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ORIGINAL PAPER

On a class of weak R-duals and the duality relations

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Abstract
The concept of R-duals of a sequence was first introduced with the motivation to obtain a general version of duality principle in Gabor analysis. Since then, various R-duals (types II, III, IV) and some relaxations of the R-dual setup have been introduced and studied by some mathematicians. All these "R-duals" provide a powerful tool in the analysis of duality relations in general frame theory. It is of independent interest in mathematics and far beyond the duality principle in Gabor analysis. Observe that the underlying sequences of a R-dual are a pair of orthonormal bases. In this paper we introduce the concept of weak R-duals based on a pair of Parseval frames. It is a new relaxation of the R-dual setup. We obtain a characterization of frames based on their weak R-duals, and prove that the weak R-dual of a frame (Riesz basis) is a frame sequence (frame). We also characterize (unitarily) equivalent frames in terms of weak R-duals. Finally, we present an explicit expression of the canonical duals of weak R-duals.

Keywords Frame · Riesz basis · R-dual · Weak R-dual · Duality principle

Mathematics Subject Classification 47A80 · 42C15 · 41A58

1 Introduction

The notion of frame dates back to [11], and was formally introduced by Duffin and Schaeffer in studying nonharmonic Fourier series [8]. But it had not generated much interest until the ground breaking work [6] by Daubechies, Grossmann and Meyer. Since then the theory of frames has been growing rapidly. Now it has seen great

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achievements in a variety of areas throughout mathematics and engineering [2,12–14].

Let $\mathcal{H}$ be a separable Hilbert space, and $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ an at most countable sequence in $\mathcal{H}$. f is called a *frame* for $\mathcal{H}$ if there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{i\in\mathbb{N}} \left|\langle f,\ f_i\rangle\right|^2 \le B\|f\|^2 \tag{1}$$

for all $f \in \mathcal{H}$, where A and B are called lower and upper frame bounds respectively. It is called a *tight (Parseval) frame* if $A = B$ ($A = B = 1$) in (1), and a *Bessel sequence* with Bessel bound B if the right-hand side inequality in (1) holds. It is called a *frame sequence* if it is a frame for its closed linear span $\overline{\mathrm{span}}\{f_i\}_{i\in\mathbb{N}}$. It is called a *Riesz sequence* in $\mathcal{H}$ if there exist constants $0 < A \le B < \infty$ such that

$$A\sum_{i\in\mathbb{N}} |c_i|^2 \le \left\|\sum_{i\in\mathbb{N}} c_i f_i\right\|^2 \le B\sum_{i\in\mathbb{N}} |c_i|^2 \tag{2}$$

for all finitely supported sequences $\mathrm{c} = \{c_i\}_{i\in\mathbb{N}}$, where A and B are called Riesz bounds. And it is called a *Riesz basis* for $\mathcal{H}$ if it is a Riesz sequence and $\overline{\mathrm{span}}\{f_i\}_{i\in\mathbb{N}} = \mathcal{H}$. For a Bessel sequence $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ in $\mathcal{H}$, we denote by T_{f} its synthesis operator, i.e.,

$$T_{\mathrm{f}}\mathrm{c} = \sum_{i=1}^{\infty} c_i f_i \quad \text{for } \mathrm{c} = \{c_i\}_{i\in\mathbb{N}} \in l^2(\mathbb{N}),$$

by T_{f}^* the adjoint operator of T_{f}, i.e.,

$$T_{\mathrm{f}}^*\xi = \{\langle \xi,\ f_i\rangle\}_{i\in\mathbb{N}} \quad \text{for } \xi \in \mathcal{H},$$

and by S_{f} the associated frame operator, i.e., $S_{\mathrm{f}} = T_{\mathrm{f}}T_{\mathrm{f}}^*$. Two frames $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ and $\mathrm{g} = \{g_i\}_{i\in\mathbb{N}}$ for $\mathcal{H}$ are said to be *equivalent* (*unitarily equivalent*) if there exists an invertible bounded operator (a unitary operator) U on $\mathcal{H}$ such that $Uf_i = g_i$ for each $i \in \mathbb{N}$. Recall that a sequence is a Riesz sequence (Riesz basis) if and only if it is an exact frame sequence (frame), i.e., it is a frame sequence (frame), but removing an arbitrary element from it cannot leave frame sequence (frame) for the initial space. For basics on frames, see e.g., [2,12,15].

Reference [2, Corollary 3.7.4] reduces the verification that a sequence in $\mathcal{H}$ is a Riesz sequence to a calculation of a countable collection of numbers. And by the beginning argument in [2, Sect. 3], at least conceptually, it is easier to check that sequence is a Riesz sequence than to check the frame property.

The frame literature contains several results relating frames and Riesz sequences. For example, in finite-dimensional setting, given a $n \times m$ matrix, its columns constitute a frame for $\mathbb{C}^n$ if and only if its rows form a Riesz sequence in $\mathbb{C}^m$. One of the prominent connections is the duality principle in Gabor analysis which was discovered almost simultaneously by three groups of researchers: Janssen[16], Daubechies, Landau, and

Landau[7], and Ron and Shen[17]. In [12], it was called the Ron-Shen duality principle. Given $g \in L^2(\mathbb{R})$ and two parameters $a, b > 0$, define the associated Gabor system $\mathcal{G}(g, a, b)$ by

$$\mathcal{G}(g, a, b) = \{E_{mb}T_{na}g : m, n \in \mathbb{Z}\},$$

where

$$T_{na}f(\cdot) = f(\cdot - na) \textit{ and } E_{mb}f(\cdot) = e^{2\pi imb\cdot}f(\cdot)$$

for $f \in L^2(\mathbb{R})$ and $m, n \in \mathbb{Z}$. The Ron-Shen duality principle states that, for each $g \in L^2(\mathbb{R})$ and $a, b > 0$ with $ab \le 1$, $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbb{R})$ if and only if $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ is a Riesz sequence in $L^2(\mathbb{R})$. Another important result is the Wexler-Raz biorthogonality relations[21] (see also [12, Theorem 7.3.1]). Partly motivated by the above results, Casazza et al. in [1] introduced the notion of R-duals in general Hilbert spaces. They characterized exactly properties of a sequence in terms of its R-dual sequence, which yields duality relations for the abstract frame setting. In particular, they also proved that $\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ is exactly one of its R-dual sequences if $\mathcal{G}(g, a, b)$ is a tight frame for $L^2(\mathbb{R})$. Christensen et al. in [3] derived conditions for a sequence to be a R-dual of a given frame, and considered a relaxation of the R-dual setup. Later, Xiao and Zhu in [22] extended the concept of R-duals to Banach spaces, and Christensen et al. in [4] presented some characterizations of R-dual sequences in Banach spaces. Chuang and Zhao in [5] characterized a class of R-duals. Stoeva and Christensen in [18,19] introduced R-duals of type II and III and showed that for tight frames these classes coincide with the R-duals. In particular, they proved that for a Gabor frame $\mathcal{G}(g, a, b)$, $\frac{1}{\sqrt{ab}}\mathcal{G}(g, \frac{1}{b}, \frac{1}{a})$ is exactly one of its R-duals of type III. We may refer to [9,10] for other related results.

Definition 1 Let $\mathcal{H}$ be a separable Hilbert space, and $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}}$ and $\mathrm{h} = \{h_i\}_{i\in\mathbb{N}}$ be orthonormal bases for $\mathcal{H}$. Given $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$ satisfying $\sum_{j=1}^{\infty} \left|\langle f_j, e_i\rangle\right|^2 < \infty$ for each $i \in \mathbb{N}$, define $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ by

$$\omega_i = \sum_{j=1}^{\infty}\langle f_j, e_i\rangle h_j \quad \text{for each } i \in \mathbb{N}. \tag{3}$$

Then ω is called the R-dual sequence of f with respect to e and h.

In this paper, we propose a kind of new R-duals called weak R-duals. Recall that a Parseval frame is a frame most like an orthonormal basis since if admits an orthonormal basis-like expansion, i.e., if e is a Parseval frame for a Hilbert space $\mathcal{H}$, then

$$f = \sum_{i\in\mathbb{N}} \langle f, e_i\rangle e_i \qquad \text{for } f \in \mathcal{H}.$$

A natural problem is to ask what we will obtain if orthonormal bases e and h in Definition 1 are replaced by two Parseval frames. This motivates us to introduce the following definition of weak R-duals.

Let $\mathcal{H}$ be a separable Hilbert space. For an infinite matrix $\mathcal{A} = (a_{ij})_{i,j\in\mathbb{N}}$ and a sequence $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ in $\mathcal{H}$, we write

$$\mathcal{A}\mathrm{f} = \left\{ \sum_{j\in\mathbb{N}} a_{ij} f_j \right\}_{i\in\mathbb{N}}$$

if every $\sum_{j\in\mathbb{N}} a_{ij} f_j$ with $i \in \mathbb{N}$ is well defined. Given sequences $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ and $\mathrm{g} = \{g_i\}_{i\in\mathbb{N}}$ in $\mathcal{H}$, we define the matrix $M(\mathrm{f}, \mathrm{g})$ by

$$M(\mathrm{f}, \mathrm{g}) = (\langle f_j,\ g_i\rangle)_{i,j\in\mathbb{N}},$$

i.e., the (i, j)-entry of $M(\mathrm{f}, \mathrm{g})$ is $\langle f_j,\ g_i\rangle$. For simplicity, we write $M(\mathrm{f}, \mathrm{f}) = M(\mathrm{f})$ which is exactly the Gram matrix associated with f. For general f and g, we call $M(\mathrm{f}, \mathrm{g})$ the *mixed Gram matrix* associated with f and g. We denote by I the identity matrix, by A^t and A^* its transpose and conjugate transpose for a matrix A, respectively, by $l_0(\mathbb{N})$ the set of finitely supported sequences defined on $\mathbb{N}$, and by $\mathscr{B}(l^2(\mathbb{N}))$ the set of bounded operators on $l^2(\mathbb{N})$.

Definition 2 Let $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}}$ and $\mathrm{h} = \{h_i\}_{i\in\mathbb{N}}$ be Parseval frames for $\mathcal{H}$, and let $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that

$$\sum_{j=1}^{\infty} \left|\langle f_j,\ e_i\rangle\right|^2 < \infty \quad \text{for each } i \in \mathbb{N}, \tag{4}$$

and

$$(M(\mathrm{h}) - I)M(\mathrm{f}, \mathrm{e})^t = 0. \tag{5}$$

Define the sequence ω by

$$\omega = \{\omega_i\}_{i\in\mathbb{N}} = M(\mathrm{f}, \mathrm{e})\mathrm{h}. \tag{6}$$

Then ω is called a weak R-dual sequence of the sequence f with respect to e and h.

Remark 1 (i) (4) shows that every row vector of $M(\mathrm{f}, \mathrm{e})$ belongs to $l^2(\mathbb{N})$. This implies that ω and $(M(\mathrm{h}) - I)M(\mathrm{f}, \mathrm{e})^t$ are both well defined since $M(\mathrm{h}) \in \mathscr{B}(l^2(\mathbb{N}))$ by Lemma 2.

(ii) The condition (5) is a technical condition for establishing the link between the synthesis and analysis operators of ω and f. This can be seen in Lemma 5.

(iii) Observe that, whenever h is an orthonormal basis, we have that $M(\mathrm{h}) = I$, and thus (5) holds. In this case, we do not require that e must be an orthonormal basis. Hence, “weak R-dual” is a genuine generalization of “R-dual”. Example 1 below provides us with another example satisfying (5) for the case that e and h are both Parseval frames but neither of them is an orthonormal basis. In particular, the “weak R-dual sequences” reduces to “R-dual sequences” if e and h are orthonormal bases for $\mathcal{H}$.

By a standard argument, we have

Example 1 Let $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}}$ be a Parseval frame for $\mathcal{H}$ and $\{\widetilde{h}_i\}_{i\in\mathbb{N}}$ an orthonormal basis for $\mathcal{H}$. Define $\mathrm{h} = \{h_i\}_{i\in\mathbb{N}}$ by

$$h_{2i-1} = h_{2i} = \frac{1}{\sqrt{2}}\widetilde{h}_i \qquad \text{for } i \in \mathbb{N}.$$

Then h is a Parseval frame for $\mathcal{H}$, and (4) and (5) hold for an arbitrary Bessel sequence $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ satisfying

$$f_{2i-1} = f_{2i} \quad \text{for } i \in \mathbb{N}.$$

The authors in [1] established duality principles based on R-duals. Precisely, they characterized frame properties of initial sequences and equivalence of frames using their R-duals, presented an explicit expression of the canonical duals of R-duals. Recall from [1, Theorem 2] that a sequence is a frame if and only if its R-dual is a Riesz sequence. It turns out that it is not the case for weak R-duals. See Theorems 1 and 2 for details. In particular, Theorem 2 gives the new dual relation that the weak R-dual of a frame (Riesz basis) is a frame sequence (frame). Corollaries 1, 3 and Remark 3 demonstrate that weak R-dual is a genuine extension of R-dual. It should be of independent interest and hopefully will motivate new research. On the other hand, we characterize (unitarily) equivalent frames in terms of weak R-duals similarly to [1, Theorem 17, 18], and derive an explicit expression of the canonical dual of a weak R-dual.

The rest of this paper is organized as follows. Section 2 focuses on dual relations based on weak R-duals. Section 3 is devoted to expressing the canonical dual of a weak R-dual.

2 Duality relations

This section focuses on duality relations based on weak R-duals. For this purpose, we first need to introduce some lemmas.

The following lemma can be obtained by the same procedure as in [15, Theorem 3.33] which dealt with finite dimensional case.

Lemma 1 *Two frames* $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ *and* $\mathrm{g} = \{g_i\}_{i\in\mathbb{N}}$ *for* $\mathcal{H}$ *are equivalent if and only if* $ker(T_\mathrm{f}) = ker(T_\mathrm{g})$.

Lemma 2 (i) $M(\mathrm{f}, \mathrm{g}) = T_\mathrm{g}^* T_\mathrm{f}$ *for arbitrary Bessel sequences* $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ *and* $\mathrm{g} = \{g_i\}_{i\in\mathbb{N}}$ *in* $\mathcal{H}$, *and thus* $M(\mathrm{f}, \mathrm{g}) \in \mathscr{B}(l^2(\mathbb{N}))$.

(ii) $M(\mathrm{f}, \mathrm{g})$ *is a bounded invertible operator on* $l^2(\mathbb{N})$ *for arbitrary Riesz bases* $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ *and* $\mathrm{g} = \{g_i\}_{i\in\mathbb{N}}$ *for* $\mathcal{H}$.

Proof (i) By a standard argument, we have that

$$\left\langle T_\mathrm{g}^* T_\mathrm{f} c,\ d\right\rangle = \left\langle T_\mathrm{f} c,\ T_\mathrm{g} d\right\rangle = \langle M(\mathrm{f}, \mathrm{g})c,\ d\rangle$$

for $c, d \in l_0(\mathbb{N})$. This implies that $M(\mathrm{f}, \mathrm{g}) = T_{\mathrm{g}}^* T_{\mathrm{f}}$ due to $l_0(\mathbb{N})$ being dense in $l^2(\mathbb{N})$.

(ii) Since T_{f} and T_{g} are bijections from $l^2(\mathbb{N})$ onto $\mathcal{H}$ if f and g are Riesz bases, (i) implies (ii).

Lemma 3 *Let* $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}}$ *and* $\mathrm{h} = \{h_i\}_{i\in\mathbb{N}}$ *be Parseval frames for* $\mathcal{H}$*. Then, given sequences* $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ *and* $\omega = \{\omega_i\}_{i\in\mathbb{N}}$*,* ω *is the weak R-dual sequence of the sequence* f *with respect to* e *and* h *if and only if* f *is the weak R-dual sequence of* ω *with respect to* h *and* e*.*

Proof By the symmetry, we only need to prove the necessity. Suppose ω is the weak R-dual sequence of the sequence f with respect to e and h. Then

$$(M(\mathrm{h}) - I)M(\mathrm{f}, \mathrm{e})^t = 0, \tag{7}$$

$$\omega = M(\mathrm{f}, \mathrm{e})\mathrm{h}. \tag{8}$$

It follows that

$$\langle f_i, e_j\rangle = \sum_{k=1}^{\infty} \langle f_k, e_j\rangle \langle h_k, h_i\rangle = \langle \omega_j, h_i\rangle \quad \text{for } i, j \in \mathbb{N} \tag{9}$$

by a simple computation. This implies that

$$\sum_{j=1}^{\infty} |\langle \omega_j, h_i\rangle|^2 = \sum_{j=1}^{\infty} |\langle f_i, e_j\rangle|^2 = \|f_i\|^2 < \infty \quad \text{for } i \in \mathbb{N}, \tag{10}$$

and

$$f_i = \sum_{j=1}^{\infty} \langle f_i, e_j\rangle e_j = \sum_{j=1}^{\infty} \langle \omega_j, h_i\rangle e_j \quad \text{for } i \in \mathbb{N} \tag{11}$$

due to e being a Parseval frame. It follows from (9) and (11) that

$$\langle \omega_j, h_i\rangle = \langle f_i, e_j\rangle = \sum_{k=1}^{\infty} \langle \omega_k, h_i\rangle \langle e_k, e_j\rangle \quad \text{for } i, j \in \mathbb{N} \tag{12}$$

which is equivalent to $(M(\mathrm{e}) - I)M(\omega, \mathrm{h})^t = 0$. Collecting (10)–(12) leads to the necessity.

Remark 2 When $\{e_i\}_{i\in\mathbb{N}}$ and $\{h_i\}_{i\in\mathbb{N}}$ are orthonormal bases, it is obvious that $M(\mathrm{h}) = I$, therefore Lemma 3 above will degenerate Lemma 1 in [1].

Lemma 4 *Let* $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}}$ *and* $\mathrm{h} = \{h_i\}_{i\in\mathbb{N}}$ *be Parseval frames for* $\mathcal{H}$*. Assume that* $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ *is the weak R-dual of* $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ *with respect to* e *and* h*. Then, for all* $\mathrm{a} = \{a_i\}_{i\in\mathbb{N}}$, $\mathrm{b} = \{b_i\}_{i\in\mathbb{N}} \in l_0(\mathbb{N})$, *we have*

$$\left\|\sum_{j=1}^{\infty} a_j \omega_j\right\|^2 = \sum_{i=1}^{\infty} |\langle \phi, f_i\rangle|^2 \ \textit{and} \ \left\|\sum_{i=1}^{\infty} b_i f_i\right\|^2 = \sum_{j=1}^{\infty} |\langle g, \omega_j\rangle|^2$$

where $\phi = \sum_{j=1}^{\infty} \overline{a_j} e_j$ *and* $g = \sum_{i=1}^{\infty} \overline{b_i} h_i$.

Proof We only need to prove first equation because the second equation automatically holds by Lemma 3. Fix $\phi = \sum_{j=1}^{\infty} \overline{a_j} e_j$ with $a \in l_0(\mathbb{N})$. Since ω is the weak R-dual sequence of the sequence f with respect to e and h, we have

$$\omega = M(\mathrm{f}, \mathrm{e})\mathrm{h} \tag{13}$$

and

$$(M(\mathrm{h}) - I)M(\mathrm{f}, \mathrm{e})^t = 0. \tag{14}$$

From (13), it follows that

$$\omega_j = \sum_{k=1}^{\infty} \langle f_k, e_j \rangle h_k \quad \text{for } j \in \mathbb{N}. \tag{15}$$

By (14), we have

$$\langle f_i, e_j \rangle = \sum_{k=1}^{\infty} \langle f_k, e_j \rangle \langle h_k, h_i \rangle \quad \text{for } i, j \in \mathbb{N}. \tag{16}$$

It follows that

$$\begin{aligned} \langle f_i, \phi \rangle &= \sum_{j=1}^{\infty} a_j \langle f_i, e_j \rangle \\ &= \sum_{j=1}^{\infty} a_j \left\langle \sum_{k=1}^{\infty} \langle f_k, e_j \rangle h_k, h_i \right\rangle, \end{aligned}$$

and thus

$$\langle f_i, \phi \rangle = \left\langle \sum_{j=1}^{\infty} a_j \omega_j, h_i \right\rangle \quad \text{for } i \in \mathbb{N} \tag{17}$$

by (15). It follows

$$\sum_{i=1}^{\infty} |\langle f_i, \phi \rangle|^2 = \left\| \sum_{j=1}^{\infty} a_j \omega_j \right\|^2 \tag{18}$$

due to h being a Parseval frame. □

Proposition 5 in [1] established the connection between a sequence and its R-dual sequence. As an immediate consequence of Lemma 4, we have following lemma. It extends Proposition 5 in [1], and establishes the connection between a sequence and its weak R-dual sequence.

Lemma 5 *Let* e $= \{e_i\}_{i\in\mathbb{N}}$ *and* h $= \{h_i\}_{i\in\mathbb{N}}$ *be Parseval frames for* $\mathcal{H}$. *Assume that* $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ *is the weak R-dual of* f $= \{f_i\}_{i\in\mathbb{N}}$ *with respect to* e *and* h. *Then* f *is a Bessel sequence in* $\mathcal{H}$ *with bound* B *if and only if* ω *is a Bessel sequence in* $\mathcal{H}$ *with bound* B. *In this case, for all* a $= \{a_i\}_{i\in\mathbb{N}}$, b $= \{b_i\}_{i\in\mathbb{N}} \in l^2(\mathbb{N})$, *we have*

$$\left\|\sum_{j=1}^{\infty} a_j\omega_j\right\|^2 = \sum_{i=1}^{\infty} |\langle \phi,\ f_i\rangle|^2 \ \text{and}\ \left\|\sum_{i=1}^{\infty} b_i f_i\right\|^2 = \sum_{j=1}^{\infty} \left|\langle g,\ \omega_j\rangle\right|^2,$$

where $\phi = \sum_{j=1}^{\infty} \overline{a_j} e_j$ *and* $g = \sum_{i=1}^{\infty} \overline{b_i} h_i$.

Lemma 6 *Let* f $= \{f_i\}_{i\in\mathbb{N}}$ *and* h $= \{h_i\}_{i\in\mathbb{N}}$ *be Bessel sequences in* $\mathcal{H}$, *and* e $= \{e_i\}_{i\in\mathbb{N}}$ *be complete in* $\mathcal{H}$. *Define*

$$\omega_j = \sum_{i=1}^{\infty} \langle f_i,\ e_j\rangle h_i \quad \text{for } j \in \mathbb{N}.$$

Then, for $g \in \mathcal{H}$, *we have that* $g \in (span\{\omega_j\}_{j\in\mathbb{N}})^{\perp}$ *if and only if* $\{\langle h_i,\ g\rangle\}_{i\in\mathbb{N}} \in ker(T_{\mathrm{f}})$.

Proof The proof is similar to that of [1, Lemma 3]. For completeness, we give it here. By a simple computation, we have

$$\langle g,\ \omega_j\rangle = \left\langle e_j,\ \sum_{i=1}^{\infty} \langle h_i,\ g\rangle\, f_i\right\rangle \quad \text{for } j \in \mathbb{N}.$$

This implies that $g \in (span\{\omega_j\}_{j\in\mathbb{N}})^{\perp}$ if and only if

$$\left\langle e_j,\ \sum_{i=1}^{\infty} \langle h_i,\ g\rangle\, f_i\right\rangle = 0 \quad \text{for } j \in \mathbb{N}.$$

It is in turn equivalent to $\{\langle h_i,\ g\rangle\}_{i\in\mathbb{N}} \in ker(T_{\mathrm{f}})$ by the completeness of e in $\mathcal{H}$. □

Lemma 7 *Let* f $= \{f_i\}_{i\in\mathbb{N}}$ *and* g $= \{g_i\}_{i\in\mathbb{N}}$ *be frames for* $\mathcal{H}$, *then* f *is unitarily equivalent to* g *if and only if*

$$\left\|\sum_{i=1}^{\infty} c_i f_i\right\|^2 = \left\|\sum_{i=1}^{\infty} c_i g_i\right\|^2 \ \text{for c} = \{c_i\}_{i\in\mathbb{N}} \in l^2(\mathbb{N}). \tag{19}$$

Proof Neccssity. Suppose $\mathcal{A}$ is a unitary operator on $\mathcal{H}$ satisfying $\mathcal{A} f_i = g_i$ for $i \in \mathbb{N}$. Then

$$\mathcal{A}(\sum_{i=1}^{\infty} c_i f_i) = \sum_{i=1}^{\infty} c_i g_i \quad \text{for c} \in l^2(\mathbb{N}).$$

This leads to (19) by the unitarity of $\mathcal{A}$.

Sufficiency. Suppose (19) holds. Then $ker(T_{\mathrm{f}}) = ker(T_{\mathrm{g}})$. This implies that, for $\mathrm{c}, \mathrm{d} \in l^2(\mathbb{N})$,

$$\sum_{i=1}^{\infty} c_i f_i = \sum_{i=1}^{\infty} d_i f_i \text{ if and only if } \sum_{i=1}^{\infty} c_i g_i = \sum_{i=1}^{\infty} d_i g_i.$$

For an arbitrary $f \in \mathcal{H}$, there exists $\mathrm{c} \in l^2(\mathbb{N})$ such that $f = \sum_{i=1}^{\infty} c_i f_i$ since f is a frame for $\mathcal{H}$. Define the operator $\mathcal{A}$ on $\mathcal{H}$ by

$$\mathcal{A}(\sum_{i=1}^{\infty} c_i f_i) = \sum_{i=1}^{\infty} c_i g_i \quad \text{for } \mathrm{c} \in l^2(\mathbb{N}). \tag{20}$$

Then $\mathcal{A}$ is a well-defined isometry by (19) and the above arguments. Also observing that g is a frame, we have that $\mathcal{A}$ is also onto. So $\mathcal{A}$ is a unitary operator on $\mathcal{H}$. By (20), we also have $\mathcal{A} f_i = g_i$ for $i \in \mathbb{N}$. The proof is completed.

Theorem 1 *Let* $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}}$ *and* $\mathrm{h} = \{h_i\}_{i\in\mathbb{N}}$ *be Parseval frames for* $\mathcal{H}$, *and* $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ *be a Bessel sequence in* $\mathcal{H}$. *Assume that* $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ *is the weak R-dual of* f *with respect to* e *and* h. *Then* f *is a frame for* $\mathcal{H}$ *with bounds* A *and* B *if and only if*

$$A \sum_{j=1}^{\infty} |a_j|^2 \le \left\| \sum_{j=1}^{\infty} \overline{a_j} \omega_j \right\|^2 \le B \sum_{j=1}^{\infty} |a_j|^2 \quad \textit{for } \mathrm{a} = \{a_i\}_{i\in\mathbb{N}} \in (ker(T_{\mathrm{e}}))^{\perp}. \tag{21}$$

Proof By Lemma 5.5.5 in [2] and Lemma 5, we have

$$\|T_{\mathrm{e}}\mathrm{a}\|^2 = \sum_{j=1}^{\infty} |a_j|^2 \tag{22}$$

and

$$\left\| \sum_{j=1}^{\infty} \overline{a_j} \omega_j \right\|^2 = \sum_{i=1}^{\infty} |\langle T_{\mathrm{e}}\mathrm{a}, \; f_i\rangle|^2 \tag{23}$$

for $\mathrm{a} = \{a_i\}_{i\in\mathbb{N}} \in (ker(T_{\mathrm{e}}))^{\perp}$, respectively. Therefore, (21) is equivalent to

$$A \, \|T_{\mathrm{e}}\mathrm{a}\|^2 \le \sum_{i=1}^{\infty} |\langle T_{\mathrm{e}}\mathrm{a}, \; f_i\rangle|^2 \le B \, \|T_{\mathrm{e}}\mathrm{a}\|^2 \text{ for } \mathrm{a} \in (ker(T_{\mathrm{e}}))^{\perp}.$$

It is in turn equivalent to

$$A \, \|f\|^2 \le \sum_{i=1}^{\infty} |\langle f, \; f_i\rangle|^2 \le B \, \|f\|^2 \text{ for } f \in \mathcal{H}.$$

Since $T_{\mathrm{e}}((ker(T_{\mathrm{e}}))^{\perp}) = \mathcal{H}$. This finishes the proof. □

Corollary 1 *Let* $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}}$ *and* $\mathrm{h} = \{h_i\}_{i\in\mathbb{N}}$ *be Parseval frames for* $\mathcal{H}$. *Assume that* $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ *is the weak R-dual of* $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ *with respect to* e *and* h. *Then the following statements hold:*

- (i) *If* e *is an orthonormal basis for* $\mathcal{H}$, *then* f *is a frame for* $\mathcal{H}$ *with bounds* A *and* B *if and only if* ω *is a Riesz sequence in* $\mathcal{H}$ *with bounds* A *and* B.
- (ii) *If* e *is not an orthonormal basis for* $\mathcal{H}$, *then* ω *cannot be a Riesz sequence.*

Proof (i) $ker(T_{\mathrm{e}}) = \{0\}$ if e is an orthonormal basis for $\mathcal{H}$. This leads to (i) by Theorem 1.

(ii) $ker(T_{\mathrm{e}}) \neq \{0\}$ if e is not an orthonormal basis. It follows that $ker(T_{\omega}) \neq \{0\}$ by (27). This leads to (ii).

Remark 3 In Corollary 1(i), h need not be an orthonormal basis. So Corollary 1(i) is a genuine extension of [1, Theorem 2].

Theorem 2 *Let* $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}}$ *and* $\mathrm{h} = \{h_i\}_{i\in\mathbb{N}}$ *be Parseval frames for* $\mathcal{H}$. *Assume that* $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ *is the weak R-dual of* $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ *with respect to* e *and* h. *Then the following statements hold:*

- (i) *If* f *is a frame for* $\mathcal{H}$ *with bounds* A *and* B, *then* ω *is a frame sequence in* $\mathcal{H}$ *with the same bounds.*
- (ii) *If* f *is a Riesz basis for* $\mathcal{H}$ *with bounds* A *and* B, *then* ω *is a frame for* $\mathcal{H}$ *with the same bounds.*

Proof (i) First we prove that

$$T_{\omega}\overline{\mathrm{c}} = T_{\mathrm{h}}\overline{T_{\mathrm{f}}^{*}T_{\mathrm{e}}\mathrm{c}} \quad \text{for } \mathrm{c} = \{c_i\}_{i\in\mathbb{N}} \in l^2(\mathbb{N}). \tag{24}$$

By Lemma 5, ω is a Bessel sequence in $\mathcal{H}$. This implies that T_{ω} is well defined and a bounded operator from $l^2(\mathbb{N})$ to $\mathcal{H}$. Observing that T_{h}, T_{f}^{*} and T_{e} are also bounded operators, in order to get (24), we only need to prove that

$$T_{\omega}\overline{\mathrm{c}} = T_{\mathrm{h}}\overline{T_{\mathrm{f}}^{*}T_{\mathrm{e}}\mathrm{c}} \quad \text{for } \mathrm{c} = \{c_i\}_{i\in\mathbb{N}} \in l_0(\mathbb{N}). \tag{25}$$

Next we prove (25). Since ω is the weak R-dual of f with respect to e and h, we have

$$\omega_j = \sum_{i=1}^{\infty} \langle f_i, e_j \rangle h_i \quad \text{for } j \in \mathbb{N}. \tag{26}$$

It follows that

$$\begin{aligned}T_\omega \overline{\mathrm{c}} &= \sum_{j=1}^{\infty} \overline{c_j} \omega_j \\ &= \sum_{j=1}^{\infty} \overline{c_j} \sum_{i=1}^{\infty} \langle f_i, e_j \rangle h_i \\ &= \sum_{i=1}^{\infty} \left\langle f_i, \sum_{j=1}^{\infty} c_j e_j \right\rangle h_i\end{aligned}$$

equivalently,

$$T_\omega \overline{\mathrm{c}} = T_{\mathrm{h}} \overline{T_{\mathrm{f}}^* T_{\mathrm{e}} \mathrm{c}}$$

for $\mathrm{c} \in l_0(\mathbb{N})$. This shows that (25) holds, and thus (24) holds.

By (24), we have

$$\{\overline{\mathrm{c}} : \mathrm{c} \in ker(T_{\mathrm{e}})\} \subset ker(T_\omega). \tag{27}$$

This leads to

$$(ker(T_\omega))^\perp \subset \{\overline{\mathrm{c}} : \mathrm{c} \in (ker(T_{\mathrm{e}}))^\perp\}. \tag{28}$$

By Theorem 1,

$$A \sum_{j=1}^{\infty} |a_j|^2 \leq \left\| \sum_{j=1}^{\infty} a_j \omega_j \right\|^2 \leq B \sum_{j=1}^{\infty} |a_j|^2$$

for $\mathrm{a} \in \{\overline{\mathrm{c}} : \mathrm{c} \in (ker(T_{\mathrm{e}}))^\perp\}$. This implies that

$$A \sum_{j=1}^{\infty} |a_j|^2 \leq \left\| \sum_{j=1}^{\infty} a_j \omega_j \right\|^2 \leq B \sum_{j=1}^{\infty} |a_j|^2 \quad \text{for } \mathrm{a} \in (ker(T_\omega))^\perp$$

by (28). Therefore, ω is a frame sequence in $\mathcal{H}$ with bounds A and B by Lemma 5.5.5 in [2].

(ii) Suppose f is a Riesz basis for $\mathcal{H}$ with bounds A and B, then ω is a frame sequence in $\mathcal{H}$ with the same bounds. So, in order to get (ii), we only need to prove it is complete in $\mathcal{H}$. Suppose $g \in \mathcal{H}$ satisfies

$$\langle \omega_j, g \rangle = 0 \text{ for all } j \in \mathbb{N}.$$

Then, using (26) we have

$$0 = \langle \omega_j, g \rangle = \left\langle \sum_{i=1}^{\infty} \langle h_i, g \rangle f_i, e_j \right\rangle$$

for $j \in \mathbb{N}$ by a simple computation. This implies that

$$\sum_{i=1}^{\infty} \langle h_i, g \rangle f_i = 0$$

since e is complete in $\mathcal{H}$. Thus $g = 0$ by the fact that f is a Riesz basis and h is a frame for $\mathcal{H}$. □

Observing that the Eq. (5) in Definition 2 automatically holds if h is an orthonormal basis for $\mathcal{H}$ (even if e is not an orthonormal basis for $\mathcal{H}$). As an immediate consequence of Theorem 2, we have the following corollary:

Corollary 2 *Let* $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}}$ *be a Parseval frame for* $\mathcal{H}$ *and* $\mathrm{h} = \{h_i\}_{i\in\mathbb{N}}$ *be an orthonormal basis for* $\mathcal{H}$*. Then the following statements hold:*

(i) *If* $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ *is a frame for* $\mathcal{H}$ *with bounds* A *and* B*, then its weak R-dual sequence* $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ *with respect to* e *and* h *is a frame sequence in* $\mathcal{H}$ *with the same bounds.*

(ii) *If* $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ *is a Riesz basis for* $\mathcal{H}$ *with bounds* A *and* B*, then its weak R-dual sequence* $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ *with respect to* e *and* h *is a frame for* $\mathcal{H}$ *with the same bounds.*

The following is an example of Theorem 2(i).

Example 2 Let $\{\widetilde{h}_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Define $\mathrm{h} = \{h_i\}_{i\in\mathbb{N}}$ by

$$h_{2i-1} = h_{2i} = \frac{1}{\sqrt{2}}\widetilde{h}_i \quad \text{for } i \in \mathbb{N},$$

and $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}} = \mathrm{h}$. Then e and h are Parseval frames for $\mathcal{H}$. Take $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ by

$$f_{4i-3} = f_{4i-2} = \widetilde{h}_i,\ f_{4i-1} = f_{4i} = 0 \quad \text{for } i \in \mathbb{N}.$$

Then it is easy to check that f is a frame for $\mathcal{H}$ satisfying

$$f_{2i-1} = f_{2i} \quad \text{for } i \in \mathbb{N}.$$

This implies that (4) and (5) hold by Example 1. By Theorem 2(i), the weak R-dual sequence $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ of the sequence f with respect to e and h is a frame sequence. We claim that it is not a frame although it is a frame sequence. Indeed, by Definition 2, ω has the form

$$\omega_i = \sum_{j=1}^{\infty} \langle f_j, e_i \rangle h_j \quad \text{for } i \in \mathbb{N}.$$

By a simple computation, we have

$$\omega_{2i-1} = \omega_{2i} = \widetilde{h}_{2i-1} \quad \text{for } i \in \mathbb{N}.$$

Obviously, $\langle \widetilde{h}_{2k}, \omega_i \rangle = 0 \quad \text{for } i, k \in \mathbb{N}$. This shows that ω cannot be complete in $\mathcal{H}$. Thus it is not a frame.

By Theorems 1 and 2(ii), we have the following corollary.

Corollary 3 *Let* $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}}$ *be an orthonormal basis for* $\mathcal{H}$ *and* $\mathrm{h} = \{h_i\}_{i\in\mathbb{N}}$ *be a Parseval frame for* $\mathcal{H}$*. Assume that* $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ *is a Riesz basis for* $\mathcal{H}$ *and* $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ *is the weak R-dual of* f *with respect to* e *and* h*. Then* ω *is a Riesz basis for* $\mathcal{H}$*.*

The following theorem generalizes [1, Proposition 17] to the weak R-dual case. It characterizes the equivalence between two frames in terms of corresponding weak R-duals.

Theorem 3 *Let* $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}}$ *be a Parseval frame for* $\mathcal{H}$ *and* $\mathrm{h} = \{h_i\}_{i\in\mathbb{N}}$ *be an orthonormal basis for* $\mathcal{H}$*. Assume that* $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ *and* $\mathrm{g} = \{g_i\}_{i\in\mathbb{N}}$ *are frames for* $\mathcal{H}$*, and that* $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ *and* $\gamma = \{\gamma_i\}_{i\in\mathbb{N}}$ *are the weak R-duals of* f *and* g *with respect to* e *and* h*, respectively. Then* f *is equivalent to* g *if and only if*

$$\overline{span}\{\omega_i\}_{i\in\mathbb{N}} = \overline{span}\{\gamma_i\}_{i\in\mathbb{N}} \tag{29}$$

Proof By Lemma 1, f is equivalent to g if and only if

$$ker(T_{\mathrm{f}}) = ker(T_{\mathrm{g}}). \tag{30}$$

So, to finish the proof, we only need to prove the equivalence between (30) and (29).

Suppose (30) holds. Next we prove

$$(span\{\omega_j\}_{j\in\mathbb{N}})^{\perp} = (span\{\gamma_j\}_{j\in\mathbb{N}})^{\perp}$$

which implies (29). Let $\varphi \in (span\{\omega_j\}_{j\in\mathbb{N}})^{\perp}$. Then

$$\{\langle h_i, \varphi\rangle\}_{i\in\mathbb{N}} \in ker(T_{\mathrm{f}})$$

by Lemma 6, and thus

$$\{\langle h_i, \varphi\rangle\}_{i\in\mathbb{N}} \in ker(T_{\mathrm{g}})$$

by (30). Again by Lemma 6, it follows that $\varphi \in (span\{\gamma_j\}_{j\in\mathbb{N}})^{\perp}$. Thus

$$(span\{\omega_j\}_{j\in\mathbb{N}})^{\perp} \subset (span\{\gamma_j\}_{j\in\mathbb{N}})^{\perp}$$

by the arbitrariness of φ. Similarly, we can prove the converse inclusion. Therefore,

$$(span\{\omega_j\}_{j\in\mathbb{N}})^{\perp} = (span\{\gamma_j\}_{j\in\mathbb{N}})^{\perp}.$$

Now we prove that (29) implies (30). By (29), we have

$$(span\{\omega_j\}_{j\in\mathbb{N}})^{\perp} = (span\{\gamma_j\}_{j\in\mathbb{N}})^{\perp}.$$

By Lemma 6, this implies that, for $\xi \in \mathcal{H}$,

$$\{\langle h_i, \xi\rangle\}_{i\in\mathbb{N}} \in ker(T_{\mathrm{f}}) \text{ if and only if } \{\langle h_i, \xi\rangle\}_{i\in\mathbb{N}} \in ker(T_{\mathrm{g}}). \tag{31}$$

Define $\Theta : \mathcal{H} \to l^2(\mathbb{N})$ by

$$\Theta\xi = \{\langle h_i, \xi\rangle\}_{i\in\mathbb{N}} \quad \text{for } \xi \in \mathcal{H}.$$

Then Θ is a bijection since h is an orthonormal basis for $\mathcal{H}$. So the statement (31) implies that, for every $\xi \in \mathcal{H}$,

$$\Theta\xi \in ker(T_{\mathrm{f}}) \text{ if and only if } \Theta\xi \in ker(T_{\mathrm{g}}),$$

equivalently,

$$\xi \in \Theta^{-1}ker(T_{\mathrm{f}}) \text{ if and only if } \xi \in \Theta^{-1}ker(T_{\mathrm{g}}).$$

That is to say $\Theta^{-1}ker(T_{\mathrm{f}}) = \Theta^{-1}ker(T_{\mathrm{g}})$. Therefore $ker(T_{\mathrm{f}}) = ker(T_{\mathrm{g}})$. □

Theorem 4 *Let* $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}}$ *and* $\mathrm{h} = \{h_i\}_{i\in\mathbb{N}}$ *be Parseval frames for* $\mathcal{H}$*. Assume that* $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ *and* $\mathrm{g} = \{g_i\}_{i\in\mathbb{N}}$ *are frames for* $\mathcal{H}$*, and that* $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ *and* $\gamma = \{\gamma_i\}_{i\in\mathbb{N}}$ *are the weak R-duals of* f *and* g *with respect to* e *and* h*, respectively. Then* f *is unitarily equivalent to* g *if and only if* $S_\omega = S_\gamma$*.*

Proof $S_\omega = S_\gamma$ if and only if

$$\langle S_\omega\phi, \phi\rangle = \left\langle S_\gamma\phi, \phi\right\rangle \quad \text{for } \phi \in \mathcal{H}. \tag{32}$$

Since h is a frame for $\mathcal{H}$, for every $\phi \in \mathcal{H}$ there exists $\mathrm{c} \in l^2(\mathbb{N})$ such that $\phi = \sum_{i=1}^{\infty} \overline{c_i} h_i$. Applying Lemma 5 to ϕ, we have

$$\left\| \sum_{i=1}^{\infty} c_i f_i \right\|^2 = \langle S_\omega\phi, \phi\rangle$$

and

$$\left\| \sum_{i=1}^{\infty} c_i g_i \right\|^2 = \left\langle S_\gamma\phi, \phi\right\rangle.$$

Therefore, (32) is equivalent to

$$\left\| \sum_{i=1}^{\infty} c_i f_i \right\|^2 = \left\| \sum_{i=1}^{\infty} c_i g_i \right\|^2 \quad \text{for } \mathrm{c} \in l^2(\mathbb{N}),$$

where the arbitrariness of c follows from that of ϕ. This is in turn equivalent to the fact that f is unitarily equivalent to g by Lemma 7. The proof is completed. □

3 An expression of the canonical duals of weak R-duals

When e and h are orthonormal bases for $\mathcal{H}$ and f is a frame for $\mathcal{H}$ with frame operator S_{f}, the canonical dual of R-dual sequence ω can be represented as $\sum_{j=1}^{\infty} \left\langle S_{\mathrm{f}}^{-1} f_j, e_i\right\rangle h_j$.

In this section, we give an expression of canonical dual of weak R-dual sequence. For this purpose, we first give some lemmas.

Applying (4.9-6), (4.9-7) and Theorem 4.9-A in [20], by a standard argument, we have the following lemma.

Lemma 8 *For an arbitrary* $T \in \mathscr{B}(\mathcal{H})$*, if one of* $range(T)$*,* $range(T^*)$*,* $range(TT^*)$ *and* $range(T^*T)$ *is a closed subspace of* $\mathcal{H}$*, so are the other three.*

Lemma 9 *Let* $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}}$ *be a frame sequence in* $\mathcal{H}$*. Then* $M(\mathrm{e})$ *is a bounded operator with closed range.*

Proof Let τ be a unitary operator from $\mathcal{H}$ onto $l^2(\mathbb{N})$. Then T_{e} is a bounded operator with closed range, and so is $T_{\mathrm{e}}\tau$. This implies that $(T_{\mathrm{e}}\tau)^*(T_{\mathrm{e}}\tau) = \tau^* T_{\mathrm{e}}^* T_{\mathrm{e}}\tau$ is also a bounded operator with closed range by Lemma 8. Also observe that τ is unitary and $M(\mathrm{e}) = T_{\mathrm{e}}^* T_{\mathrm{e}}$. It follows that $M(\mathrm{e})$ is a bounded operator with closed range.

Lemma 10 *Let* $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}}$ *be a frame for* $\mathcal{H}$ *and* $\mathcal{A} : \mathcal{H} \to \mathcal{K}$ *be a bounded operator with closed range, then*

$$\overline{span}\{\mathcal{A}e_j\}_{j\in\mathbb{N}} = range(\mathcal{A}).$$

Proof It is obvious that $\overline{span}\{\mathcal{A}e_j\}_{j\in\mathbb{N}} \subset range(\mathcal{A})$, we only need to prove $\overline{span}\{\mathcal{A}e_j\}_{j\in\mathbb{N}} \supset range(\mathcal{A})$. For arbitrary $f \in range(\mathcal{A})$, we have $\mathcal{A}g = f$ for some $g \in \mathcal{H}$. Since e is a frame for $\mathcal{H}$, we have $g = \sum_{i=1}^{\infty} a_i e_i$ for some $\mathrm{a} = \{a_i\}_{i\in\mathbb{N}} \in l^2(\mathbb{N})$. Let $a_i^{(n)} = \begin{cases} a_i, & \text{if } i \le n; \\ 0, & \text{if } i > n \end{cases}$, then $g = \lim_{n\to\infty} \sum_{i=1}^{\infty} a_i^{(n)} e_i$. Thus $f = \mathcal{A}g = \lim_{n\to\infty} \sum_{i=1}^{\infty} a_i^{(n)} \mathcal{A}e_i \in \overline{span}\{\mathcal{A}e_j\}_{j\in\mathbb{N}}$. The proof is completed. □

Lemma 11 *Let* $M_1, M_2 \subset \mathcal{H}$*,* $\overline{M_1} = \overline{M_2}$*, and* $\mathcal{A} : \mathcal{H} \to \mathcal{K}$ *be a bounded operator, then* $\overline{\mathcal{A}M_1} = \overline{\mathcal{A}M_2}$*.*

Proof By the symmetry, we only need to prove $\mathcal{A}M_1 \subset \overline{\mathcal{A}M_2}$. For $x \in M_1$, we have $x \in \overline{M_2}$ since $\overline{M_1} = \overline{M_2}$. This implies that $x = \lim_{n\to\infty} x_n$ for some sequence $\{x_n\}_{n\in\mathbb{N}}$ in M_2. It leads to

$$\mathcal{A}x = \lim_{n\to\infty} \mathcal{A}x_n \in \overline{\mathcal{A}M_2}. \tag{33}$$

Thus $\mathcal{A}M_1 \subset \overline{\mathcal{A}M_2}$ by the arbitrariness of x.

The following lemma is a generalization of Proposition 12 and Corollary 1 in [1] which deals with the case of R-duals.

Lemma 12 *Let* $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}}$ *and* $\mathrm{h} = \{h_i\}_{i\in\mathbb{N}}$ *be Parseval frames for* $\mathcal{H}$*, and* $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ *a Bessel sequence in* $\mathcal{H}$*. Assume that* $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ *is the weak R-dual of* f *with respect to* e *and* h*. Then the following statements hold:*

(i) *For all* $j, k \in \mathbb{N}$*,*

$$\langle \omega_j, \omega_k \rangle = \left\langle S_{\mathrm{f}}^{\frac{1}{2}} e_k, S_{\mathrm{f}}^{\frac{1}{2}} e_j \right\rangle;$$

(ii) *For all* $\mathrm{a} = \{a_i\}_{i\in\mathbb{N}} \in l^2(\mathbb{N})$,

$$\left\| \sum_{j=1}^{\infty} a_j \omega_j \right\| = \left\| \sum_{j=1}^{\infty} \overline{a_j} S_{\mathrm{f}}^{\frac{1}{2}} e_j \right\|.$$

Proof (i) By Definition 2, we have $M(\mathrm{f}, \mathrm{e})^t = M(\mathrm{h})M(\mathrm{f}, \mathrm{e})^t$. It is equivalent to

$$\left\langle f_i, e_j \right\rangle = \sum_{k=1}^{\infty} \left\langle f_k, e_j \right\rangle \langle h_k, h_i \rangle \quad \text{for } i, j \in \mathbb{N}. \tag{34}$$

On the other hand, we have

$$\left\langle \omega_j, \omega_k \right\rangle = \left\langle \sum_{i=1}^{\infty} \left\langle f_i, e_j \right\rangle h_i, \sum_{m=1}^{\infty} \langle f_m, e_k \rangle h_m \right\rangle \tag{35}$$

$$= \sum_{i=1}^{\infty} \left\langle f_i, e_j \right\rangle \overline{\sum_{m=1}^{\infty} \langle f_m, e_k \rangle \langle h_m, h_i \rangle}. \tag{36}$$

Collecting (34) and (36) leads to

$$\begin{aligned} \left\langle \omega_j, \omega_k \right\rangle &= \sum_{i=1}^{\infty} \left\langle f_i, e_j \right\rangle \overline{\langle f_i, e_k \rangle} \\ &= \left\langle S_{\mathrm{f}} e_k, e_j \right\rangle \\ &= \left\langle S_{\mathrm{f}}^{\frac{1}{2}} e_k, S_{\mathrm{f}}^{\frac{1}{2}} e_j \right\rangle. \end{aligned}$$

(ii) By (i), we have

$$\begin{aligned} \left\| \sum_{j=1}^{\infty} a_j \omega_j \right\|^2 &= \sum_{j,k=1}^{\infty} a_j \overline{a_k} \left\langle \omega_j, \omega_k \right\rangle \\ &= \sum_{j,k=1}^{\infty} a_j \overline{a_k} \left\langle S_{\mathrm{f}}^{\frac{1}{2}} e_k, S_{\mathrm{f}}^{\frac{1}{2}} e_j \right\rangle \\ &= \left\| \sum_{j=1}^{\infty} \overline{a_j} S_{\mathrm{f}}^{\frac{1}{2}} e_j \right\|^2. \end{aligned}$$

Lemma 13 *Let* $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}}$ *and* $\mathrm{h} = \{h_i\}_{i\in\mathbb{N}}$ *be Parseval frames for* $\mathcal{H}$. *Two Bessel sequences* $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ *and* $\mathrm{g} = \{g_i\}_{i\in\mathbb{N}}$ *in* $\mathcal{H}$ *form a pair of dual frames for* $\mathcal{H}$ *if and only if*

$$\langle \omega_i, \gamma_k \rangle = \langle e_k, e_i \rangle \text{ for } i, k \in \mathbb{N}. \tag{37}$$

whenever $\gamma = \{\gamma_i\}_{i\in\mathbb{N}}$ *is a weak R-dual of* g *with respect to* e *and* h*, and*

$$\omega_i = \sum_{j=1}^{\infty}\left\langle f_j,\ e_i\right\rangle h_j \textit{ for } i \in \mathbb{N}.$$

Proof For the weak R-duals ω of f, γ of g, we have

$$\begin{aligned}\langle \omega_i,\ \gamma_k\rangle &= \left\langle \sum_{j=1}^{\infty}\left\langle f_j,\ e_i\right\rangle h_j,\ \sum_{j=1}^{\infty}\left\langle g_j,\ e_k\right\rangle h_j \right\rangle \\ &= \sum_{j=1}^{\infty}\left\langle f_j,\ e_i\right\rangle \sum_{m=1}^{\infty}\overline{\langle g_m,\ e_k\rangle \left\langle h_m,\ h_j\right\rangle}\end{aligned}$$

for $i, k \in \mathbb{N}$. On the other hand, by Definition 2, we have $M(\mathrm{g}, \mathrm{e})^t = M(\mathrm{h})M(\mathrm{g}, \mathrm{e})^t$, equivalently,

$$\left\langle g_j,\ e_k\right\rangle = \sum_{m=1}^{\infty}\langle g_m,\ e_k\rangle \left\langle h_m,\ h_j\right\rangle \quad \text{for } j, k \in \mathbb{N}. \tag{38}$$

It follows that

$$\begin{aligned}\langle \omega_i,\ \gamma_k\rangle &= \sum_{j=1}^{\infty}\left\langle f_j,\ e_i\right\rangle \overline{\left\langle g_j,\ e_k\right\rangle} \\ &= \left\langle e_k,\ \sum_{j=1}^{\infty}\left\langle e_i,\ f_j\right\rangle g_j \right\rangle\end{aligned}$$

for $i, k \in \mathbb{N}$. Therefore, (37) holds if and only if

$$e_i = \sum_{j=1}^{\infty}\left\langle e_i,\ f_j\right\rangle g_j \quad \text{for } i \in \mathbb{N}, \tag{39}$$

equivalently, f and g form a pair of dual frames by the completeness of e in $\mathcal{H}$. The proof is completed.

Recall from Theorem 2 and Proposition 14 in [1] that, if $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ is a frame, and $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}}$ and $\mathrm{h} = \{h_i\}_{i\in\mathbb{N}}$ are orthonormal bases for $\mathcal{H}$, then the R-dual $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ of f associated with e and h is a Riesz sequence in $\mathcal{H}$, and its canonical dual $\{S_{\omega}^{-1}\omega_i\}_{i\in\mathbb{N}}$ can be represented as

$$S_{\omega}^{-1}\omega_i = \sum_{j=1}^{\infty}\left\langle S_{\mathrm{f}}^{-1} f_j,\ e_i\right\rangle h_j.$$

Theorem 2 shows that, if $\mathrm{e} = \{e_i\}_{i\in\mathbb{N}}$ and $\mathrm{h} = \{h_i\}_{i\in\mathbb{N}}$ are Parseval frames for $\mathcal{H}$, $\mathrm{f} = \{f_i\}_{i\in\mathbb{N}}$ is a frame for $\mathcal{H}$, and $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ be the weak R-dual of f with respect to e and h, then ω is a frame sequence which need not to be a Riesz sequence. So it is a natural problem to express the canonical dual of ω in this case. The following theorem gives an explicit expression in terms of pseudo-inverse of $M(\mathrm{e})$.

Theorem 5 *Let* $\mathrm{e}=\{e_i\}_{i\in\mathbb{N}}$ *and* $\mathrm{h}=\{h_i\}_{i\in\mathbb{N}}$ *be Parseval frames for* $\mathcal{H}$. *Let* $\mathrm{f}=\{f_i\}_{i\in\mathbb{N}}$ *be a frame for* $\mathcal{H}$ *with frame operator* S_{f} *and* $\omega=\{\omega_i\}_{i\in\mathbb{N}}$ *be the weak R-dual of* f *with respect to* e *and* h. *Define* ω_i^* *for each* $i\in\mathbb{N}$ *by*

$$\omega_i^*=\sum_{j=1}^{\infty}\left\langle S_{\mathrm{f}}^{-1}f_j,\, e_i\right\rangle h_j.$$

Then

$$\{S_\omega^{-1}\omega_i\}_{i\in\mathbb{N}}=M(\mathrm{e})^{\dagger}\{\omega_i^*\}_{i\in\mathbb{N}}. \tag{40}$$

Proof Since $T_{\mathrm{f}}^*S_{\mathrm{f}}^{-1}$ and T_{f}^* are bounded operators with closed range and e is Parseval frame for $\mathcal{H}$, we have

$$\overline{span}\{T_{\mathrm{f}}^*S_{\mathrm{f}}^{-1}e_i\}_{i\in\mathbb{N}}=range(T_{\mathrm{f}}^*S_{\mathrm{f}}^{-1})=range(T_{\mathrm{f}}^*)=\overline{span}\{T_{\mathrm{f}}^*e_i\}_{i\in\mathbb{N}}$$

by Lemma 10. It follows that

$$\overline{span}\{\overline{T_{\mathrm{f}}^*S_{\mathrm{f}}^{-1}e_i}\}_{i\in\mathbb{N}}=\overline{span}\{\overline{T_{\mathrm{f}}^*e_i}\}_{i\in\mathbb{N}},$$

and thus

$$\overline{span}\{T_{\mathrm{h}}\overline{T_{\mathrm{f}}^*S_{\mathrm{f}}^{-1}e_i}\}_{i\in\mathbb{N}}=\overline{span}\{T_{\mathrm{h}}\overline{T_{\mathrm{f}}^*e_i}\}_{i\in\mathbb{N}}$$

by Lemma 11. Also observing that

$$\omega_i^*=T_{\mathrm{h}}\overline{T_{\mathrm{f}}^*S_{\mathrm{f}}^{-1}e_i}\text{ and }\omega_i=T_{\mathrm{h}}\overline{T_{\mathrm{f}}^*e_i}$$

leads to

$$\overline{span}\{\omega_i^*\}_{i\in\mathbb{N}}=\overline{span}\{\omega_i\}_{i\in\mathbb{N}}. \tag{41}$$

By Lemma 2, $M(\{S_{\mathrm{f}}^{-1}f_j\}_{j\in\mathbb{N}},\mathrm{e})\in\mathscr{B}(l^2(\mathbb{N}))$. So its transpose operator $M(\{S_{\mathrm{f}}^{-1}f_j\}_{j\in\mathbb{N}},\mathrm{e})^t\in\mathscr{B}(l^2(\mathbb{N}))$. For $\mathrm{c}\in l_0(\mathbb{N})$, it is easy to check that

$$\sum_{i=1}^{\infty}c_i\omega_i^*=T_{\mathrm{h}}M(\{S_{\mathrm{f}}^{-1}f_j\}_{j\in\mathbb{N}},\mathrm{e})^t\mathrm{c}.$$

It follows that

$$\begin{aligned}\left\|\sum_{i=1}^{\infty}c_i\omega_i^*\right\|&\le\|T_{\mathrm{h}}\|\left\|M(\{S_{\mathrm{f}}^{-1}f_j\}_{j\in\mathbb{N}},\mathrm{e})^t\mathrm{c}\right\|\\&\le\|T_{\mathrm{h}}\|\left\|M(\{S_{\mathrm{f}}^{-1}f_j\}_{j\in\mathbb{N}},\mathrm{e})^t\right\|\|\mathrm{c}\|\end{aligned}$$

for c $\in l_0(\mathbb{N})$. This implies that $\{\omega_i^*\}_{i\in\mathbb{N}}$ is a Bessel sequence in $\overline{span}\{\omega_i\}_{i\in\mathbb{N}}$ by (41). By Lemma 9, $M(\mathrm{e})$ is a bounded operator with closed range which shows that $M(\mathrm{e})^\dagger$ is well defined and bounded. Therefore, $M(\mathrm{e})^\dagger\{\omega_i^*\}_{i\in\mathbb{N}}$ in (40) is well defined. Next we prove (40) to finish the proof. By Theorem 2, ω is a frame sequence in $\mathcal{H}$. And by Lemma 13,

$$M(\mathrm{e}) = \overline{M(\omega, \omega^*)}.$$

It follows that

$$M(\mathrm{e})\bar{\mathrm{a}} = \overline{M(\omega, \omega^*)\mathrm{a}}.$$

for a $\in l^2(\mathbb{N})$. This is in turn equivalent to

$$M(\mathrm{e})\bar{\mathrm{a}} = \{\langle \omega_i^*, \varphi\rangle\}_{i\in\mathbb{N}} \tag{42}$$

with $\varphi = \sum_{j=1}^{\infty} a_j\omega_j$ and a $\in l^2(\mathbb{N})$ by a standard computation. In other words, (42) tells us that, for $\varphi \in \overline{span}\{\omega_i\}_{i\in\mathbb{N}}$, a is a solution to the equation

$$\varphi = \sum_{j=1}^{\infty} a_j\omega_j \tag{43}$$

in $l^2(\mathbb{N})$ if and only if it is a solution to (42) in $l^2(\mathbb{N})$. This leads to the fact that (42) and (43) share the same unique $l^2(\mathbb{N})$-solution with the minimal norm. Also observe that (42) implies that this solution is a $= \overline{M(\mathrm{e})^\dagger\{\langle \omega_i^*, \varphi\rangle\}_{i\in\mathbb{N}}}$, and that (43) implies that this solution is a $= \{\langle \varphi, S_\omega^{-1}\omega_i\rangle\}_{i\in\mathbb{N}}$. Therefore, we have

$$\left\{\left\langle \varphi, S_\omega^{-1}\omega_i\right\rangle\right\}_{i\in\mathbb{N}} = \overline{M(\mathrm{e})^\dagger\{\langle \omega_i^*, \varphi\rangle\}_{i\in\mathbb{N}}} \quad \text{for } \varphi \in \overline{span}\{\omega_i\}_{i\in\mathbb{N}}. \tag{44}$$

Let $M(\mathrm{e})^\dagger = (b_{ij})_{i,j\in\mathbb{N}}$. Then, from (44), we have

$$\begin{aligned}\left\{\left\langle \varphi, S_\omega^{-1}\omega_i\right\rangle\right\}_{i\in\mathbb{N}} &= \left\{\sum_{j=1}^{\infty}\overline{b_{ij}}\left\langle \varphi, \omega_j^*\right\rangle\right\}_{i\in\mathbb{N}} \\ &= \left\{\left\langle \varphi, \sum_{j=1}^{\infty} b_{ij}\omega_j^*\right\rangle\right\}_{i\in\mathbb{N}}\end{aligned}$$

for $\varphi \in \overline{span}\{\omega_i\}_{i\in\mathbb{N}}$. This leads to (40) by the arbitrariness of φ.

Remark 4 In particular, if e is an orthonormal basis for $\mathcal{H}$ (even if h is not) in Theorem 5, then $M(\mathrm{e}) = I$, and thus $\{S_\omega^{-1}\omega_i\}_{i\in\mathbb{N}} = \{\omega_i^*\}_{i\in\mathbb{N}}$. So Theorem 5 is a genuine generalization of Proposition 14 in [1].

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