\beta_i) - f(x_k)| + |f(x_k) - f(\alpha_i)|)^{p_n} \\
&\quad \le (V(f, p_n \uparrow \infty, \phi, [x_{k-1}, x_k]) + V(f, p_n \uparrow \infty, \phi, [x_k, x_{k+1}]))^{p_n} < \left(\frac{2\varepsilon}{16}\right)^{p_n}.
\end{aligned}$$

Since $\frac{1}{p_n} \le 1$, we obtain

$$\begin{aligned}
&\left(\sum_{i=1}^{s} |f(\beta_i) - f(\alpha_i)|^{p_n}\right)^{1/p_n} \\
&\quad = \left(\sum_{k=1}^{m} \left(\sum_{i \in A_k} |f(\beta_i) - f(\alpha_i)|^{p_n} + \sum_{i \in B_k} |f(\beta_i) - f(\alpha_i)|^{p_n}\right)\right)^{1/p_n} \\
&\quad < \left(\sum_{k=1}^{m} \left(\frac{6\varepsilon}{16}\right)^{p_n} + \sum_{k=1}^{m} \left(\frac{2\varepsilon}{16}\right)^{p_n}\right)^{1/p_n} \le \left(\sum_{k=1}^{m} \left(\frac{6\varepsilon}{16}\right)^{p_n}\right)^{1/p_n} + \left(\sum_{k=1}^{m} \left(\frac{2\varepsilon}{16}\right)^{p_n}\right)^{1/p_n}.
\end{aligned}$$

Note that $\frac{1}{\phi(n)} \le \beta_i - \alpha_i < \delta \le \frac{1}{\phi(r)}$, hence $n > r$. Since $m^{\frac{1}{p_r}} < 2$, the last term does not exceed to

$$m^{\frac{1}{p_n}} \cdot \frac{8\varepsilon}{16} < m^{\frac{1}{p_r}} \cdot \frac{\varepsilon}{2} < \varepsilon.$$

Remark 3.3 and Proposition 4.3 imply that $AC(p_n \uparrow \infty, \phi) \subset BV(p_n \uparrow \infty, \phi)$.

Proposition 4.4 *If f is absolute continuous, then $f \in AC(p_n \uparrow \infty, \phi)$.*

Proof Let $\varepsilon > 0$, then there exists $\delta > 0$ such that for every non-overlapping intervals $(\alpha_i, \beta_i), i = 1, 2, \ldots, m$, is satisfying inequality

$$\sum_{i=1}^{m} |f(\beta_i) - f(\alpha_i)| < \varepsilon$$

when $\sum_{i=1}^{m}(\beta_i - \alpha_i) < \delta$.

If $x_2 - x_1 < \delta$ then for each partition $x_1 = t_0 < t_1 < \cdots < t_k = x_2$ we have $\sum_{i=1}^{k}(t_i - t_{i-1}) = x_2 - x_1 < \delta$, hence

$$\left(\sum_{i=1}^{k} |f(t_i) - f(t_{i-1})|^{p_n}\right)^{1/p_n} \leq \sum_{i=1}^{k} |f(t_i) - f(t_{i-1})| < \varepsilon.$$

The last inequality implies $V(f, p_n \uparrow \infty, \phi, [x_1, x_2]) \leq \varepsilon$. By Proposition 4.3 $f \in AC(p_n \uparrow \infty, \phi)$. □

Proposition 4.5 *If $\phi(n)^{\frac{1}{p_n}}$ is bounded then every continuous function on $[a, b]$ is $((p_n), \phi)$-absolute continuous.*

Proof Let $\phi(n)^{\frac{1}{p_n}} \leq C$ where C is a positive constant and f be continuous on $[a, b]$. Therefore, f is uniformly continuous. If $\varepsilon > 0$ is given then there exists δ $(0 < \delta < 1)$ such that

$$|f(t_1) - f(t_2)| < \frac{\varepsilon}{2C}, \quad if \quad t_1 - t_2 < \delta.$$

Let $[x_1, x_2] \subset [a, b]$ and $x_2 - x_1 < \delta$. If $x_1 = t_0 < t_1 < \cdots < t_m = x_2$ is an arbitrary partition, where $t_i - t_{i-1} \geq \frac{1}{\phi(n)}$, then it is clear that $m \leq \delta\phi(n) < \phi(n)$ and

$$\left(\sum_{1}^{m} |f(t_i) - f(t_{i-1})|^{p_n}\right)^{1/p_n} < \left(\sum_{i=1}^{m} \left(\frac{\varepsilon}{2C}\right)^{p_n}\right)^{1/p_n} = m^{\frac{1}{p_n}} \frac{\varepsilon}{2C} \leq \phi(n)^{\frac{1}{p_n}} \frac{\varepsilon}{2C} \leq \frac{\varepsilon}{2}.$$

Hence $V(f, p_n \uparrow \infty, \phi, [x_1, x_2]) < \varepsilon$ and by Proposition 4.3 f is $((p_n), \phi)$-absolute continuous.

Proposition 4.6 *If $\phi(n)^{\frac{1}{p_n}}$ is not bounded then there exists a continuous function f which is not $((p_n), \phi)$-absolute continuous.*

Proof Since $\phi(n)^{1/p_n}$ is not bounded then for every positive integer k there exists a positive integer n_k such that

$$\left[\frac{\phi(n_k)}{4k(k+1)}\right]^{1/p_{n_k}} > k.$$

Let $c_k = \frac{1}{k}, k = 1, 2, \ldots,$ and $\lambda_k = \left[\frac{\phi(n_k)}{4k(k+1)}\right]$. Consider the following continuous function on [0, 1]:

$$f(t) = \begin{cases} \lambda_k^{-1/p_{n_k}} \sin(2\lambda_k \pi \frac{t-c_{k+1}}{c_k - c_{k+1}}), \; if \quad t \in [c_{k+1}, c_k], \\ 0, \; if \quad t = 0. \end{cases}$$

Let $x_i^k = c_{k+1} + i\frac{c_k - c_{k+1}}{4\lambda_k}, i = 0, 1, \ldots, 4\lambda_k$. It is clear that $x_0^k = c_{k+1}, x_{4\lambda_k}^k = c_k$ and

$$f(x_i^k) = \lambda_k^{-1/p_{n_k}} \sin\left(i\frac{\pi}{2}\right); \tag{5}$$

$$x_i^k - x_{i-1}^k = \frac{c_k - c_{k+1}}{4\lambda_k} \geq \frac{1}{k(k+1)} : \frac{4\phi(n_k)}{4k(k+1)} = \frac{1}{\phi(n_k)}. \tag{6}$$

From (5) and (6) we obtain

$$V(f, p_n \uparrow \infty, \phi, [c_{k+1}, c_k]) \geq \left(\sum_{i=1}^{4\lambda_k} |f(x_i^k) - f(x_{i-1}^k)|^{p_{n_k}}\right)^{1/p_{n_k}}$$
$$= \left(\sum_{i=1}^{4\lambda_k} \left(\lambda_k^{-1/p_{n_k}}\right)^{p_{n_k}}\right)^{1/p_{n_k}} = \lambda_k^{-1/p_{n_k}} \cdot (4\lambda_k)^{1/p_{n_k}} > 1.$$

Lemma 4.3 implies f is not $((p_n), \phi)$-absolute continuous.

Lemma 4.7 *Let $\{f_k\}_{i=1}^{\infty}$ be a sequence of functions from $AC(p_n \uparrow \infty, \phi, [a, b])$ which is convergent in variation to f, then $f \in AC(p_n \uparrow \infty, \phi, [a, b])$.*

Proof Let $\varepsilon > 0$ be given, then there exists N such that if $k > N$

$$V(f_k - f, p_n \uparrow \infty, \phi, [a, b]) < \frac{\varepsilon}{4}.$$

Let $k_0 > N$. Since f_{k_0} is $((p_n), \phi)$-absolute continuous then there exists $\delta > 0$ such that

$$V(f_{k_0}, p_n \uparrow \infty, \phi, [t_1, t_2]) < \frac{\varepsilon}{4},$$

where $t_2 - t_1 < \delta$. Hence by Proposition 2.1(a) and Remark 3.2 we have

$$V(f, p_n \uparrow \infty, \phi, [t_1, t_2]) \leq V(f - f_{k_0}, p_n \uparrow \infty, \phi, [t_1, t_2]) + V(f_{k_0},$$
$$p_n \uparrow \infty, \phi, [t_1, t_2]) \leq \varepsilon.$$

Thus, by Proposition 4.3 f is in $AC(p_n \uparrow \infty, \phi, [a, b])$.

In Lemma 4.7 convergence in variation can not be replaced with uniform convergence. Indeed, Fejer $(C, 1)$ means of the continuous function f (constructed in Lemma 4.6) with respect to trigonometric system converges uniformly to f, but $f \notin AC(p_n \uparrow \infty, \phi)$.

Lemma 4.8 *Let f be a function on $[a, b]$, $[c, d] \subset [a, b]$ and $f(c) = f(d) = 0$. If*

$$g(t) = \begin{cases} f(t), \ t \in [c, d], \\ \quad 0, \ t \in [a, b]\backslash[c, d], \end{cases}$$

then

$$V(g, p_n \uparrow \infty, \phi, [a, b]) \leq 5 \cdot V(f, p_n \uparrow \infty, \phi, [c, d]).$$

Proof By Lemma 2.3, for an arbitrary partition of $[a, b]$ where $t_{k-1} < c \leq t_k$ and $t_{r-1} \leq d < t_r$, we get

$$\begin{aligned}
\left(\sum_{i=1}^{m} |g(t_i) - g(t_{i-1})|^{p_n}\right)^{\frac{1}{p_n}} &\leq \left(\sum_{i=k+1}^{r-1} |g(t_i) - g(t_{i-1})|^{p_n}\right)^{\frac{1}{p_n}} + |g(t_k) - g(t_{k-1})| \\
&+ |g(t_r) - f(t_{r-1})| = \left(\sum_{i=k+1}^{r-1} |f(t_i) - f(t_{i-1})|^{p_n}\right)^{\frac{1}{p_n}} + |f(t_k) - f(c)| \\
&+ |f(d) - f(t_{r-1})| \\
&\leq 5 \cdot V(f, p_n \uparrow \infty, \phi, [c, d]).
\end{aligned}$$

Lemma 4.9 *Let f be a function on $[a, b]$ and $\{c_i\}_1^\infty$ be a sequence such that $c_i \downarrow a$, $c_1 = b$ and $f(c_i) = 0$, $i = 1, 2, \ldots,$ then*

$$V(f, p_n \uparrow \infty, \phi, [a, b]) \leq 5 \cdot \sum_{i=1}^{\infty} V(f, p_n \uparrow \infty, \phi, [c_{i+1}, c_i]).$$

Proof Let

$$f_i(t) = \begin{cases} f(t), \ t \in [c_{i-1}, c_i], \\ \quad 0, \ t \notin [c_{i-1}, c_i]. \end{cases}$$

It is clear that $f = \sum_{i=1}^{\infty} f_i$ on $[a, b]$ and by Proposition 2.1(d)

$$\begin{aligned}
V(f, p_n \uparrow \infty, \phi, [a, b]) &\leq \liminf_{k\to\infty} V\left(\sum_{i=1}^{k} f_i, \ p_n \uparrow \infty, \phi, [a, b]\right) \\
&\leq \liminf_{k\to\infty} \sum_{i=1}^{k} V(f_i, p_n \uparrow \infty, \phi, [a, b]) \leq \sum_{i=1}^{\infty} V(f_i, p_n \uparrow \infty, \phi, [a, b]).
\end{aligned}$$

By Lemma 4.8 we get

$$V(f, p_n \uparrow \infty, \phi, [a, b]) \leq 5 \cdot \sum_{i=1}^{\infty} V(f, p_n \uparrow \infty, \phi, [c_{i+1}, c_i]).$$

Lemma 4.10 *Let f be a periodic function with a period h. Then for every a*

$$V(f, p_n \uparrow \infty, \phi, [a, a + mh]) \leq 4m^{\frac{1}{p_r}} \cdot V(f, p_n \uparrow \infty, \phi, [a, a + h]),$$

where $mh < \frac{1}{\phi(r-1)}$.

Proof Using periodicity of f, it is clear that for each t_1 and t_2 from $[a, a + mh]$ we have

$$|f(t_1) - f(t_2)| \leq V(f, p_n \uparrow \infty, \phi, [a, a + h]).$$

Let $a = t_0 < t_1 < \cdots < t_s = a + mh$ be an arbitrary partition, such that $|t_i - t_{i-1}| \geq \frac{1}{\phi(n)}$, $i = 1, 2, \ldots, s$. It is clear that $n \geq r$. Suppose

$$A_k := \{i : [t_{i-1}, t_i] \subset [a + (k-1)h, a + kh]\}, \quad k = 1, 2, \ldots, m.$$

By Lemma 2.3 we have

$$\sum_{i \in A_k} |f(t_{i-1}) - f(t_i)|^{p_n} \leq (3V(f, p_n \uparrow \infty, \phi, [a, a + h]))^{p_n}.$$

Let

$$B_k := \{i : t_{i-1} < x_k < t_i\}, \quad k = 1, 2, \ldots, m.$$

B_k consists at most of one point. If $i \in B_k$ then

$$|f(t_i) - f(t_{i-1})| \leq V(f, p_n \uparrow \infty, \phi, [a, a + h]).$$

We obtain

$$\begin{aligned}
&\left(\sum_{i=1}^{s} |f(t_i) - f(t_{i-1})|^{p_n}\right)^{1/p_n} \\
&\leq \left(\sum_{k=1}^{m} \left(\sum_{i \in A_k} |f(t_{i-1}) - f(t_i)|^{p_n} + \sum_{i \in B_k} |f(t_{i-1}) - f(t_i)|^{p_n}\right)\right)^{1/p_n} \\
&\leq \left(\sum_{k=1}^{m} (3V(f, p_n \uparrow \infty, \phi, [a, a + h]))^{p_n} + (V(f, p_n \uparrow \infty, \phi, [a, a + h]))^{p_n}\right)^{1/p_n} \\
&\leq 4m^{\frac{1}{p_n}} V(f, p_n \uparrow \infty, \phi, [a, a + h]) \leq 4m^{\frac{1}{p_r}} V(f, p_n \uparrow \infty, \phi, [a, a + h]).
\end{aligned}$$

Proposition 4.11 *For every $q \geq 1$ there exists a function which is $((p_n), \phi)$-absolute continuous but it is not in V_q, where V_q is the class of functions of Wiener-Young [10] q-th generalization of total variation.*

Proof Let $c_k = \frac{1}{k}$, $k = 1, 2, \ldots$, and consider the following function f on $[0, 1]$:

$$f(t) = \begin{cases} \frac{1}{k^2} \sin(2k^{[3q]}\pi \frac{t-c_{k+1}}{c_k - c_{k+1}}), \; if \quad t \in [c_{k+1}, c_k], \\ 0, \; if \quad t = 0, \end{cases}$$

This function is periodic with period $h = \frac{c_k - c_{k+1}}{k^{[3q]}}$ on $[c_{k+1}, c_k]$, $f(c_i) = 0, i = 1, 2, ..,$ and

$$V(f, p_n \uparrow \infty, \phi, [c_{k+1}, c_{k+1} + h]) \leq \frac{4}{k^2}. \tag{7}$$

Let r be the least positive integer for which $p_r > 3q$, then $2 - \frac{[3q]}{p_r} > 1$. It is clear that $c_{k+1} + k^{[3q]}h = c_k$. If $\frac{1}{k} - \frac{1}{k+1} > \frac{1}{\phi(r-1)}$ then (7) and Lemma 4.10 imply that

$$\begin{aligned} V(f, p_n \uparrow \infty, \phi, [c_{k+1}, c_k]) &\leq 4k^{\frac{[3q]}{p_r}} \cdot V(f, p_n \uparrow \infty, \phi, [c_{k+1}, c_{k+1} + h]) \\ &\leq 4k^{\frac{[3q]}{p_r}} \cdot \frac{4}{k^2} \leq \frac{16}{k^{2-[3q]/p_r}}. \end{aligned} \tag{8}$$

Let

$$s_k(t) = \begin{cases} f(t), \; t \in [c_{k+1}, 1], \\ 0, \; t \notin [c_{k+1}, 1], \end{cases}$$

Lemma 4.8 implies that

$$V(f - s_k, p_n \uparrow \infty, \phi, [a, b]) \leq 5 \cdot V(f, p_n \uparrow \infty, \phi, [0, c_{k+1}]).$$

(8) and Lemma 4.9 imply that the right side of the last inequality does not exceed to

$$25 \cdot \sum_{j=k+1}^{\infty} V(f, p_n \uparrow \infty, \phi, [c_{j+1}, c_j]) \leq 25 \cdot \sum_{j=k+1}^{\infty} \frac{16}{k^{2-[3q]/p_r}} \to 0, \quad k \to 0.$$

Since s_k is absolute continuous we conclude that s_k is in $AC(p_n \uparrow \infty, \phi, [0, 1])$ and by lemma 4.7 f is $((p_n), \phi)$-absolute continuous.

Let $x_i^k = c_{k+1} + i\frac{h}{4}$, $i = 0, 1, \ldots, 4k^{[3q]}$. Then $f(x_i^k) = \frac{1}{k^2} \sin\left(i\frac{\pi}{2}\right)$ and

$$\sum_{i=1}^{4k^{[3q]}} |f(x_i^k) - f(x_{i-1}^k)|^q = \sum_{i=1}^{4k^{[3q]}} \left(\frac{1}{k^2}\right)^q = 4k^{[3q]} \cdot \frac{1}{k^{2q}} > 1. \tag{9}$$

For every positive integer M there exists a positive integer N, such that $N^{\frac{1}{q}} \geq M$. If we consider the points $x_i^k,\ i = 0, 1, \ldots, 4k^{[3q]}\ ,\ k = 1, 2, \ldots, N$, by (9) we get

$$V_q(f, [0, 1]) \geq \left(\sum_{k=1}^{N} \sum_{i=1}^{4k^{[3q]}} |f(x_i^k) - f(x_{i-1}^k)|^q \right)^{1/q} > \left(\sum_{k=1}^{N} 1 \right)^{1/q} = N^{1/q} \geq M.$$

This means that f is not in $V_q([0, 1])$.

5 Approximation by Steklov functions

Lemma 5.1 *Let $\{g_h : h \geq 0\}$ be a set of functions on $[a, b]$ and satisfies conditions:*

(i) For every $\varepsilon > 0$ there exists a positive integer N, such that if $h > N$ then for arbitrary $t \in [a, b]$ we have $g_h(t) < \varepsilon$;
(ii) For every positive number ε there exists a fixed partition $a = x_0 < x_1 < \cdots < x_r = b$ and a positive integer N such that for every $h > N$ we have

$$V(g_h, p_n \uparrow \infty, \phi, [x_{i-1}, x_i]) < \varepsilon, \quad i = 1, 2, \ldots, r.$$

Then,

$$V(g_h, p_n \uparrow \infty, \phi, [a, b]) \to 0, \quad h \to \infty.$$

Proof Let $\varepsilon > 0$ is given. By condition (ii) there exists a fixed $a = x_0 < x_1 < \cdots < x_r = b$ partition and a positive integer N_1 such that for every $h > N_1$ we have

$$V(g_h, p_n \uparrow \infty, \phi, [x_{i-1}, x_i]) < \frac{\varepsilon}{10}, \ i = 1, 2, \ldots r. \tag{10}$$

Let l be the least integer such that $r^{\frac{1}{p_l}} \leq 2$ and $\phi(l) \geq \frac{5}{b-a}$.

By condition (i) there exists N_2 such that if $h > N_2$ then

$$g_h(t) < \frac{\varepsilon}{2(b-a)\phi(l)}, \quad t \in [a, b]. \tag{11}$$

We must show that

$$V(g_h, p_n \uparrow \infty, \phi, [a, b]) < \varepsilon,$$

when $h > N = \max\{N_1, N_2\}$.

Let $a = t_0 < t_1 < \cdots < t_m = b$ be an arbitrary partition of the interval $[a, b]$ such that $|t_i - t_{i-1}| \geq \frac{1}{\phi(n)}$, $i = 1, 2, \ldots, m$. Consider two cases.

Case 1 $n \leq l$, then $b - a = \sum_{i=1}^m (t_i - t_{i-1}) \geq \sum_1^m \frac{1}{\phi(n)} = \frac{m}{\phi(n)}$, hence

$$m \leq (b-a)\phi(n) \leq (b-a)\phi(l). \tag{12}$$

By (11)

$$|g_h(t_i) - g_h(t_{i-1})| \leq |g_h(t_i)| + |g_h(t_{i-1})| \\ \leq \frac{\varepsilon}{2(b-a)\phi(l)} + \frac{\varepsilon}{2(b-a)\phi(l)} = \frac{\varepsilon}{(b-a)\phi(l)}.$$

By the last inequality and (12) we obtain

$$\left(\sum_{i=1}^{m} |g_h(t_i) - g_h(t_{i-1})|^{p_n}\right)^{\frac{1}{p_n}} \leq m^{\frac{1}{p_n}} \frac{\varepsilon}{(b-a)\phi(l)} < m \frac{\varepsilon}{(b-a)\phi(l)} \leq \varepsilon.$$

Case 2 $n > l$. Let

$$A_k := \{i : [t_{i-1}, t_i] \subset [x_{k-1}, x_k]\}, \quad k = 1, 2, \ldots, r.$$

By (10) and Lemma 2.3 we get

$$\sum_{i \in A_k} |g_h(t_{i-1}) - g_h(t_i)|^{p_n} \leq (3V(g_h, p_n \uparrow \infty, \phi, [x_{k-1}, x_k]))^{p_n} < \left(\frac{3\varepsilon}{10}\right)^{p_n}. \tag{13}$$

Let

$$B_k := \{i : t_{i-1} < x_k < t_i\}, \quad k = 1, 2, \ldots, r.$$

Note that B_k consists at most of one point. If $i \in B_k$ then

$$|g_h(t_i) - g_h(t_{i-1})| < \frac{\varepsilon}{(b-a)\phi(l)}.$$

By the last inequality and (13) we obtain

$$\left(\sum_{i=1}^{m} |g_h(t_i) - g_h(t_{i-1})|^{p_n}\right)^{\frac{1}{p_n}}$$
$$= \left(\sum_{k=1}^{r} \left(\sum_{i \in A_k} |g_h(t_{i-1}) - g_h(t_i)|^{p_n} + \sum_{i \in B_k} |g_h(t_{i-1}) - g_h(t_i)|^{p_n}\right)\right)^{\frac{1}{p_n}}$$
$$< \left\{\sum_{k=1}^{r} \left(\left(\frac{3\varepsilon}{10}\right)^{p_n} + \left(\frac{\varepsilon}{(b-a)\phi(l)}\right)^{p_n}\right)\right\}^{\frac{1}{p_n}} \leq r^{\frac{1}{p_l}} \left(\frac{3\varepsilon}{10} + \frac{\varepsilon}{(b-a)\phi(l)}\right) \leq \varepsilon.$$

This means that

$$V(g_h, p_n \uparrow \infty, \phi, [a, b]) \to 0, \quad h \to \infty.$$

Lemma 5.2 *Let f be a periodic function with period $b-a$ and there exist $\varepsilon > 0$ and $\delta > 0$ such that*

$$V(f, p_n \uparrow \infty, \phi, [t_1, t_2]) < \varepsilon,$$

for every $t_2 - t_1 < \delta$, $[t_1, t_2] \subset [a, b]$. Then for every real h

$$V(f^h, p_n \uparrow \infty, \phi, [t_1, t_2]) < 8\varepsilon,$$

where $f^h(t) = f(h+t)$ and $t_2 - t_1 < \delta$, $[t_1, t_2] \subset [a, b]$.

Proof Since f is periodic, we can consider only the case when $0 < h < b-a$. We have two cases:

If $b \notin [t_1 + h, t_2 + h]$, by periodicity of f we get

$$V(f^h, p_n \uparrow \infty, \phi, [t_1, t_2]) = V(f, p_n \uparrow \infty, \phi, [t_1 + h, t_2 + h]) < \varepsilon.$$

If $b \in [t_1 + h, t_2 + h]$ then by Remark 3.3

$$\begin{aligned} V(f^h, p_n \uparrow \infty, \phi, [t_1, t_2]) &= V(f, p_n \uparrow \infty, \phi, [t_1 + h, t_2 + h]) \\ &\leq 4V(f, p_n \uparrow \infty, \phi, [t_1 + h, b] + 4V(f, p_n \uparrow \infty, \phi, [b, t_2 + h])) < 8\varepsilon. \end{aligned}$$

Lemma 5.3 *If a function f is $((p_n), \phi)$-absolute continuous on $[a, b]$, periodic with period $b-a$, then $V(f^h - f, p_n \uparrow \infty, \phi, [a, b]) \to 0$, $h \to 0+$, where $f^h(t) = f(h+t)$.*

Proof Let $g_{1/h} := f^h - f$. Now we show that $g_{1/h}$ satisfies conditions of Lemma 5.1.

(1) Since f is $((p_n), \phi)$-absolute continuous , it is uniformly continuous on $[a, b]$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(t+h) - f(t)| < \varepsilon$, when $h < \delta$.
(2) Let $\varepsilon > 0$ be given. Since f is $((p_n), \phi)$ absolute continuous, there exists $\eta > 0$ such that $V(f, p_n \uparrow \infty, \phi, [t_1, t_2]) < \frac{\varepsilon}{9}$ when $t_2 - t_1 < \eta$.

Let $a = x_0 < \cdots < x_m = b$ be a partition of $[a, b]$ such that $x_i - x_{i-1} < \eta$, then $V(f, p_n \uparrow \infty, \phi, [x_{i-1}, x_i]) < \frac{\varepsilon}{9}$, $i = 1, 2, \ldots, m$, and by Lemma 5.2

$$\begin{aligned} &V(g_{\frac{1}{h}}, p_n \uparrow \infty, \phi, [x_{i-1}, x_i]) \\ &\quad \leq V(f^h, p_n \uparrow \infty, \phi, [x_{i-1}, x_i]) + V(f, p_n \uparrow \infty, \phi, [x_{i-1}, x_i]) < \varepsilon. \end{aligned}$$

We get that $g_{\frac{1}{h}}$ satisfies conditions of Lemma 5.1, that implies $V(f^h - f, p_n \uparrow \infty, \phi, [a, b]) \to 0$, for $h \to 0+$.

Proposition 5.4 *Let $f \in AC(p_n \uparrow \infty, \phi, [a.b])$ be periodic with period $b-a$. Then the sequence f_k of the Steklov functions of f, defined by the formula*

$$f_k(t) = k \int_t^{t+\frac{1}{k}} f(\tau) d\tau,$$

is convergent in variation to $f(t)$.

Proof

$$f_k(t) - f(t) = k\int_0^{\frac{1}{k}} (f(t+\tau) - f(t))\,d\tau.$$

Let $\varepsilon > 0$ be given, then by Lemma 5.3 there exists $\delta > 0$ such that

$$V(f^h - f, p_n \uparrow \infty, \phi, [a,b]) < \varepsilon, \quad h < \delta, \tag{14}$$

where $f^h(t) = f(h+t)$. Since $|x|^p$, $p \geq 1$, is a convex function, by Jensen inequality

$$\left| k\int_0^{\frac{1}{k}} g(t)dt \right|^{p_n} \leq k\int_0^{\frac{1}{k}} |g(t)|^{p_n} dt. \tag{15}$$

Suppose $\frac{1}{k} < \delta$ and $a = t_0 < t_1 < \cdots < t_m = b$ be an arbitrary partition of the interval $[a,b]$ such that $|t_i - t_{i-1}| \geq \frac{1}{\phi(n)}$, $i = 1, 2, \ldots, m$. By (14) and (15) we get

$$\left(\sum_{i=1}^{m} |(f_k - f)(t_i) - (f_k - f)(t_{i-1})|^{p_n}\right)^{1/p_n}$$
$$= \left(\sum_1^m \left| k\int_0^{\frac{1}{k}} (f(t_i+\tau) - f(t_i) - f(t_{i-1}+\tau) + f(t_{i-1}))d\tau \right|^{p_n}\right)^{1/p_n}$$
$$\leq \left(\sum_1^m k\int_0^{\frac{1}{k}} |f(t_i+\tau) - f(t_i) - f(t_{i-1}+\tau) + f(t_{i-1})|^{p_n} d\tau\right)^{1/p_n}$$
$$\leq \left(k\int_0^{\frac{1}{k}} V(f^\tau - f, p_n \uparrow \infty, \phi, [a,b])^{p_n} d\tau\right)^{1/p_n} < \left(k\int_0^{\frac{1}{k}} \varepsilon^{p_n} d\tau\right)^{1/p_n} = \varepsilon.$$

Hence $V(f_k - f, p_n \uparrow \infty, \phi, [a,b]) < \varepsilon$, when $\frac{1}{k} < \delta$.

If f is an integrable function on $[a,b]$ then its Steklov functions f_k are absolute continuous, hence, $f_k \in AC(p_n \uparrow \infty, \phi, [a,b])$. Therefore, By Lemma 4.7, if $V(f_k - f, p_n \uparrow \infty, \phi) \to 0$ then $f \in AC(p_n \uparrow \infty, \phi, [a.b])$.

6 Approximation by singular integrals

Now we shall consider the problem of approximation in variation of periodic function f which is $((p_n), \phi)$-absolute continuous on $[a,b]$, by integrals of the form

$$I_q(t) = \int_a^b K_q(\tau) f(t+\tau)d\tau.$$

Lemma 6.1 *Let f be a periodic function with period $b-a$, K_q be a function such that $\int_a^b |K(t)|dt = \theta$, and $I(t) = \int_a^b K(\tau)f(t+\tau)dt$. Then*

$$V(I, p_n \uparrow \infty, \phi, [c,d]) \leq \theta \cdot \sup_{\tau \in [a,b]} V(f^\tau, p_n \uparrow \infty, \phi, [c,d])$$

for every closed interval $[c,d]$, where $f^\tau(t) = f(t+\tau)$.

Proof Let $c = t_0 < t_1 < \cdots < t_m = d$ be an arbitrary partition such that $|t_i - t_{i-1}| \geq \frac{1}{\phi(n)}$, $i = 1, 2, \ldots, m$. Then

$$\left(\sum_{i=1}^m |I(t_i) - I(t_{i-1})|^{p_n}\right)^{1/p_n} = \left(\sum_1^m \left|\int_a^b K(\tau)(f(t_i+\tau) - f(t_{i-1}+\tau))d\tau\right|^{p_n}\right)^{1/p_n}$$
$$\leq \left(\sum_1^m \left(\int_a^b |K(\tau)| \cdot |(f(t_i+\tau) - f(t_{i-1}+\tau))|d\tau\right)^{p_n}\right)^{1/p_n}.$$

By Jensen' inequality, the last term does not exceed to

$$\left(\sum_{i=1}^m \theta^{p_n} \cdot \frac{\int_a^b |K(\tau)| \cdot |f(t_i+\tau) - f(t_{i-1}+\tau)|^{p_n} d\tau}{\theta}\right)^{1/p_n}$$
$$= \frac{\theta}{\theta^{1/p_n}} \left(\int_a^b |K(\tau)| \cdot \sum_{i=1}^m |f(t_i+\tau) - f(t_{i-1}+\tau)|^{p_n} d\tau\right)^{1/p_n}$$
$$\leq \frac{\theta}{\theta^{1/p_n}} \cdot \sup_{\tau \in [a,b]} V(f^\tau, p_n \uparrow \infty, \phi, [c,d]) \cdot \left(\int_a^b |K(\tau)|d\tau\right)^{1/p_n}$$
$$= \theta \cdot \sup_{\tau \in [a,b]} V(f^\tau, p_n \uparrow \infty, \phi, [c,d]).$$

Hence, we get

$$V(I, p_n \uparrow \infty, \phi, [c,d]) \leq \theta \cdot \sup_{\tau \in [a,b]} V(f^\tau, p_n \uparrow \infty, \phi, [c,d]).$$

Proposition 6.2 *Let $\int_a^b |k_q(t)|dt = \theta_q$, $q = 1, 2 \ldots$, and (θ_q) is bounded; f is $((p_n), \phi)$-absolute continuous, periodic with period $b-a$ and $I_q(t) = \int_a^b K_q(\tau)f(t+\tau)dt$. If for some ξ the sequence of functions $I_q(t)$ converges uniformly to $f^\xi(t)$ then*

$$V(I_q - f^\xi, p_n \uparrow \infty, \phi, [a,b]) \to 0, \ q \to \infty,$$

where $f^\xi(t) = f(t+\xi)$.

Proof It is sufficient to show that the sequence $I_q - f^\xi$, $q = 1, 2, \ldots,$ satisfies condition $ii)$ of Lemma 5.1.

Let $\theta_q \le C,\ \ q = 1, 2, \dots,$ and $\varepsilon > 0$. Since f is $((p_n), \phi)$ absolute continuous, there exists $\eta > 0$ such that $V(f, p_n \uparrow \infty, \phi, [t_1, t_2]) < \frac{\varepsilon}{8(C+1)}$ for every $t_2 - t_1 < \eta$.

Soppuse $a = x_0 < \cdots < x_m = b$ be a partition of $[a, b]$ such that $x_i - x_{i-1} < \eta$, then $V(f, p_n \uparrow \infty, \phi, [x_{i-1}, x_i]) < \frac{\varepsilon}{8(C+1)}$, $i = 1, 2, \dots, m$, and by Lemma 5.2

$$V(f^h, p_n \uparrow \infty, \phi, [x_{i-1}, x_i]) < \frac{\varepsilon}{(C+1)},\ i = 1, 2, \dots ,$$

for every real h. By Lemma 6.1

$$V(I_q, p_n \uparrow \infty, \phi, [x_{i-1}, x_i]) \le \theta_q \cdot \sup_{\tau \in [a,b]} V(f^\tau, p_n \uparrow \infty, \phi, [x_{i-1}, x_i]) \le \frac{C\varepsilon}{C+1}.$$

By the last two inequalities we obtain

$$\begin{aligned} &V(I_q - f^\xi, p_n \uparrow \infty, \phi, [x_{i-1}, x_i]) \le \\ &\quad V(I_q, p_n \uparrow \infty, \phi, [x_{i-1}, x_i]) \\ &\quad + V(f^\xi, p_n \uparrow \infty, \phi, [x_{i-1}, x_i]) < \frac{C\varepsilon}{C+1} + \frac{\varepsilon}{(C+1)} = \varepsilon, \end{aligned}$$

for every $i = 1, 2, \dots, m$. □

Corollary 6.3 *Let f be a periodic function with period 2π and $\sigma_n^\alpha(f)$ be (C, α), $\alpha > 0$, means of Fourier series of f with respect to the trigonometric system. Then $\sigma_n^\alpha(f)$ is convergent in variation to f if and only if $f \in AC(p_n \uparrow \infty, \phi)$.*

Sufficiency follows from Proposition 6.2.

Necessity. Since $\sigma_n^\alpha(f)$ is absolute continuous then $\sigma_n^\alpha(f) \in AC(p_n \uparrow \infty, \phi)$. By Lemma 4.7, if $\sigma_n^\alpha(f)$ is convergent in variation to f then f is $((p_n), \phi)$-absolute continuous.

Corollary 6.4 *Let $K_q(t) \ge 0$, $\int_a^b K_q(t)dt \to 1$ as $q \to \infty$ and $\int_{a+\delta}^{b-\delta} K_q(t)dt \to 0$ as $q \to \infty$ for each $0 < \delta < \frac{1}{2}(b-a)$ and f is periodic with period $b - a$.*
If $f \in AC(p_n \uparrow \infty, \phi, [a, b])$ then $V(I_q - f^a, p_n \uparrow \infty, \phi, [a, b]) \to 0$, where $f^a(t) = f(t + a)$.

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