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Derivations and cohomologies of Lipschitz algebras

Kazuhiro Kawamura[1]

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Abstract

For a compact metric space (M, d), Lip$M$ denotes the Banach algebra of all complex-valued Lipschitz functions on (M, d). Motivated by a classical result of de Leeuw, we give a canonical construction of a compact Hausdorff space $\hat{M}$ and a continuous surjection $\pi : \hat{M} \to M$ which may viewed as a metric analogue of the unit sphere bundle over a Riemannian manifold. It is shown that, for each $n \geq 1$ the continuous Hochschild cohomology $\mathrm{H}^n(\mathrm{Lip}M, C(\hat{M}))$ has the infinite rank as a Lip$M$-module, if the metric space (M, d) admits a local geodesic structure, for example, if M is a compact Riemannian manifold or a non-positively curved metric space. Here $C(\hat{M})$ denotes the algebra of all complex-valued continuous functions on $\hat{M}$. On the other hand, if the coefficient $C(\hat{M})$ is replaced with $C(M)$, then it is shown that $\mathrm{H}^1(\mathrm{Lip}M, C(M)) = 0$ for each compact Lipschitz manifold M.

Keywords Lipschitz algebra · Hochschild cohomology · De Leeuw map · Tangent bundle · Stone–Čech compactifications

Mathematics Subject Classification 46H99 · 55N35 · 58A99 · 16W99

1 Introduction, main result and preliminaries

For a Banach algebra A and a Banach A-bimodule X, let $C^n(A, X)$ be the continuous n-cochains of A to X

$$C^n(A, X) = \left\{ f : A^n \to X \mid f \text{ is a bounded } n\text{-linear map} \right\}$$

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✉ Kazuhiro Kawamura
kawamura@math.tsukuba.ac.jp

[1] Institute of Mathematics, University of Tsukuba, Ibaraki 305-8571, Japan

with $C^0(A,X) = X$. The coboundary operator $\delta^n : C^n(A,X) \to C^{n+1}(A,X)$ is defined by

$$\delta^n f(a_1,\dots a_{n+1}) = a_1 \cdot f(a_2,\dots,a_{n+1}) + \sum_{i=1}^{n}(-1)^i f(a_1,\dots,a_i a_{i+1},\dots,a_{n+1}) \\ +(-1)^{n+1} f(a_1,\dots,a_n)\cdot a_{n+1} \tag{1.1}$$

for $f \in C^n(A,X)$ and $a_1,\dots,a_{n+1} \in A$. Then $\delta^{n+1}\circ\delta^n = 0$ and $Z^n(A,X) = \operatorname{Ker}\delta^n \supset B^n(A,X) = \operatorname{Im}\delta^{n-1}$. The continuous Hochschild cohomology of A with coefficient X is defined by $\mathrm{H}^n(A,X) = Z^n(A,X)/B^n(A,X)$ (see [1,5,6]). When A is a commutative Banach algebra, $C^n(A,X)$ is a left A-module by the action

$$(a\cdot f)(a_1,\dots,a_n) = a\cdot f(a_1,\dots,a_n), \quad f\in A,\ a,a_1,\dots,a_n \in A$$

and the coboundary operator $\delta^n : C^n(A,X) \to C^{n+1}(A,X)$ is an A-module homomorphism, which induces a left A-module structure on $\mathrm{H}^n(A,X)$.

For a Banach algebra A and a Banach A-bimodule X, a bounded linear operator $D : A \to X$ is called a *derivation* if it follows the Leibniz rule:

$$D(ab) = a\cdot Db + Da\cdot b, \quad a,b\in A. \tag{1.2}$$

The space of all continuous derivations $A \to X$ is denoted by $\mathfrak{D}(A,X)$. An *inner derivation* is a derivation $D : A \to X$ defined by $Da = a\cdot x - x\cdot a$ $(a\in A)$ for some $x\in X$. The first cohomology $\mathrm{H}^1(A,X)$ is isomorphic to the space of derivations modulo the inner derivations.

The present paper studies continuous Hochschild cohomologies of Lipschitz algebras over compact metric spaces. For a compact metric space (M,d), let $\mathrm{Lip}M$ be the Banach algebra of all complex-valued Lipschitz functions $f : M \to \mathbb{C}$ with the norm

$$\|f\|_L = \|f\|_\infty + L(f)$$

where $\|f\|_\infty = \sup_{p\in M}|f(p)|$, the sup norm, and

$$L(f) := \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)} \mid x,y\in X,\ x\neq y\right\},$$

the Lipschitz constant of f. In a previous paper [8] the author proved that, for each $n\geq 1$, $\mathrm{H}^n(\mathrm{Lip}M,\mathbb{C})$ is an infinite dimensional $\mathbb{C}$-linear space when M contains a certain point-sequence which converges to a point $p\in M$. Here $\mathbb{C}$ is endowed with a $\mathrm{Lip}M$-bimodule structure given by:

$$f\cdot z = z\cdot f = f(p)z, \quad f\in\mathrm{Lip}M, \quad z\in\mathbb{C}. \tag{1.3}$$

The above result relies only on the local geometry of M at p and a question arises whether the same holds if the coefficient $\mathbb{C}$ is replaced with an appropriate continuous function algebra over M with a Lip$M$-module structure. The present paper gives an answer to the question.

For a compact metric space (M, d), let $\tilde{M} = M \times M \backslash \Delta M$, where $\Delta M = \{(x, x) \mid x \in M\} \subset M \times M$. Let $\beta\tilde{M}$ be the Stone–Čech compactification of $\tilde{M}$ (see [20]). Since $M \times M$ is another compactification of $\tilde{M}$, there exists a continuous surjection $\pi : \beta\tilde{M} \to M \times M$ such that $\pi|\pi^{-1}(\tilde{M}) : \pi^{-1}(\tilde{M}) \to \tilde{M}$ is a homeomorphism. Let

$$\begin{aligned} \hat{M} = \pi^{-1}(\Delta M) \\ \text{with the restriction of the map } \pi,\ \pi|\hat{M} : \hat{M} \to \Delta M. \end{aligned} \tag{1.4}$$

The restriction $\pi|\hat{M}$ is also denoted by $\pi : \hat{M} \to \Delta M$. In what follows we identify the space ΔM with M via the diagonal map $\Delta_M : M \to \Delta M$ and the map $(\Delta_M)^{-1} \circ \pi$ is also denoted by $\pi : \hat{M} \to M$. As will be explained in Sect. 3, the space $\hat{M}$ may be regarded as an analogue of the unit sphere bundle of the tangent bundle over a Riemannian manifold. For a point $\omega \in \hat{M}$, a point derivation $D_\omega : \mathrm{Lip}M \to \mathbb{C}$ is defined as an analogue of the directional derivative of smooth functions.

The Banach space $C(\hat{M})$ of all complex-valued continuous functions on $\hat{M}$ with the sup norm admits a Banach Lip$M$-bimodule structure given by

$$\begin{aligned} (f \cdot \varphi)(\omega) =& (\varphi \cdot f)(\omega) = f(\pi(\omega))\varphi(\omega), \\ f \in& \mathrm{Lip}M,\ \varphi \in C(\hat{M}),\ \omega \in \hat{M}. \end{aligned} \tag{1.5}$$

Our first result is on the continuous Hochschild cohomology $\mathrm{H}^*(\mathrm{Lip}M, C(\hat{M}))$. A map $\gamma : [a, b] \to M$ of the interval $[a, b]$ to a metric space (M, d) is called a *geodesic* if $d(\gamma(s), \gamma(t)) = |s - t|$ for each $s, t \in [a, b]$. By abuse of terminology the image of γ, denoted by Imγ, is also called a geodesic.

Definition 1.1 A metric space (M, d) is said to satisfy the condition (G) if there exists a positive number $\delta > 0$ such that

(∗) for each $x, y \in M$ with $d(x, y) \leq \delta$, there exists a unique geodesic $\gamma_{xy} : [0, d(x, y)] \to M$ such that $\gamma_{xy}(0) = x$, $\gamma_{xy}(d(x, y)) = y$.

Besides Riemannian manifolds, all CAT(κ) metric spaces (see [3]) are examples of spaces satisfying the condition (G).

Theorem 1.2 *Let (M, d) be a compact metric space satisfying the condition (G). Then for each $n \geq 1$, the cohomology* $\mathrm{H}^n(\mathrm{Lip}M, C(\hat{M}))$ *has the infinite* Lip$M$*-rank in the sense that, for each $N \geq 1$, there exist* Lip$M$*-linearly independent N elements in* $\mathrm{H}^n(\mathrm{Lip}M, C(\hat{M}))$.

The main result of [8] may be viewed as a local version of the above theorem. The above theorem should also be compared with the homological dimension theorems of Ogneva [14,15], Kleshchev [10] and Pugach [18]; the global homological dimension of the Frechét algebra $C^\infty(M)$ of the smooth functions on a smooth manifold M is equal to dim M [14,15], while the global homological dimension of $C^n(M)$ of the Banach algebra of the C^n-functions on M is infinity for each n, $1 \le n < \infty$. A long standing open problem is to decide the global homological dimension of $C([0,1]) = C^0([0,1])$ [5, Chap. V, section 2.5].

Our proof is conceptually motivated by the classical Hochschild–Kostant–Rosenberg theorem [13,16,17]. The space $\mathfrak{D}(\mathrm{Lip}M, C(\hat{M}))$ of all derivations $\mathrm{Lip}M \to C(\hat{M})$ is a $\mathrm{Lip}M$-module under the action

$$(f \cdot D)g(\omega) = f(\pi(\omega))Dg(\omega), \quad f, g \in \mathrm{Lip}M, \omega \in \hat{M}.$$

We take the n-fold exterior product $\wedge^n_{\mathrm{Lip}M}\mathfrak{D}(\mathrm{Lip}M, C(\hat{M}))$ of the $\mathrm{Lip}M$-module $\mathfrak{D}(\mathrm{Lip}M, C(\hat{M}))$, define a homomorphism $\Omega_n : \wedge^n_{\mathrm{Lip}M}\mathfrak{D}(\mathrm{Lip}M, C(\hat{M})) \to \mathrm{H}^n(\mathrm{Lip}M, C(\hat{M}))$ by

$$\Omega_n(D_1 \wedge \cdots \wedge D_n)(a_1, \ldots, a_n) = \det((D_i a_j)_{1 \le i, j \le n}),$$
$$D_1, \ldots, D_n \in \mathfrak{D}(\mathrm{Lip}M, C(\hat{M})), a_1, \ldots, a_n \in \mathrm{Lip}M \tag{1.6}$$

and prove that the image $\mathrm{Im}\,\Omega_n$ contains arbitrarily large number of $\mathrm{Lip}M$-linearly independent elements of $\mathrm{H}^n(\mathrm{Lip}M, C(\hat{M}))$ when the space M satisfies the condition (G). The notion of alternating n-cocycle due to Johnson [7] plays the crucial role in the proof.

The above idea naturally leads to the study of the cohomology with $C(M)$-coefficient $\mathrm{H}^n(\mathrm{Lip}M, C(M))$. The situation is rather different than that of the smooth-function setup and we prove the following theorem. A homeomorphism $h : S_1 \to S_2$ between metric spaces (S_1, d_1) and (S_2, d_2) is called a *bi-Lipschitz homeomorphism* (a lipeomorphism in [11]) if h and h^{-1} are both Lipschitz maps. A topological embedding $\alpha : D \to M$ of a metric space D into a metric space M is called a *bi-Lipschitz embedding* if $\alpha : D \to \mathrm{Im}\alpha$ is a bi-Lipschitz homeomorphism. Throughout $\mathbb{R}^m$ is assumed to be endowed with the standard Euclidean metric. Let $D^m = \{x \in \mathbb{R}^m \mid \|x\| \le 1\}$ and $\mathrm{int}D^m = \{x \in D^m \mid \|x\| < 1\}$.

Theorem 1.3 *Let (M, d) be a compact metric space such that, for each point $p \in M$, there exists a bi-Lipschitz embedding $\alpha : D^{m(p)} \to M$ of $D^{m(p)}$ into M ($m(p)$ may depend on p) such that $p \in \alpha(D^m)$ and $\alpha(\mathrm{int}D^{m(p)})$ is open in M. Then we have*

$$\mathrm{H}^1(\mathrm{Lip}M, C(M)) = \mathfrak{D}(\mathrm{Lip}M, C(M)) = 0.$$

In particular the conclusion holds for each compact Lipschitz manifold M.

Theorem 1.2 is proved in Sect. 2 and Theorem 1.3 is proved in Sect. 3 after developing the sphere-bundle-analogue mentioned above.

The rest of this section fixes notation and recalls some basic results. For a compact metric space (M, d), let $\pi : \hat{M} \to \Delta M$ be the map defined in (1.4). For a Lipschitz function $f : M \to \mathbb{C}$, let $\Phi_f : \tilde{M} \to \mathbb{C}$ be the function defined by

$$\Phi_f(x, y) = \frac{f(x) - f(y)}{d(x, y)}, \quad (x, y) \in \tilde{M}.$$

By the Lipschitz condition, Φ_f is a bounded continuous function on $\tilde{M}$ and hence admits the unique extension, called the *de Leeuw map* [2,4,19,22]

$$\beta\Phi_f : \beta\tilde{M} \to \mathbb{C},$$

to the Stone-Čech compactification of $\tilde{M}$ which restricts to the map

$$\hat{\Phi}_f := \beta\Phi_f|\hat{M} : \hat{M} \to \mathbb{C} \tag{1.7}$$

on the space $\hat{M}$. This defines a pairing $\hat{\Phi} : \hat{M} \times \mathrm{Lip}M \to \mathbb{C}$ by

$$\hat{\Phi}(\omega, f) = \hat{\Phi}_f(\omega), \quad \omega \in \hat{M}, f \in \mathrm{Lip}M$$

such that

$$|\hat{\Phi}(\omega, f)| \le L(f) \le \|f\|_L, \quad \omega \in \hat{M}, f \in \mathrm{Lip}M. \tag{1.8}$$

It is convenient to introduce the notation

$$D_\omega f = \hat{\Phi}_f(\omega), \quad \omega \in \hat{M},\ f \in \mathrm{Lip}M. \tag{1.9}$$

The map $\hat{\Phi}$ (or D_ω in the above notation) induces two maps

$$D : \mathrm{Lip}M \to C(\hat{M}), \quad T : \hat{M} \to (\mathrm{Lip}M)^*$$

defined by

$$\begin{aligned} Df(\omega) &= D_\omega f = \hat{\Phi}_f(\omega), \\ T(\omega)(f) &= D_\omega f = \hat{\Phi}_f(\omega), \quad \omega \in \hat{M}, \quad f \in \mathrm{Lip}M. \end{aligned} \tag{1.10}$$

Observe that (1.8) guarantees that $T(\omega) \in (\mathrm{Lip}M)^*$ for each $\omega \in \hat{M}$. The map D is a $\|\cdot\|_L - \|\cdot\|_\infty$-bounded linear operator and T is continuous if $(\mathrm{Lip}M)^*$ is endowed with the weak*-topology. We use the map D in the proof of Theorem 1.2 and T will be used in the discussion on the space $\hat{M}$ in Sect. 3. It follows from the proof of [19, Theorem 9.8] that $D : \mathrm{Lip}M \to C(\hat{M})$ satisfies

$$D(fg) = (\pi^* g)Df + (\pi^* f)Dg, \quad f, g \in \mathrm{Lip}M, \tag{1.11}$$

that is, D is a derivation of $\mathrm{Lip}M$ to the $\mathrm{Lip}M$-module $C(\hat{M})$ (cf. 1.5). A *point derivation* $D : \mathrm{Lip}M \to \mathbb{C}$ at a point $p \in M$ is a bounded linear functional on $\mathrm{Lip}M$ such that

$$D(fg) = f(p)Dg + g(p)Df, \quad f, g \in \mathrm{Lip}M.$$

The space of all point derivations at p is denoted by $\mathfrak{D}_p(\mathrm{Lip}M)$. The next result, which also follows from of [19, Theorem 9.8], explains the role of the operator defined by (1.9).

Theorem 1.4 (cf. [19, Theorem 9.8]) *Let (M, d) be a compact metric space and let $\pi : \hat{M} \to M$ be the map defined in (1.4).*

1. *For each $p \in M$ and for each $\omega \in \pi^{-1}(p) \subset \hat{M}$, $D_\omega : \mathrm{Lip}M \to \mathbb{C}$ is a continuous point derivation at p.*
2. *The weak $*$-closure of the linear span of $\{D_\omega \mid \omega \in \pi^{-1}(p)\}$ is equal to the space $\mathfrak{D}_p(\mathrm{Lip}M)$.*

We use the classical extension theorem of McShane [12].

Theorem 1.5 [12] *Let (K, d) be a metric space and let E be a subset of K. For each bounded real-valued Lipschitz function $f : E \to \mathbb{R}$, there exists a Lipschitz function $F : K \to \mathbb{R}$ such that*

1. *$F|E = f$,*
2. *$\|F\|_\infty = \|f\|_\infty$ and $L(F) = L(f)$.*

Next we recall the notion of alternating cocycles due to Johnson. Let $\mathfrak{S}_n$ be the nth symmetric group. For a Banach algebra A and a Banach A-bimodule X, the continuous n-cochains $C^n(A, X)$ is an $\mathfrak{S}_n$-module by the action

$$(\sigma F)(a_1, \ldots, a_n) = F(a_{\sigma(1)}, \ldots, a_{\sigma(n)}), \quad \sigma \in \mathfrak{S}_n,\ a_1, \ldots, a_n \in A.$$

An n-chain F is said to be *alternating* if $\sigma F = (\mathrm{sgn}\sigma)F$, where sgn σ denotes the signature of $\sigma \in \mathfrak{S}_n$. The subspace of all continuous alternating n-cocycles is denoted by $Z^n_{\mathrm{alt}}(A, X)$. An n-chain $F \in C^n(A, X)$ is called an *n-derivation* if

$$\begin{aligned} &F(a_1, \ldots, a_{i-1}, b_i c_i, a_{i+1}, \ldots, a_n) \\ &\quad = b_i \cdot F(a_1, \ldots, a_{i-1}, c_i, a_{i+1}, \ldots, a_n) \\ &\qquad + F(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n) \cdot c_i \end{aligned} \tag{1.12}$$

for each $i = 1, \ldots, n$ and for each $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, b_i, c_i \in A$.

Theorem 1.6 [7, Theorem 2.3, Propostion 2.9, Corollary 2.10] *Let A be a commutative Banach algebra and let X be a symmetric Banach A-bimodule.*

1. *An n-cochain $F \in C^n(A, X)$ is an alternating n-cocycle if and only if it is an alternating n-derivation.*
2. *The restriction $q_n|Z^n_{\mathrm{alt}}(A, X) : Z^n_{\mathrm{alt}}(A, X) \to H^n(A, X)$ of the natural quotient map $q_n : Z^n(A, X) \to H^n(A, X)$ to $Z^n_{\mathrm{alt}}(A, X)$ is injective.*

2 Proof of Theorem 1.2

This section is devoted to prove Theorem 1.2. The proof is divided into several steps. In Step 1, we give a construction of derivations Lip$M \to C(\hat{M})$. Step 2 supplies a construction of Lipschitz functions associated with a convergent point-sequence of M. Step 3 proves the theorem for $n = 1$ and the proof for $n > 1$ will be given in Step 4.

We start with a general discussion on maps induced on the Stone–Čech compactification of a space. Let M be a compact metric space and let $\pi : \beta\tilde{M} \to M \times M$ be the continuous surjection defined in (1.4) with the restriction $\pi : \hat{M} \to M$ (recall the identification $M \approx \Delta M$). Let N be a closed, hence compact, neighborhood of the diagonal set ΔM and let $F : N \to N$ be a continuous map such that $F(\Delta M) = F^{-1}(\Delta M) = \Delta M$. Let $\tilde{N} = N \backslash \Delta M$ and let $\tilde{F} = F|\tilde{N} : \tilde{N} \to \tilde{N}$ be the restriction of F. The map $\tilde{F}$ admits a unique extension $\beta\tilde{F} : \beta\tilde{N} \to \beta\tilde{N}$. Since N is another compactification of $\tilde{N}$, there exists the canonical continuous surjection $\pi_N : \beta\tilde{N} \to N$ such that $\pi_N|\pi_N^{-1}(\tilde{N}) : \pi_N^{-1}(\tilde{N}) \to \tilde{N}$ is a homeomorphism. Notice that $\beta\tilde{F}$ is the unique map such that

$$\beta\tilde{F}|\pi_N^{-1}(\tilde{N}) = \pi_N^{-1} \circ \tilde{F} \circ \pi_N|\pi_N^{-1}(\tilde{N}). \tag{2.1}$$

Lemma 2.1 1. *We have the inclusion*

$$\hat{M} = \pi^{-1}(\Delta M) \subset \beta\tilde{N} \subset \beta\tilde{M}$$

and $\pi_N = \pi|\beta\tilde{N}$.

2. $\pi_N \circ \beta\tilde{F} = F \circ \pi_N$.
3. *The restriction* $\beta\tilde{F}|\hat{M}$ *of* $\beta\tilde{F}$ *to* $\hat{M}$ *induces a map* $\hat{F} : \hat{M} \to \hat{M}$ *such that* $\pi \circ \hat{F} = (F|\Delta M) \circ \pi$.

Proof 1. Since N is closed in M, $\tilde{N}$ is closed in $\tilde{M}$ and by [20, Proposition 1.48], the Stone-Čech compactification $\beta\tilde{N}$ is the closure of $\tilde{N}$ in $\beta\tilde{M}$: $\beta\tilde{N} = \mathrm{cl}_{\beta\tilde{M}}\tilde{N}$. In particular $\beta\tilde{N} \subset \beta\tilde{M}$ and we have $\pi_N = \pi|\beta\tilde{N}$. It follows from this that $\pi^{-1}(\Delta M) \subset \beta\tilde{N}$.

2. We have from (2.1) that $\pi_N \circ \beta\tilde{F}|\pi_N^{-1}(\tilde{N}) = \tilde{F} \circ \pi_N|\pi_N^{-1}(\tilde{N})$ and the desired equality follows from the denseness of $\pi_N^{-1}(\tilde{N})$ in $\beta\tilde{M}$.
3. is a direct consequence of (1) and (2).

□

For a map $F : N \to N$ as above, we define a bounded linear map F^*D : Lip$M \to C(\hat{M})$ by

$$((F^*D)f)(\omega) = D_{\hat{F}(\omega)}f, \quad \omega \in \hat{M},\ f \in \mathrm{Lip}M.$$

Lemma 2.2 *If $F|\Delta M = \mathrm{id}_{\Delta M}$, then the operator $F^* D : \mathrm{Lip}M \to C(\hat{M})$ is a derivation.*

Proof It suffices to verify the Leibniz rule. Fix Lipschitz functions $f, g \in \mathrm{Lip}M$ and a point $\omega \in \hat{M}$. We have, by (1.11), the assumption $F|\Delta M = \mathrm{id}_{\Delta M}$ and (3) of Lemma 2.1, the following equalities:

$$\begin{aligned}((F^* D) fg)(\omega) &= D_{\hat{F}(\omega)} fg \\ &= \pi^* f(\hat{F}(\omega)) D_{\hat{F}(\omega)} g + \pi^* g(\hat{F}(\omega)) D_{\hat{F}(\omega)} f \\ &= f(\pi(\hat{F}(\omega)) D_{\hat{F}(\omega)} g + g(\pi(\hat{F}(\omega)) D_{\hat{F}(\omega)} f \\ &= f(\pi(\omega))(F^* D) g(\omega) + g(\pi(\omega))(F^* D) f(\omega).\end{aligned}$$

Recalling the Lip$M$-module structure of $C(\hat{M})$ ((1.5)) we obtain the conclusion. □

Proof of Theorem 1.2 Step 1. Let (M, d) be a compact metric space satisfying the condition (G) with a positive number $\delta > 0$ that meets the condition $(*)$ of Definition 1.1. We may and will assume that $\delta < 1$. Let

$$W = \{(x, y) \mid d(x, y) \leq \delta\} \tag{2.2}$$

and for each $(x, y) \in W$, let γ_{xy} be the unique geodesic joining x with y. In what follows it is convenient to take the parametrization of γ_{xy} as

$$\gamma_{xy} : \left[-\frac{d(x, y)}{2}, \frac{d(x, y)}{2}\right] \to M, \quad \gamma_{xy}\left(-\frac{d(x, y)}{2}\right) = x, \quad \gamma_{xy}\left(\frac{d(x, y)}{2}\right) = y.$$

Also let $m_{xy} = \gamma_{xy}(0)$, the midpoint of x and y. For $w(x, y) = d(x, y)/2$, the above parametrization of γ_{xy} is given by

$$\gamma_{xy} : [-w(x, y), w(x, y)] \to M. \quad \gamma_{xy}(-w(x, y)) = x, \gamma_{xy}(w(x, y)) = y.$$

We make a convention that $\gamma_{xx} = m_{xx} = \{x\}$ and $w(x, x) = 0$. Let $\kappa : [0, \delta] \to [0, 1]$ be the function defined by

$$\kappa(t) = t/\delta, \quad t \in [0, \delta]. \tag{2.3}$$

It satisfies

$$\kappa^{-1}(0) = \{0\}, \quad \kappa^{-1}(1) = \{\delta\}, \quad \kappa'(t) > 0. \tag{2.4}$$

The argument in Step 1 depends only on (2.4) and the explicit form (2.3) will be used in later steps. Let $H : W \to W$ be the map defined by

$$H(x, y) = \big(\gamma_{xy}(-w(x, y)\kappa(w(x, y))), \gamma_{xy}(w(x, y)\kappa(w(x, y)))\big), \quad (x, y) \in W. \tag{2.5}$$

Let $\xi(x,y) = \gamma_{xy}(-w(x,y)\kappa(w(x,y)))$ and $\eta(x,y) = \gamma_{xy}(w(x,y)\kappa(w(x,y)))$ so that $H(x,y) = (\xi(x,y), \eta(x,y))$. The map H has the following properties.

(a) For each $(x,y) \in W$, we have

(a.1) the points $\xi(x,y), \eta(x,y)$ are on the geodesic γ_{xy},
(a.2) $m_{\xi(x,y)\eta(x,y)} = m_{xy}$,
(a.3) $w(\xi(x,y), \eta(x,y)) = d(\xi(x,y), m_{xy}) = d(\eta(x,y), m_{xy}) = \kappa(w(x,y))\, w(x,y)$,
(a.4) $\gamma_{\xi(x,y)\eta(x,y)} = \gamma_{xy}|[-\kappa(w(x,y))w(x,y), \kappa(w(x,y))w(x,y)]$.

(b) $H|\Delta M = \mathrm{id}_{\Delta M},\ H|\partial W = \mathrm{id}_{\partial W}$ and $H^{-1}(\Delta M) = H(\Delta M) = \Delta M$,
(c) If $d(x,y) < \delta$, then $\lim_{n\to\infty} H^n(x,y) = (m_{xy}, m_{xy})$, where H^n denotes the n-fold iteration of H.

□

Proof (a.1)–(a.3) are direct consequences of the definition. (a.4) follows from the uniqueness of the geodesic joining $\xi(x,y)$ and $\eta(x,y)$. (b) follows from the definition (2.5) and (2.4). Note that $d(x,y) = \delta$ if $(x,y) \in \partial W$. To verify (c) let $w^i = w(H^i(x,y))$. By induction we can see directly that $w^{i+1} < w^i$ and $\kappa(w^{i+1}) < \kappa(w^i)$ due to (2.4). Then we see from (a.3) that

$$\begin{aligned} w^{i+1} &= \kappa(w^i)w^i = \kappa(w^i)\kappa(w^{i-1})\cdots\kappa(w^1)w^1 \\ &\le \kappa(w^1)^i w^1. \end{aligned}$$

Since $w(x,y) = d(x,y)/2 \le \delta/2 < 1$, we have $\kappa(w^1) = \kappa(w(x,y)) < 1$ and $\lim_i w^i = 0$. This and (a.2) imply the condition (c).

We apply Lemma 2.1 to the map $H : W \to W$ defined on the closed neighbourhood W of ΔM and obtain a sequence of linear operators

$$\left\{ (H^n)^* D : \mathrm{Lip} M \to C(\hat{M}) \mid n \ge 1 \right\}.$$

We see from Lemma 2.2 and the condition (b) that $(H^n)^* D$ is a derivation. Our goal is to prove that the above forms a Lip$M$-linearly independent sequence of derivations. Step 2. Fix a point p of M and take a geodesic $\gamma : [0,\delta] \to M$ such that $\gamma(0) = p$. Take a sequence $S_0 = \{x_k, y_k \mid k \ge 1\}$ of points on the geodesic Imγ which satisfies the following conditions:

(d.1) $\lim_k x_k = \lim_k y_k = p$, $x_k \ne y_k$ for each k,
(d.2) $d(x_1, p) < \delta$ and, for each $k \ge 1$, $d(x_{k+1}, p) < d(y_k, p) < d(x_k, p)$,
(d.3) for each $k \ge 1$, $d(x_{k+1}, y_{k+1}) < d(x_k, y_k)$.

For a fixed integer $N \ge 1$, we examine the sequence $\{H^\nu(x_k, y_k) \mid k \ge 1,\ 1 \le \nu \le N\}$ of points of W. The following statements are consequences of (a)–(c) above and will be used later.

(e) For each k, the geodesic $\gamma_{x_k y_k}$ is the geodesic segment in γ joining x_k and y_k, denoted by $\overline{x_k y_k}$ for simplicity.

(f) For $i \geq 0$, let $(x_k^i, y_k^i) = H^i(x_k, y_k)$ with $(x_k^0, y_k^0) = (x_k, y_k)$. Then the points x_k^{i+1} and y_k^{i+1} are on the geodesic $\overline{x_k^i y_k^i}$ so that $d(x_k^i, m_k) \downarrow 0$ and $d(y_k^i, m_k) \downarrow 0$ as $i \to \infty$.

The next lemma describes a general procedure to find a Lipschitz function that detects the derivation $(H^i)^* D$.

Lemma 2.3 (cf. [8, Lemma 2.2]) *Under the above notation, for each $N \geq 1$ and for each $i \in \{1, \dots, N\}$, there exist an integer $k_0 \geq 1$ and a real-valued Lipschitz function $f \in \mathrm{Lip} M$ such that*

1. $L(f) = 1$,
2. *for each $k \geq k_0$ we have $|\Phi_f(x_k^i, y_k^i)| \geq 1/4$ for each $i = 1, \dots, N$,*
3. *for each $k \geq k_0$ and for each $j \in \{1, \dots, N\}$ with $j \neq i$, we have $\Phi_f(x_k^j, y_k^j) = 0$.*

Proof First we make some preliminary estimates on the distance $d(x_k^i, y_\ell^j)$. Let $d_k = d(x_k, y_k)$, $w_k = w(x_k, y_k) = d_k/2$ and $m_k = m_{x_k y_k}$. Also for $j \geq 1$, let $w_k^j = d(x_k^j, y_k^j)/2 = d(x_k^j, m_k) = d(y_k^j, m_k)$. Under this notation we have

$$w_k^j = \delta (w_k/\delta)^{2^i} \tag{2.6}$$

In fact, $w_k^1 = \kappa(w(x_k, y_k)) w(x_k, y_y) = w_k^2/\delta$, and $w_k^{j+1} = \kappa(w_k^j) w_k^j = \delta^{-1}(w_k^j)^2$, from which (2.6) follows by an induction. Let

$$\varepsilon_k^j = \frac{d(x_k^j, x_k^{j+1})}{d(x_k, y_k)} = \frac{d(x_k^j, x_k^{j+1})}{d_k}, \quad j \geq 0. \tag{2.7}$$

We have by (2.6)

$$\begin{aligned} \varepsilon_k^j &= \frac{1}{d_k} \left(d\left(x_k^j, m_k\right) - d\left(x_k^{j+1}, m_k\right) \right) \\ &= \frac{1}{d_k} \delta \left(\frac{w_k}{\delta}\right)^{2^j} \left(1 - \left(\frac{w_k}{\delta}\right)^{2^j}\right). \end{aligned} \tag{2.8}$$

Let $r_k = w_k/\delta$. We use (2.8) to see

$$\frac{\varepsilon_k^j}{\varepsilon_k^{j-1}} = (r_k)^{2^{j-1}} \cdot \frac{1 - r_k^{2^j}}{1 - r_k^{2^{j-1}}}$$

for each $j \geq 1$. Since $w_k = d(x_k, y_k)/2 < d(x_k, p)/2 \leq \delta/2$, we see $0 < r_k < 1$ and thus, for each $j \geq 1$, we obtain

$$\lim_{k \to \infty} \frac{\varepsilon_k^j}{\varepsilon_k^{j-1}} = 0.$$

Also by (d.1) we see $\lim_k w_k = 0$. Take a large $k_0 \geq 1$ such that

$$\begin{aligned} &\frac{\varepsilon_k^j}{\varepsilon_k^{j-1}} \leq 1, \quad k \geq k_0, \ 1 \leq j \leq N \quad \text{and} \\ &r_k = w_k/\delta \leq 1/2, \quad k \geq k_0. \end{aligned} \tag{2.9}$$

Fix an integer $N \geq 1$, let $S_k^N = \{x_k^j, y_k^j \mid 0 \leq j \leq N\}$ and $S^N = \cup_{k \geq k_0} S_k^N \cup \{p\}$. We fix $i \in \{1, \ldots, N\}$ and define a function $f : S^N \to [0, \infty)$ by:

$$\begin{aligned} &f(p) = 0, \\ &f(x_k^i) = \varepsilon_k^i d_k = d\left(x_k^i, x_k^{i+1}\right), \quad \text{(see (2.7))} \\ &f(y_k^i) = 0, \\ &f(x_k^j) = f(y_k^j) = 0, \quad k \geq k_0, 0 \leq j \leq N, j \neq i. \end{aligned} \tag{2.10}$$

We first verify that the function f is a Lipschitz function on S^N with the Lipschitz constant 1 which satisfies the condition (2) and (3).

In order to estimate

$$\Phi_f(x_k^i, x_\ell^j) = \frac{f(x_k^i) - f(x_\ell^j)}{d(x_k^i, x_\ell^j)},$$

we may assume that $k \leq \ell$. First we observe

$$\Phi_f(x_k^i, x_k^{i+1}) = \frac{f(x_k^i) - f(x_k^{i+1})}{d(x_k^i, x_k^{i+1})} = 1 \tag{2.11}$$

and by (2.9)

$$0 \leq \Phi_f(x_k^i, x_k^{i-1}) = \frac{f(x_k^i) - f(x_k^{i-1})}{d(x_k^i, x_k^{i-1})} = \frac{\varepsilon_k^i}{\varepsilon_k^{i-1}} \leq 1. \tag{2.12}$$

For j with $0 \leq j \leq i-2$, we see $d(x_k^i, x_k^j) = d(x_k^i, x_k^{i-1}) + d(x_k^{i-1}, x_k^j) \geq d(x_k^i, x_k^{i-1})$ by (f). Hence we have by (2.12),

$$0 \leq \Phi_f(x_k^i, x_k^j) = \frac{f(x_k^i) - f(x_k^j)}{d(x_k^i, x_k^j)} \leq \frac{f(x_k^i)}{d(x_k^i, x_k^{i-1})} \leq 1 \tag{2.13}$$

Similarly by using (2.11) we have for j with $i + 2 \leq j \leq N$,

$$0 \leq \Phi_f\left(x_k^i, x_k^j\right) \leq 1. \tag{2.14}$$

Next we estimate $\Phi_f(x_k^i, x_\ell^j)$ for $\ell > k$. By definition $|\Phi_f(x_k^i, x_\ell^i)| = \frac{|\varepsilon_k^i d_k - \varepsilon_\ell^i d_\ell|}{d(x_k^i, x_\ell^i)}$, and we see

$$\varepsilon_k^i d_k \geq \varepsilon_\ell^i d_\ell.$$

In fact, we have, by (2.8), $\varepsilon_k^i d_k = \delta r_k^{2^i}(1 - r_k^{2^i})$ and $\varepsilon_\ell^i d_\ell = \delta r_\ell^{2^i}(1 - r_\ell^{2^i})$. Also by (2.9) we have $r_\ell = w_\ell/\delta \leq w_k/\delta = r_k \leq 1/2$ and hence $r_\ell^{2^i} \leq r_k^{2^i} \leq 1/2$, from which we obtain the desired inequality.

Also by (d.2) we have $d(x_k^i, x_\ell^i) = d(x_k^i, x_k^{i+1}) + d(x_k^{i+1}, x_\ell^i) \geq d(x_k^i, x_k^{i+1})$. Hence we obtain, by (2.11),

$$\begin{aligned} |\Phi_f(x_k^i, x_\ell^i)| &= \frac{|\varepsilon_k^i d_k - \varepsilon_\ell^i d_\ell|}{d(x_k^i, x_\ell^i)} = \frac{\varepsilon_k^i d_k - \varepsilon_\ell^i d_\ell}{d(x_k^i, x_\ell^i)} \\ &\leq \frac{\varepsilon_k^i d_k}{d(x_k^i, x_\ell^i)} \leq \frac{\varepsilon_k^i d_k}{d(x_k^i, x_k^{i+1})} = 1. \end{aligned} \tag{2.15}$$

Similarly we have for $\ell > k$,

$$\begin{aligned} &|\Phi_f(x_k^i, x_\ell^j)| \leq 1, \quad 0 \leq j \leq N, j \neq i \\ &|\Phi_f(x_k^i, y_\ell^j)| \leq 1, \quad 0 \leq j \leq N. \end{aligned} \tag{2.16}$$

Combining (2.11)–(2.16), we obtain $L(f) = 1$ on S^N.

In order to prove (2), we estimate $\Phi_f(x_k^i, y_k^i) = \frac{\varepsilon_k^i d_k}{d(x_k^i, y_k^i)}$. First we see

$$\begin{aligned} d(x_k^i, y_k^i) &= d(x_k . y_k) - \sum_{j=0}^{i-1} \left(d(x_k^j, x_k^{j+1}) + d(y_k^j, y_k^{j+1}) \right) \\ &= d(x_k, y_k) - 2\sum_{j=0}^{i-1} \varepsilon_k^j d_k = d_k \left(1 - 2\sum_{j=0}^{i-1} \varepsilon_k^j \right). \end{aligned} \tag{2.17}$$

Using (2.8) with $d_k = 2w_k$, we compute

$$2\sum_{j=0}^{i-1} \varepsilon_k^j = \frac{\delta}{w_k} \sum_{j=0}^{i-1} r_k^{2^j}(1 - r_k^{2^j}) = \frac{\delta}{w_k} \left(r_k - r_k^{2^i} \right).$$

Hence we obtain, by $w_k = \delta r_k$ (see 2.9),

$$2d_k \sum_{j=0}^{i-1} \varepsilon_k^j = d_k \frac{\delta}{w_k} \left(r_k - r_k^{2^i} \right) = d_k \left(1 - r_k^{2^i - 1} \right)$$

and by (2.17), we have

$$d(x_k^i, y_k^i) = d_k r_k^{2^i-1}.$$

Thus we obtain

$$\begin{aligned}\Phi_f(x_k^i, y_k^i) &= \frac{\varepsilon_k^i d_k}{d_k r_k^{2^i-1}} = \frac{r_k^{2^i}(1-r_k^{2^i})}{r_k^{2^i-1}} \frac{\delta}{d_k} \\ &= \frac{r_k}{d_k}\delta(1-r_k^{2^i}) = \frac{w_k(1-r_k^{2^i})}{d_k} \\ &= (1-r_k^{2^i})/2.\end{aligned}$$

Using $r_k^{2^i} = (\frac{w_k}{\delta})^{2^i} \leq \frac{w_k}{\delta} \leq 1/2$ we see that the last term of the above is at least $1/4$. Hence we obtain

$$\Phi_f(x_k^i, y_k^i) \geq 1/4, \tag{2.18}$$

which proves (2). (3) directly follows from the definition (2.10). Finally we apply Theorem 1.5 to the above f to obtain a Lipschitz extension $\bar{f} : M \to \mathbb{R}$ such that $L(\bar{f}) = L(f) = 1$, the desired condition (1). The function $\bar{f}$ satisfies (2) and (3) as well. This completes the proof of lemma. □

Step 3. We prove the theorem for $n = 1$. Since $C(\hat{M})$ is a symmetric Lip$M$-module, we have $\mathrm{H}^1(\mathrm{Lip}M, C(\hat{M})) = \mathfrak{D}(\mathrm{Lip}M, C(\hat{M}))$. In order to prove that $\mathfrak{D}(\mathrm{Lip}M, C(\hat{M}))$ has the infinite Lip$M$-rank, we take the map $H : M \times M \to M \times M$ in Step 1, fix an integer $N \geq 1$ and consider the N derivations

$$H^*D, \dots, (H^N)^*D : \mathrm{Lip}M \to C(\hat{M}),$$

and assume that, for $\varphi_1, \dots, \varphi_N \in \mathrm{Lip}M$, the equality

$$\sum_{j=1}^{N} \varphi_j(\pi(\omega))(H^j)^* D_\omega f = 0 \tag{2.19}$$

holds for each $\omega \in \hat{M}$ and for each $f \in \mathrm{Lip}M$. We fix $i \in \{1, \dots, N\}$ and show that $\varphi_i \equiv 0$. Pick an arbitrary point $p \in M$, take a geodesic γ, choose a sequence $\{x_k, y_k \mid k \geq 1\}$ of points on γ such that

$$\begin{aligned}&\gamma : [0, \delta] \to M, \text{ with } \gamma(0) = p \text{ and} \\ &\text{the sequence } \{x_k, y_k \mid k \geq 1\} \text{ satisfies (d.1)-(d.3),}\end{aligned} \tag{2.20}$$

and apply Lemma 2.3 to find an integer $k_0 \geq 1$ and a Lipschitz function f satisfying the conditions of the lemma.

Let ω be an accumulation point of the set $\{(x_k, y_k) \mid k \geq k_0\} \subset \beta\tilde{W}$. Then $\pi(\omega)$, as a point of $M \times M$, is an accumulation point of the set $\{(x_k, y_k) \mid k \geq k_0\} \subset M \times M$, that is, the singleton (p, p). Recalling the identification $M \approx \Delta M$ via the diagonal map, we have $\pi(\omega) = p$. Also $\hat{H}^j(\omega) = (\beta \tilde{H}^j)(\omega) = (\beta \tilde{H})^j(\omega)$ is an accumulation point of $\{H^j(x_k, y_k) = (x_k^j, y_k^j) \mid k \geq k_0\}$. This and the conditions 2 and 3 of Lemma 2.3 imply

$$\begin{aligned} D_{\hat{H}^i}(\omega) f &= \beta \Phi_f(\hat{H}^i(\omega)) \geq 1/4 \text{ and} \\ D_{\hat{H}^j}(\omega) f &= \beta \Phi_f(\hat{H}^j(\omega)) = 0, \quad 1 \leq j \leq N, j \neq i. \end{aligned}$$

Therefore from (2.19) we have

$$0 = \sum_{j=1}^{N} \varphi_j(\pi(\omega))((H^j)^* D)_\omega f = \varphi_i(p) D_{\hat{H}^i(\omega)} f$$

which shows $\varphi_i(p) = 0$ as required.

This completes the proof of the theorem for $n = 1$.

Step 4. This step finishes the proof of theorem, proving the case $n > 1$, by carrying out the idea stated in Sect. 1. Rather than considering the homomorphism Ω_n in (1.6), we proceed directly as follows. Let $Z^n_{\text{alt}}(\text{Lip}M, C(\hat{M}))$ be the space of the alternating n-cocycles on $\text{Lip}M$ with coefficient $C(\hat{M})$. By Theorem 1.3 we have an injection $Z^n_{\text{alt}}(\text{Lip}M, C(\hat{M})) \to \text{H}^n(\text{Lip}M, C(\hat{M}))$ and thus it suffices to prove that $Z^n_{\text{alt}}(\text{Lip}M, C(\hat{M}))$ has the infinite $\text{Lip}M$-rank.

Fix an arbitrary integer $N \geq 1$. For $\nu = 1, \ldots, N$ and $i = 1, \ldots, n$, let

$$H_{\nu,i} = H^{(\nu-1)n+i} : W \to W$$

and define the n-cochain $d_\nu \in C^n(\text{Lip}M, C(\hat{M}))$ by

$$d_\nu(a_1, \ldots, a_n)(\omega) = \det\left((H^*_{\nu,i} D) a_j(\omega)\right) = \det\left((D_{\hat{H}_{\nu,i}(\omega)} a_j)_{1 \leq i,j \leq n}\right). \tag{2.21}$$

It follows from the definition that d_ν is an alternating cochain. By Lemma 2.2, $D_{\hat{H}_{\nu,i}}(\omega)$ is a derivation, from which it follows that d_ν is an n-derivation. Thus by (1) of Theorem 1.6 we see that d_ν is an alternating cocycle: $d_\nu \in Z^n_{\text{alt}}(\text{Lip}M, C(\hat{M}))$.

Assume that, for $\varphi_1, \ldots, \varphi_N \in \text{Lip}M$, the equality

$$\sum_{\nu=1}^{N} \varphi_\nu(\pi(\omega)) d_\nu(a_1, \ldots, a_n)(\omega) = 0 \tag{2.22}$$

holds for each $\omega \in \hat{M}$ and for each $a_1, \ldots, a_n \in \text{Lip}M$. We fix $\mu \in \{1, \ldots, N\}$ and show $\varphi_\mu \equiv 0$. Take an arbitrary point p of M and choose a geodesic γ and a sequence $\{x_k, y_k \mid k \geq 1\}$ as in (2.20). Applying Lemma 2.3 we obtain an integer $k_0 \geq 1$ and a sequence $\{f_j \mid 1 \leq j \leq n\}$ of Lipschitz functions such that

$$L(f_j) = 1, \quad 1 \le j \le n,$$
$$|\Phi_{f_j}(H_{\mu,j}(x_k, y_k))| \ge 1/4, \quad k \ge k_0,\ 1 \le j \le n, \tag{2.23}$$
$$\Phi_{f_j}(H_{\mu,t}(x_k, y_k)) = 0, \quad k \ge k_0,\ 1 \le t \le n, t \ne j, \tag{2.24}$$
$$\Phi_{f_j}(H_{\nu,t}(x_k, y_k)) = 0, \quad k \ge k_0,\ 1 \le \nu \le N, \nu \ne \mu,\ 1 \le t \le n. \tag{2.25}$$

Let ω be an accumulation point of $\{(x_k, y_k) \mid k \ge k_0\} \subset \tilde{W}$. As in Step 3, we see $\pi(\omega) = p$ and $\hat{H}_{\nu,i}(\omega)$ is an accumulation point of $\{H_{\nu,i}(x_k, y_k) \mid k \ge k_0\}$ for each ν and i with $1 \le \nu \le N,\ 1 \le i \le n$. Thus by (2.23) and (2.24) we find a nonzero c_i such that

$$D_{\hat{H}_{\mu,i}(\omega)} f_j = \hat{\Phi}_{f_j}(\hat{H}_{\nu,i}(\omega)) = \beta\Phi_{f_j}(\hat{H}_{\nu,i}(\omega)) = \delta_{ij} c_i.$$

Also by (2.25) $D_{\hat{H}_{\nu,i}(\omega)} f_j = 0$ for each $\nu \ne \mu$. Hence by (2.22) we have

$$\begin{aligned} 0 &= \sum_{\nu=1}^{N} \varphi_\nu(\pi(\omega)) d_\nu(a_1, \ldots, a_n)(\omega) \\ &= \varphi_\mu(\pi(\omega)) d_\mu(f_1, \ldots, f_n)(\omega) = \varphi_\mu(p) c_1 \cdots c_n, \end{aligned}$$

which implies $\varphi_\mu(p) = 0$ as desired.

This completes Step 4 and hence completes the proof of the theorem. □

3 The space $\hat{M}$ and Proof of Theorem 1.3

Here we compare the point derivation D_ω for a point $\omega \in \hat{M}$ [see (1.9) and Theorem 1.4] with the derivation by tangent vectors of compact smooth manifolds. The comparison indicates that the space $\hat{M}$ may be regarded, to certain extent, as a Lipschitz analogue of the unit sphere bundle of a Riemannian manifold.

Let (M, g) be a compact Riemannian manifold with the metric d induced by g. By the compactness of M, there exists a $\delta > 0$ such that, for each pair p, q of points of M with $d(p, q) \le \delta$, there exists a unique geodesic $\gamma_{pq} : [0, d(p, q)] \to M$ such that

$$\gamma_{pq}(0) = p, \quad \gamma_{pq}(d(p, q)) = q, \quad \|\dot{\gamma}_{pq}(t)\| \equiv 1. \tag{3.1}$$

As in (2.2), let $W = \{(p, q) \in M \times M \mid d(p, q) \le \delta\}$ and let $\tilde{W} = W \setminus \Delta M$. By Lemma 2.1, we have the inclusion $\hat{M} \subset \beta\tilde{W} \subset \beta\tilde{M}$ and the canonical surjection $\pi_W : \beta\tilde{W} \to W$ is the restriction of $\pi : \beta\tilde{M} \to M \times M$. In what follows π_W is simply denoted by $\pi : \beta\tilde{W} \to W$. Let $\tau : TM \to M$ be the tangent bundle of M and let $SM = \{v \in TM \mid \|v\| = 1\}$, the unit sphere bundle. We define a map $V : \tilde{W} \to SM$ by

$$V(p, q) = \dot{\gamma}_{qp}(0) \in S_pM, \quad (p, q) \in \tilde{W}. \tag{3.2}$$

By the uniqueness of the geodesic γ_{qp} (3.1), the map V is a well-defined continuous map to the compact space SM and hence extends uniquely to the Stone-Čech compactification: $\beta V : \beta\tilde{W} \to SM$ which restricts to:

$$\hat{V} := \beta V|\hat{M} : \hat{M} \to SM.$$

As in Sect. 1, let $\Delta_M : M \to \Delta M \subset M \times M$ be the diagonal map. We have

Lemma 3.1 *We have the equality*

$$\Delta_M \circ \tau \circ \hat{V} = \pi.$$

Proof For a point $\omega \in \hat{M} \subset \beta\tilde{W}$ there exists a net $(p_\alpha, q_\alpha)_\alpha$ of points of $\tilde{W}$ such that $\lim_\alpha(p_\alpha, q_\alpha) = \omega$ in $\beta\tilde{W}$. By the continuity of βV we have

$$\hat{V}(\omega) = \lim_\alpha V(p_\alpha, q_\alpha) = \lim_\alpha \dot{\gamma}_{q_\alpha p_\alpha}(0).$$

Noticing $\tau(\dot{\gamma}_{q_\alpha p_\alpha}(0)) = q_\alpha$, we have

$$\Delta_M(\tau(\hat{V}(\omega))) = \Delta_M\left(\lim_\alpha \tau V(p_\alpha, q_\alpha)\right) = \left(\lim_\alpha q_\alpha, \lim_\alpha q_\alpha\right).$$

On the other hand $\pi(\omega) = \lim_\alpha(p_\alpha, q_\alpha) = (\lim_\alpha p_\alpha, \lim_\alpha q_\alpha)$. Since $\omega \in \pi^{-1}(\Delta M)$ we have by [19, Lemma 9.6] that $\lim_\alpha p_\alpha = \lim_\alpha q_\alpha$. Hence we have $\Delta_M(\tau(\hat{V}(\omega))) = \pi(\omega)$, as desired. □

In Sect. 1 the map $T : \hat{M} \to (\mathrm{Lip}M)^*$ was defined by $(T(\omega))(f) = D_\omega f$ for $\omega \in \hat{M}$, $f \in \mathrm{Lip}M$. The map is continuous when $(\mathrm{Lip}M)^*$ is endowed with the weak*-topology. Restricting $T(\omega)$ to the subspace $C^1(M)$ of $\mathrm{Lip}M$ consisting of the C^1-functions on M we obtain a composition

$$T : \hat{M} \to (\mathrm{Lip}M)^* \to (C^1(M))^*$$

which is continuous when $(C^1(M))^*$ is endowed with the weak*-topology. On the other hand we have a map $\theta : SM \to (C^1(M))^*$ given by

$$(\theta(v))(f) = vf, \quad v \in SM, \quad f \in C^1(M). \tag{3.3}$$

See [21, 1.21] for the action of tangent vectors on C^1-functions. The map θ is related to the map T by the next lemma. For $\xi \in (C^1(M))^*$ and $f \in C^1(M)$, $\xi(f)$ is also denoted by $\langle \xi, f \rangle$.

Lemma 3.2 1. $\theta \circ \hat{V} = T$, *that is, for each* $\omega \in \hat{M}$ *and for each* $f \in C^1(M)$, *we have*

$$D_\omega f = \hat{V}(\omega) f.$$

2. $\mathrm{Im}\,\theta = \mathrm{Im}\,T$.

Proof 1. For a point $\omega \in \hat{M}$ take a net $((p_\alpha, q_\alpha))_\alpha$ of points of $\tilde{W}$ such that $\omega = \lim_\alpha((p_\alpha, q_\alpha))$. By the continuity of θ, we have, for each $f \in C^1(M)$,

$$\begin{aligned}\langle(\theta \circ \hat{V})(\omega), f\rangle &= \langle\theta(\lim_\alpha V(p_\alpha, q_\alpha)), f\rangle \\ &= \lim_\alpha\langle\theta(\dot{\gamma}_{q_\alpha p_\alpha}(0)), f\rangle = \lim_\alpha \dot{\gamma}_{q_\alpha p_\alpha}(0) f \\ &= \lim_\alpha \frac{d}{dt}\Big|_{t=0} f(\gamma_{q_\alpha p_\alpha}(t)).\end{aligned}$$

The complex-valued function f is written as $f = u + iv$ for real-valued C^1-functions u and v. Applying the mean value theorem to u and v, we have

$$\begin{aligned}f(p_\alpha) - f(q_\alpha) &= f(\gamma_{q_\alpha p_\alpha}(d(q_\alpha, p_\alpha))) - f(\gamma_{q_\alpha p_\alpha}(0)) \\ &= \left(\frac{d(u \circ \gamma_{q_\alpha p_\alpha})}{dt}(\rho_\alpha) + i\frac{d(v \circ \gamma_{q_\alpha p_\alpha})}{dt}(\sigma_\alpha)\right) d(p_\alpha, q_\alpha)\end{aligned} \tag{3.4}$$

for some $\rho_\alpha, \sigma_\alpha \in (0, d(p_\alpha, q_\alpha))$. Since $\omega \in \hat{M}$ we have again by [19, Lemma 9.6] that $\lim_\alpha d(p_\alpha, q_\alpha) = 0$. By (3.4) we have

$$\Phi_f(p_\alpha, q_\alpha) = \frac{d(u \circ \gamma_{q_\alpha p_\alpha})}{dt}(\rho_\alpha) + i\frac{d(v \circ \gamma_{q_\alpha p_\alpha})}{dt}(\sigma_\alpha) \tag{3.5}$$

Taking the limit in (3.5) and using $\lim_\alpha(\frac{d}{dt}u \circ \gamma_{q_\alpha p_\alpha})(\rho_\alpha) = (\frac{d}{dt}u \circ \gamma_{q_\alpha p_\alpha})(0)$, $\lim_\alpha(\frac{d}{dt}v \circ \gamma_{q_\alpha p_\alpha})(\sigma_\alpha) = (\frac{d}{dt}v \circ \gamma_{q_\alpha p_\alpha})(0)$, we have

$$\begin{aligned}D_\omega f &= \hat{\Phi}_f(\omega) = \lim_\alpha \Phi_f(p_\alpha, q_\alpha) \\ &= \lim_\alpha \frac{d(f \circ \gamma_{q_\alpha p_\alpha})}{dt}(0) = \hat{V}(\omega) f.\end{aligned}$$

This proves (1).

2. From (1) we see $\mathrm{Im}T \subset \mathrm{Im}\theta$. In order to prove the reverse inclusion, let $v \in S_p M$ with $\|v\| = 1$ and take the geodesic $\gamma_v : [0, \delta] \to M$ such that

$$\gamma_v(0) = p, \quad \dot{\gamma}_v(0) = v.$$

Note that the point $(\gamma_v(t), p)$ is in $\tilde{W}$ for each $t \in (0, \delta]$. Using $\|\dot{\gamma}_v\| \equiv 1$, we see $d(\gamma_v(t), p) = t$. Thus for each $f \in C^1(M)$ and for each $t \in (0, \delta]$,

$$\begin{aligned}\Phi_f((\gamma_v(t), p)) &= \frac{f(\gamma_v(t)) - f(\gamma_v(0))}{d(\gamma_v(t), p)} \\ &= \frac{1}{d(\gamma_v(t), p)} \frac{d(f \circ \gamma_v)}{dt}(\rho_t) \cdot d(\gamma_v(t), p) \\ &= \frac{d(f \circ \gamma_v)}{dt}(\rho_t)\end{aligned}$$

for some $\rho_t \in (0,t)$. Let $\omega \in \hat{M}$ be an accumulation point of $\{(\gamma_v(t), p) \mid t \in (0,\delta]\}$. Since the net $(\frac{d(f\circ\gamma_v)}{dt}(\rho_t))_{t\in(0,\delta]}$ converges to $\frac{d(f\circ\gamma_v)}{dt}(0) = \dot{\gamma}(0)f = vf$, we have

$$D_\omega f = \hat{\Phi}_f(\omega) = vf,$$

as desired.

□

In view of these lemmas we may regard $\hat{M}$ as a Lipschitz-analogue of the unit sphere bundle SM of a Riemannian manifold M. Let $\Gamma(M, SM)$ be the space of smooth sections of the bundle $\tau : SM \to M$

$$\Gamma(M, SM) = \{\sigma : M \to SM \mid \tau \circ \sigma = \mathrm{id}_M\}$$

and let $\Theta : \Gamma(M, SM) \to \mathfrak{D}(C^1(M), C(M))$ be the map defined by

$$\Theta(\sigma)(f)(p) = \sigma(p)f, \quad \sigma \in \Gamma(M, SM),\ f \in C^1(M),\ p \in M. \tag{3.6}$$

A standard argument shows that the image $\mathrm{Im}\Theta$ is non-zero and finitely generated as a $C^1(M)$-module. The map Θ yields the map θ in Lemma 3.2 when localized at a point p: To be more precise, let $\epsilon_p : \Gamma(M, SM) \to S_pM$ and $e_p : \mathfrak{D}(C^1(M), C(M)) \to \mathfrak{D}_p(M)$ be the evaluation maps defined by

$$\begin{aligned} &\epsilon_p(\sigma) = \sigma(p), \quad \sigma \in \Gamma(M, SM), \\ &e_p(D)(f) = (Df)(p), \quad D \in \mathfrak{D}(C^1(M), C(M)), \quad f \in C^1(M). \end{aligned} \tag{3.7}$$

Then we have

$$e_p \circ \Theta = \theta \circ \epsilon_p.$$

Here the similarity between the spaces $\hat{M}$ and SM breaks down: every continuous map $\sigma : M \to \hat{M}$ of a path-connected compact metric space M must be a constant map, because the space $\hat{M} = \beta\tilde{M}\backslash\tilde{M}$, being a remainder of the Stone–Čech compactification of a non-psuedo-compact Lindelöf space $\tilde{M}$, contains no metrizable compact connected subsets which are not singletons [9] and hence the image $\sigma(M)$ must be a singleton. In particular there exists no continuous map $\sigma : M \to \hat{M}$ such that $\pi \circ \sigma = \mathrm{id}_M$ for such a space. This prevents us from defining a map which corresponds to Θ (3.6) to obtain elements of $\mathfrak{D}(\mathrm{Lip}M, C(M))$. More strongly, Theorem 1.3 states that there exists no non-zero derivations $\mathrm{Lip}M \to C(M)$ when M is a compact Lipschitz manifold. Combining [8, Theorem 3.5] we see that the map $e_p : \mathfrak{D}(\mathrm{Lip}M, C(M)) \to \mathfrak{D}_p(M)$ that corresponds to (3.7) reduces to the trivial map $0 \to$ (an ∞-dimensional space).

The rest of this section is devoted to the proof of Theorem 1.3. First the theorem is proved for $M = [0,1]^m \subset \mathbb{R}^m$ and the result is combined with Theorem 1.5 to prove the general case. We start with several lemmas. For simplicity let $I = [0,1]$.

For $i \in \{1, \dots, n\}$ and $a \in \mathrm{int} I$, the subspace $H = \{(t_1, \dots, t_m) \in I^m \mid t_i = a\}$ is called a *coordinate section*. For two points $x, y \in I^m$, $\overline{xy}$ denotes the segment joining x with y. For a subset S of I^m, $\mathrm{int} S$ denotes the interior of S in I^m. Notice that for each derivation $D : \mathrm{Lip} M \to C(M)$ we have

$$Dc = 0 \tag{3.8}$$

for each constant function $c \in \mathrm{Lip} M$.

Lemma 3.3 *For $a \in \mathrm{int} I$ and $i = 1, \dots, m$, let $H^{a,i} = \{(t_1, \dots, t_m) \in I^m \mid t_i = a\}$ be a coordinate section of I^m and let*

$$H_+^{a,i} = \{(t_1, \dots, t_m) \in I^m \mid t_i \geq a\}, \quad H_-^{a,i} = \{(t_1, \dots, t_m) \in I^m \mid t_i \leq a\}.$$

For a Lipschitz function $f \in \mathrm{Lip} I^m$ with $f | H^{a,i} \equiv 0$, let

$$f_+(x) = \begin{cases} f(x) & \text{if } x \in H_+^{a,i}, \\ 0 & \text{if } x \in H_-^{a,i}, \end{cases} \tag{3.9}$$

and

$$f_-(x) = \begin{cases} f(x) & \text{if } x \in H_-^{a,i}, \\ 0 & \text{if } x \in H_+^{a,i}. \end{cases} \tag{3.10}$$

Then f_+ and f_- are Lipschitz functions such that $f = f_+ + f_-$ and $f_+ \cdot f_- = 0$.

Proof Let $x \in H_+^{a,i}$ and $y \in H_-^{a,i}$ and take the point $m \in \overline{xy} \cap H^{a,i}$. We have

$$\frac{|f_+(x) - f_+(y)|}{\|x - y\|} = \frac{|f_+(x)|}{\|x - y\|} \leq \frac{|f_+(x)|}{\|x - m\|} \leq L(f).$$

Thus f_+ is a Lipschitz function. Similarly f_- is a Lipschitz function. The last equalities follow directly from the definition. □

Lemma 3.4 *Let $D : \mathrm{Lip} I^m \to C(I^m)$ be a derivation and let B be a convex body in I^m. For each $f \in \mathrm{Lip} I^m$ with $f | B \equiv 0$, we have $Df | B \equiv 0$.*

Proof Let $g(x) = d(x, \overline{I^m \setminus B}),\ x \in I^m$. It is straightforward to see

$$g \in \mathrm{Lip} I^m, \quad g^{-1}(0) = \overline{I^m \setminus B}, \quad fg \equiv 0. \tag{3.11}$$

Restricting the equality $0 = D(fg) = f \cdot Dg + g \cdot Df$ to $\mathrm{int} B$, we obtain

$$(g | \mathrm{int} B) \cdot (Df | \mathrm{int} B) = 0$$

and hence $Df | \mathrm{int} B = 0$. By the continuity of Df we have $Df | B = 0$. □

Lemma 3.5 *Let H be a coordinate section of I^m and let $D : \mathrm{Lip}I^m \to C(I^m)$ be a derivation. For each function $f \in \mathrm{Lip}M$ with $f|H \equiv 0$, we have $Df|H \equiv 0$.*

Proof We may assume $H = H^{a,m} = \{(t_1, \ldots, t_{m-1}, a) \mid t_i \in I, 1 \le i \le m-1\}$ for some $a \in \mathrm{int}I$. Let $H_+ = H_+^{a,i}$, $H_- = H_-^{a,i}$, and let $f \in \mathrm{Lip}M$ with $f|H \equiv 0$. We see that the functions f_+ and f_- defined by (3.9) and (3.10) are Lipschitz such that $f = f_+ + f_-$, $f_+f_- = 0$ due to Lemma 3.3. From the equality $0 = D(f_+f_-) = f_+Df_- + f_-Df_+$ we see

$$f(x)Df_-(x) = 0, \quad x \in H_+, \tag{3.12}$$

$$f(y)Df_+(y) = 0, \quad y \in H_-. \tag{3.13}$$

We take an arbitrary $p \in H$ and prove $Df(p) = 0$ by considering two cases.

Case 1. There exists an $\varepsilon > 0$ such that, for the rectangular neighbourhood $B_\varepsilon = \prod_{i=1}^m [p_i - \varepsilon, p_i + \varepsilon]$, we have either

$$f|B_\varepsilon \cap H_+ \equiv 0 \quad \text{or } f|B_\varepsilon \cap H_- \equiv 0.$$

Applying Lemma 3.4 to the convex body $B_\varepsilon \cap H_+$ or $B_\varepsilon \cap H_-$, we conclude $Df|B_\varepsilon \cap H_+ \equiv 0$ or $Df|B_\varepsilon \cap H_- \equiv 0$. In particular we have $Df(p) = 0$.

Case 2. There exist two sequences $(x_k)_{k\ge1}$ and $(y_k)_{k\ge1}$ such that

(i) $x_k \in H_+$, $y_k \in H_-$ and $f(x_k) \ne 0 \ne f(y_k)$ for each $k \ge 1$,
(ii) $\lim_k x_k = \lim_k y_k = p$.

By (3.12), (3.13) and (i) above, we have $Df_-(x_k) = Df_+(y_k) = 0$ for each k and hence by continuity of $Df_\pm$ we see $Df_-(p) = Df_+(p) = 0$. Then we see $Df(p) = Df_+(p) + Df_-(p) = 0$.

Since p is an arbitrary point of H we have $Df|H \equiv 0$. □

Remark 3.6 The above lemma holds also for $m = 1$ in which case H is a singleton in $\mathrm{int}I$.

Proof of Theorem 1.3 Step 1. As before let $I = [0, 1]$. First we prove the theorem for $M = I^m$ by induction on m.

(i) $m = 1$. Let $D : \mathrm{Lip}I \to C(I)$ be a derivation, let $f \in \mathrm{Lip}I$, and take a point $p \in \mathrm{int}I$. Let $f_p : I \to \mathbb{C}$ be the function defined by $f_p(t) = f(t) - f(p)$, $t \in I$. By (3.8) we have $Df_p = Df$. Since $f_p(p) = 0$, we have $(Df_p)(p) = 0$ by Lemma 3.5 and Remark 3.6. Thus we obtain $Df(p) = Df_p(p) = 0$. Since p is an arbitrary point of $\mathrm{int}I$ we see by continuity that $Df \equiv 0$ on I.
(ii) Assume that theorem holds for m and let $D : \mathrm{Lip}I^{m+1} \to C(I^{m+1})$ be a derivation. Take a point $a = (a_1, \ldots, a_{m+1}) \in \mathrm{int}I^{m+1}$ and take the coordinate section $H = \{(t_1, \ldots, t_m, a_{m+1}) \mid t_i \in I, 1 \le i \le m\}$. The space H is isometric to I^m and the inclusion of H into I^{m+1} is denoted by $\iota : H \to I^{m+1}$. Let $R : I^{m+1} \to H$ be the projection defined by

$$R(t_1, \ldots, t_{m+1}) = (t_1, \ldots, t_m, a_{m+1}), \quad (t_1, \ldots, t_{m+1}) \in I^{m+1}.$$

The map R is a Lipschitz map. We define an operator $d : \mathrm{Lip}H \to C(H)$ by $d = \iota^* \circ D \circ R^*$ which is explicitly given by

$$df = D(f \circ R)|H, \quad f \in \mathrm{Lip}H.$$

We show that d is a derivation. Indeed using $R|H = \mathrm{id}_H$ we have

$$\begin{aligned} d(fg) &= D(fg \circ R)|H = D((f \circ R) \cdot (g \circ R))|H \\ &= (f \circ R|H) \cdot D(g \circ R)|H + (g \circ R|H) \cdot (f \circ R)|H \\ &= fD(g \circ R)|H + gD(f \circ R)|H = fdg + gdf. \end{aligned}$$

By the inductive hypothesis and the isometry $H \equiv I^m$ we see $d = 0$. Thus for each $h \in \mathrm{Lip}H$, we have

$$dh = D(h \circ R)|H = 0. \tag{3.14}$$

For an arbitrary $f \in \mathrm{Lip}I^{m+1}$, consider the function g_f given by

$$g_f = f - (f|H) \circ R$$

which is a Lipschitz function on I^{m+1} such that $g_f|H \equiv 0$. By Lemma 3.5 we see $(Dg_f)|H \equiv 0$ and thus by (3.14) we have

$$Df|H = D((f|H) \circ R)|H = 0.$$

In particular $Df(a) = 0$. Since a is an arbitrary point of $\mathrm{int}I$ we see by continuity of Df that $Df \equiv 0$ on I^{m+1}.
This finishes the inductive step and Step 1 is completed.

Step 2. For a proof of general M, we use the next lemma. The standard Euclidean metric on I^m is denoted by ρ.

Lemma 3.7 *Let $D : \mathrm{Lip}M \to C(M)$ be a derivation. Let $\alpha : I^m \to M$ be a bi-Lipschitz embedding of I^m into a compact metric space (M, d) such that $\alpha(\mathrm{int}I^m)$ is open in M. For each $f \in \mathrm{Lip}M$ with $f|\alpha(I^m) \equiv 0$, we have $Df|\alpha(I^m) \equiv 0$.*

Proof For an $\epsilon \in (0, 1)$, let $\epsilon I^m = [\epsilon, 1 - \epsilon]^m$. We define a function $g : M \to [0, \infty)$ by

$$g(x) = \begin{cases} d(\alpha^{-1}(x), \overline{I^m \setminus \epsilon I^m}), & \text{if } x \in \alpha(I^m), \\ 0, & \text{if } x \notin \alpha(I^m). \end{cases}$$

Notice that

$$g|\alpha(\overline{I^m \setminus \epsilon I^m}) \equiv 0 \tag{3.15}$$

and hence the above function is well-defined.

In order to see that g is a Lipschitz function, first notice that $t \mapsto d(t, \overline{I^m \setminus \epsilon I^m})$ is a Lipschitz function on I^m. Since α is a bi-Lipschitz embedding we see that $g|\alpha(I^m)$ is a Lipschitz function. This and (3.15) imply that g is a locally Lipschitz function. By the compactness of M we conclude that $g \in \mathrm{Lip}M$ (see [11, p. 85]). Also by the definition $g(q) \neq 0$ for each $q \in \alpha(\mathrm{int}(\epsilon I^m))$.

For each $f \in \mathrm{Lip}M$ with $f|\alpha(I^m) \equiv 0$, we have $fg \equiv 0$ and thus

$$0 = D(fg)|\alpha(\epsilon I^m) = f \cdot Dg|\alpha(\epsilon I^m) + g \cdot Df|\alpha(\epsilon I^m) = g \cdot Df|\alpha(\epsilon I^m),$$

which implies $Df|\alpha(\mathrm{int}(\epsilon I^m)) = 0$. Since ϵ is an arbitrary number in $(0, 1)$ we see that $Df|\alpha(I^m) \equiv 0$. □

In order to finish the proof of Theorem, let M be a compact metric space as in the hypothesis and let $D : \mathrm{Lip}M \to C(M)$ be a continuous derivation. Fix a point $p \in M$. Take a bi-Lipschitz embedding $\alpha : I^m \to M$ such that $p \in \alpha(I^m)$ and $\alpha(\mathrm{int}I^m)$ is open in M. First we show that there exists a Lipschitz map $R : M \to \alpha(I^m)$ such that $R|\alpha(I^m) = \mathrm{id}_{\alpha(I^m)}$.

To show the above, let $\mathrm{proj}_j : I^m \to I$ be the projection to the j-th factor $(1 \le j \le m)$. The map $\mathrm{proj}_j \circ \alpha^{-1} : \alpha(I^m) \to I$ is a Lipschitz function and we apply Theorem 1.5 to obtain a Lipschitz function $r_j : M \to I$ such that $r_j|\alpha(I^m) = \mathrm{proj}_j \circ \alpha^{-1}$. Define $r : M \to I^m$ by $r(x) = (r_j(x))_{1 \le j \le m}$ and let

$$R = \alpha \circ r : M \to \alpha(I^m).$$

Then the map R is the desired Lipschitz map (see [11, Lemma 5.6]).

Take a function $f \in \mathrm{Lip}M$ and let g_f be the function given by

$$g_f = f - ((f|\alpha(I^m)) \circ R)$$

which is a Lipschitz function such that $g_f|\alpha(I^m) \equiv 0$. By Lemma 3.7 we see $Dg_f|\alpha(I^m) \equiv 0$. Thus we see

$$Df|\alpha(I^m) = D\left((f|\alpha(I^m)) \circ R\right)|\alpha(I^m). \tag{3.16}$$

We notice that the Lipschitz homeomorphism $\alpha : I^m \to \alpha(I^m)$ induces algebraic isomorphisms $\alpha^* : \mathrm{Lip}(\mathrm{Im}\alpha) \to \mathrm{Lip}(I^m)$ and $\alpha^* : C(\alpha(I^m)) \to C(I^m)$. It follows from this and Step 1 that the derivation $d : \mathrm{Lip}(\alpha(I^m)) \to C(\mathrm{Im}\alpha)$ defined by

$$dg = D(g \circ R)|\alpha(I^m), \quad g \in \mathrm{Lip}(\alpha(I^m))$$

is the zero-homomorphism. It implies $D(f|\alpha(I^m) \circ R)|\alpha(I^m) = 0$ for each $f \in \mathrm{Lip}M$. Combining this with (3.16) we have $Df|\alpha(I^m) = 0$ and thus $Df(p) = 0$, as required.

This completes the proof of theorem. □

For a compact metric space M as in Theorem 1.3 and $n \ge 2$, take an alternating n-cochain $F \in Z^n_{\mathrm{alt}}(\mathrm{Lip}(M), C(M))$. By (1) of Theorem 1.6, F is an n-derivation. Fixing arbitrary Lipschitz functions $f_1, \ldots, f_{n-1} \in \mathrm{Lip}(M)$, we have the linear operator

$f \mapsto F(f_1, \ldots, f_{n-1}, f)$ that is a derivation due to (1.12). It follows from the proof of Theorem 1.3 that the operator is zero and we conclude:

Corollary 3.8 *Let M be a compact metric space as in Theorem 1.3. Then we have $Z^n_{\mathrm{alt}}(\mathrm{Lip}(M), C(M)) = 0$ for each $n \geq 2$.*

It is not known to the author whether the cohomology $\mathrm{H}^n(\mathrm{Lip}(M), C(M))$ is trivial for each $n \geq 2$ and for each compact metric space M in Theorem 1.3.

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