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ORIGINAL PAPER

On (p, r, s)-summing Bloch maps and Lapresté norms

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Abstract

The theory of (p, r, s)-summing and (p, r, s)-nuclear linear operators on Banach spaces was developed by Pietsch in his book on operator ideals (Pietsch in Operator ideals, North-Holland Mathematical Library, North-Holland Publishing Co., Amsterdam, 1980, Chapters 17 and 18) Due to recent advances in the theory of ideals of Bloch maps, we extend these concepts to Bloch maps from the complex open unit disc $\mathbb{D}$ into a complex Banach space X. Variants for (r, s)-dominated Bloch maps of classical Pietsch's domination and Kwapień's factorization theorems of (r, s)-dominated linear operators are presented. We define analogues of Lapresté's tensor norms on the space of X-valued Bloch molecules on $\mathbb{D}$ to address the duality of the spaces of (p^*, r, s)-summing Bloch maps from $\mathbb{D}$ into X^*. The class of (p, r, s)-nuclear Bloch maps is introduced and analysed to give examples of (p, r, s)-summing Bloch maps.

Keywords Summing operators · Vector-valued Bloch maps · Pietsch domination · Kwapień factorization

Mathematics Subject Classification 30H30 · 46E15 · 46E40 · 47B38

Dedicated to the memory of Professor Albrecht Pietsch (1934–2024).

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Introduction

The concept of absolutely p-summing linear operators between Banach spaces for $0 < p \leq \infty$ was introduced by Pietsch [13] and the notion of absolutely (p, r)-summing operators for $0 < r \leq p \leq \infty$ is due to Mitjagin and Pełczyński [11] though the famous factorization theorem for (p, r)-dominated operators was proved by Kwapień [9].

In his famous monograph about operator ideals [13], Pietsch introduced a more general multi-index concept with the definition of (p, r, s)-summing operators for $0 < p, r, s \leq \infty$ and $1/p \leq 1/r + 1/s$. The study of the duality of these operator spaces was addressed with the introduction of suitable norms on the tensor product of Banach spaces by Chevet [5], Saphar [15] and Lapresté [10].

In other settings, (p, r, s)-summing maps have been dealed by some authors as, for example, Chávez-Domínguez [4] for Lipschitz maps, and Achour [1], Bernardino, Pellegrino, Seoane-Sepúlveda and Souza [2] and Fernández-Unzueta and García-Hernández [7] for multilinear operators and polynomials.

Our main purpose in this paper is to introduce and establish the most notable properties of a notion of (p, r, s)-summing Bloch maps from the complex open unit disc $\mathbb{D}$ into a complex Banach space X.

Let $\mathcal{H}(\mathbb{D}, X)$ be the space of all holomorphic maps from $\mathbb{D}$ into X. Let us recall that a map $f \in \mathcal{H}(\mathbb{D}, X)$ is called *Bloch* if

$$\rho_{\mathcal{B}}(f) = \sup\left\{(1 - |z|^2)\left\|f'(z)\right\| : z \in \mathbb{D}\right\} < \infty.$$

The linear space of all Bloch maps from $\mathbb{D}$ into X, under the Bloch seminorm $\rho_{\mathcal{B}}$, is denoted by $\mathcal{B}(\mathbb{D}, X)$. The *normalized Bloch space* $\widehat{\mathcal{B}}(\mathbb{D}, X)$ is the closed subspace of $\mathcal{B}(\mathbb{D}, X)$ formed by all those maps f for which $f(0) = 0$, under the *Bloch norm* $\rho_{\mathcal{B}}$. For simplicity, we write $\widehat{\mathcal{B}}(\mathbb{D})$ instead of $\widehat{\mathcal{B}}(\mathbb{D}, \mathbb{C})$. We denote by $\widehat{\mathcal{B}}(\mathbb{D}, \mathbb{D})$ the set of all holomorphic functions $h : \mathbb{D} \to \mathbb{D}$ for which $h(0) = 0$.

We now introduce some notation. For Banach spaces X and Y, $\mathcal{L}(X, Y)$ denotes the Banach space of all continuous linear operators from X into Y, equipped with the operator canonical norm. As usual, X^* denotes the dual space $\mathcal{L}(X, \mathbb{K})$, and J_X the canonical injection of X into X^{**}. B_X stands for the closed unit ball of X. Given $1 \leq p \leq \infty$, p^* denotes the *conjugate index of* p defined by $p^* = p/(p-1)$ if $p \neq 1$, $p^* = \infty$ if $p = 1$, and $p^* = 1$ if $p = \infty$.

Let X be a Banach space, $n \in \mathbb{N}$ and a finite set of vectors $(x_i)_{i=1}^n$ in X. For any $1 \leq p \leq \infty$, the *strong p-norm of* $(x_i)_{i=1}^n$ is defined by

$$\left\|(x_i)_{i=1}^n\right\|_p = \begin{cases} \left(\displaystyle\sum_{i=1}^n \|x_i\|^p\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \displaystyle\max_{1\leq i\leq n} \|x_i\| & \text{if } p = \infty, \end{cases}$$

and the *weak p-norm of* $(x_i)_{i=1}^n$ by

$$\omega_p\left((x_i)_{i=1}^n\right) = \sup_{x^* \in B_{X^*}} \left\| (x^*(x_i))_{i=1}^n \right\|_p .$$

According to Pietsch [14, 17.1.1], given Banach spaces X, Y and $0 < p, r, s \leq \infty$ such that $1/p \leq 1/r + 1/s$, an operator $T \in \mathcal{L}(X, Y)$ is (p, r, s)*-summing* if there exists a constant $C \geq 0$ such that

$$\left\| (y_i^*(T(x_i)))_{i=1}^n \right\|_p \leq C\omega_r\left((x_i)_{i=1}^n\right)\omega_s\left((y_i^*)_{i=1}^n\right)$$

for any $n \in \mathbb{N}$, $(x_i)_{i=1}^n$ in X and $(y_i^*)_{i=1}^n$ in Y^*. The least of all constants C for which such an inequality holds is denoted by $\pi_{(p,r,s)}(T)$, and the linear space of all such operators is represented by $\Pi_{(p,r,s)}(X, Y)$.

We now propose a Bloch version of the notion of (p, r, s)-summing linear operators. Towards this end, we introduce a third norm: given two finite sets of points $(\lambda_i)_{i=1}^n$ in $\mathbb{C}$ and $(z_i)_{i=1}^n$ in $\mathbb{D}$, the *weak Bloch p-norm of* $(\lambda_i, z_i)_{i=1}^n$ is defined by

$$\omega_p^{\widehat{\mathcal{B}}}\left((\lambda_i, z_i)_{i=1}^n\right) = \sup_{g \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left\| (\lambda_i g'(z_i))_{i=1}^n \right\|_p .$$

In particular, we write $\omega_p^{\widehat{\mathcal{B}}}\left((z_i)_{i=1}^n\right)$ instead of $\omega_p^{\widehat{\mathcal{B}}}\left((\lambda_i, z_i)_{i=1}^n\right)$ if $\lambda_i = 1$ for all $i \in \{1, \ldots, n\}$.

Definition 0.1 Let X be a complex Banach space and let $1 \leq p, r, s \leq \infty$ such that $1/p \leq 1/r + 1/s$. We say that a map $f \in \mathcal{H}(\mathbb{D}, X)$ is (p, r, s)*-summing Bloch* if there is a constant $C \geq 0$ such that for any $n \in \mathbb{N}$, $(\lambda_i)_{i=1}^n$ in $\mathbb{C}$, $(z_i)_{i=1}^n$ in $\mathbb{D}$ and $(x_i^*)_{i=1}^n$ in X^*, we have

$$\left\| (\lambda_i x_i^*(f'(z_i)))_{i=1}^n \right\|_p \leq C\omega_r^{\widehat{\mathcal{B}}}\left((\lambda_i, z_i)_{i=1}^n\right)\omega_s\left((x_i^*)_{i=1}^n\right).$$

The smallest such constants C is denoted by $\pi_{(p,r,s)}^{\mathcal{B}}(f)$. The linear space of all such maps is denoted by $\Pi_{(p,r,s)}^{\mathcal{B}}(\mathbb{D}, X)$, and $\Pi_{(p,r,s)}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$ stands for its subspace formed by all those maps f for which $f(0) = 0$. A map (p, r, s)-summing Bloch map f from $\mathbb{D}$ into X is called (r, s)*-dominated Bloch* whenever $1/p = 1/r + 1/s$.

We now describe the contents of this paper. In parallelism with the theory of absolutely (p, r, s)-summing operators, we prove that $[\Pi_{(p,r,s)}^{\widehat{\mathcal{B}}}, \pi_{(p,r,s)}^{\mathcal{B}}]$ is a Banach ideal of normalized Bloch maps. We also show that the space $(\Pi_{(p,r,s)}^{\mathcal{B}}(\mathbb{D}, X), \pi_{(p,r,s)}^{\mathcal{B}})$ is Möbius-invariant in an approach to Complex Analysis.

For $1 \leq p, r, s < \infty$ such that $1/p = 1/r + 1/s$, our main result in this paper gathers both variants for (r, s)-dominated Bloch maps of Pietsch's domination and Kwapień's factorization theorems for (r, s)-dominated linear operators (see [14, Theorems 7.4.2 and 7.4.3]).

In order to address the duality of the $\Pi^{\widehat{\mathcal{B}}}_{(p,r,s)}$-spaces, we introduce Bloch analogues of Lapresté norms [10] on the space of X-valued Bloch molecules of $\mathbb{D}$, denoted by $\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}$. For $1 \leq p, r, s \leq \infty$ and $1/\theta := 1/p + 1/r + 1/s \geq 1$, we prove that $\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}$ is a Bloch reasonable θ-crossnorm on such a space so that, whenever $1/p^* \leq 1/r + 1/s$, $(\Pi^{\widehat{\mathcal{B}}}_{(p^*,r,s)}(\mathbb{D}, X^*), \pi^{\mathcal{B}}_{(p^*,r,s)})$ is isometrically isomorphic to $(\mathcal{G}(\mathbb{D})\widehat{\otimes}_{\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}} X, \mu^{\widehat{\mathcal{B}}}_{(p,r,s)})^*$, where $\mathcal{G}(\mathbb{D})$ is the Bloch-free Banach space of $\mathbb{D}$.

In order to give examples of (p, r, s)-summing Bloch maps, the concept of (p, r, s)-nuclear Bloch maps from $\mathbb{D}$ into X for $1 \leq p, r, s \leq \infty$ such that $1+1/p \geq 1/r+1/s$ is introduced and it is proved that the space formed by such Bloch maps is an θ-Banach normalized Bloch ideal where $1/\theta := 1/p + 1/r^* + 1/s^*$.

1 Results

From now on and unless otherwise stated, we will suppose that X is a complex Banach space and $1 \leq p, r, s \leq \infty$ with $1/p \leq 1/r + 1/s$.

1.1 Inclusions

We first show that the new functions introduced are actually Bloch functions.

Given semi-normed spaces (X, ρ_X) and (Y, ρ_Y), we will write $(X, \rho_X) \leq (Y, \rho_Y)$ to point out that $X \subseteq Y$ and $\rho_Y(x) \leq \rho_X(x)$ for all $x \in X$.

Proposition 1.1 $(\Pi^{\mathcal{B}}_{(p,r,s)}(\mathbb{D}, X), \pi^{\mathcal{B}}_{(p,r,s)}) \leq (\mathcal{B}(\mathbb{D}, X), \rho_{\mathcal{B}})$.

Proof Let $f \in \Pi^{\mathcal{B}}_{(p,r,s)}(\mathbb{D}, X)$. For each $z \in \mathbb{D}$, Hahn–Banach Theorem provides a functional $x^*_z \in B_{X^*}$ such that $\left|x^*_z(f'(z))\right| = \left\|f'(z)\right\|$. Taking $n = 1$, $\lambda_1 = (1-|z|^2)$, $z_1 = z$ and $x^*_1 = x^*_z$, we have

$$\begin{aligned}(1-|z|^2)\left\|f'(z)\right\| &= (1-|z|^2)\left|x^*_z(f'(z))\right| \\ &\leq \pi^{\mathcal{B}}_{(p,r,s)}(f)\omega^{\widehat{\mathcal{B}}}_r\left((\lambda_i, z_i)_{i=1}^n\right)\omega_s\left((x^*_i)_{i=1}^n\right) \\ &\leq \pi^{\mathcal{B}}_{(p,r,s)}(f).\end{aligned}$$

Hence $f \in \mathcal{B}(\mathbb{D}, X)$ with $\rho_{\mathcal{B}}(f) \leq \pi^{\mathcal{B}}_{(p,r,s)}(f)$. □

We now prove that the concept of (p, r, s)-summing Bloch maps extends that of p-summing Bloch maps introduced in [3].

For any $1 \leq p \leq \infty$, let us recall that a map $f \in \mathcal{H}(\mathbb{D}, X)$ is *p-summing Bloch* if there is a constant $C \geq 0$ such that

$$\left\|(\lambda_i f'(z_i))_{i=1}^n\right\|_p \leq C\omega^{\widehat{\mathcal{B}}}_p\left((\lambda_i, z_i)_{i=1}^n\right)$$

for any $n \in \mathbb{N}$, $(\lambda_i)_{i=1}^n$ in $\mathbb{C}$ and $(z_i)_{i=1}^n$ in $\mathbb{D}$. The infimum of all constants C for which such an inequality holds, denoted by $\pi^{\mathcal{B}}_p(f)$, defines a seminorm on the linear space, denoted by $\Pi_p^{\mathcal{B}}(\mathbb{D}, X)$, of all p-summing Bloch maps from $\mathbb{D}$ into X. Furthermore, $\pi_p^{\mathcal{B}}$ is a norm on the subspace $\Pi_p^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$ formed by all those maps $f \in \Pi_p^{\mathcal{B}}(\mathbb{D}, X)$ for which $f(0) = 0$.

Proposition 1.2 $(\Pi_{p,p,\infty}^{\mathcal{B}}(\mathbb{D}, X), \pi_{p,p,\infty}^{\mathcal{B}}) = (\Pi_p^{\mathcal{B}}(\mathbb{D}, X), \pi_p^{\mathcal{B}})$.

Proof Let $f \in \Pi_{(p,p,\infty)}^{\mathcal{B}}(\mathbb{D}, X)$. Given $n \in \mathbb{N}$, $(\lambda_i)_{i=1}^n$ in $\mathbb{C}$ and $(z_i)_{i=1}^n$ in $\mathbb{D}$, we have

$$\begin{aligned}\left\|(\lambda_i f'(z_i))_{i=1}^n\right\|_p &= \left\|(\lambda_i x_i^*(f'(z_i)))_{i=1}^n\right\|_p \\ &\leq \pi_{(p,p,\infty)}^{\mathcal{B}}(f)\omega_p^{\widehat{\mathcal{B}}}\left((\lambda_i, z_i)_{i=1}^n\right)\omega_\infty\left((x_i^*)_{i=1}^n\right) \\ &\leq \pi_{(p,p,\infty)}^{\mathcal{B}}(f)\omega_p^{\widehat{\mathcal{B}}}\left((\lambda_i, z_i)_{i=1}^n\right)\end{aligned}$$

where we have taken $x_i^* \in B_{X^*}$ such that $\left|x_i^*(f'(z_i))\right| = \left\|f'(z_i)\right\|$ for each $i \in \{1, \ldots, n\}$ by Hahn–Banach Theorem. Hence $f \in \Pi_p^{\mathcal{B}}(\mathbb{D}, X)$ with $\pi_p^{\mathcal{B}}(f) \leq \pi_{(p,p,\infty)}^{\mathcal{B}}(f)$.

Conversely, let $f \in \Pi_p^{\mathcal{B}}(\mathbb{D}, X)$. Let $n \in \mathbb{N}$, $(\lambda_i)_{i=1}^n$ in $\mathbb{C}$, $(z_i)_{i=1}^n$ in $\mathbb{D}$ and $(x_i^*)_{i=1}^n$ in X^*. For each $i \in \{1, \ldots, n\}$, Hahn–Banach Theorem provides a functional $y_i^{**} \in B_{X^{**}}$ such that $\left|y_i^{**}(x_i^*)\right| = \left\|x_i^*\right\|$. We obtain

$$\begin{aligned}\left\|(\lambda_i x_i^*(f'(z_i)))_{i=1}^n\right\|_p &\leq \left\|(\lambda_i \left\|x_i^*\right\| f'(z_i))_{i=1}^n\right\|_p \\ &\leq \left\|(\lambda_i f'(z_i))_{i=1}^n\right\|_p \left\|(x_i^*)_{i=1}^n\right\|_\infty \\ &\leq \pi_p^{\mathcal{B}}(f)\omega_p^{\widehat{\mathcal{B}}}\left((\lambda_i, z_i)_{i=1}^n\right)\left\|(x_i^*)_{i=1}^n\right\|_\infty \\ &= \pi_p^{\mathcal{B}}(f)\omega_p^{\widehat{\mathcal{B}}}\left((\lambda_i, z_i)_{i=1}^n\right)\left\|(y_i^{**}(x_i^*))_{i=1}^n\right\|_\infty \\ &\leq \pi_p^{\mathcal{B}}(f)\omega_p^{\widehat{\mathcal{B}}}\left((\lambda_i, z_i)_{i=1}^n\right)\omega_\infty\left((x_i^*)_{i=1}^n\right),\end{aligned}$$

and thus $f \in \Pi_{(p,p,\infty)}^{\mathcal{B}}(\mathbb{D}, X)$ with $\pi_{(p,p,\infty)}^{\mathcal{B}}(f) \leq \pi_p^{\mathcal{B}}(f)$. □

1.2 Banach Bloch ideal property

Given $\theta \in (0, 1]$ and a linear space X over $\mathbb{K}$, recall that a *θ-norm* on X is a function $\mu\colon X \to \mathbb{R}$ satisfying that $x = 0$ whenever $\mu(x) = 0$, $\mu(\lambda x) = |\lambda|\mu(x)$ for all $\lambda \in \mathbb{K}$ and $x \in X$, and $\mu(x + y)^\theta \leq \mu(x)^\theta + \mu(y)^\theta$ for all $x, y \in X$. We say that (X, μ) is an *θ-normed space*, and it is said that (X, μ) is an *θ-Banach space* if every Cauchy sequence in (X, μ) converges in (X, μ).

Following [8, Definition 5.11], a *θ-normed (θ-Banach) normalized Bloch ideal*, denoted as $[\mathcal{I}^{\widehat{\mathcal{B}}}, \|\cdot\|_{\mathcal{I}^{\widehat{\mathcal{B}}}}]$, is a subclass $\mathcal{I}^{\widehat{\mathcal{B}}}$ equipped with a θ-norm $\|\cdot\|_{\mathcal{I}^{\widehat{\mathcal{B}}}}$ of the class of all normalized Bloch maps $\widehat{\mathcal{B}}$ endowed with the Bloch norm $\rho_{\mathcal{B}}$ such that for each complex Banach space X, the components $\mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$ satisfy the following properties:

(P1) $(\mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D}, X), \|\cdot\|_{\mathcal{I}^{\widehat{\mathcal{B}}}})$ is a θ-normed (θ-Banach) space and $\rho_{\mathcal{B}}(f) \leq \|f\|_{\mathcal{I}^{\widehat{\mathcal{B}}}}$ for $f \in \mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$.

(P2) For any $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x \in X$, the map $g \cdot x : \mathbb{D} \to X$, given by $(g \cdot x)(z) = g(z)x$ if $z \in \mathbb{D}$, is in $\mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$ and $\|g \cdot x\|_{\mathcal{I}^{\widehat{\mathcal{B}}}} = \rho_{\mathcal{B}}(g)\,\|x\|$.

(P3) The *ideal property*: if $f \in \mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$, $h \in \widehat{\mathcal{B}}(\mathbb{D}, \mathbb{D})$ and $T \in \mathcal{L}(X, Y)$ where Y is a complex Banach space, then $T \circ f \circ h$ belongs to $\mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D}, Y)$ and $\|T \circ f \circ h\|_{\mathcal{I}^{\widehat{\mathcal{B}}}} \leq \|T\|\,\|f\|_{\mathcal{I}^{\widehat{\mathcal{B}}}}$.

In the case $\theta = 1$, we remove any reference to θ.

Proposition 1.3 $\left[\Pi^{\widehat{\mathcal{B}}}_{(p,r,s)}, \pi^{\mathcal{B}}_{(p,r,s)}\right]$ *is a Banach normalized Bloch ideal.*

Proof Let X be a complex Banach space and let $n \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, $z_i \in \mathbb{D}$ and $x_i^* \in X^*$ for all $i \in \{1, \ldots, n\}$.

(P1) Given $f \in \Pi^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, X)$, it is clear that $\pi^{\mathcal{B}}_{(p,r,s)}(f) \geq 0$. If $\pi^{\mathcal{B}}_{(p,r,s)}(f) = 0$, then $\rho_{\mathcal{B}}(f) = 0$ by Proposition 1.1, and so $f = 0$. For any $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}\left\|(\lambda_i x_i^*((\lambda f)'(z_i)))_{i=1}^n\right\|_p &= |\lambda|\left\|(\lambda_i x_i^*(f'(z_i)))_{i=1}^n\right\|_p \\ &\leq |\lambda|\,\pi^{\mathcal{B}}_{(p,r,s)}(f)\omega_r^{\widehat{\mathcal{B}}}\left((\lambda_i, z_i)_{i=1}^n\right)\omega_s\left((x_i^*)_{i=1}^n\right),\end{aligned}$$

and thus $\lambda f \in \Pi^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, X)$ with $\pi^{\mathcal{B}}_{(p,r,s)}(\lambda f) \leq |\lambda|\,\pi^{\mathcal{B}}_{(p,r,s)}(f)$. If $\lambda = 0$, this implies that $\pi^{\mathcal{B}}_{(p,r,s)}(\lambda f) = 0 = |\lambda|\,\pi^{\mathcal{B}}_{(p,r,s)}(f)$. If $\lambda \neq 0$, we have $\pi^{\mathcal{B}}_{(p,r,s)}(f) = \pi^{\mathcal{B}}_{(p,r,s)}(\lambda^{-1}(\lambda f)) \leq \left|\lambda^{-1}\right|\pi^{\mathcal{B}}_{(p,r,s)}(\lambda f)$, hence $|\lambda|\,\pi^{\mathcal{B}}_{(p,r,s)}(f) \leq \pi^{\mathcal{B}}_{(p,r,s)}(\lambda f)$, and so $\pi^{\mathcal{B}}_{(p,r,s)}(\lambda f) = |\lambda|\,\pi^{\mathcal{B}}_{(p,r,s)}(f)$.

For any $f_1, f_2 \in \Pi^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, X)$, we have

$$\begin{aligned}\left\|(\lambda_i x_i^*((f_1 + f_2)'(z_i)))_{i=1}^n\right\|_p &\leq \left\|(\lambda_i x_i^*(f_1'(z_i)))_{i=1}^n\right\|_p + \left\|(\lambda_i x_i^*(f_2'(z_i)))_{i=1}^n\right\|_p \\ &\leq \left(\pi^{\mathcal{B}}_{(p,r,s)}(f_1) + \pi^{\mathcal{B}}_{(p,r,s)}(f_2)\right)\omega_r^{\widehat{\mathcal{B}}}\left((\lambda_i, z_i)_{i=1}^n\right) \\ &\quad \times \omega_s\left((x_i^*)_{i=1}^n\right),\end{aligned}$$

and thus $f_1 + f_2 \in \Pi^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, X)$ with $\pi^{\mathcal{B}}_{(p,r,s)}(f_1 + f_2) \leq \pi^{\mathcal{B}}_{(p,r,s)}(f_1) + \pi^{\mathcal{B}}_{(p,r,s)}(f_2)$. Consequently, $(\Pi^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, X), \pi^{\mathcal{B}}_{(p,r,s)})$ is a normed space.

To show its completeness, let (f_i) be a sequence in $\Pi^{\mathcal{B}}_{(p,r,s)}(\mathbb{D}, X)$ such that $\sum_{i=1}^\infty \pi^{\mathcal{B}}_{(p,r,s)}(f_i) < \infty$. Since $\rho_{\mathcal{B}} \leq \pi^{\mathcal{B}}_{(p,r,s)}$ on $\Pi^{\mathcal{B}}_{(p,r,s)}(\mathbb{D}, X)$ by Proposition 1.1, and $(\widehat{\mathcal{B}}(\mathbb{D}, X), \rho_{\mathcal{B}})$ is a Banach space, there exists $f = \sum_{i=1}^\infty f_i \in \widehat{\mathcal{B}}(\mathbb{D}, X)$ in the norm $\rho_{\mathcal{B}}$. We will prove that $\sum_{i=1}^\infty f_i = f$ in the norm $\pi^{\mathcal{B}}_{(p,r,s)}$. Given $m \in \mathbb{N}$, $\lambda_k \in \mathbb{C}$,

$z_k \in \mathbb{D}$ and $x_k^* \in X^*$ for all $k \in \{1, \ldots, m\}$, we have

$$\begin{aligned}\left\|\left(\lambda_k x_k^*\left(\left(\sum_{i=1}^n f_i\right)'(z_k)\right)\right)_{k=1}^m\right\|_p &\leq \pi_{(p,r,s)}^{\mathcal{B}}\left(\sum_{i=1}^n f_i\right)\omega_r^{\widehat{\mathcal{B}}}\left((\lambda_k, z_k)_{k=1}^m\right)\\ &\quad\times\omega_s\left((x_k^*)_{k=1}^m\right)\\ &\leq\left(\sum_{i=1}^n \pi_{(p,r,s)}^{\mathcal{B}}(f_i)\right)\omega_r^{\widehat{\mathcal{B}}}\left((\lambda_k, z_k)_{k=1}^m\right)\\ &\quad\times\omega_s\left((x_k^*)_{k=1}^m\right)\end{aligned}$$

for all $n \in \mathbb{N}$, and by taking limits with $n \to \infty$ yields

$$\begin{aligned}\left\|\left(\lambda_k x_k^*\left(\left(\sum_{i=1}^\infty f_i\right)'(z_k)\right)\right)_{k=1}^m\right\|_p &\leq\left(\sum_{i=1}^\infty \pi_{(p,r,s)}^{\mathcal{B}}(f_i)\right)\omega_r^{\widehat{\mathcal{B}}}\left((\lambda_k, z_k)_{k=1}^m\right)\\ &\quad\times\omega_s\left((x_k^*)_{k=1}^m\right).\end{aligned}$$

Hence $f \in \Pi_{(p,r,s)}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$ with $\pi_{(p,r,s)}^{\mathcal{B}}(f) \leq \sum_{n=1}^\infty \pi_{(p,r,s)}^{\mathcal{B}}(f_n)$. Moreover,

$$\pi_{(p,r,s)}^{\mathcal{B}}\left(f - \sum_{i=1}^n f_i\right) = \pi_{(p,r,s)}^{\mathcal{B}}\left(\sum_{i=n+1}^\infty f_i\right) \leq \sum_{i=n+1}^\infty \pi_{(p,r,s)}^{\mathcal{B}}(f_i)$$

for all $n \in \mathbb{N}$, and therefore $\sum_{i=1}^\infty f_i = f$ in the norm $\pi_{(p,r,s)}^{\mathcal{B}}$.

(P2) Let $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x \in X$. It is immediate that $g \cdot x \in \widehat{\mathcal{B}}(\mathbb{D}, X)$ with $\rho_{\mathcal{B}}(g \cdot x) = \rho_{\mathcal{B}}(g)\,\|x\|$. For $g = 0$ or $x = 0$, (P2) is clear. If $g \neq 0$ and $x \neq 0$, the generalized Hölder's inequality gives

$$\begin{aligned}||(\lambda_i x_i^*((g \cdot x)'(z_i)))_{i=1}^n||_p &= \rho_{\mathcal{B}}(g)\,\|x\|\left\|\left(\lambda_i\left(\frac{g}{\rho_{\mathcal{B}}(g)}\right)'(z_i)x_i^*\left(\frac{x}{\|x\|}\right)\right)_{i=1}^n\right\|_p\\ &= \rho_{\mathcal{B}}(g)\,\|x\|\left\|\left(\lambda_i\left(\frac{g}{\rho_{\mathcal{B}}(g)}\right)'(z_i)J_X\left(\frac{x}{\|x\|}\right)(x_i^*)\right)_{i=1}^n\right\|_p\\ &\leq \rho_{\mathcal{B}}(g)\,\|x\|\left\|\left(\lambda_i\left(\frac{g}{\rho_{\mathcal{B}}(g)}\right)'(z_i)\right)_{i=1}^n\right\|_r\left\|\left(J_X\left(\frac{x}{\|x\|}\right)(x_i^*)\right)_{i=1}^n\right\|_s\\ &\leq \rho_{\mathcal{B}}(g)\,\|x\|\,\omega_r^{\widehat{\mathcal{B}}}\left((\lambda_i, z_i)_{i=1}^n\right)\omega_s\left((x_i^*)_{i=1}^n\right),\end{aligned}$$

and thus $g \cdot x \in \Pi_{(p,r,s)}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$ with $\pi_{(p,r,s)}^{\mathcal{B}}(g \cdot x) \leq \rho_{\mathcal{B}}(g)\,\|x\|$. Conversely,

$$\rho_{\mathcal{B}}(g)\,\|x\| = \rho_{\mathcal{B}}(g \cdot x) \leq \pi_{(p,r,s)}^{\mathcal{B}}(g \cdot x)$$

by using Proposition 1.1.

(P3) Let $f \in \Pi^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, X)$, $T \in \mathcal{L}(X, Y)$ and $h \in \widehat{\mathcal{B}}(\mathbb{D}, \mathbb{D})$. Clearly, $(T \circ f \circ h)(0) = 0$ and $T \circ f \circ h \in \mathcal{H}(\mathbb{D}, Y)$ with

$$(T \circ f \circ h)' = T \circ (f \circ h)' = T \circ [h' \cdot (f' \circ h)].$$

Let $n \in \mathbb{N}$, $(\lambda_i)_{i=1}^n$ in $\mathbb{C}$, $(z_i)_{i=1}^n$ in $\mathbb{D}$ and $(y_i^*)_{i=1}^n$ in Y^*. We have

$$\begin{aligned}
\|(\lambda_i y_i^*((T \circ f \circ h)'(z_i)))_{i=1}^n\|_p &= \left\|(\lambda_i y_i^*(T(h'(z_i)f'(h(z_i)))))_{i=1}^n\right\|_p \\
&\leq \|T\| \left\|(\lambda_i h'(z_i) y_i^*(f'(h(z_i))))_{i=1}^n\right\|_p \\
&\leq \|T\| \, \pi^{\mathcal{B}}_{(p,r,s)}(f) \omega_r^{\widehat{\mathcal{B}}}\left((\lambda_i h'(z_i), h(z_i))_{i=1}^n\right) \omega_s\left((y_i^*)_{i=1}^n\right) \\
&\leq \|T\| \, \pi^{\mathcal{B}}_{(p,r,s)}(f) \omega_r^{\widehat{\mathcal{B}}}\left((\lambda_i, z_i)_{i=1}^n\right) \omega_s\left((y_i^*)_{i=1}^n\right),
\end{aligned}$$

where it is applied that $\rho_{\mathcal{B}}(g \circ h) \leq \rho_{\mathcal{B}}(g)$ for all $g \in \widehat{\mathcal{B}}(\mathbb{D})$ by the Pick–Schwarz Lemma. So $T \circ f \circ h \in \Pi^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, X)$ with $\pi^{\mathcal{B}}_{(p,r,s)}(T \circ f \circ h) \leq \|T\| \, \pi^{\mathcal{B}}_{(p,r,s)}(f)$. □

1.3 Pietsch's domination and Kwapień's factorization

For $1 \leq p, r, s < \infty$ such that $1/p = 1/r + 1/s$, we present a result gathering both variants for (p, r, s)-summing Bloch maps of Pietsch's domination theorem [14, Theorem 7.4.2] and Kwapień's factorization theorem [14, Theorem 7.4.3] for (r, s)-dominated linear operators.

Given a Banach space X, we will denote by $\mathcal{P}(B_{X^*})$ the set of all regular Borel probability measures μ on B_{X^*} with the topology w^*.

Theorem 1.4 *Let $1 \leq p, r, s < \infty$ be with $1/p = 1/r + 1/s$ and $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$. The following statements are equivalent:*

(i) $f \in \Pi^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, X)$.

(ii) *(Pietsch's domination). There exist a constant $C > 0$ and measures $\mu \in \mathcal{P}(B_{\widehat{\mathcal{B}}(\mathbb{D})})$ and $\nu \in \mathcal{P}(B_{X^{**}})$ such that*

$$\left|x^*(f'(z))\right| \leq C \left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left|g'(z)\right|^r d\mu(g)\right)^{\frac{1}{r}} \left(\int_{B_{X^{**}}} \left|x^{**}(x^*)\right|^s d\nu(x^{**})\right)^{\frac{1}{s}}$$

for all $z \in \mathbb{D}$ and $x^* \in X^*$.

(iii) *(Kwapień's factorization). There exist a Banach space Z, a closed subspace $Y \subseteq Z$, a map $h \in \Pi_r^{\widehat{\mathcal{B}}}(\mathbb{D}, Z)$ with $h'(\mathbb{D}) \subseteq Y$ and an operator $T \in \mathcal{L}(Y, X)$ with $T^* \in \Pi_s(X^*, Y^*)$ such that $f' = T \circ h'$.*

In this case,

$$\pi^{\mathcal{B}}_{(p,r,s)}(f) = \inf\{C : C \text{ as in } (ii)\} = \inf\left\{\pi_s(T^*)\pi_r^{\mathcal{B}}(h) : f' = T \circ h'\right\}$$

and, in addition, both infimums are attained.

Proof $(i) \Leftrightarrow (ii)$: We will apply a general Pietsch domination theorem (see [12, Theorem 4.6]). Define the functions

$$\begin{aligned} R_1 &: B_{\widehat{\mathcal{B}}(\mathbb{D})} \times \mathbb{D} \times \mathbb{C} \to [0, \infty[, \qquad R_1(g, z, \lambda) = |\lambda| \left|g'(z)\right|, \\ R_2 &: B_{X^{**}} \times X^* \to [0, \infty[, \qquad R_2(x^{**}, x^*) = \left|x^{**}(x^*)\right|, \\ S &: \widehat{\mathcal{B}}(\mathbb{D}, X) \times \mathbb{D} \times \mathbb{C} \times X^* \to [0, \infty[, \qquad S(f, z, \lambda, x^*) = |\lambda| \left|x^*(f'(z))\right|. \end{aligned}$$

Note that R_1, R_2 and S satisfy the properties (1)–(2) preceding to [12, Definition 4.4]:

1. For each $z \in \mathbb{D}$, $\lambda \in \mathbb{C}$ and $x^* \in X^*$, the maps

$$\begin{aligned} (R_1)_{z,\lambda} &: B_{\widehat{\mathcal{B}}(\mathbb{D})} \to [0, \infty[\qquad (R_1)_{z,\lambda}(g) = R_1(g, z, \lambda), \\ (R_2)_{x^*} &: B_{X^{**}} \to [0, \infty[\qquad (R_2)_{x^*}(x^{**}) = R_2(x^{**}, x^*), \end{aligned}$$

   are continuous.
2. The equalities

$$\begin{aligned} R_1(g, z, \beta_1\lambda) &= \beta_1 R_1(g, z, \lambda), \\ R_2(x^{**}, \beta_2 x^*) &= \beta_2 R_2(x^{**}, x^*), \\ S(f, z, \beta_1\lambda, \beta_2 x^*) &= \overline{\beta}_1 \beta_2 S(f, z, \lambda, x^*), \end{aligned}$$

   hold for all $g \in B_{\widehat{\mathcal{B}}(\mathbb{D})}$, $x^{**} \in B_{X^{**}}$, $z \in \mathbb{D}$, $\lambda \in \mathbb{C}$, $x^* \in X^*$, $\beta_1, \beta_2 \in [0, 1]$ and $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$.

Now, in view of Definition 4.4 and Theorem 4.6 in [12], we have that f is (p, r, s)-summing Bloch if and only if f is R_1, R_2-S abstract (r, s)-summing if and only if there is a constant $C > 0$ and measures $\mu \in \mathcal{P}(B_{\widehat{\mathcal{B}}(\mathbb{D})})$ and $\nu \in \mathcal{P}(B_{X^{**}})$ such that

$$S(f, z, \lambda, x^*) \leq C \left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}} R_1(g, z, \lambda)^r d\mu(g) \right)^{\frac{1}{r}} \left(\int_{B_{X^{**}}} R_2(x^{**}, x^*)^s d\nu(x^{**}) \right)^{\frac{1}{s}}$$

for all $z \in \mathbb{D}$, $\lambda \in \mathbb{C}$ and $x^* \in X^*$, and this means that

$$\left|x^*(f'(z))\right| \leq C \left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left|g'(z)\right|^r d\mu(g) \right)^{\frac{1}{r}} \left(\int_{B_{X^{**}}} \left|x^{**}(x^*)\right|^s d\nu(x^{**}) \right)^{\frac{1}{s}}$$

for all $z \in \mathbb{D}$ and $x^* \in X^*$. In this case, $\pi^{\mathcal{B}}_{(p,r,s)}(f) = \min\{C : C \text{ as in } (ii)\}$.

$(ii) \Rightarrow (iii)$: Let $\iota_{\mathbb{D}} : \mathbb{D} \to C(B_{\widehat{\mathcal{B}}(\mathbb{D})})$ be defined by $\iota_{\mathbb{D}}(z)(g) = g'(z)$ for all $z \in \mathbb{D}$ and $g \in B_{\widehat{\mathcal{B}}(\mathbb{D})}$, and let $j_r : C(B_{\widehat{\mathcal{B}}(\mathbb{D})}) \to L_r(\mu)$ be the canonical injection. In the light of [3, Lemma 1.5], we can find a map $h \in \widehat{\mathcal{B}}(\mathbb{D}, L_r(\mu))$ with $\rho_{\mathcal{B}}(h) = 1$ such that $h' = j_r \circ \iota_{\mathbb{D}}$. Moreover, $h \in \Pi^{\widehat{\mathcal{B}}}_r(\mathbb{D}, L_r(\mu))$ with $\pi^{\mathcal{B}}_r(h) = 1$. Consider the linear subspace $Y = \overline{\text{lin}}(h'(\mathbb{D})) \subseteq L_r(\mu)$ and the operator $T \in \mathcal{L}(Y, X)$ defined by $T(h'(z)) = f'(z)$ for all $z \in \mathbb{D}$. Using (ii), we have

$$\begin{aligned}\left\|T^*(x^*)\right\| &= \sup\left\{\left|T^*(x^*)(h'(z))\right| : z \in \mathbb{D},\ \left\|h'(z)\right\| \leq 1\right\}\\ &= \sup\left\{\left|x^*(T(h'(z)))\right| : z \in \mathbb{D},\ \left\|h'(z)\right\| \leq 1\right\}\\ &= \sup\left\{\left|x^*(f'(z))\right| : z \in \mathbb{D},\ \left\|h'(z)\right\| \leq 1\right\}\\ &\leq C\left(\int_{B_{X^{**}}} \left|x^{**}(x^*)\right|^s d\nu(x^{**})\right)^{\frac{1}{s}}\end{aligned}$$

for all $x^* \in X^*$, and thus $T^* \in \Pi_s(X^*, Y^*)$ with $\pi_s(T^*) \leq C$. Hence (iii) holds and $\pi_s(T^*)\pi_r^{\mathcal{B}}(h) \leq C$. Taking the infimum over all such constants C, it follows that $\pi_s(T^*)\pi_r^{\mathcal{B}}(h) \leq \inf\{C\colon C \text{ as in } (ii)\}$.

$(iii) \Rightarrow (ii)$: Suppose there exist maps h and T as in (iii). For any $z \in \mathbb{D}$ and $x^* \in X^*$, we have

$$\left|x^*(f'(z))\right| = \left|x^*((T \circ h')(z))\right| = \left|T^*(x^*)(h'(z))\right| \leq \left\|T^*(x^*)\right\| \left\|h'(z)\right\|.$$

By both Pietsch domination theorems for p-summing linear operators [14, Theorem 7.3.2] and p-summing Bloch maps [3, Theorem 1.4], there are measures $\nu \in \mathcal{P}(B_{X^{**}})$ and $\mu \in \mathcal{P}(B_{\widehat{\mathcal{B}}(\mathbb{D})})$ such that

$$\left\|T^*(x^*)\right\| \leq \pi_s(T^*)\left(\int_{B_{X^{**}}} \left|x^{**}(x^*)\right|^s d\nu(x^{**})\right)^{\frac{1}{s}}$$

and

$$\left\|h'(z)\right\| \leq \pi_r^{\mathcal{B}}(h)\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left|g'(z)\right|^r d\mu(g)\right)^{\frac{1}{r}}.$$

Hence we have

$$\begin{aligned}\left|x^*(f'(z))\right| &\leq \pi_s(T^*)\pi_r^{\mathcal{B}}(h)\\ &\quad\times\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left|g'(z)\right|^r d\mu(g)\right)^{\frac{1}{r}}\left(\int_{B_{X^{**}}} \left|x^{**}(x^*)\right|^s d\nu(x^{**})\right)^{\frac{1}{s}},\end{aligned}$$

and this proves (ii) with $\pi_s(T^*)\pi_r^{\mathcal{B}}(h) \in \{C\colon C \text{ as in } (ii)\}$. It follows that $\inf\{C\colon C \text{ as in } (ii)\} \leq \inf\left\{\pi_s(T^*)\pi_r^{\mathcal{B}}(h)\colon f' = T \circ h'\right\}$. □

1.4 Möbius invariance

The *Möbius group of* $\mathbb{D}$, denoted by $\mathrm{Aut}(\mathbb{D})$, consists of all biholomorphic bijections $\phi\colon \mathbb{D} \to \mathbb{D}$. Each $\phi \in \mathrm{Aut}(\mathbb{D})$ has the form $\phi = \lambda\phi_a$ with $\lambda \in \mathbb{T}$ and $a \in \mathbb{D}$, where

$$\phi_a(z) = \frac{a - z}{1 - \overline{a}z} \qquad (z \in \mathbb{D}).$$

Given a complex Banach space X, let us recall that a linear space $\mathcal{A}(\mathbb{D}, X) \subseteq \mathcal{H}(\mathbb{D}, X)$ endowed with a seminorm $p_{\mathcal{A}}$ is *Möbius-invariant* if:

(i) $\mathcal{A}(\mathbb{D}, X) \subseteq \mathcal{B}(\mathbb{D}, X)$ and there exists $C \geq 0$ such that $\rho_{\mathcal{B}}(f) \leq C p_{\mathcal{A}}(f)$ for all $f \in \mathcal{A}(\mathbb{D}, X)$,
(ii) $f \circ \phi \in \mathcal{A}(\mathbb{D}, X)$ with $p_{\mathcal{A}}(f \circ \phi) = p_{\mathcal{A}}(f)$ for all $\phi \in \mathrm{Aut}(\mathbb{D})$ and $f \in \mathcal{A}(\mathbb{D}, X)$.

We have the following interesting fact.

Proposition 1.5 *The space* $(\Pi^{\mathcal{B}}_{(p,r,s)}(\mathbb{D}, X), \pi^{\mathcal{B}}_{(p,r,s)})$ *is Möbius-invariant.*

Proof By Proposition 1.1, $\Pi^{\mathcal{B}}_{(p,r,s)}(\mathbb{D}, X) \subseteq \mathcal{B}(\mathbb{D}, X)$ and $\rho_{\mathcal{B}}(f) \leq \pi^{\mathcal{B}}_{(p,r,s)}(f)$ for all $f \in \Pi^{\mathcal{B}}_{(p,r,s)}(\mathbb{D}, X)$. On the other hand, a proof similar to that of (P3) in Proposition 1.3 yields that if $f \in \Pi^{\mathcal{B}}_{(p,r,s)}(\mathbb{D}, X)$ and $\phi \in \mathrm{Aut}(\mathbb{D})$, then $f \circ \phi \in \Pi^{\mathcal{B}}_{(p,r,s)}(\mathbb{D}, X)$ with $\pi^{\mathcal{B}}_{p}(f \circ \phi) \leq \pi^{\mathcal{B}}_{p}(f)$, and from this fact it is inferred that $\pi^{\mathcal{B}}_{p}(f) = \pi^{\mathcal{B}}_{p}((f \circ \phi) \circ \phi^{-1}) \leq \pi^{\mathcal{B}}_{p}(f \circ \phi)$. □

1.5 Lapresté norms on Bloch molecules

Our approach on the duality of the spaces $(\Pi^{\widehat{\mathcal{B}}}_{(p,r,s)}, \pi^{\mathcal{B}}_{(p,r,s)})$ requires the introduction of Bloch analogues of Lapresté norms [10] on the tensor product of Banach spaces (a generalization of the Chevet–Saphar norms [5, 15] on such tensor products). Given two linear spaces E and F, the tensor product space $E \otimes F$ equipped with a norm α will be denoted by $E \otimes_\alpha F$, and the completion of $E \otimes_\alpha F$ by $E \widehat{\otimes}_\alpha F$.

Towards this end, we first recall some concepts and results borrowed from [8]. For each $z \in \mathbb{D}$, a *Bloch atom of* $\mathbb{D}$ is the functional $\gamma_z \in \widehat{\mathcal{B}}(\mathbb{D})^*$ given by $\gamma_z(f) = f'(z)$ for all $f \in \widehat{\mathcal{B}}(\mathbb{D})$. The called *Bloch molecules of* $\mathbb{D}$ are the elements of the space

$$\mathrm{lin}(\{\gamma_z \colon z \in \mathbb{D}\}) \subseteq \widehat{\mathcal{B}}(\mathbb{D})^*,$$

and the *Bloch-free Banach space of* $\mathbb{D}$ is the space

$$\mathcal{G}(\mathbb{D}) = \overline{\mathrm{lin}}(\{\gamma_z \colon z \in \mathbb{D}\}) \subseteq \widehat{\mathcal{B}}(\mathbb{D})^*.$$

Theorem 1.6 [8]

(i) The map $\Gamma\colon z \in \mathbb{D} \mapsto \gamma_z \in \mathcal{G}(\mathbb{D})$ *is holomorphic with* $\|\gamma_z\| = 1/(1 - |z|^2)$.
(ii) The map $\Lambda\colon \widehat{\mathcal{B}}(\mathbb{D}) \to \mathcal{G}(\mathbb{D})^*$, *given by* $\Lambda(g)(\gamma_z) = g'(z)$ *for all* $z \in \mathbb{D}$ *and* $g \in \widehat{\mathcal{B}}(\mathbb{D})$, *is an isometric isomorphism.*

(iii) *For each* $h \in \widehat{\mathcal{B}}(\mathbb{D}, \mathbb{D})$, *there exists a unique* $\widehat{h} \in \mathcal{L}(\mathcal{G}(\mathbb{D}), \mathcal{G}(\mathbb{D}))$ *such that* $\widehat{h} \circ \Gamma = h' \cdot (\Gamma \circ h)$. *Furthermore,* $||\widehat{h}|| \leq 1$.

(iv) *For each complex Banach space* X *and each* $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$, *there is a unique* $S_f \in \mathcal{L}(\mathcal{G}(\mathbb{D}), X)$ *such that* $S_f \circ \Gamma = f'$ *and* $\|S_f\| = p_{\mathcal{B}}(f)$.

(v) $f \mapsto S_f$ *is an isometric isomorphism of* $\widehat{\mathcal{B}}(\mathbb{D}, X)$ *onto* $\mathcal{L}(\mathcal{G}(\mathbb{D}), X)$. □

Given a complex Banach space X, the space of X-*valued Bloch molecules of* $\mathbb{D}$ is defined as

$$\operatorname{lin}(\Gamma(\mathbb{D})) \otimes X = \operatorname{lin}(\{\gamma_z \otimes x \colon z \in \mathbb{D},\ x \in X\}) \subseteq \widehat{\mathcal{B}}(\mathbb{D}, X^*)^*,$$

where $\gamma_z \otimes x \colon \widehat{\mathcal{B}}(\mathbb{D}, X^*) \to \mathbb{C}$ is the functional given by

$$(\gamma_z \otimes x)(f) = \left\langle f'(z), x \right\rangle \qquad \left(f \in \widehat{\mathcal{B}}(\mathbb{D}, X^*)\right).$$

Each element $\gamma \in \operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$ can be expressed as $\gamma = \sum_{i=1}^n \lambda_i \gamma_{z_i} \otimes x_i$ for some n in $\mathbb{N}$, $(\lambda_i)_{i=1}^n$ in $\mathbb{C}$, $(z_i)_{i=1}^n$ in $\mathbb{D}$ and $(x_i)_{i=1}^n$ in X, and its action is

$$\gamma(f) = \sum_{i=1}^n \lambda_i \left\langle f'(z_i), x_i \right\rangle.$$

Definition 1.7 Let $1 \leq p, r, s \leq \infty$ and $\gamma \in \operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$. Define

$$\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma) = \inf \left\{ \left\| (\lambda_i)_{i=1}^n \right\|_p \omega_r^{\widehat{\mathcal{B}}}\left((z_i)_{i=1}^n\right) \omega_s\left((x_i)_{i=1}^n\right) \right\},$$

the infimum being taken over all the representations of γ as $\sum_{i=1}^n \lambda_i \gamma_{z_i} \otimes x_i$ with n in $\mathbb{N}$, $(\lambda_i)_{i=1}^n$ in $\mathbb{C}$, $(z_i)_{i=1}^n$ in $\mathbb{D}$ and $(x_i)_{i=1}^n$ in X.

Following [3, Definition 2.5], we say that a θ-norm α on $\operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$ with $\theta \in (0, 1]$ is a *Bloch reasonable crossnorm* if:

(i) $\alpha(\gamma_z \otimes x) \leq \|\gamma_z\| \|x\|$ for all $z \in \mathbb{D}$ and $x \in X$,

(ii) For $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x^* \in X^*$, the linear functional $g \otimes x^*$ on $\operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$ given by $(g \otimes x^*)(\gamma_z \otimes x) = g'(z)x^*(x)$ is bounded on $\operatorname{lin}(\Gamma(\mathbb{D})) \otimes_\alpha X$ with $\|g \otimes x^*\| \leq \rho_{\mathcal{B}}(g) \|x^*\|$.

The proof of the following result is based on [10, Theorem 1.1].

Theorem 1.8 *Let* $1 \leq p, r, s \leq \infty$ *and* $1/\theta := 1/p + 1/r + 1/s \geq 1$. *Then* $\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}$ *is a Bloch reasonable* θ*-crossnorm on* $\operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$.

Proof Let $\gamma \in \operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$ and let $\sum_{i=1}^n \lambda_i \gamma_{z_i} \otimes x_i$ be a representation of γ. Clearly, $\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma) \geq 0$. Given $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\lambda\gamma) &\leq \left\| (\lambda\lambda_i)_{i=1}^n \right\|_p \omega_r^{\widehat{\mathcal{B}}}\left((z_i)_{i=1}^n\right) \omega_s\left((x_i)_{i=1}^n\right) \\ &= |\lambda| \left\| (\lambda_i)_{i=1}^n \right\|_p \omega_r^{\widehat{\mathcal{B}}}\left((z_i)_{i=1}^n\right) \omega_s\left((x_i)_{i=1}^n\right)\end{aligned}$$

If $\lambda = 0$, we obtain $\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\lambda\gamma) = 0 = |\lambda|\,\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma)$. For $\lambda \neq 0$, since the preceding inequality holds for every representation of γ, we deduce that $\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\lambda\gamma) \leq |\lambda|\,\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma)$. For the converse inequality, note that

$$\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma) = \mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\lambda^{-1}(\lambda\gamma)) \leq |\lambda^{-1}|\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\lambda\gamma),$$

thus $|\lambda|\,\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma) \leq \mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\lambda\gamma)$ and hence $\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\lambda\gamma) = |\lambda|\,\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma)$.

We now prove that $\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma) = 0$ implies $\gamma = 0$. Applying that $\theta \leq 1$ and the generalized Hölder's inequality, we obtain

$$\begin{aligned}\left|\sum_{i=1}^{n}\lambda_i h'(z_i)y^*(x_i)\right| &\leq \left|\sum_{i=1}^{n}|\lambda_i|^{\theta}\left|h'(z_i)\right|^{\theta}\left|y^*(x_i)\right|^{\theta}\right|^{\frac{1}{\theta}} \\ &\leq \left(\sum_{i=1}^{n}|\lambda_i|^{p}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}\left|h'(z_i)\right|^{r}\right)^{\frac{1}{r}}\left(\sum_{i=1}^{n}\left|y^*(x_i)\right|^{s}\right)^{\frac{1}{s}} \\ &\leq \left\|(\lambda_i)_{i=1}^{n}\right\|_{p}\omega_r^{\widehat{\mathcal{B}}}\left((z_i)_{i=1}^{n}\right)\omega_s\left((x_i)_{i=1}^{n}\right)\end{aligned}$$

for any $h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}$ and $y^* \in B_{X^*}$. Since the quantity $\left|\sum_{i=1}^{n}\lambda_i h'(z_i)y^*(x_i)\right|$ does not depend on the representation of γ since

$$\sum_{i=1}^{n}\lambda_i h'(z_i)y^*(x_i) = \left(\sum_{i=1}^{n}\lambda_i\gamma_{z_i}\otimes x_i\right)(h\cdot y^*) = \gamma(h\cdot y^*),$$

taking the infimum over all representations of γ we deduce that

$$\left|\sum_{i=1}^{n}\lambda_i h'(z_i)y^*(x_i)\right| \leq \mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma)$$

for any $h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}$ and $y^* \in B_{X^*}$. Now, if $\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma) = 0$, we have

$$\left(\sum_{i=1}^{n}\lambda_i y^*(x_i)\gamma_{z_i}\right)(h) = \sum_{i=1}^{n}\lambda_i h'(z_i)y^*(x_i) = 0$$

for all $h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}$ and $y^* \in B_{X^*}$. For each $y^* \in B_{X^*}$, it is $\sum_{i=1}^{n}\lambda_i y^*(x_i)\gamma_{z_i} = 0$, and since $\Gamma(\mathbb{D})$ is linearly independent in $\mathcal{G}(\mathbb{D})$ by [8, Remark 2.8], it follows that $\lambda_i y^*(x_i) = 0$ for all $i \in \{1, \ldots, n\}$, hence $\lambda_i x_i = 0$ for all $i \in \{1, \ldots, n\}$ since B_{X^*} separates the points of X, and thus $\gamma = \sum_{i=1}^{n}\lambda_i\gamma_{z_i}\otimes x_i = 0$.

To prove the triangular inequality of $\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}$, let $\gamma_j \in \operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$ for $j = 1, 2$ and $\varepsilon > 0$. For $j = 1, 2$, by homogeneity we can choose a representation

$$\gamma_j = \sum_{i=1}^{n} \lambda_{j,i} \gamma_{z_{j,i}} \otimes x_{j,i}$$

for some n in $\mathbb{N}$, $\left(\lambda_{j,i}\right)_{i=1}^{n}$ in $\mathbb{C}$, $\left(z_{j,i}\right)_{i=1}^{n}$ in $\mathbb{D}$ and $\left(x_{j,i}\right)_{i=1}^{n}$ in X, so that

$$\begin{aligned}
\left\| (\lambda_{j,i})_{i=1}^{n} \right\|_p &\leq \left(\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma_j)^{\theta} + \varepsilon \right)^{\frac{1}{p}}, \\
\omega_r^{\widehat{\mathcal{B}}} \left((z_{j,i})_{i=1}^{n} \right) &\leq \left(\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma_j)^{\theta} + \varepsilon \right)^{\frac{1}{r}}, \\
\omega_s \left((x_{j,i})_{i=1}^{n} \right) &\leq \left(\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma_j)^{\theta} + \varepsilon \right)^{\frac{1}{s}}.
\end{aligned}$$

We can joint these representations of γ_1 and γ_2 to obtain a representation of $\gamma_1 + \gamma_2$ in the form $\sum_{i,j=1}^{n} \lambda_{j,i} \gamma_{z_{j,i}} \otimes x_{j,i}$ such that

$$\begin{aligned}
\left\| (\lambda_{j,i})_{i,j=1}^{n} \right\|_p &\leq \left(\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma_1)^{\theta} + \mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma_2)^{\theta} + 2\varepsilon \right)^{\frac{1}{p}}, \\
\omega_r^{\widehat{\mathcal{B}}} \left((z_{j,i})_{i,j=1}^{n} \right) &\leq \left(\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma_1)^{\theta} + \mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma_2)^{\theta} + 2\varepsilon \right)^{\frac{1}{r}}, \\
\omega_s \left((x_{j,i})_{i,j=1}^{n} \right) &\leq \left(\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma_1)^{\theta} + \mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma_2)^{\theta} + 2\varepsilon \right)^{\frac{1}{s}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma_1 + \gamma_2) &\leq \left\| (\lambda_{j,i})_{i,j=1}^{n} \right\|_p \omega_r^{\widehat{\mathcal{B}}} \left((z_{j,i})_{i,j=1}^{n} \right) \omega_s \left((x_{j,i})_{i,j=1}^{n} \right) \\
&\leq \left(\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma_1)^{\theta} + \mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma_2)^{\theta} + 2\varepsilon \right)^{\frac{1}{\theta}},
\end{aligned}$$

and since ε was arbitrary, we deduce that

$$\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma_1 + \gamma_2)^{\theta} \leq \mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma_1)^{\theta} + \mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma_2)^{\theta}.$$

To finish, we will show that $\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}$ is a Bloch reasonable crossnorm on $\operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$. First, given $z \in \mathbb{D}$ and $x \in X$, taking $n = 1$, $\lambda_1 = 1$, $z_1 = z$ and $x_1 = x$, we have

$$\begin{aligned}
\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma_z \otimes x) &\leq \left\| (\lambda_i)_{i=1}^{n} \right\|_p \omega_r^{\widehat{\mathcal{B}}} \left((z_i)_{i=1}^{n} \right) \omega_s \left((x_i)_{i=1}^{n} \right) \\
&\leq \frac{1}{1 - |z|^2} \|x\| = \|\gamma_z\| \|x\|.
\end{aligned}$$

Second, given $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x^* \in X^*$ with $g \neq 0 \neq x^*$, using that $\theta \leq 1$ and the generalized Hölder's inequality, one has

$$\begin{aligned}
\left|(g \otimes x^*)(\gamma)\right| &= \left|\sum_{i=1}^{n} \lambda_i (g \otimes x^*)(\gamma_{z_i} \otimes x_i)\right| = \left|\sum_{i=1}^{n} \lambda_i g'(z_i) x^*(x_i)\right| \\
&\leq \rho_{\mathcal{B}}(g) \left\|x^*\right\| \left\| \left(\lambda_i \left(\frac{g}{\rho_{\mathcal{B}}(g)}\right)'(z_i) \left(\frac{x^*}{\|x^*\|}\right)(x_i) \right)_{i=1}^{n} \right\|_1 \\
&\leq \rho_{\mathcal{B}}(g) \left\|x^*\right\| \left\| \left(\lambda_i \left(\frac{g}{\rho_{\mathcal{B}}(g)}\right)'(z_i) \left(\frac{x^*}{\|x^*\|}\right)(x_i) \right)_{i=1}^{n} \right\|_\theta \\
&\leq \rho_{\mathcal{B}}(g) \left\|x^*\right\| \left\|(\lambda_i)_{i=1}^n\right\|_p \left\| \left(\left(\frac{g}{\rho_{\mathcal{B}}(g)}\right)'(z_i) \right)_{i=1}^{n} \right\|_r \left\| \left(\left(\frac{x^*}{\|x^*\|}\right)(x_i) \right)_{i=1}^{n} \right\|_s \\
&\leq \rho_{\mathcal{B}}(g) \left\|x^*\right\| \left\|(\lambda_i)_{i=1}^n\right\|_p \omega_r^{\widehat{\mathcal{B}}} \left((z_i)_{i=1}^n\right) \omega_s \left((x_i)_{i=1}^n\right).
\end{aligned}$$

It follows that $|(g \otimes x^*)(\gamma)| \leq \rho_{\mathcal{B}}(g) \|x^*\| \mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma)$ by taking infimum over all the representations of γ. Hence $g \otimes x^* \in (\mathrm{lin}(\Gamma(\mathbb{D})) \otimes_{\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}} X)^*$ with $\|g \otimes x^*\| \leq \rho_{\mathcal{B}}(g) \|x^*\|$. □

1.6 Duality

We will prove that the dual of $\mathcal{G}(\mathbb{D}) \widehat{\otimes}_{\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}} X$ can be canonically identified as the space $\Pi^{\widehat{\mathcal{B}}}_{(p^*,r,s)}(\mathbb{D}, X^*)$ with the norm $\pi^{\mathcal{B}}_{(p^*,r,s)}$ whenever $1 \leq p, r, s \leq \infty$ such that $1/p^* \leq 1/r + 1/s$.

The following easy lemma will be needed.

Lemma 1.9 *Let X be a Banach space, $n \in \mathbb{N}$, $(x_i^*)_{i=1}^n$ in X^* and $1 \leq p \leq \infty$. Then*

$$\sup_{x^{**} \in \mathcal{B}_{X^{**}}} \left\|(x^{**}(x_i^*))_{i=1}^n\right\|_p = \sup_{x \in \mathcal{B}_X} \left\|(x_i^*(x))_{i=1}^n\right\|_p .$$

Proof Since $x_i^*(x) = J_X(x)(x_i^*)$ for $i = 1, \ldots, n$, the inequality $\geq$ is immediate. Conversely, let $\varepsilon > 0$. For each $x^{**} \in \mathcal{B}_{X^{**}}$, Helly's Lemma gives an $y \in X$ such that $\|y\| \leq 1 + \varepsilon$ and $x_i^*(y) = x^{**}(x_i^*)$ for all $i \in \{1, \ldots, n\}$, and therefore

$$\begin{aligned}
\left\|(x^{**}(x_i^*))_{i=1}^n\right\|_p &= (1+\varepsilon) \left\| \left(x_i^* \left(\frac{y}{1+\varepsilon} \right) \right)_{i=1}^{n} \right\|_p \\
&\leq (1+\varepsilon) \sup_{x \in \mathcal{B}_X} \left\|(x_i^*(x))_{i=1}^n\right\|_p .
\end{aligned}$$

It follows that

$$\sup_{x^{**}\in\mathcal{B}_{X^{**}}} \left\|(x^{**}(x_i^*))_{i=1}^n\right\|_p \leq (1+\varepsilon) \sup_{x\in\mathcal{B}_X} \left\|(x_i^*(x))_{i=1}^n\right\|_p$$

and since ε was arbitrary, we obtain the inequality $\leq$. □

Theorem 1.10 *Let $1 \leq p, r, s \leq \infty$ such that $1/p^* \leq 1/r + 1/s$. Then the spaces $(\Pi^{\widehat{\mathcal{B}}}_{(p^*,r,s)}(\mathbb{D}, X^*), \pi^{\mathcal{B}}_{(p^*,r,s)})$ and $(\mathcal{G}(\mathbb{D})\widehat{\otimes}_{\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}} X)^*$ are isometrically isomorphic via the canonical pairing*

$$\Lambda(f)(\gamma) = \sum_{i=1}^n \lambda_i \left\langle f'(z_i), x_i\right\rangle$$

for all $f \in \Pi^{\widehat{\mathcal{B}}}_{p^*,r,s}(\mathbb{D}, X^*)$ and $\gamma = \sum_{i=1}^n \lambda_i \gamma_{z_i} \otimes x_i \in \operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$. Moreover, on the closed unit ball of $(\Pi^{\widehat{\mathcal{B}}}_{(p^*,r,s)}(\mathbb{D}, X^*), \pi^{\mathcal{B}}_{(p^*,r,s)})$, the weak topology coincides with the topology of pointwise $\sigma(X^*, X)$-convergence.*

Proof We will only prove the result whenever $1 < p < \infty$, and the other cases can be proved similarly.

Let $f \in \Pi^{\widehat{\mathcal{B}}}_{(p^*,r,s)}(\mathbb{D}, X^*)$ and let $\Lambda_0(f)\colon \operatorname{lin}(\Gamma(\mathbb{D})) \otimes X \to \mathbb{C}$ be the linear functional given by

$$\Lambda_0(f)(\gamma) = \sum_{i=1}^n \lambda_i \left\langle f'(z_i), x_i\right\rangle$$

for any $\gamma = \sum_{i=1}^n \lambda_i \gamma_{z_i} \otimes x_i \in \operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$. We claim that $\Lambda_0(f) \in (\operatorname{lin}(\Gamma(\mathbb{D})) \otimes_{\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}} X)^*$ with $\|\Lambda_0(f)\| \leq \pi^{\mathcal{B}}_{(p^*,r,s)}(f)$. Indeed, Hölder's inequality and an application of Lemma 1.9 yield

$$\begin{aligned}
|\Lambda_0(f)(\gamma)| &\leq \sum_{i=1}^n |\lambda_i| \left|\left\langle f'(z_i), x_i\right\rangle\right| \\
&\leq \left(\sum_{i=1}^n |\lambda_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n \left|\left\langle f'(z_i), x_i\right\rangle\right|^{p^*}\right)^{\frac{1}{p^*}} \\
&= \left(\sum_{i=1}^n |\lambda_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n \left|\left\langle J_X(x_i), f'(z_i)\right\rangle\right|^{p^*}\right)^{\frac{1}{p^*}} \\
&\leq \left\|(\lambda_i)_{i=1}^n\right\|_p \pi^{\mathcal{B}}_{(p^*,r,s)}(f)\omega_r^{\widehat{\mathcal{B}}}\left((z_i)_{i=1}^n\right)\omega_s\left((J_X(x_i))_{i=1}^n\right) \\
&= \left\|(\lambda_i)_{i=1}^n\right\|_p \pi^{\mathcal{B}}_{(p^*,r,s)}(f)\omega_r^{\widehat{\mathcal{B}}}\left((z_i)_{i=1}^n\right)\omega_s\left((x_i)_{i=1}^n\right).
\end{aligned}$$

Taking infimum over all the representations of γ, we deduce that

$$|\Lambda_0(f)(\gamma)| \leq \pi^{\mathcal{B}}_{(p^*,r,s)}(f)\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}(\gamma),$$

and since γ was arbitrary, this proves our claim.

Since $\mathrm{lin}(\Gamma(\mathbb{D}))$ is a norm-dense linear subspace of $\mathcal{G}(\mathbb{D})$ and $\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}$ is a θ-norm on $\mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$, then $\mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$ is a norm-dense linear subspace of $\mathcal{G}(\mathbb{D})\widehat{\otimes}_{\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}} X$. Hence there is a unique continuous map $\Lambda(f)$ from $\mathcal{G}(\mathbb{D})\widehat{\otimes}_{\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}} X$ into $\mathbb{C}$ extending $\Lambda_0(f)$. Further, $\Lambda(f)$ is linear and $\|\Lambda(f)\| = \|\Lambda_0(f)\|$.

Let $\Lambda\colon \Pi^{\widehat{\mathcal{B}}}_{(p^*,r,s)}(\mathbb{D}, X^*) \to (\mathcal{G}(\mathbb{D})\widehat{\otimes}_{\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}} X)^*$ be so defined. In view of [3, Corollary 2.3], Λ_0 is injective and linear from $\Pi^{\widehat{\mathcal{B}}}_{(p^*,r,s)}(\mathbb{D}, X^*)$ into $(\mathcal{G}(\mathbb{D}) \otimes X)^*$, and therefore so is Λ. To prove that Λ is a surjective isometry, let $\varphi \in (\mathcal{G}(\mathbb{D})\widehat{\otimes}_{\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}} X)^*$ and define $F_\varphi\colon \mathbb{D} \to X^*$ by

$$\left\langle F_\varphi(z), x\right\rangle = \varphi(\gamma_z \otimes x) \qquad (z \in \mathbb{D},\ x \in X).$$

As in the proof of [3, Proposition 2.4], there exists $f_\varphi \in \widehat{\mathcal{B}}(\mathbb{D}, X^*)$ with $\rho_{\mathcal{B}}(f_\varphi) \leq \|\varphi\|$ such that $f'_\varphi = F_\varphi$.

We now prove that $f_\varphi \in \Pi^{\widehat{\mathcal{B}}}_{(p^*,r,s)}(\mathbb{D}, X^*)$. Fix $n \in \mathbb{N}$, $(\lambda_i)_{i=1}^n$ in $\mathbb{C}$, $(z_i)_{i=1}^n$ in $\mathbb{D}$ and $(x_i^{**})_{i=1}^n$ in X^{**}. Let $\varepsilon > 0$. By Helly's Lemma, for each $i \in \{1, \ldots, n\}$, we can find $x_i \in X$ with $\|x_i\| \leq (1+\varepsilon)\left\|x_i^{**}\right\|$ and $\left\langle f'_\varphi(z_i), x_i\right\rangle = \left\langle x_i^{**}, f'_\varphi(z_i)\right\rangle$. Clearly, the map $T: \mathbb{C}^n \to \mathbb{C}$, defined by

$$T(t_1, \ldots, t_n) = \sum_{i=1}^n t_i\lambda_i \left\langle x_i^{**}, f'_\varphi(z_i)\right\rangle, \quad \forall (t_1, \ldots, t_n) \in \mathbb{C}^n,$$

is linear and continuous on $(\mathbb{C}^n, ||\cdot||_p)$ with

$$\|T\| = \left(\sum_{i=1}^n |\lambda_i|^{p^*} \left|\left\langle x_i^{**}, f'_\varphi(z_i)\right\rangle\right|^{p^*}\right)^{\frac{1}{p^*}}.$$

For any $(t_1, \ldots, t_n) \in \mathbb{C}^n$ with $||(t_1, \ldots, t_n)||_p \leq 1$, we have

$$\begin{aligned}
|T(t_1, \ldots, t_n)| &= \left|\varphi\left(\sum_{i=1}^n t_i\lambda_i\gamma_{z_i} \otimes x_i\right)\right| \\
&\leq \|\varphi\|\, \mu^{\widehat{\mathcal{B}}}_{(p,r,s)}\left(\sum_{i=1}^n t_i\lambda_i\gamma_{z_i} \otimes x_i\right) \\
&\leq \|\varphi\| \left\|(t_i)_{i=1}^n\right\|_p \omega_r^{\widehat{\mathcal{B}}}\left((\lambda_i, z_i)_{i=1}^n\right) \omega_s\left((x_i)_{i=1}^n\right).
\end{aligned}$$

For each $i \in \{1,\dots,n\}$, Hahn–Banach Theorem provides $x_i^{***} \in B_{X^{***}}$ such that $\left|x_i^{***}(x_i^{**})\right| = \left\|x_i^{**}\right\|$. Note that $\omega_s\left((x_i)_{i=1}^n\right) \le \omega_s\left((x_i^{**})_{i=1}^n\right)$ because

$$\begin{aligned}\left\|(x^*(x_i))_{i=1}^n\right\|_s &\le (1+\varepsilon)\left\|(x_i^{**})_{i=1}^n\right\|_s = (1+\varepsilon)\left\|(x_i^{***}(x_i^{**}))_{i=1}^n\right\|_s \\ &\le (1+\varepsilon)\omega_s\left((x_i^{**})_{i=1}^n\right)\end{aligned}$$

for all $x^* \in B_{X^*}$. Therefore we can write

$$\left(\sum_{i=1}^n |\lambda_i|^{p^*}\left|\left\langle x_i^{**}, f'_\varphi(z_i)\right\rangle\right|^{p^*}\right)^{\frac{1}{p^*}} \le (1+\varepsilon)\left\|\varphi\right\|\omega_r^{\widehat{\mathcal{B}}}\left((\lambda_i, z_i)_{i=1}^n\right)\omega_s\left((x_i^{**})_{i=1}^n\right).$$

By letting ε tend to zero gives $f_\varphi \in \Pi^{\widehat{\mathcal{B}}}_{(p^*,r,s)}(\mathbb{D}, X^*)$ with $\pi^{\mathcal{B}}_{(p^*,r,s)}(f_\varphi) \le \|\varphi\|$.

Finally, for any $\gamma = \sum_{i=1}^n \lambda_i\gamma_{z_i} \otimes x_i \in \operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$, we get

$$\begin{aligned}\Lambda(f_\varphi)(\gamma) &= \sum_{i=1}^n \lambda_i\left\langle f'_\varphi(z_i), x_i\right\rangle = \sum_{i=1}^n \lambda_i\varphi(\gamma_{z_i} \otimes x_i) \\ &= \varphi\left(\sum_{i=1}^n \lambda_i\gamma_{z_i} \otimes x_i\right) = \varphi(\gamma).\end{aligned}$$

Hence $\Lambda(f_\varphi) = \varphi$ on a dense subspace of $\mathcal{G}(\mathbb{D})\widehat{\otimes}_{\mu^{\widehat{\mathcal{B}}}_{(p,r,s)}} X$ and, consequently, $\Lambda(f_\varphi) = \varphi$. Moreover, $\pi^{\mathcal{B}}_{(p^*,r,s)}(f_\varphi) \le \|\varphi\| = \left\|\Lambda(f_\varphi)\right\|$.

The assertion about the weak* topology can be proved with the same argument as in the proof of Theorem 2.8 in [3]. □

2 (*p*, *r*, *s*)-Nuclear Bloch maps

In order to present examples of (p,r,s)-summing Bloch maps, we introduce the class of (p,r,s)-nuclear Bloch maps.

Let X be a complex Banach space and $1 \le p \le \infty$. Let $\ell_p(X)$ be the Banach space of all p-summable sequences $(x_n)_{n=1}^\infty$ in X, with the norm

$$\left\|(x_n)_{n=1}^\infty\right\|_p = \begin{cases}\left(\sum_{n=1}^\infty \|x_n\|^p\right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty, \\ \max_{n\in\mathbb{N}} \|x_n\| & \text{if } p = \infty,\end{cases}$$

and let $\ell_p^\omega(X)$ be the Banach space of all weakly p-summable sequences $(x_n)_{n=1}^\infty$ in X, with the norm

$$\omega_p\left((x_n)_{n=1}^\infty\right) = \sup_{x^*\in\mathcal{B}_{X^*}}\left\|(x^*(x_n))_{n=1}^\infty\right\|_p.$$

As usual, we will write ℓ_p and ℓ_p^ω instead of $\ell_p(\mathbb{C})$ and $\ell_p^\omega(\mathbb{C})$, respectively.

By [14, Definition 18.1.1], given Banach spaces X, Y and $0 < p, r, s \leq \infty$ with $1 + 1/p \geq 1/r + 1/s$, an operator $T \in \mathcal{L}(X, Y)$ is (p, r, s)-*nuclear* if $T = \sum_{n=1}^\infty \lambda_n x_n^* \cdot y_n$ in the operator canonical norm of $\mathcal{L}(X, Y)$, where $(\lambda_n)_{n=1}^\infty \in \ell_p$, $(x_n^*)_{n=1}^\infty \in \ell_{s^*}^\omega(X^*)$ and $(y_n)_{n=1}^\infty \in \ell_{r^*}^\omega(Y)$. In the case $p = \infty$, we take $(\lambda_n)_{n=1}^\infty \in c_0$. It is said that $\sum_{n=1}^\infty \lambda_n x_n^* \cdot y_n$ is a (p, r, s)-*nuclear representation of* T. Define

$$\nu_{(p,r,s)}(T) = \inf\{\|(\lambda_n)_{n=1}^\infty\|_p \omega_{s^*}((x_n^*)_{n=1}^\infty)\omega_{r^*}((y_n)_{n=1}^\infty)\},$$

where the infimum is taken over all (p, r, s)-nuclear representations of T. Let $\mathcal{N}_{(p,r,s)}(X, Y)$ be the set of all (p, r, s)-nuclear operators from X into Y.

The corresponding version for Bloch maps could be the following.

Definition 2.1 Let $1 \leq p, r, s \leq \infty$ such that $1 + 1/p \geq 1/r + 1/s$. A map $f \in \mathcal{H}(\mathbb{D}, X)$ is said to be (p, r, s)-nuclear Bloch if $f = \sum_{n=1}^\infty \lambda_n g_n \cdot x_n$ in the Bloch norm $\rho_{\mathcal{B}}$, where $(\lambda_n)_{n=1}^\infty \in \ell_p$, $(g_n)_{n=1}^\infty \in \ell_{s^*}^\omega(\widehat{\mathcal{B}}(\mathbb{D}))$ and $(x_n)_{n=1}^\infty \in \ell_{r^*}^\omega(X)$. For $p = \infty$, we choose $(\lambda_n)_{n=1}^\infty \in c_0$. We say that $\sum_{n=1}^\infty \lambda_n g_n \cdot x_n$ is a (p, r, s)-*nuclear Bloch representation of* f and we set

$$\nu_{(p,r,s)}^{\mathcal{B}}(f) = \inf\{\|(\lambda_n)_{n=1}^\infty\|_p \omega_{s^*}((g_n)_{n=1}^\infty)\omega_{r^*}((x_n)_{n=1}^\infty)\},$$

where the infimum is taken over all (p, r, s)-nuclear Bloch representations of f. Let $\mathcal{N}_{(p,r,s)}^{\mathcal{B}}(\mathbb{D}, X)$ be the set of all (p, r, s)-nuclear Bloch maps from $\mathbb{D}$ into X, and let $\mathcal{N}_{(p,r,s)}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$ be its subset formed by all those maps f for which $f(0) = 0$.

Putting $1/\theta := 1/p + 1/r^* + 1/s^*$, $\mathcal{N}_{(p,r,s)}(X, Y)$ is a θ-Banach operator ideal under the norm

$$\nu_{(p,r,s)}(T) = \inf\{\|(\lambda_n)_{n=1}^\infty\|_p \omega_{s^*}((x_n^*)_{n=1}^\infty)\omega_{r^*}((y_n)_{n=1}^\infty)\},$$

by taking the infimum is taken over all (p, r, s)-nuclear representations of T (see [14, Theorem 18.1.2]).

In order to establish a Bloch variant of this result, we first study the linearization of (p, r, s)-summing Bloch maps and (p, r, s)-nuclear Bloch maps.

Proposition 2.2 *Let $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$ and assume that $S_f \in \Pi_{(p,r,s)}(\mathcal{G}(\mathbb{D}), X)$. Then $f \in \Pi_{(p,r,s)}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$ and $\pi_{(p,r,s)}^{\mathcal{B}}(f) \leq \pi_{(p,r,s)}(S_f)$.*

Proof Given $n \in \mathbb{N}$, $(\lambda_i)_{i=1}^n$ in $\mathbb{C}$, $(z_i)_{i=1}^n$ in $\mathbb{D}$ and $(x_i^*)_{i=1}^n$ in X^*, using Theorem 1.6 we have

$$\begin{aligned}\left\|(\lambda_i x_i^*(f'(z_i)))_{i=1}^n\right\|_p &= \left\|(x_i^*(S_f(\lambda_i \gamma_{z_i})))_{i=1}^n\right\|_p \\ &\leq \pi_{(p,r,s)}(S_f)\omega_r\left((\lambda_i \gamma_{z_i})_{i=1}^n\right)\omega_s\left((x_i^*)_{i=1}^n\right)\end{aligned}$$

and since

$$\begin{aligned}\omega_r\left((\lambda_i\gamma_{z_i})_{i=1}^n\right) &= \sup_{\phi\in B_{\mathcal{G}(\mathbb{D})^*}} \left\|(\phi(\lambda_i\gamma_{z_i}))_{i=1}^n\right\|_r \\ &= \sup_{g\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left\|(\Lambda(g)(\lambda_i\gamma_{z_i}))_{i=1}^n\right\|_r \\ &= \sup_{g\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left\|(\lambda_i g'(z_i))_{i=1}^n\right\|_r = \omega_r^{\widehat{\mathcal{B}}}((\lambda_i, z_i)_{i=1}^n),\end{aligned}$$

the result is proven. □

Theorem 2.3 *Let* $1 \leq p, r, s \leq \infty$ *such that* $1+1/p \geq 1/r+1/s$ *and let* $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$. *The following assertions are equivalent:*

(i) $f : \mathbb{D} \to X$ *is a* (p, r, s)*-nuclear Bloch map.*
(ii) $S_f : \mathcal{G}(\mathbb{D}) \to X$ *is a* (p, r, s)*-nuclear linear operator.*

In this case, $\nu^{\mathcal{B}}_{(p,r,s)}(f) = \nu_{(p,r,s)}(S_f)$.

Proof $(i) \Rightarrow (ii)$: Assume that $f \in \mathcal{N}^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, X)$ and let $\sum_{n=1}^{\infty} \lambda_n g_n \cdot x_n$ be a (p, r, s)-nuclear Bloch representation of f. First, note that if $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x \in X$, we have that $\Lambda(g) \cdot x \in \mathcal{L}(\mathcal{G}(\mathbb{D}), X)$ and

$$(g \cdot x)'(z) = g'(z)x = \Lambda(g)(\gamma_z)x = (\Lambda(g) \cdot x)(\gamma_z) = (\Lambda(g) \cdot x \circ \Gamma)(z)$$

for all $z \in \mathbb{D}$, and thus Theorem 1.6 gives $S_{g\cdot x} = \Lambda(g) \cdot x$. Since

$$\rho_{\mathcal{B}}\left(f - \sum_{k=1}^{n} \lambda_k g_k \cdot x_k\right) = \left\|S_f - \sum_{k=1}^{n} \lambda_k S_{g_k\cdot x_k}\right\| = \left\|S_f - \sum_{k=1}^{n} \lambda_k \Lambda(g_k) \cdot x_k\right\|$$

for all $n \in \mathbb{N}$, it follows that $S_f = \sum_{n=1}^{\infty} \lambda_n \Lambda(g_n) \cdot x_n$ in the operator norm. Moreover, note that

$$\begin{aligned}\omega_{s^*}((\Lambda(g_n))_{n=1}^{\infty}) &= \sup_{\phi\in B_{\mathcal{G}(\mathbb{D})^{**}}} \left\|(\phi(\Lambda(g_n)))_{n=1}^{\infty}\right\|_{s^*} \\ &= \sup_{\phi\in B_{\mathcal{G}(\mathbb{D})^{**}}} \left\|(\Lambda^*(\phi)(g_n))_{n=1}^{\infty}\right\|_{s^*} \\ &= \sup_{\varphi\in B_{\widehat{\mathcal{B}}(\mathbb{D})^*}} \left\|(\varphi(g_n))_{n=1}^{\infty}\right\|_{s^*} = \omega_{s^*}((g_n)_{n=1}^{\infty}),\end{aligned}$$

where $\Lambda^* : \mathcal{G}(\mathbb{D})^{**} \to \widehat{\mathcal{B}}(\mathbb{D})^*$ is the adjoint operator of $\Lambda : \widehat{\mathcal{B}}(\mathbb{D}) \to \mathcal{G}(\mathbb{D})^*$. Hence $S_f \in \mathcal{N}_{(p,r,s)}(\mathcal{G}(\mathbb{D}), X)$ with

$$\nu_{(p,r,s)}(S_f) \leq \|(\lambda_n)_{n=1}^{\infty}\|_p \omega_{s^*}((g_n)_{n=1}^{\infty})\omega_{r^*}((x_n)_{n=1}),$$

and passing to the infimum over all (p, r, s)-nuclear Bloch decompositions of f, we conclude that $\nu_{(p,r,s)}(S_f) \leq \nu^{\mathcal{B}}_{(p,r,s)}(f)$.

$(ii) \Rightarrow (i)$ is proven with a reasoning similar to the previous one. □

We are ready to establish a Bloch version of Theorem 18.1.2 in [14].

Corollary 2.4 *Let $1 \leq p, r, s \leq \infty$ such that $1 + 1/p \geq 1/r + 1/s$ and let $1/\theta := 1/p + 1/r^* + 1/s^*$. Then $[\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,r,s)}, \nu^{\mathcal{B}}_{(p,r,s)}]$ is a θ-Banach normalized Bloch ideal.*

Proof Let X be a complex Banach space.

(P1): Let $\lambda \in \mathbb{C}$ and $f, g \in \mathcal{N}^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, X)$. Using Theorems 1.6, 2.3 and [14, Theorem 18.1.2], we obtain that $\nu^{\mathcal{B}}_{(p,r,s)}$ is a norm on $\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, X)$:

$$\begin{aligned}
\nu^{\mathcal{B}}_{(p,r,s)}(\lambda f) &= \nu_{(p,r,s)}(S_{\lambda f}) = \nu_{(p,r,s)}(\lambda S_f) = |\lambda|\, \nu_{(p,r,s)}(S_f) = |\lambda|\, \nu^{\mathcal{B}}_{(p,r,s)}(f),\\
\nu^{\mathcal{B}}_{(p,r,s)}(f+g)^\theta &= \nu_{(p,r,s)}(S_{f+g})^\theta = \nu_{(p,r,s)}(S_f + S_g)^\theta\\
&\leq \nu_{(p,r,s)}(S_f)^\theta + \nu_{(p,r,s)}(S_g)^\theta = \nu^{\mathcal{B}}_{(p,r,s)}(f)^\theta + \nu^{\mathcal{B}}_{(p,r,s)}(g)^\theta,\\
\nu^{\mathcal{B}}_{(p,r,s)}(f) &= 0 \Rightarrow \nu_{(p,r,s)}(S_f) = 0 \Rightarrow S_f = 0 \Rightarrow f' = S_f \circ \Gamma = 0 \Rightarrow f = 0.
\end{aligned}$$

To see that the norm $\nu^{\mathcal{B}}_{(p,r,s)}$ is complete, note that another application of those theorems assures that $f \mapsto S_f$ is an isometric isomorphism of $(\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, X), \nu^{\mathcal{B}}_{(p,r,s)})$ onto $(\mathcal{N}_{(p,r,s)}(\mathcal{G}(\mathbb{D}), X), \nu_{(p,r,s)})$, and moreover

$$\rho_{\mathcal{B}}(f) = \left\| S_f \right\| \leq \nu_{(p,r,s)}(S_f) = \nu^{\mathcal{B}}_{(p,r,s)}(f).$$

(P2): Let $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x \in X$. By the operator ideal property of $[\mathcal{N}_{(p,r,s)}, \nu_{(p,r,s)}]$ and Theorem 1.6, $S_{g\cdot x} = \Lambda(g) \cdot x \in \mathcal{N}_{(p,r,s)}(\mathcal{G}(\mathbb{D}), X)$ with $\nu_{(p,r,s)}(S_{g\cdot x}) = \|\Lambda(g)\|\, \|x\| = \rho_{\mathcal{B}}(g)\, \|x\|$. Hence $g \cdot x \in \mathcal{N}^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, X)$ with $\nu^{\mathcal{B}}_{(p,r,s)}(g \cdot x) = \rho_{\mathcal{B}}(g)\, \|x\|$ by Theorem 2.3.

(P3): Let $f \in \mathcal{N}^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, X)$, $T \in \mathcal{L}(X, Y)$ and $h \in \widehat{\mathcal{B}}(\mathbb{D}, \mathbb{D})$. Since $T \circ S_f \circ \widehat{h} \in \mathcal{L}(\mathcal{G}(\mathbb{D}), Y)$ and

$$\begin{aligned}
(T \circ f \circ h)' &= T \circ [h' \cdot (f' \circ h)] = T \circ [h' \cdot (S_f \circ \Gamma \circ h)]\\
&= T \circ [S_f(h' \cdot (\Gamma \circ h))] = T \circ [S_f \circ (\widehat{h} \circ \Gamma)]\\
&= (T \circ S_f \circ \widehat{h}) \circ \Gamma,
\end{aligned}$$

one has that $S_{T\circ f\circ h} = T \circ S_f \circ \widehat{h}$ by Theorem 1.6. Since $S_f \in \mathcal{N}_{(p,r,s)}(\mathcal{G}(\mathbb{D}), X)$ by Theorem 2.3, we get that $S_{T\circ f\circ h} \in \mathcal{N}_{(p,r,s)}(\mathcal{G}(\mathbb{D}), Y)$ with $\nu_{(p,r,s)}(S_{T\circ f\circ h}) \leq \|T\|\, \nu_{(p,r,s)}(S_f)||\widehat{h}||$ by the operator ideal property of $[\mathcal{N}_{(p,r,s)}, \nu_{(p,r,s)}]$, and thus $T \circ f \circ h \in \mathcal{N}^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, Y)$ with $\nu^{\mathcal{B}}_{(p,r,s)}(T \circ f \circ h) \leq \|T\|\, \nu^{\mathcal{B}}_{(p,r,s)}(f)$ by Theorems 1.6 and 2.3. □

We conclude arriving at the objective of this section.

Corollary 2.5 *Let* $1 \leq p, r, s \leq \infty$ *such that* $1/p \leq 1/r + 1/s \leq 1 + 1/p$. *Then* $(\mathcal{N}^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, X), \nu^{\mathcal{B}}_{(p,r,s)}) \leq (\Pi^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, X), \pi^{\mathcal{B}}_{(p,r,s)})$.

Proof Let $f \in \mathcal{N}^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, X)$. Then $S_f \in \mathcal{N}_{(p,r,s)}(\mathcal{G}(\mathbb{D}), X)$ with $\nu_{(p,r,s)}(S_f) = \nu^{\mathcal{B}}_{(p,r,s)}(f)$ by Theorem 2.3. Since

$$(\mathcal{N}_{(p,r,s)}(\mathcal{G}(\mathbb{D}), X), \nu_{(p,r,s)}) \leq (\Pi_{(p,r,s)}(\mathcal{G}(\mathbb{D}), X), \pi_{(p,r,s)}),$$

it follows that $S_f \in \Pi_{(p,r,s)}(\mathcal{G}(\mathbb{D}), X)$ with $\pi_{(p,r,s)}(S_f) \leq \nu_{(p,r,s)}(S_f)$. By Proposition 2.2, $f \in \Pi^{\widehat{\mathcal{B}}}_{(p,r,s)}(\mathbb{D}, X)$ and $\pi^{\mathcal{B}}_{(p,r,s)}(f) \leq \pi_{(p,r,s)}(S_f) \leq \nu^{\mathcal{B}}_{(p,r,s)}(f)$. □

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