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ORIGINAL PAPER

Asymptotics of the eigenvalues of seven-diagonal Toeplitz matrices of a special form

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Abstract

This work is devoted to the construction of a uniform asymptotics in the dimension of the matrix n tending to infinity of all eigenvalues in the case of a seven-diagonal Toeplitz matrix with a symbol having a zero of the sixth order, while the cases of symbols with zeros of the second and fourth orders were considered earlier. On the other hand, the results obtained refine the results of the classical work of Parter and Widom on the asymptotics of the extreme eigenvalues. We also note that the obtained formulas showed high computational efficiency both in sense of accuracy (already for relatively small values of n) and in sense of speed.

Keywords Toeplitz matrices · Eigenvectors · Asymptotic expansions

Mathematics Subject Classification 15B05 · 15A18

1 Introduction

Let $a(t)$ be a Lebesgue integrable function defined on the unit circle $\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$. We denote by $T_n(a)$ the Toeplitz matrix $T_n(a) := (a_{j-k})_{j,k=1}^{n-1}$, where $n \in \mathbb{N}$ is a natural number, and a_l denotes the l-th coefficient of the Fourier series of the

Dedicated to Ilya Spitkovsky on his occasion of his 70th birthday.

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function a. Note that the Toeplitz matrix can be viewed as an operator from a finite dimensional vector space. The function $a(t)$ is called the symbol of the Toeplitz matrix (Toeplitz operator) $T_n(a)$. This paper is devoted to finding asymptotic formulas for the eigenvalues of the Toeplitz matrix with the symbol $a(t) = (t - 2 + \frac{1}{t})^3$.

Toeplitz matrices, as well as closely related Toeplitz operators, have been intensively studied for various classes of symbols over the past, about a hundred years [7, 8, 13, 19, 20]). The importance of this subject is largely due to the numerous applications of Toeplitz matrices in numerical methods of differential and integral equations, probability theory, statistical physics (see, for example, [10, 11, 14, 15]). As mentioned above, this work is devoted to finding asymptotic formulas for the eigenvalues of the Toeplitz matrix with the symbol $a(t) = (t - 2 + \frac{1}{t})^3$. Toeplitz matrices with this symbol are self-adjoint matrices. However, the study of non-self-adjoint Toeplitz matrices, whose symbol is the cube of the linear Laurent polynomial and it has a five-power derivative at the end of the interval equal to zero, can also be reduced to this case. We note that all the asymptotic formulas for the eigenvalues obtained in this paper, in essence, admit an uniform estimate for the remainder term, with respect to the number of eigenvalue. It should be said that the symbol under consideration has specific properties: it is a real, symmetric function, and the first five derivatives of the symbol are equal to zero at the point $t = 1$. The last condition, namely the vanishing of the first five derivatives, significantly complicates the problem of finding an asymptotic formula for the eigenvalues, since in this case the general research methods developed in the previous works are inapplicable (see works [2–6, 9], which present general approaches to finding the asymptotics of the eigenvalues for various classes of Toeplitz matrices). In addition, the case we are considering is more complicated than that considered in the work [1]. It should be mentioned that our asymptotic formulas give a better approximation for the first eigenvalues than the classical formulas obtained by Parter [16]. Note that although we are considering a special case of the symbol described above, nevertheless, it seems to us, the method we use can also be applied to study the asymptotic formulas of Toeplitz matrices with arbitrary symbol defined above mentioned properties. In other words, we hope that the methods developed in this article can also be extended to Toeplitz band matrices with an arbitrary real symmetric symbol that allows its derivatives to vanish up to the fifth order at some marked point.

Our investigation is based on the formula for the determinant of the Toeplitz matrix, obtained in [12] and after some transformations of this formula we reduce the finding of eigenvalues to solving ordered set of n equations. Each of these equations has a unique solution. We present an iterative algorithm and an asymptotic formula for quickly calculating the eigenvalues and exploring their location. The formulas which we use are different for the cases of even and odd eigenvalues. This fact follows from the specificity of the formula for the determinant of the Toeplitz matrix [12]. Note that the use of the fixed point method it possible to calculate eigenvalues with any given accuracy and obtain a very high speed of convergence to the exact value. We note also that the asymptotic formulas that we obtained in this article make it possible to calculate approximately the eigenvalues with high accuracy of large Toeplitz matrices (with size greater than $10^6 \times 10^6$) with the considered symbol, in the case when no other numerical methods are applicable.

The paper is organized as follows. Section 2 contains the main results of the work. Section 3 contains some auxiliary results. Section 4 contains the proof of our main results, and Sect. 5 gives numerical examples illustrating the effectiveness of the results.

2 Main results

In this section, we will present the main results of the article. We formulate a theorem describing an asymptotic formula for the eigenvalues of a Toeplitz matrix with the symbol mentioned above. The eigenvalues are calculated as the values of the function $g(\varphi) = a(e^{i\varphi})$, where $a(t) = (t - 2 + \frac{1}{t})^3$, for fixed values of the argument φ. Furthermore, the function $g(\varphi) = a(e^{i\varphi}) = -\left(2\sin\frac{\varphi}{2}\right)^6$ defined on $[0, 2\pi]$ takes the minimum value at the point π equal to $m = -2^6$ and has the following properties:

(i) The function $g : [0, 2\pi] \to \mathbb{R}$, has range $[m, 0]$, $g(\varphi - \pi) = g(\varphi)$ for $\varphi > \pi$ and $g^{(1)}(\varphi) < 0$ for $\varphi \in (0, \pi)$.
(ii) $g(\pi) := m$, $g^{(1)}(\pi) = 0$, and $g^{(2)}(\pi) > 0$.
(iii) $g(0) = 0$, $g^{(k)}(0) = 0$ $(k = 1, \ldots, 5)$, and $g^{(6)}(0) < 0$.

Thus, the structure of the asymptotic formula for the eigenvalues is such that this formula is a refinement, on the one hand, of Szego's limit theorem, which describes the limit spectrum of Toeplitz matrices as the image of the unit circle $\mathbb{T}$ under the action of the symbol, and on the other, as mentioned in the introduction, is a refinement of the results of Spitzer and Schmidt [19], which give the same answer in self-adjoint as Szego's limit theorem [8, 18].

Note that the problem is solved with respect to the variable φ, from which the eigenvalues λ are expressed by a simple substitution $\lambda = g(\varphi)$. Let's introduce some functions. All functions will be defined on the interval $\varphi \in (0, \pi)$.

$$\begin{aligned} &\beta(\varphi) := \arccos\left(1 - (1-\cos\varphi)e^{\frac{2\pi i}{3}}\right), \quad \gamma(\varphi) = \overline{\beta(\varphi)}, \\ &c(\varphi) := \Re(\beta(\varphi)), \quad b(\varphi) := \Im(\beta(\varphi)). \end{aligned} \tag{2.1}$$

$$\begin{aligned} C_1(\varphi) &:= \frac{\sin(\gamma)}{\sin(\varphi)} e^{\frac{\pi i}{3}}, \quad C_2(\varphi) := \frac{\sin(\beta)}{\sin(\varphi)} e^{\frac{2\pi i}{3}}, \\ B(\varphi) &:= \Re\left(\frac{\sin(\beta)}{\sin(\varphi)} e^{\frac{-\pi i}{3}}\right), \quad C(\varphi) := -\Im\left(\frac{\sin(\beta)}{\sin(\varphi)} e^{\frac{-\pi i}{3}}\right). \end{aligned} \tag{2.2}$$

Arccos is multivalued function, $\beta(\varphi)$ is one of its regular branches. The existence of this branch when $\varphi \in (0, \pi)$ will be shown in the Sect. 3.

Theorem 2.1 *Let $\lambda = g(\varphi)$. Then the equation* $\det T_n(a - g(\varphi)) = 0$ *is equivalent to the following set of equations:*

$$\varphi = \frac{2}{n+3}\left[\pi j + \arctan f(\varphi, n)\right], \tag{2.3}$$

$$j \in \left\{1, 2, \dots, \left[\frac{n+1}{2}\right]\right\}$$

and

$$\varphi = \frac{2}{n+3}\left[\pi j + \frac{\pi}{2} + \arctan h(\varphi, n)\right], \tag{2.4}$$

$$j \in \left\{1, 2, \dots, \left[\frac{n}{2}\right]\right\}.$$

where

$$\begin{aligned} f(\varphi, n) &= C_1(\varphi)\tan\left(\frac{n+3}{2}\gamma\right) - C_2(\varphi)\tan\left(\frac{n+3}{2}\beta\right), \\ h(\varphi, n) &= -C_1(\varphi)\frac{1}{\tan\left(\frac{n+3}{2}\gamma\right)} + C_2(\varphi)\frac{1}{\tan\left(\frac{n+3}{2}\beta\right)} \end{aligned} \tag{2.5}$$

Remark 2.1 Note that the functions $f(\varphi, n)$ and $h(\varphi, n)$ are real-valued and can be written as:

$$\begin{aligned} f(\varphi, n) &= 2\frac{B(\varphi)\sin((n+3)c(\varphi)) + C(\varphi)\sinh((n+3)b(\varphi))}{\cos((n+3)c(\varphi)) + \cosh((n+3)b(\varphi))}, \\ h(\varphi, n) &= 2\frac{-B(\varphi)\sin((n+3)c(\varphi)) + C(\varphi)\sinh((n+3)b(\varphi))}{-\cos((n+3)c(\varphi)) + \cosh((n+3)b(\varphi))}. \end{aligned}$$

Let us introduce the following notation:

$$F = F(\varphi, j, n) = \frac{2}{n+3}\left[\pi j + \arctan f(\varphi, n)\right], \tag{2.6}$$

$$H = H(\varphi, j, n) = \frac{2}{n+3}\left[\pi j + \frac{\pi}{2} + \arctan h(\varphi, n)\right]. \tag{2.7}$$

Theorem 2.2 *If n is sufficiently large then*

(i) *For any $j \in \{1, \dots, [\frac{n+1}{2}]\}$ the Eq. (2.3) has exactly one root φ_{2j-1} on the interval $(\frac{\pi(2j-1)}{n+3}, \frac{\pi(2j+1)}{n+3})$. Moreover, the solution can be found using the recursive formula $\varphi_{2j-1}^{(k+1)} = F(\varphi_{2j-1}^{(k)}, n)$, where $\varphi_{2j-1}^{(0)} = d_{1,j} := \frac{2\pi j}{n+3}$, and we can write the following estimate:*

$$\left|\varphi_{2j-1}^{(k)} - \varphi_{2j-1}\right| \le \frac{5\pi}{n+3}(0.62)^k, \tag{2.8}$$

If at the same time $j > \dfrac{\ln(2(n+3))}{\pi} + \dfrac{1}{2}$,

$$\left|\varphi_{2j-1}^{(k)} - \varphi_{2j-1}\right| = O\left(\frac{1}{(n+3)^{k+1}}\right), \tag{2.9}$$

where k is iteration number.

(ii) *For any* $j \in \{1, \dots, [\frac{n}{2}]\}$ *the Eq.* (2.4) *has exactly one root* φ_{2j} *on the interval* $(\frac{2\pi j}{n+3}, \frac{2\pi(j+1)}{n+3})$*. Moreover, the solution can be found using the recursive formula* $\varphi_{2j}^{(k+1)} = H(\varphi_{2j}^{(k)}, n)$*, where* $\varphi_{2j}^{(0)} = d_{2,j} := \frac{\pi(2j+1)}{n+3}$*, and we can write the following estimate:*

$$\left|\varphi_{2j}^{(k)} - \varphi_{2j}\right| \le \frac{5\pi}{n+3}(0.62)^k. \tag{2.10}$$

If at the same time $j > \dfrac{\ln(2(n+3))}{\pi}$

$$\left|\varphi_{2j}^{(k)} - \varphi_{2j}\right| = O\left(\frac{1}{(n+3)^{k+1}}\right), \tag{2.11}$$

where k is iteration number.

Let $q := \frac{n+3}{2}$. Recall that the parameter $d_{1,j} := d_{2j-1} = \frac{2\pi j}{n+3}$. Then φ can be represented as $\varphi = d_{1,j} + \frac{u}{q}$, and Eq. (2.3) can be rewritten as:

$$u = \arctan f\left(d_{1,j} + \frac{u}{q}, n\right), \tag{2.12}$$

where $u \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Recall that the parameter $d_{2,j} := d_{2j} = \frac{\pi(2j+1)}{n+3}$, so if $\varphi = d_{2,j} + \frac{w}{q}$, then Eq. (2.4) can be rewritten as:

$$w = \arctan h\left(d_{2,j} + \frac{w}{q}, n\right), \tag{2.13}$$

where $w \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Theorem 2.3 *Let* $a(t) = (t - 2 + \frac{1}{t})^3$ *and n is sufficiently large. If* $j > \dfrac{2\ln(n+3)}{\pi} + 1$ *then*

(i)

$$\varphi_{2j-1} = d_{1,j} + \frac{2u_{1,j}^*}{n+3} + \frac{4u_{2,j}^*}{(n+3)^2} + O\left(\frac{1}{n^3}\right), \tag{2.14}$$

where

$$u_{1,j}^{*}=\arctan\left(-i\left(C_{1}\left(d_{1,j}\right)+C_{2}\left(d_{1,j}\right)\right)\right) \tag{2.15}$$

and

$$u_{2,j}^{*}=-i\frac{C_{1}'(d_{1,j})+C_{2}'(d_{1,j})}{1+\left(-iC_{1}\left(d_{1,j}\right)-iC_{2}\left(d_{1,j}\right)\right)^{2}}u_{1,j}^{*}. \tag{2.16}$$

(ii)

$$\varphi_{2j}=d_{2,j}+\frac{2w_{1,j}^{*}}{n+3}+\frac{4w_{2,j}^{*}}{(n+3)^{2}}+O\left(\frac{1}{n^{3}}\right), \tag{2.17}$$

where

$$w_{1,j}^{*}=\arctan\left(-i\left(C_{1}\left(d_{2,j}\right)+C_{2}\left(d_{2,j}\right)\right)\right)$$

and

$$w_{2,j}^{*}=-i\frac{C_{1}'(d_{2,j})+C_{2}'(d_{2,j})}{1+\left(-iC_{1}\left(d_{2,j}\right)-iC_{2}\left(d_{2,j}\right)\right)^{2}}w_{1,j}^{*}.$$

Theorem 2.4 *Let* $j<\dfrac{2\ln(n+3)}{\pi}+1$. *Then*

$$\varphi_{2j-1}=\frac{2\pi j}{n+3}+O\left(\left(\frac{j}{(n+3)}\right)^{3}\right), \tag{2.18}$$

and

$$\varphi_{2j}=\frac{\pi(2j+1)}{n+3}+\frac{2w_{1}^{*}}{n+3}+O\left(\left(\frac{j}{(n+3)}\right)^{3}\right) \tag{2.19}$$

where $w_{1,j}^{*}$ *is the unique solution to the equation*

$$w_{1}=\arctan\left(\frac{2(-1)^{j+1}\cos\left(w_{1}\right)}{(-1)^{j}\sin\left(w_{1}\right)+\cosh\left(\left(qd_{2,j}+w_{1}\right)\sqrt{3}\right)}\right) \tag{2.20}$$

on the interval $(-\frac{\pi}{2},\frac{\pi}{2})$.

Theorem 2.5 *Let* $a(t)=(t-2+\frac{1}{t})^{3}$. *Then starting from some* n

(i) *If* $j > \dfrac{2\ln(n+3)}{\pi} + 1$ *then*

$$\lambda_{2j-1}^{(n)} = g(d_{1,j}) + g'(d_{1,j})\frac{2u_{1,j}^*}{n+3} + \frac{4u_{2,j}^* g'(d_{1,j}) + 2(u_{1,j}^*)^2 g''(d_{1,j})}{(n+3)^2} + O\left(\frac{1}{n^3}\right), \tag{2.21}$$

and

$$\lambda_{2j}^{(n)} = g(d_{2,j}) + g'(d_{2,j})\frac{2w_{1,j}^*}{n+3} + \frac{4w_{2,j}^* g'(d_{2,j}) + 2(w_{1,j}^*)^2 g''(d_{2,j})}{(n+3)^2} + O\left(\frac{1}{n^3}\right), \tag{2.22}$$

where $u_{1,j}^*$, $u_{2,j}^*$, $w_{1,j}^*$ *and* $w_{2,j}^*$ *are defined in the same way as in the Theorem* 2.3.

(ii) *If* $j < \dfrac{2\ln(n+3)}{\pi} + 1$ *then*

$$\lambda_{2j-1}^{(n)} = g(d_{1,j}) + O\left(\frac{j^3}{n^3}\right), \tag{2.23}$$

$$\lambda_{2j}^{(n)} = g(d_{2,j}) + g'(d_{2,j})\frac{2w_{1,j}^*}{n+3} + \frac{2(w_{1,j}^*)^2 g''(d_{2,j})}{(n+3)^2} + O\left(\frac{j^3}{n^3}\right), \tag{2.24}$$

where $w_{1,j}^*$ *is defined in the same way as in the Theorem* 2.4.

The following result gives us the asymptotic formulas for the extreme eigenvalues near zero.

Theorem 2.6 *Let* $g(\varphi) = a(e^{i\varphi}) = -\left(2\sin\frac{\varphi}{2}\right)^6$ *and* $j = o(n^{\frac{2}{3}})$ *as* $n \to \infty$.

(i) *If* $j > \dfrac{2\ln(n+3)}{\pi} + 1$, *then*

$$\lambda_{2j-1}^{(n)} = -d_{1,j}^6 + \frac{1}{4}d_{1,j}^8 - \frac{3\sqrt{3}}{2}\frac{d_{1,j}^7}{n+3} + \Delta_1(n,j), \tag{2.25}$$

$$\lambda_{2j}^{(n)} = -d_{2,j}^6 + \frac{1}{4}d_{2,j}^8 - \frac{3\sqrt{3}}{2}\frac{d_{2,j}^7}{n+3} + \Delta_2(n,j), \tag{2.26}$$

where $|\Delta_1(n,j)| \le M_1\left(\frac{d_{1,j}^5}{n^3} + d_{1,j}^{10}\right)$, $|\Delta_2(n,j)| \le M_1\left(\frac{d_{2,j}^5}{n^3} + d_{2,j}^{10}\right)$ *where the constant* M_1 *does not depend on* j *and* n.

(ii) *If* $j < \dfrac{2\ln(n+3)}{\pi} + 1$, *then*

$$\lambda_{2j-1}^{(n)} = -d_{1,j}^6 + O(d_{1,j}^8), \tag{2.27}$$

$$\lambda_{2j}^{(n)} = -\frac{((2j+1)\pi + 2w_{1,j}^*)^6}{(n+3)^6} + O(d_{2,j}^8), \tag{2.28}$$

where $w_{1,j}^*$ *is defined in the same way as in the Theorem* 2.4.

Similar formulas can be written for broader applicability conditions.

Remark 2.2 Let $g(\varphi) = a(e^{i\varphi}) = -\left(2\sin\frac{\varphi}{2}\right)^6$ and $\frac{j}{n} \to 0$ as $n \to \infty$.

(i) If $j > \dfrac{2\ln(n+3)}{\pi} + 1$, then

$$\lambda_{2j-1}^{(n)} = -d_{1,j}^6 + \frac{1}{4}d_{1,j}^8 + \Delta_3(n,j), \tag{2.29}$$

$$\lambda_{2j}^{(n)} = -d_{2,j}^6 + \frac{1}{4}d_{2,j}^8 + \Delta_4(n,j), \tag{2.30}$$

where $|\Delta_3(n,j)| \le M_2\left(\frac{d_{1,j}^7}{n} + d_{1,j}^{10}\right)$, $|\Delta_4(n,j)| \le M_2\left(\frac{d_{2,j}^7}{n} + d_{2,j}^{10}\right)$ where the constant M_2 does not depend in j and n.

(ii) If $j < \dfrac{2\ln(n+3)}{\pi} + 1$, then

$$\lambda_{2j-1}^{(n)} = -d_{1,j}^6 + O(d_{1,j}^8), \tag{2.31}$$

$$\lambda_{2j}^{(n)} = -\frac{((2j+1)\pi + 2w_{1,j}^*)^6}{(n+3)^6} + O(d_{2,j}^8), \tag{2.32}$$

where $w_{1,j}^*$ is defined in the same way as in the Theorem 2.4.

3 Chebyshev polynomial

To solve this problem, we need to solve the equation $\det T_n(a - g(\varphi)) = 0, \varphi \in (0,\pi)$. To find the determinant we will use the results obtained in the paper [12]. Let's define Chebyshev polynomials $\{Q_n\}$, $\{U_n\}$, $\{V_n\}$, $\{W_n\}$, which satisfy the same recurrent formula

$$Q_{n+1}(x) = 2xQ_n(x) - Q_{n-1}(x),\ n = 1, 2, \ldots$$

and the different initial conditions are:

$$Q_0(x) = U_0(x) = 1, \quad 2Q_1(x) = U_1(x) = 2x,$$
$$W_0(x) = V_0(x) = 1, \quad W_1(x) = V_1(x) + 2 = 2x + 1.$$

It is easy to check that these polynomials satisfy the following conditions

$$\begin{aligned} Q_n(\cos\theta) &= \cos n\theta, \quad U_n(\cos\theta) = \frac{\sin{(n+1)\theta}}{\sin\theta}, \\ V_n(\cos\theta) &= \frac{\cos{(n+\frac{1}{2})\theta}}{\cos\frac{\theta}{2}}, \quad W_n(\cos\theta) = \frac{\sin{(n+\frac{1}{2})\theta}}{\sin\frac{1}{2}\theta}. \end{aligned} \tag{3.1}$$

In [12], for the generating polynomial $a(t) = \sum_{k=-r}^{r} a_k t^k$, where $a_r \neq 0$, $a_k = a_{-k}$, the following theorem was proved.

Theorem 3.1 ([12, Theorem 1]) *Let ξ_j and $\frac{1}{\xi_j}$ be the (distinct) zeros of the polynomial $g_1(t) = t^r a(t)$. Then, for all $p \geq 1$ $\det T_{2p}$ equals*

$$\frac{a_r^{2p}}{2^{r(r-1)}} \times \frac{\begin{vmatrix} V_p(\alpha_1) & \dots & V_p(\alpha_r) \\ \vdots & \ddots & \vdots \\ V_{p+r-1}(\alpha_1) & \cdots & V_{p+r-1}(\alpha_r) \end{vmatrix}}{\prod_{1\leq i\leq j\leq r}(\alpha_j - \alpha_i)} \times \frac{\begin{vmatrix} W_p(\alpha_1) & \dots & W_p(\alpha_r) \\ \vdots & \ddots & \vdots \\ W_{p+r-1}(\alpha_1) & \cdots & W_{p+r-1}(\alpha_r) \end{vmatrix}}{\prod_{1\leq i\leq j\leq r}(\alpha_j - \alpha_i)}$$

and $\det T_{2p+1}$ *equals*

$$\frac{(-1)^r a_r^{2p+1}}{2^{r(r-2)}} \times \frac{\begin{vmatrix} U_p(\alpha_1) & \dots & U_p(\alpha_r) \\ \vdots & \ddots & \vdots \\ U_{p+r-1}(\alpha_1) & \cdots & U_{p+r-1}(\alpha_r) \end{vmatrix}}{\prod_{1\leq i\leq j\leq r}(\alpha_j - \alpha_i)} \times \frac{\begin{vmatrix} Q_{p+1}(\alpha_1) & \dots & Q_{p+1}(\alpha_r) \\ \vdots & \ddots & \vdots \\ Q_{p+r}(\alpha_1) & \cdots & Q_{p+r}(\alpha_r) \end{vmatrix}}{\prod_{1\leq i\leq j\leq r}(\alpha_j - \alpha_i)},$$

where $\alpha_k = \frac{1}{2}(\xi_k + \frac{1}{\xi_k})$ $(k = 1, \dots, r)$ are the zeros of the polynomial $h_1(x) = a_0 + 2\sum_{k=1}^{r} a_k Q_k(x)$.

In our case $g_1(t) = (t^2 - 2t + 1)^3 - \lambda t^3$, taking into account that $\lambda = g(\varphi) = (2\cos\varphi - 2)^3$ it is easy to get that:

$$\begin{aligned} \alpha_1 &= \cos\varphi, \\ \alpha_2 &= 1 + (\cos\varphi - 1)e^{\frac{2\pi i}{3}}, \\ \alpha_3 &= 1 + (\cos\varphi - 1)e^{\frac{-2\pi i}{3}}. \end{aligned}$$

Now we introduce auxiliary functions that are defined on the interval $\varphi \in [0, \pi]$:

$$B_c = B_c(\varphi) = \sqrt{\cos^2\varphi - 3\cos\varphi + 3} = |\alpha_2|, \tag{3.2}$$

$$\psi_c = \psi_c(\varphi) = \arctan \frac{\sqrt{3}(\cos\varphi - 1)}{3 - \cos\varphi} = \arg(\alpha_2), \tag{3.3}$$

$$B_s = B_s(\varphi) = |(1-\alpha_2^2)^{\frac{1}{2}}| = \sqrt[4]{(1-\cos\varphi)^2(7 - 4\cos\varphi + \cos^2\varphi)}, \tag{3.4}$$

$$\psi_s = \psi_s(\varphi) = \arg\left((1-\alpha_2^2)^{\frac{1}{2}}\right) = \frac{1}{2}\left(\pi + \arctan \frac{\sqrt{3}(3-\cos\varphi)}{(-1-\cos\varphi)}\right). \tag{3.5}$$

Obviously, $(1-\alpha_2^2)^{\frac{1}{2}}$ has two regular branches, and in formula (3.5) we have chosen one of them.

Lemma 3.1 *Let* $\varphi \in [0, \pi]$. *Then*

1. $B_s(\varphi)$ *is increasing function, and* $B_s(\varphi) \in [0, 2\sqrt[4]{3}]$.
2. $\psi_s(\varphi)$ *is decreasing function, and* $\psi_s(\varphi) \in [\frac{\pi}{4}, \frac{\pi}{3}]$.
3. $B_c(\varphi)$ *is increasing function, and* $B_c(\varphi) \in [1, \sqrt{7}]$.
4. $\psi_c(\varphi)$ *is decreasing function, and* $\psi_c(\varphi) \in [-\arctan \frac{\sqrt{3}}{2}, 0]$.
5. $B_c \cos(\psi_c) - B_s \sin(\psi_s) > 0$.
6. $B_c \cos(\psi_c) + B_s \sin(\psi_s) > 0$.

Proof To prove the first four points, we differentiate the corresponding functions, and decompose them into multipliers. The values at the edges of the interval are found by simple substitution:

$$B_s' = \frac{\sin(\varphi)(9 - 7\cos(\varphi) + 2\cos^2(\varphi))}{2(1-\cos(\varphi))^{\frac{1}{2}}(7 - 4\cos(\varphi) + \cos^2(\varphi))^{\frac{3}{4}}} > 0. \tag{3.6}$$

So, the function $B_s(\varphi)$ is increasing.

$$\psi_s' = -\frac{\sqrt{3}\sin(\varphi)}{2(7 - 4\cos(\varphi) + \cos^2(\varphi))} < 0. \tag{3.7}$$

So, the function $\psi_s(\varphi)$ is decreasing.

$$B_c' = \frac{\sin(\varphi)(3 - 2\cos(\varphi))}{2\sqrt{(3 - 3\cos(\varphi) + 3\cos^2(\varphi))}} > 0. \tag{3.8}$$

So, the function $B_c(\varphi)$ is increasing.

$$\psi_c' = -\frac{\sqrt{3}\sin(\varphi)}{2(3 - 3\cos(\varphi) + \cos^2(\varphi))} < 0. \tag{3.9}$$

So, the function $\psi_c(\varphi)$ is decreasing.
To prove points 5 and 6 we show that $B_c > B_s$ and $\cos(\psi_c) > \sin(\psi_s)$, from which the statement of this item will follow. Since $B_c > 0$ and $B_s > 0$, therefore $B_c > B_s$ is equivalent to $B_c^4 - B_s^4 > 0$.

$$B_c^4 - B_s^4 = 2 - \cos^2\varphi > 0.$$

Now we show that $\cos(\psi_c) > \sin(\psi_s)$. $\cos(\psi_c) = \sin(\frac{\pi}{2} + \psi_c)$, at the same time, from the points (2 and 4), it follows that $(\frac{\pi}{2} + \psi_c)$ and ψ_s, are in the first quadrant, so $\sin(\frac{\pi}{2} + \psi_c) > \sin(\psi_s)$ is equivalent to $\frac{\pi}{2} + \psi_c - \psi_s > 0$. It's not hard to get that

$$\psi_c' - \psi_s' = -\frac{\sqrt{3}\sin(\varphi)(4 - \cos(\varphi))}{2(7 - 4\cos(\varphi) + \cos^2(\varphi))(3 - 3\cos(\varphi) + \cos^2(\varphi))} < 0.$$

Which means $\frac{\pi}{2} + \psi_c - \psi_s > \frac{\pi}{2} + \psi_c(\pi) - \psi_s(\pi) = \frac{\pi}{2} - \arctan\frac{\sqrt{3}}{2} - \frac{\pi}{4} > 0$, so $\cos(\psi_c) > \sin(\psi_s)$. This means that the statement (5) is true. From the statements (1–4), it follows that both terms in the expression $B_c\cos(\psi_c) + B_s\sin(\psi_s)$ is positive, which means that the statement(6) is also true. □

Next, we show that there are such regular functions $\beta = \beta(\varphi)$ and $\gamma = \gamma(\varphi)$, $\varphi \in (0, \pi]$ which will satisfy the equations:

$$\begin{aligned} \cos\beta &= 1 + (\cos\varphi - 1)e^{\frac{2\pi i}{3}} = \alpha_2, \\ \cos\gamma &= 1 + (\cos\varphi - 1)e^{\frac{-2\pi i}{3}} = \alpha_3. \end{aligned} \tag{3.10}$$

To do this, it is enough to show that each of the multifunctions $E_2 = -i\,\mathrm{Log}(\alpha_2 + i(1 - \alpha_2^2)^{\frac{1}{2}})$ and $E_3 = -i\,\mathrm{Log}(\alpha_3 + i(1 - \alpha_3^2)^{\frac{1}{2}})$ has at least one regular branch for $\varphi \in (0, \pi]$. Given the notation (3.2)–(3.5), it is not difficult to make sure that

$$\alpha_3 = B_c e^{-i\psi_c} = \overline{\alpha_2}$$

and also

$$(1 - \alpha_3^2)^{\frac{1}{2}} = B_s e^{-i\psi_s}.$$

Note that in this case, we choose one of the two regular branches. With this in mind, it is sufficient to show that each of the functions $\tilde{E}_2 = -i\,\mathrm{Log}(\tilde{B}_c e^{i\tilde{\psi}_c} + i\tilde{B}_s e^{i\tilde{\psi}_s})$ and $\tilde{E}_3 = -i\,\mathrm{Log}(\tilde{B}_c e^{-i\tilde{\psi}_c} + i\tilde{B}_s e^{-i\tilde{\psi}_s})$, has at least one regular branch for $\varphi \in \Omega_\varepsilon$.

Lemma 3.2 *There is such a region $\Omega \supset (0, \pi]$ that the multifunctions*

$$\tilde{E}_2 = -i\,\mathrm{Log}(\tilde{B}_c e^{i\tilde{\psi}_c} + i\tilde{B}_s e^{i\tilde{\psi}_s})$$

and

$$\tilde{E}_3 = -i\,\mathrm{Log}(\tilde{B}_c e^{-i\tilde{\psi}_c} + i\tilde{B}_s e^{-i\tilde{\psi}_s})$$

have regular branches $\tilde{\beta}(\varphi)$ and $\tilde{\gamma}(\varphi)$ in this region.

Proof Since the functions $\tilde{B}_c, \tilde{\psi}_c, \tilde{B}_s, \tilde{\psi}_c$ are regular in $\hat{\Omega}$, the function $\tilde{B}_c e^{i\tilde{\psi}_c} + i\tilde{B}_s e^{i\tilde{\psi}_s}$ is also regular in $\hat{\Omega}$ and continuously extendable to a point $\varphi = 0$. Since the interval $[0, \pi]$ is compact and taking into account assertion 5 of the Lemma 3.1, it follows that there exists a simply connected region $\Omega_\beta((0,\pi] \subset \Omega_\beta \subset \hat{\Omega})$ such that for all $\varphi \in \Omega_\beta$ $\Re(\tilde{B}_c e^{i\tilde{\psi}_c} + i\tilde{B}_s e^{i\tilde{\psi}_s}) > 0$ and hence $\tilde{B}_c e^{i\tilde{\psi}_c} + i\tilde{B}_s e^{i\tilde{\psi}_s} \neq 0$, so the multifunction $\tilde{E}_2$ have a regular branch in Ω_β, moreover, these branches are continuously extended to the point $\varphi = 0$. The function Log has an infinite number of regular branches, but to choose one of them, it is enough to determine its value at one point, let's put $\tilde{\beta}(\varphi) = 0$. Similarly, the multifunction $\tilde{E}_3$ have a regular branch in some simply connected region $\Omega_\gamma((0,\pi] \subset \Omega_\gamma \subset \hat{\Omega})$, with the value $\tilde{\gamma}(0) = 0$. Let's put $\Omega = \Omega_\beta \cap \Omega_\gamma$, so $(0,\pi] \in \Omega$ and the lemma is proved. □

Remark 3.1 In Lemma 2.1 we not only proved the existence of regular branches for the functions $\tilde{E}_2$ and $\tilde{E}_3$ but also chose specific branches $\beta(\varphi)$ and $\gamma(\varphi)$ which satisfy the equalities (3.10), and are regular on the interval $(0,\pi]$ and continuous on the interval $[0,\pi]$ and

$$\begin{aligned}\cos(\beta) &= B_c e^{i\psi_c},\\ \sin(\beta) &= B_s e^{i\psi_s},\\ \cos(\gamma) &= B_c e^{-i\psi_c},\\ \sin(\gamma) &= B_s e^{-i\psi_s}.\end{aligned} \tag{3.11}$$

4 Proof of the main results

Lemma 4.1 *Let $\varphi \in [0,\pi]$ and φ small enough. Then*

1. $c = \frac{1}{2}\varphi - \frac{1}{16}\varphi^3 + O(\varphi^5)$.
2. $b = \frac{\sqrt{3}}{2}\varphi - \frac{\sqrt{3}}{48}\varphi^3 + O(\varphi^5)$.

Proof

$$(\cos\beta)' = -\beta'\sin\beta,$$

so

$$\beta' = -\frac{(\cos\beta)'}{\sin\beta},$$

from where it is not difficult to get that

$$\beta' = \frac{\sin\varphi}{\sin\beta}e^{\frac{2\pi i}{3}}. \tag{4.1}$$

Similarly, we find derivatives up to the fifth order. To reduce expressions, we will use the fact that $\sin^2 t = 1 - \cos^2 t$, as well as the formula (2.1). As a result, we get

$$\beta'' = \frac{\sqrt{3}(1-\cos(\varphi))^2}{\sin^3(\beta)}e^{\frac{\pi i}{6}}. \tag{4.2}$$

$$\beta''' = \frac{\sqrt{3}e^{\frac{5\pi i}{6}}(1-\cos(\varphi))^2\sin(\varphi)}{\sin^5(\beta)}\left[1+(1-\cos\varphi)e^{\frac{2\pi i}{3}}\right], \tag{4.3}$$

$$\beta^{(4)} = -\frac{\sqrt{3}i(1-\cos(\varphi))^4}{\sin^7(\beta)}\left[-1+(4+5\cos\varphi)e^{\frac{2\pi i}{3}}+(2-\cos\varphi-\cos^2\varphi)e^{\frac{-2\pi i}{3}}\right], \tag{4.4}$$

$$\beta^{(5)} = \frac{\sqrt{3}e^{\frac{\pi i}{6}}(1-\cos(\varphi))^4\sin(\varphi)}{\sin^9(\beta)}\left[4-9\cos(\varphi)+3\cos^2(\varphi)+\cos^3(\varphi)+(-9+18\cos\varphi)e^{\frac{2\pi i}{3}} +(21-6\cos\varphi-15\cos^2\varphi)e^{\frac{-2\pi i}{3}}\right]. \tag{4.5}$$

From where it is easy to get

$$\beta'_+(0) = \lim_{\varphi\to 0+0}\beta' = e^{\frac{\pi i}{3}}, \tag{4.6}$$

$$\beta''_+(0) = \lim_{\varphi\to 0+0}\beta'' = 0. \tag{4.7}$$

$$\beta'''_+(0) = -\frac{3}{8}-\frac{\sqrt{3}}{8}i, \tag{4.8}$$

$$\beta^{(4)}_+(0) = 0, \tag{4.9}$$

$$\lim_{\varphi\to 0+0}\beta^{(5)} = \frac{15}{16}-\frac{\sqrt{3}}{4}i. \tag{4.10}$$

Since $\beta^{(5)}$ is continuous on the interval $(0,\pi]$ and its limit is finite as $\varphi\to 0$, it is bounded on $(0,\pi]$. Taking into account (4.6), (4.7), (4.8) and (4.9) if $\varphi\in(0,\pi]$:

$$\beta(\varphi) = \varphi\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)-\varphi^3\left(\frac{3}{48}+\frac{\sqrt{3}i}{48}\right)+O(\varphi^5). \tag{4.11}$$

From where we get the statement of the Lemma. □

Proof of Theorem 2.1 If $n=2p$ then by Theorem 3.1 we have

$$det(T_{2p}(a-g(\varphi))) = \frac{1}{2^6}\times\frac{\begin{vmatrix} V_p(\cos\varphi) & V_p(\cos\beta) & V_p(\cos\gamma)\\ V_{p+1}(\cos\varphi) & V_{p+1}(\cos\beta) & V_{p+1}(\cos\gamma)\\ V_{p+2}(\cos\varphi) & V_{p+2}(\cos\beta) & V_{p+2}(\cos\gamma)\end{vmatrix}}{(\cos\gamma-\cos\beta)(\cos\gamma-\cos\varphi)(\cos\beta-\cos\varphi)}$$
$$\times\frac{\begin{vmatrix} W_p(\cos\varphi) & W_p(\cos\beta) & W_p(\cos\gamma)\\ W_{p+1}(\cos\varphi) & W_{p+1}(\cos\beta) & W_{p+1}(\cos\gamma)\\ W_{p+2}(\cos\varphi) & W_{p+2}(\cos\beta) & W_{p+2}(\cos\gamma)\end{vmatrix}}{(\cos\gamma-\cos\beta)(\cos\gamma-\cos\varphi)(\cos\beta-\cos\varphi)}.$$

(4.12)

It is easy to check that for $\varphi \in (0, \pi)$ $\cos\varphi$, $\cos\gamma$, $\cos\beta$ are pairwise distinct, which means that the equation $\det(T_{2p}(a - g(\varphi))) = 0$ is equivalent to the equation:

$$\begin{vmatrix} V_p(\cos\varphi) & V_p(\cos\beta) & V_p(\cos\gamma) \\ V_{p+1}(\cos\varphi) & V_{p+1}(\cos\beta) & V_{p+1}(\cos\gamma) \\ V_{p+2}(\cos\varphi) & V_{p+2}(\cos\beta) & V_{p+2}(\cos\gamma) \end{vmatrix} \times \begin{vmatrix} W_p(\cos\varphi) & W_p(\cos\beta) & W_p(\cos\gamma) \\ W_{p+1}(\cos\varphi) & W_{p+1}(\cos\beta) & W_{p+1}(\cos\gamma) \\ W_{p+2}(\cos\varphi) & W_{p+2}(\cos\beta) & W_{p+2}(\cos\gamma) \end{vmatrix} = 0. \tag{4.13}$$

Taking into account the properties (3.1), the Eq. (4.13) will take the form:

$$\begin{vmatrix} \frac{\cos((p+\frac{1}{2})\varphi)}{\cos\frac{\varphi}{2}} & \frac{\cos((p+\frac{1}{2})\beta)}{\cos\frac{\beta}{2}} & \frac{\cos((p+\frac{1}{2})\gamma)}{\cos\frac{\gamma}{2}} \\ \frac{\cos((p+\frac{3}{2})\varphi)}{\cos\frac{\varphi}{2}} & \frac{\cos((p+\frac{3}{2})\beta)}{\cos\frac{\beta}{2}} & \frac{\cos((p+\frac{3}{2})\gamma)}{\cos\frac{\gamma}{2}} \\ \frac{\cos((p+\frac{5}{2})\varphi)}{\cos\frac{\varphi}{2}} & \frac{\cos((p+\frac{5}{2})\beta)}{\cos\frac{\beta}{2}} & \frac{\cos((p+\frac{5}{2})\gamma)}{\cos\frac{\gamma}{2}} \end{vmatrix} \times \begin{vmatrix} \frac{\sin((p+\frac{1}{2})\varphi)}{\sin\frac{\varphi}{2}} & \frac{\sin((p+\frac{1}{2})\beta)}{\sin\frac{\beta}{2}} & \frac{\sin((p+\frac{1}{2})\gamma)}{\sin\frac{\gamma}{2}} \\ \frac{\sin((p+\frac{3}{2})\varphi)}{\sin\frac{\varphi}{2}} & \frac{\sin((p+\frac{3}{2})\beta)}{\sin\frac{\beta}{2}} & \frac{\sin((p+\frac{3}{2})\gamma)}{\sin\frac{\gamma}{2}} \\ \frac{\sin((p+\frac{5}{2})\varphi)}{\sin\frac{\varphi}{2}} & \frac{\sin((p+\frac{5}{2})\beta)}{\sin\frac{\beta}{2}} & \frac{\sin((p+\frac{5}{2})\gamma)}{\sin\frac{\gamma}{2}} \end{vmatrix} = 0. \tag{4.14}$$

It is not difficult to check that $\sin(\varphi) \neq 0$, $\sin(\beta) \neq 0$ and $\sin(\gamma) \neq 0$ if $\varphi \in (0, \pi)$. Then, since $n = 2p$, the set of solutions to the Eq. (4.14) coincides with the union of the sets of solutions to the equations:

$$\begin{vmatrix} \cos(\frac{n+1}{2}\varphi) & \cos(\frac{n+1}{2}\beta) & \cos(\frac{n+1}{2}\gamma) \\ \cos(\frac{n+3}{2}\varphi) & \cos(\frac{n+3}{2}\beta) & \cos(\frac{n+3}{2}\gamma) \\ \cos(\frac{n+5}{2}\varphi) & \cos(\frac{n+5}{2}\beta) & \cos(\frac{n+5}{2}\gamma) \end{vmatrix} = 0 \tag{4.15}$$

and

$$\begin{vmatrix} \sin(\frac{n+1}{2}\varphi) & \sin(\frac{n+1}{2}\beta) & \sin(\frac{n+1}{2}\gamma) \\ \sin(\frac{n+3}{2}\varphi) & \sin(\frac{n+3}{2}\beta) & \sin(\frac{n+3}{2}\gamma) \\ \sin(\frac{n+5}{2}\varphi) & \sin(\frac{n+5}{2}\beta) & \sin(\frac{n+5}{2}\gamma) \end{vmatrix} = 0. \tag{4.16}$$

If $n = 2p + 1$, similar reasoning will lead to the same Eqs. (4.15) and (4.16). In the formula (4.15), instead of the first and third lines, we write down their half sum and half difference, then we get:

$$\begin{vmatrix} \cos(\frac{n+1}{2}\varphi) & \cos(\frac{n+1}{2}\beta) & \cos(\frac{n+1}{2}\gamma) \\ \cos(\frac{n+3}{2}\varphi) & \cos(\frac{n+3}{2}\beta) & \cos(\frac{n+3}{2}\gamma) \\ \cos(\frac{n+5}{2}\varphi) & \cos(\frac{n+5}{2}\beta) & \cos(\frac{n+5}{2}\gamma) \end{vmatrix} = 0 \Leftrightarrow$$
$$\begin{vmatrix} \cos(\frac{n+3}{2}\varphi)\cos\varphi & \cos(\frac{n+3}{2}\beta)\cos\beta & \cos(\frac{n+3}{2}\gamma)\cos\gamma \\ \cos(\frac{n+3}{2}\varphi) & \cos(\frac{n+3}{2}\beta) & \cos(\frac{n+3}{2}\gamma) \\ \sin(\frac{n+3}{2}\varphi)\sin\varphi & \sin(\frac{n+3}{2}\beta)\sin\beta & \cos(\frac{n+3}{2}\gamma)\sin\gamma \end{vmatrix} = 0 \Leftrightarrow$$
$$\begin{vmatrix} 0 & \cos(\frac{n+3}{2}\beta)[\cos\beta - \cos\varphi] & \cos(\frac{n+3}{2}\gamma)[\cos\gamma - \cos\varphi] \\ \cos(\frac{n+3}{2}\varphi) & \cos(\frac{n+3}{2}\beta) & \cos(\frac{n+3}{2}\gamma) \\ \sin(\frac{n+3}{2}\varphi)\sin\varphi & \sin(\frac{n+3}{2}\beta)\sin\beta & \sin(\frac{n+3}{2}\gamma)\sin\gamma \end{vmatrix} = 0 \tag{4.17}$$

Let us expand the determinant over the first column and denote for brevity $q = \frac{n+3}{2}$

$$\begin{aligned} &\sin(q\varphi)\sin\varphi\cos(q\beta)\cos q\gamma)[\cos\beta - \cos\varphi - \cos\gamma + \cos\varphi] = \\ &\cos(q\varphi)[\cos(q\beta)\sin(q\gamma)\sin\gamma[\cos\beta - \cos\varphi] \\ &-\cos(q\gamma)\sin(q\beta)\sin\beta[\cos\gamma - \cos\varphi]] \end{aligned}$$

From where we get

$$\tan(q\varphi) = \frac{\cos(q\beta)\sin(q\gamma)\sin\gamma\frac{[\cos\beta - \cos\varphi]}{[\cos\beta - \cos(\gamma)]} - \cos(q\gamma)\sin(q\beta)\sin\beta\frac{[\cos\gamma - \cos\varphi]}{[\cos\beta - \cos(\gamma)]}}{\sin\varphi\cos(q\beta)\cos(q\gamma)} \tag{4.18}$$

It is not difficult to make sure that

$$\begin{aligned} \cos(\beta) - \cos(\gamma) &= (\cos(\varphi) - 1)\sqrt{3}i, \\ \cos(\varphi) - \cos(\beta) &= (\cos(\varphi) - 1)\left(\frac{3}{2} - \frac{\sqrt{3}i}{2}\right), \\ \cos(\varphi) - \cos(\gamma) &= (\cos(\varphi) - 1)\left(\frac{3}{2} + \frac{\sqrt{3}i}{2}\right). \end{aligned}$$

So:

$$
\begin{aligned}
\frac{\cos(\beta)-\cos(\varphi)}{\cos(\beta)-\cos(\gamma)} &= \frac{1}{2}+\frac{\sqrt{3}i}{2} = e^{\frac{\pi i}{3}},\\
\frac{\cos(\gamma)-\cos(\varphi)}{\cos(\beta)-\cos(\gamma)} &= -\frac{1}{2}+\frac{\sqrt{3}i}{2} = e^{\frac{2\pi i}{3}}.
\end{aligned} \tag{4.19}
$$

Then

$$
\tan q\varphi = C_1(\varphi)\tan(q\gamma) - C_2(\varphi)\tan(q\beta) \tag{4.20}
$$

where $C_1(\varphi) = \dfrac{\sin\gamma}{\sin\varphi}e^{\frac{\pi i}{3}}$, $C_2(\varphi) = \dfrac{\sin\beta}{\sin\varphi}e^{\frac{2\pi i}{3}}$.

Similarly, the second set of formulas is obtained (2.5) □

Lemma 4.2 *Let $\varphi \in [0, \pi]$. Then*

1. *$c(\varphi)$ is increasing function.*
2. *$b(\varphi)$ is increasing function.*
3. *$c'(\varphi)$ is decreasing function, and $c'(0) = \frac{1}{2}$, $c'(\pi) = 0$.*
4. *$b'(\varphi)$ is decreasing function, and $b'(0) = \frac{\sqrt{3}}{2}$, $b'(\pi) = 0$.*
5. *$\frac{c(\varphi)}{\varphi}$ is decreasing function.*
6. *$\frac{b(\varphi)}{\varphi}$ is decreasing function, $\frac{b(\varphi)}{\varphi} > 0.5$.*
7. *$\frac{B_c}{B_s}$ is decreasing function.*
8. *$\frac{B_s}{\sin(\varphi)}$ is increasing function.*

Proof It follows from the Eq. (4.1) that

$$
\begin{aligned}
c'(\varphi) &= \Re(\beta') = \frac{\sin\varphi}{B_s}\cos\left(\frac{2\pi}{3}-\psi_s\right),\\
b'(\varphi) &= \Im(\beta') = \frac{\sin\varphi}{B_s}\sin\left(\frac{2\pi}{3}-\psi_s\right).
\end{aligned}
$$

Since $\psi_s(\varphi) \in (\frac{\pi}{4}, \frac{\pi}{3})$, then $c'(\varphi) > 0$, $b'(\varphi) > 0$ so the functions $a(\varphi)$ and $b(\varphi)$ are increasing. From the formula (4.2) and also taking into account the fact that $\psi_s \in (\frac{\pi}{4}, \frac{\pi}{3})$ we will get

$$
\begin{aligned}
c'' &= \frac{\sqrt{3}(1-\cos(\varphi))^2}{B_s^3}\cos\left(\frac{\pi}{6}-3\psi_s\right) < 0,\\
b'' &= \frac{\sqrt{3}(1-\cos(\varphi))^2}{B_s^3}\sin\left(\frac{\pi}{6}-3\psi_s\right) < 0,
\end{aligned}
$$

so the functions $c'(\varphi)$ and $b'(\varphi)$ are decreasing. From the formula (4.6) it follows that $c'(0) = \frac{1}{2}$ and $b'(0) = \frac{\sqrt{3}}{2}$. Well $c'(\pi)$ and $b'(\pi)$ can be found by a simple substitution.

Find the derivative of the function $\frac{b(\varphi)}{\varphi}$ and show that it is negative.

$$\left(\frac{b(\varphi)}{\varphi}\right)' = \frac{b'(\varphi)\varphi - b(\varphi)}{\varphi^2}.$$

In order to prove that this derivative is negative, it is sufficient to show that $b'(\varphi)\varphi - b(\varphi) < 0$.

$$(b'(\varphi)\varphi - b(\varphi))' = b''(\varphi)\varphi < 0.$$

We obtain that $b'(\varphi)\varphi - b(\varphi) \leq b(0) = 0$, this means that $\frac{b(\varphi)}{\varphi}$ is decreasing function and $\frac{b(\varphi)}{\varphi} > \frac{b(\pi)}{\pi}$. From Lemma 3.2 and equality (3.11) we will get that $\beta(\varphi) = -i\ln(B_c e^{i\psi_c} + iB_s e^{i\psi_s})$. Taking into account the Lemma 3.1 we will get:

$$\begin{aligned} \beta(\pi) &= -i\ln\left(2 - \sqrt{3}i + i(\sqrt[4]{12} + \sqrt[4]{12}i)\right) = -i\ln\left((\sqrt[4]{4} - \sqrt[4]{3})(\sqrt[4]{4} + \sqrt[4]{3}i)\right) \\ &= -i\ln e^{\ln\left((\sqrt[4]{4} - \sqrt[4]{3})\sqrt{\sqrt{4} + \sqrt{3}}\right) + i\arctan\sqrt[4]{\frac{3}{4}}} \\ &= \arctan\sqrt[4]{\frac{3}{4}} - i\ln\left((\sqrt[4]{4} - \sqrt[4]{3})\sqrt{\sqrt{4} + \sqrt{3}}\right). \end{aligned} \tag{4.21}$$

Since $b(\varphi) = \Im\beta(\varphi)$, a simple check shows that

$$\frac{b(\pi)}{\pi} = \frac{-\ln\left((\sqrt[4]{4} - \sqrt[4]{3})\sqrt{\sqrt{4} + \sqrt{3}}\right)}{\pi} > 0.5.$$

So the statement (6) is true. The statement (5) is proved similarly.
Since $\frac{B_c}{B_s} > 0$, decreasing $\frac{B_c}{B_s}$ is equivalent to decreasing $\frac{B_c^4}{B_s^4}$. By taking the derivative of the function $\frac{B_c^4}{B_s^4}$ and expand into factors we get that:

$$\left(\frac{B_c^4}{B_s^4}\right)' = -\frac{2\sin(\varphi)(3 - 3\cos(\varphi) + \cos^2(\varphi))(2 - \cos(\varphi))(3 + \cos(\varphi))}{(1 - \cos(\varphi))^3(7 - 4\cos(\varphi) + \cos^2(\varphi))^2} < 0.$$

This means that the function $\frac{B_c}{B_s}$ decreases.
Since $\frac{B_s}{\sin(\varphi)} > 0$, increasing $\frac{B_s}{\sin(\varphi)}$ is equivalent to decreasing $\frac{B_s^4}{\sin^4(\varphi)}$. By taking the derivative of the function $\frac{B_s^4}{\sin^4(\varphi)}$ and expand into factors we get that:

$$\left(\frac{B_s^4}{\sin^4(\varphi)}\right)' = \frac{6(3 - \cos(\varphi))(1 - \cos(\varphi))^3}{\sin^5(\varphi)} > 0.$$

This means that the function $\frac{B_s}{\sin(\varphi)}$ increases. □

Lemma 4.3 *For a sufficiently large n*

1. *If* $\varphi \in \left[\frac{\pi}{n+3}, \pi\right)$ *then*

$$|F'(\varphi, n)| < 0.62,$$

also

$$|F'(\varphi, n)| = O\left(e^{-\frac{(n+3)\varphi}{2}}\right) + O\left(\frac{1}{n+3}\right),$$

and in particular if $\varphi > \frac{2\ln(2(n+3))}{n+3}$ *then*

$$|F'(\varphi, n)| = O\left(\frac{1}{n+3}\right).$$

2. *If* $\varphi \in \left[\frac{2\pi}{n+3}, \pi\right)$ *then*

$$|H'(\varphi, n)| < 0.62,$$

also

$$|H'(\varphi, n)| = O\left(e^{-\frac{(n+3)\varphi}{2}}\right) + O\left(\frac{1}{n+3}\right),$$

and in particular if $\varphi > \frac{2\ln(2(n+3))}{n+3}$ *then*

$$|H'(\varphi, n)| = O\left(\frac{1}{n+3}\right).$$

Proof Let's put for brevity the entries $q = \frac{n+3}{2}$, then

$$F(\varphi, n, j) = F(\varphi, n) := \frac{1}{q}\left[\pi j + \arctan\left(f(\varphi, n)\right)\right],$$

where

$$f(\varphi, n) = C_1(\varphi)\tan(q\gamma) - C_2(\varphi)\tan(q\beta),$$

so

$$F'(\varphi, n) = \frac{1}{q}\frac{f'(\varphi, n)}{1 + f^2(\varphi, n)}.$$

$$\frac{1}{q}f'(\varphi, n) = \frac{1}{q}(f_1 + f_2), \tag{4.22}$$

where

$$f_1 = \frac{q\gamma' C_1}{\cos^2 q\gamma} - \frac{q\beta' C_2}{\cos^2 q\beta}, \quad f_2 = C_1'(\varphi)\tan(q\gamma) - C_2'(\varphi)\tan(q\beta)$$

Consider $\frac{f_1}{q}$. Taking into account the formulas (4.1), (2.2) and taking into account that $\gamma' = \overline{\beta'}$ we get that

$$C_1\gamma' = e^{\frac{-\pi i}{3}}, \quad C_2\beta' = e^{\frac{4\pi i}{3}}. \tag{4.23}$$

Then

$$\frac{f_1}{q} = \frac{1}{\cos^2 q\gamma}e^{\frac{-\pi i}{3}} - \frac{1}{\cos^2 q\beta}e^{\frac{4\pi i}{3}}.$$

Let's estimate $\frac{n+3}{2}b(\varphi)$. From Lemmas 4.2, 4.1 and condition $\varphi \in [\frac{\pi}{n+3}, \pi)$ it follows, that for sufficiently large n

$$\frac{n+3}{2}b(\varphi) \geq \frac{n+3}{2}b\left(\frac{\pi}{n+3}\right) = \frac{\sqrt{3}\pi}{4} + o\left(\frac{1}{n+3}\right) < 1.36. \tag{4.24}$$

$$\begin{aligned}\frac{f_1}{q} &\leq \left|\frac{2}{\cos^2 q\gamma}\right| = \left|\frac{8}{\left(e^{qb+iqa} + e^{-qb-iqa}\right)^2}\right| \\ &= \left|\frac{8}{e^{2qb+2iqa}}\right|\left|\frac{1}{(1+e^{-2qb-2iqa})^2}\right| \leq \left|\frac{8}{e^{2qb}}\right|\left|\frac{1}{(1-e^{-2qb})^2}\right| \\ &= O(e^{-2qb})\end{aligned} \tag{4.25}$$

In particular, due to the decreasing function in the formula (4.25), as well as taking into account (4.24), an estimate can be made

$$\left|\frac{f_1}{q}\right| < 0.61. \tag{4.26}$$

Consider the expression $\frac{f_2(\varphi, n)}{1 + f^2(\varphi, n)}$ and show its limitation. Let's first estimate $|\tan(q\gamma)| = |\tan(q\beta)|$, as well as $\Psi = \arg(\tan(q\beta)) = -\arg(\tan(q\gamma))$.

$$\tan(q\beta) = \frac{\sin(q\beta)}{\cos(q\beta)} = \frac{\sin(q\beta)\cos(q\gamma)}{\cos(q\beta)\cos(q\gamma)}$$

$$
\begin{aligned}
&= \frac{\sin(q(\beta-\gamma))+\sin(q(\beta+\gamma))}{\cos(q(\beta-\gamma))+\cos(q(\beta+\gamma))} = \frac{\sin(2qbi)+\sin(2qc)}{\cos(2qbi)+\cos(2qc)} \\
&= \frac{\sin(2qc)+i\sinh(2qb)}{\cosh(2qb)+\cos(2qc)}
\end{aligned} \tag{4.27}
$$

From where, taking into account (4.24), we get:

$$
|\tan(q\beta)| = \frac{\sqrt{\sinh^2(2qb)+\sin^2(2qc)}}{\cosh(2qb)+\cos(2qc)} < 1.2, \quad |\tan(q\beta)| > 0.8 \tag{4.28}
$$

$$
|\tan(\Psi)| = \frac{\sinh(2qb)}{|\sin(2qc)|} > 7.5. \tag{4.29}
$$

Given that $\sinh(2qb) > 0$, we get that

$$
\left|\frac{\pi}{2} - \Psi\right| < \frac{\pi}{10}. \tag{4.30}
$$

From the formula (4.23) and the formula (4.2), it is not difficult to get

$$
C_2' = \left(\frac{1}{\beta'} e^{\frac{4\pi i}{3}}\right)' = -\frac{\beta''}{(\beta')^2} e^{\frac{4\pi i}{3}} = \frac{\sqrt{3}(1-\cos(\varphi))}{(1+\cos(\varphi))\sin(\beta)} e^{\frac{7\pi i}{6}}. \tag{4.31}
$$

Similarly

$$
C_1' = \left(\frac{1}{\gamma'} e^{\frac{-\pi i}{3}}\right)' = -\frac{\gamma''}{(\gamma')^2} e^{\frac{-\pi i}{3}} = \frac{\sqrt{3}(1-\cos(\varphi))}{(1+\cos(\varphi))\sin(\gamma)} e^{\frac{-\pi i}{6}}. \tag{4.32}
$$

Taking into account the evaluation of (4.28) we get:

$$
\begin{aligned}
|f_2| &= \frac{\sqrt{3}(1-\cos(\varphi))}{(1+\cos(\varphi))B_s} |\tan(q\gamma)| e^{i(\frac{7\pi}{6}+\psi_s-\Psi)} \\
&\quad - \frac{\sqrt{3}(1-\cos(\varphi))}{(1+\cos(\varphi))B_s} |\tan(q\beta)| e^{i(\frac{-\pi}{6}-\psi_s+\Psi)} \\
&= \frac{\sqrt{3}(1-\cos(\varphi))}{(1+\cos(\varphi))B_s} |\tan(q\gamma)| \left(e^{i(\frac{7\pi}{6}+\psi_s-\Psi)} - e^{i(\frac{-\pi}{6}-\psi_s+\Psi)}\right) \\
&\le 2.4 \cdot \frac{\sqrt{3}(1-\cos(\varphi))}{(1+\cos(\varphi))B_s}
\end{aligned} \tag{4.33}
$$

$$
\begin{aligned}
1+f^2 &= 1 + \left(\frac{\sin(\gamma)}{\sin(\varphi)} e^{\frac{\pi i}{3}} \tan(q\gamma) - \frac{\sin(\beta)}{\sin(\varphi)} e^{\frac{2\pi i}{3}} \tan(q\beta)\right)^2 \\
&= 1 + \frac{B_s^2}{\sin^2(\varphi)} |\tan(q\beta)|^2 (e^{i(\frac{\pi}{3}-\psi_s-\Psi)} - e^{i(\frac{2\pi}{3}+\psi_s+\Psi)})^2 \\
&= 1 + \frac{B_s^2}{\sin^2(\varphi)} |\tan(q\beta)|^2 4\sin^2\left(\frac{\pi}{6}+\psi_s+\Psi\right).
\end{aligned} \tag{4.34}
$$

Consider 2 cases. If $\varphi \in [\frac{\pi}{n+3}, \frac{\pi}{2}]$ then

$$\left|\frac{f_2}{1+f^2}\right| \leq 2.4 \cdot \frac{\sqrt{3}(1-\cos(\varphi))}{(1+\cos(\varphi))B_s}. \tag{4.35}$$

And it is limited, since the right side of the expression (4.35) tends to 0 when $\varphi \to 0$, and at other points the denominator does not turn to 0.
If $\varphi \in (\frac{\pi}{2}, \pi]$,

$$\left|\frac{f_2}{1+f^2}\right| \leq 2.4 \cdot \frac{\sqrt{3}(1-\cos(\varphi))}{(1+\cos(\varphi))B_s} \frac{\sin^2(\varphi)}{B_s^2} \frac{1}{|\tan(q\beta)|^2 4\sin^2\left(\frac{\pi}{6}+\psi_s+\Psi\right)} \tag{4.36}$$

Since on the interval $[\frac{\pi}{n+3}, \frac{\pi}{2}]$ B_s^2 is delimited from 0, and from the estimate (4.28) and also taking into account point 2 of the Lemma 3.1 we obtain that

$$\begin{aligned}\left|\frac{f_2}{1+f^2}\right| &\leq V_0 \frac{\sin^2(\varphi)}{(1+\cos(\varphi))} \frac{1}{\sin^2\left(\frac{\pi}{6}+\psi_s+\Psi\right)} \\ &\to V_0 \frac{1}{2\sin^2\left(\frac{\pi}{6}+\frac{\pi}{4}+\Psi_0\right)}, \quad \text{when } \varphi \to \pi.\end{aligned} \tag{4.37}$$

Then, from the estimate (4.30), we obtain the boundedness of this expression on the interval $(\frac{\pi}{2}, \pi]$. Thus

$$\frac{1}{q}\frac{f_2}{1+f^2} = O\left(\frac{1}{q}\right) \tag{4.38}$$

Considering now (4.25), (4.26) and the fact that for sufficiently large n $O\left(\frac{1}{q}\right) < 0.01$ we obtain the statement of the first part of the theorem. For the function $H'(\varphi, n)$ the proof is similar. □

Proof of the Theorem 2.2 To solve the Eqs. (2.3) and (2.4), we apply Fix Point Method. Put

$$\varphi_{2j-1}^{(0)} = d_{2j-1}, \quad \varphi_{2j-1}^{(k+1)} = F(\varphi_{2j-1}^{(k)}, n) \tag{4.39}$$

and

$$\varphi_{2j}^{(0)} = d_{2j}, \quad \varphi_{2j}^{(k+1)} = H(\varphi_{2j}^{(k)}, n), \tag{4.40}$$

where

$$d_l = \frac{\pi(l+1)}{n+3}, \quad l = 1, 2, \ldots, n.$$

Since for real x $\arctan(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ then according to the formulas (2.6), (2.7) $\forall k \in \mathbb{N}$ $F(\varphi_{2j-1}^{(k)}, n) \in (\frac{\pi(2j-1)}{n+3}, \frac{\pi(2j+1)}{n+3})$ and $H(\varphi_{2j}^{(k)}, n) \in (\frac{2\pi j}{n+3}, \frac{2\pi(j+1)}{n+3})$, and therefore for each fixed j, the mapping defined by the formula (4.39) maps the interval $(\frac{\pi(2j-1)}{n+3}, \frac{\pi(2j+1)}{n+3})$ to the same interval, and for each fixed j the mapping defined by the formula (4.40) maps the interval $(\frac{2\pi j}{n+3}, \frac{2\pi(j+1)}{n+3})$ to the same interval. To get an estimate in the k-iteration, we will use the mean value theorem.

$$\left|\varphi_{2j-1}^{(k+1)} - \varphi_{2j-1}^{(k)}\right| = \left|F(\varphi_{2j-1}^{(k)}, n) - F(\varphi_{2j-1}^{(k-1)}, n)\right| = |F'(\theta_{2j-1}^{(k)}, n)| \left|\varphi_{2j-1}^{(k)} - \varphi_{2j-1}^{(k-1)}\right|,$$

where $\theta_{2j-1}^{(k)}$ some number lying between $\varphi_{2j-1}^{(k)}$ and $\varphi_{2j-1}^{(k-1)}$. If we denote $L_{2j-1} := \max|F'(\varphi, n)|, \varphi \in (\frac{\pi(2j-1)}{n+3}, \frac{\pi(2j+1)}{n+3})$ then

$$\left|\varphi_{2j-1}^{(k+1)} - \varphi_{2j-1}^{(k)}\right| \le L_{2j-1}\left|\varphi_{2j-1}^{(k)} - \varphi_{2j-1}^{(k-1)}\right| \le L_{2j-1}^{k}\left|\varphi_{2j-1}^{(1)} - \varphi_{2j-1}^{(0)}\right| \le L_{2j-1}^{k}\frac{\pi}{n+3}.$$

If $L_{2j-1} < 1$, then from Fixed Point Theory it follows that exists a point φ_{2j-1} such that $\varphi_{2j-1} = F(\varphi_{2j-1}, n)$, moreover:

$$\left|\varphi_{2j-1}^{(k)} - \varphi_{2j-1}\right| \le \sum_{l=k}^{+\infty}\left|\varphi_{2j-1}^{(l)} - \varphi_{2j-1}^{(l+1)}\right| \le \sum_{l=k}^{+\infty} L_{2j-1}^{l}\frac{\pi}{n+3} = \frac{L_{2j-1}^{k}}{1-L_{2j-1}}\frac{\pi}{n+3}. \tag{4.41}$$

Similar reasoning for $\varphi_{2j}^{(k)}$. If $L_{2j} := \max|H'(\varphi, n)| < 1, \varphi \in (\frac{2\pi j}{n+3}, \frac{2\pi(j+1)}{n+3})$, then there exists a point φ_{2j} such that $\varphi_{2j} = H(\varphi_{2j}, n)$, moreover:

$$\left|\varphi_{2j}^{(k)} - \varphi_{2j}\right| \le \frac{L_{2j}^{k}}{1-L_{2j}}\frac{\pi}{n+3}. \tag{4.42}$$

Introduce a new function:

$$F_1(\varphi, n, j) = F_1(\varphi, n) = \varphi - F(\varphi, n).$$

Then for any fixed j Eq. (2.3) can be rewritten as

$$F_1(\varphi, n) = 0. \tag{4.43}$$

From the Lemma 4.3 it follows that for sufficiently large n $F_1'(\varphi, n) = 1 - F'(\varphi, n) > 0$. Therefore, the function $F_1'(\varphi, n)$ is increasing. Furthermore

$$\begin{aligned} F\left(\frac{\pi(2j-1)}{n+3}, n\right) &= \frac{\pi(2j-1)}{n+3} - \frac{2}{n+3}\left[\pi j - \arctan\left(f\left(\frac{\pi(2j-1)}{n+3}\right)\right)\right] \\ &= \frac{2}{n+3}\left[-\frac{\pi}{2} + \arctan\left(f\left(\frac{\pi(2j-1)}{n+3}\right)\right)\right] < 0, \end{aligned} \tag{4.44}$$

and

$$F\left(\frac{\pi(2j+1)}{n+3},n\right)=\frac{\pi(2j+1)}{n+3}-\frac{2}{n+3}\left[\pi j-\arctan\left(f\left(\frac{\pi(2j+1)}{n+3}\right)\right)\right]$$
$$=\frac{2}{n+3}\left[\frac{\pi}{2}+\arctan\left(f\left(\frac{\pi(2j-1)}{n+3}\right)\right)\right]>0. \tag{4.45}$$

Therefore, the equation has exactly one root on the interval $(\frac{\pi(2j-1)}{n+3},\frac{\pi(2j+1)}{n+3})$. From the Lemma 4.3 and the inequality (4.41) it follows the estimates (2.8) and (2.9). The second part of the theorem is proved similarly. □

Proposition 1 *Let the function* $G(x,q)=G(x)$ *be differentiable on the interval* (η_1,η_2) *and* $\exists A<1:\forall x$ *and* $\forall q\ \ |G'(x,q)|<A$. *Let* $|G^*(x,q)-G(x,q)|\le M_1\frac{1}{q^2}$, *where* M_1 *is a constant independent of* q. *Then if* $x_1=x_1(q)$ *is the root of the equation*

$$G^*(x,q)=x \tag{4.46}$$

found up to $O\left(\frac{1}{q^2}\right)$, *and* $x_0=x_0(q)$ *is the root of the equation*

$$G(x,q)=x, \tag{4.47}$$

then $x_1(q)-x_0(q)=O\left(\frac{1}{q^2}\right)$.

Proof Let $x_2=x_2(q)$ be the root of the Eq. (4.46) then

$$|x_2-x_0|=|G^*(x_2)-G(x_0)|\le|G(x_2)-G(x_0)|$$
$$+M_1\frac{1}{q^2}=|G'(\xi)(x_2-x_0)|+M_1\frac{1}{q^2}, \tag{4.48}$$

where ξ is some number between x_2 and x_0, whence

$$|x_2-x_0|=\frac{M_1}{1-G'(\xi)}\frac{1}{q^2}\le\frac{M_1}{1-A}\frac{1}{q^2}. \tag{4.49}$$

Since $x_2(q)$ is the root of Eq. (4.46) then $|x_2(q)-x_1(q)|\le M_2\frac{1}{q^2}$ where M_2 is a constant independent of q. As a consequence

$$|x_1-x_0|\le\left(\frac{M_1}{1-A}+M_2\right)\frac{1}{q^2}=O\left(\frac{1}{q^2}\right).$$

□

Remark 4.1 A statement similar to Proposition 1 will be true if $O\left(\frac{1}{q^2}\right)$ is replaced everywhere by $O\left(\frac{j^2}{q^2}\right)$, where $\frac{j}{q} \to 0$ when $q \to \infty$.

Proof of the Theorem 2.3 Denote by j_m the smallest j for which the inequality $j > \frac{2\ln(n+3)}{\pi} + 1$ is satisfied. Let $d_{1,j} = \frac{\pi j}{q}$. Since $2b > \varphi$, as shown in the 6 of the Lemma 4.2, then

$$e^{2qb} > e^{q\varphi} > e^{\pi j_m - \frac{\pi}{2}} > e^{2\ln(2q)+\frac{\pi}{2}} > 4q^2.$$

In this case

$$\tan((q\gamma)) = -i\frac{1-e^{-2qb-i2qc}}{1+e^{-2qb-i2qc}} = -i(1+O(e^{-2qb})) = -i + O\left(\frac{1}{q^2}\right).$$
$$\tan((q\beta)) = -i\frac{1-e^{-2qb+i2qc}}{1+e^{-2qb+i2qc}} = -i(-1+O(e^{-2qb})) = -i + O\left(\frac{1}{q^2}\right).$$

Then the Eq. (2.12) can be rewritten as:

$$u = \arctan\left(-iC_1\left(d_{1,j}+\frac{u}{q}\right) - iC_2\left(d_{1,j}+\frac{u}{q}\right) + O\left(\frac{1}{q^2}\right)\right) \tag{4.50}$$

From where we get that there is some $\theta_{1,j} \in \mathbb{R} : |\theta_{1,j}| < |\frac{u}{q}|$ such that:

$$u = \arctan\left(-i(C_1\left(d_{1,j}\right) + C_2\left(d_{1,j}\right))\right) - i\frac{C_1'(d_{1,j}) + C_2'(d_{1,j})}{1+\left(-iC_1\left(d_{1,j}\right) - iC_2\left(d_{1,j}\right)\right)^2}\frac{u}{q}$$
$$+\Theta(d_{1,j}+\theta_{1,j})\frac{u^2}{q^2} + O\left(\frac{1}{q^2}\right), \tag{4.51}$$

where $\Theta(\varphi) = \frac{1}{2}\left[\arctan(-i(C_1+C_2))\right]''(\varphi)$.
Since $\Theta(\varphi)$ is bounded at $\varphi \in [0,\pi]$ we get that

$$u = \arctan\left(-i(C_1\left(d_{1,j}\right) + C_2\left(d_{1,j}\right))\right)$$
$$-i\frac{C_1'(d_{1,j}) + C_2'(d_{1,j})}{1+\left(-iC_1\left(d_{1,j}\right) - iC_2\left(d_{1,j}\right)\right)^2}\frac{u}{q} + O\left(\frac{1}{q^2}\right). \tag{4.52}$$

Let $u = u_1 + \frac{u_2}{q}$ where u_1 and u_2 bounded with respect to the parameter q the Eq. (4.52) takes the form:

$$u_1 + \frac{u_2}{q} = \arctan\left(-i(C_1\left(d_{1,j}\right) + C_2\left(d_{1,j}\right))\right) - i\frac{C_1'(d_{1,j}) + C_2'(d_{1,j})}{1+\left(-iC_1\left(d_{1,j}\right) - iC_2\left(d_{1,j}\right)\right)^2}\frac{u_1}{q}$$
$$+O\left(\frac{1}{q^2}\right).$$

Let

$$u_{1,j}^* = \arctan\left(-i(C_1\left(d_{1,j}\right) + C_2\left(d_{1,j}\right))\right),$$

and

$$u_{2,j}^* = -i\frac{C_1'(d_{1,j}) + C_2'(d_{1,j})}{1 + \left(-iC_1\left(d_{1,j}\right) - iC_2\left(d_{1,j}\right)\right)^2} u_{1,j}^*.$$

then the proposition 1 implies the assertion of the first part of the theorem. The second part of the theorem is proved in a similar way. □

For brevity, we define the functions $X_1^{(1)} = X_1^{(1)}(u_1, j, n)$ and $X_1^{(2)} = X_1^{(2)}(w_1, j, n)$:

$$\begin{aligned} X_1^{(1)} &= C_1(d_{1,j}) \tan\left(\frac{n+3}{2}\gamma(d_{1,j}) + \gamma'(d_{1,j})u_1\right) \\ &\quad - C_2(d_{1,j}) \tan\left(\frac{n+3}{2}\beta(d_{1,j}) + \beta'(d_{1,j})u_1\right), \\ X_1^{(2)} &= -\frac{C_1(d_{2,j})}{\tan\left(\frac{n+3}{2}\gamma(d_{2,j}) + \gamma'(d_{2,j})w_1\right)} + \frac{C_2(d_{2,j})}{\tan\left(\frac{n+3}{2}\beta(d_{2,j}) + \beta'(d_{2,j})w_1\right)}. \end{aligned}$$

Lemma 4.4 *Let* $a(t) = (t - 2 + \frac{1}{t})^3$. *If* $j \leq \dfrac{2\ln(n+3)}{\pi} + 1$ *then starting from some* n

1.

$$\varphi_{2j-1} = d_{1,j} + \frac{2u_{1,j}^*}{n+3} + \frac{4u_{2,j}^*}{(n+3)^2} + O\left(\frac{1}{n^3}\right), \tag{4.53}$$

where $u_{1,j}^*$ *is the solution of equation* $u_1 = \arctan(X_1^{(1)}(u_1))$ *and* $u_{2,j}^* = R^{(1)}(u_{1,j}^*)$ *is bounded with respect to the parameter* n *(see proof of the lemma).*

2.

$$\varphi_{2j} = d_{2,j} + \frac{2w_{1,j}^*}{n+3} + \frac{4w_{2,j}^*}{(n+3)^2} + O\left(\frac{1}{n^3}\right), \tag{4.54}$$

where $w_{1,j}^*$ *is the solution of equation* $w_1 = \arctan(X_1^{(2)}(w_1))$ *and* $w_{2,j}^* = R^{(2)}(w_{1,j}^*)$ *is bounded with respect to the parameter* n *(see proof of the lemma).*

Proof Consider the Eq. (2.12). We will find a solution to this equation in the form $u = u_1 + \frac{u_2}{q}$, where u_1 and u_2 are bounded with respect to the parameter q.

$$\begin{aligned} C_1(\varphi) &= C_1\left(d_{1,j} + \frac{u}{q}\right) = C_1(d_{1,j}) + C_1'(d_{1,j})\frac{u}{q} + O\left(\frac{1}{q^2}\right) \\ &= C_1(d_{1,j}) + C_1'(d_{1,j})\frac{u_1}{q} + O\left(\frac{1}{q^2}\right) \\ q\gamma(\varphi) &= q\gamma\left(d_{1,j} + \frac{u}{q}\right) = q\gamma(d_{1,j}) + \gamma'(d_{1,j})u + \frac{\gamma''(d_{1,j})u^2}{2q} + O\left(\frac{1}{q^2}\right) \\ &= q\gamma(d_{1,j}) + \gamma'(d_{1,j})u_1 + \left[\gamma'(d_{1,j})u_2 + \frac{\gamma''(d_{1,j})u_1^2}{2}\right]\frac{1}{q} + O\left(\frac{1}{q^2}\right) \end{aligned}$$

$$\tan q\gamma = \tan\left(q\gamma(d_{1,j}) + \gamma'(d_{1,j})u_1\right) + \frac{\gamma'(d_{1,j})u_2 + \frac{\gamma''(d_{1,j})u_1^2}{2}}{\cos^2(q\gamma(d_{1,j}) + \gamma'(d_{1,j})u_1)}\frac{1}{q} + O\left(\frac{1}{q^2}\right)$$

Then we have the following:

$$\begin{aligned} C_1\tan q\gamma = C_1(d_{1,j})\tan\left(q\gamma(d_{1,j}) + \gamma'(d_{1,j})u_1\right) + \Bigg[&\frac{C_1(d_{1,j})\gamma''(d_{1,j})u_1^2}{2\cos^2(q\gamma(d_{1,j}) + \gamma'(d_{1,j})u_1)} \\ &+ \tan\left(q\gamma(d_{1,j}) + \gamma'(d_{1,j})u_1\right)C_1'(d_{1,j})u_1 \\ &+ \frac{C_1(d_{1,j})\gamma'(d_{1,j})u_2}{\cos^2(q\gamma(d_{1,j}) + \gamma'(d_{1,j})u_1)}\Bigg]\frac{1}{q} + O\left(\frac{1}{q^2}\right) \end{aligned}$$

Similarly

$$\begin{aligned} &C_2\tan q\beta = C_2(d_{1,j})\tan\left(q\beta(d_{1,j}) + \beta'(d_{1,j})u_1\right) \\ &+ \Bigg[\frac{C_2(d_{1,j})\beta''(d_{1,j})u_1^2}{2\cos^2(q\beta(d_{1,j}) + \beta'(d_{1,j})u_1)} \\ &+ \tan\left(q\beta(d_{1,j}) + \beta'(d_{1,j})u_1\right)C_2'(d_{1,j})u_1 + \frac{C_2(d_{1,j})\beta'(d_{1,j})u_2}{\cos^2(q\beta(d_{1,j}) + \beta'(d_{1,j})u_1)}\Bigg]\frac{1}{q} + O\left(\frac{1}{q^2}\right) \end{aligned}$$

Then the Eq. (2.12) can be written as

$$u_1 + \frac{u_2}{q} = \arctan\left(X_1^{(1)} + \left(X_2^{(1)} + X_3^{(1)}u_2\right)\frac{1}{q} + O\left(\frac{1}{q^2}\right)\right),$$

where

$$X_1^{(1)} = C_1(d_{1,j})\tan\left(q\gamma(d_{1,j}) + \gamma'(d_{1,j})u_1\right) - C_2(d_{1,j})\tan\left(q\beta(d_{1,j}) + \beta'(d_{1,j})u_1\right)$$

$$X_2^{(1)} = \left[\frac{C_1(d_{1,j})\gamma''(d_{1,j})u_1^2}{2\cos^2(q\gamma(d_{1,j}) + \gamma'(d_{1,j})u_1)} + \tan\left(q\gamma(d_{1,j}) + \gamma'(d_{1,j})u_1\right)C_1'(d_{1,j})u_1\right]$$

$$-\left[\frac{C_2(d_{1,j})\beta''(d_{1,j})u_1^2}{2\cos^2(q\beta(d_{1,j})+\beta'(d_{1,j})u_1)}+\tan\left(q\beta(d_{1,j})+\beta'(d_{1,j})u_1\right)C_2'(d_{1,j})u_1\right] \tag{4.55}$$

and

$$X_3^{(1)}=\frac{C_1(d_{1,j})\gamma'(d_{1,j})}{\cos^2(q\gamma(d_{1,j})+\gamma'(d_{1,j})u_1)}-\frac{C_2(d_{1,j})\beta'(d_{1,j})}{\cos^2(q\beta(d_{1,j})+\beta'(d_{1,j})u_1)}$$

from where we get

$$u_1+\frac{u_2}{q}=\arctan X_1^{(1)}+\frac{X_2^{(1)}+X_3^{(1)}u_2}{1+(X_1^{(1)})^2}\frac{1}{q}+O\left(\frac{1}{q^2}\right). \tag{4.56}$$

Let $\tilde{u}_{1,j}^*$ be the root of the equation,

$$u_1=\arctan X_1^{(1)} \tag{4.57}$$

and $\tilde{u}_{2,j}^*$ be the root of the equation

$$u_2=\frac{X_2^{(1)}+X_3^{(1)}u_2}{1+(X_1^{(1)})^2}.$$

It is easy to get

$$\tilde{u}_{2,j}^*=\frac{X_2^{(1)}}{1+(X_1^{(1)})^2-X_3^{(1)}}=R^{(1)}(\tilde{u}_{1,j}^*). \tag{4.58}$$

Similarly for the Eq. (2.13) let's put

$$w_1=\arctan X_1^{(2)} \tag{4.59}$$

and

$$\tilde{w}_{2,j}^*=\frac{X_2^{(2)}}{1+(X_1^{(2)})^2-X_3^{(2)}}=R^{(2)}(\tilde{w}_{1,j}^*)$$

where

$$X_1^{(2)}=-\frac{C_1(d_{2,j})}{\tan\left(q\gamma(d_{2,j})+\gamma'(d_{2,j})u_1\right)}+\frac{C_2(d_{2,j})}{\tan\left(q\beta(d_{2,j})+\beta'(d_{2,j})u_1\right)}$$

$$X_2^{(2)}=\left[\frac{C_1(d_{2,j})\gamma''(d_{2,j})w_1^2}{2\sin^2(q\gamma(d_{2,j})+\gamma'(d_{2,j})w_1)}-\frac{C_1'(d_{2,j})w_1}{\tan\left(q\gamma(d_{2,j})+\gamma'(d_{2,j})w_1\right)}\right]$$

$$-\left[\frac{C_2(d_{2,j})\beta''(d_{2,j})w_1^2}{2\sin^2(q\beta(d_{2,j})+\beta'(d_{2,j})w_1)}-\frac{C_2'(d_{2,j})w_1}{\tan\left(q\beta(d_{2,j})+\beta'(d_{2,j})w_1\right)}\right]$$

$$X_3^{(2)}=\frac{C_1(d_{2,j})\gamma'(d_{2,j})}{\sin^2(q\gamma(d_{2,j})+\gamma'(d_{1,j})w_1)}-\frac{C_2(d_{2,j})\beta'(d_{2,j})}{\sin^2(q\beta(d_{2,j})+\beta'(d_{2,j})w_1)}$$

then the proposition 1 implies the assertion of the lemma. □

Proof of the Theorem 2.4 As $d_{1,j}=\dfrac{2\pi j}{n+3}=O\left(\dfrac{j}{q}\right)$ taking into account Lemma 4.1 it is not difficult to obtain that $\gamma=d_{1,j}e^{-\frac{\pi i}{3}}+O\left(\dfrac{j^3}{q^3}\right)$, $\beta=d_{1,j}e^{\frac{\pi i}{3}}+O\left(\dfrac{j^3}{q^3}\right)$. Then

$$C_1(d_{1,j})=1+O\left(\frac{j^2}{q^2}\right),\quad C_2(d_{1,j})=-1+O\left(\frac{j^2}{q^2}\right) \tag{4.60}$$

$$\begin{aligned}\tan(q\gamma(d_{1,j})+\gamma'(d_{1,j})u_1)&=\tan\left(qd_{1,j}e^{-\frac{\pi i}{3}}+u_1e^{-\frac{\pi i}{3}}+O\left(\frac{j^3}{q^2}\right)\right),\\ \tan(q\beta(d_{1,j})+\beta'(d_{1,j})u_1)&=\tan\left(qd_{1,j}e^{\frac{\pi i}{3}}+u_1e^{\frac{\pi i}{3}}+O\left(\frac{j^3}{q^2}\right)\right).\end{aligned} \tag{4.61}$$

Then, taking into account the fact that $d_{1,j}=\frac{\pi j}{q}$ we get:

$$\begin{aligned}X_1^{(1)}&=C_1(d_{1,j})\tan\left(q\gamma(d_{1,j})+\gamma'(d_{1,j})u_1\right)-C_2(d_{1,j})\tan\left(q\beta(d_{1,j})+\beta'(d_{1,j})u_1\right)\\&=\tan\left(qd_{1,j}e^{-\frac{\pi i}{3}}+u_1e^{-\frac{\pi i}{3}}+O\left(\frac{j^3}{q^2}\right)\right)\\&\quad+\tan\left(qd_{1,j}e^{\frac{\pi i}{3}}+u_1e^{\frac{\pi i}{3}}+O\left(\frac{j^3}{q^2}\right)\right)+O\left(\frac{j^2}{q^2}\right)\\&=\frac{2\sin\left(qd_{1,j}+u_1\right)}{\cos\left(qd_{1,j}+u_1\right)+\cosh\left((qd_{1,j}+u_1)\sqrt{3}\right)}+O\left(\frac{j^3}{q^2}\right)\\&=\frac{2(-1)^j\sin(u_1)}{(-1)^j\cos(u_1)+\cosh\left((qd_{1,j}+u_1)\sqrt{3}\right)}+O\left(\frac{j^3}{q^2}\right).\end{aligned} \tag{4.62}$$

Whence we obtain that $u_{1,j}^*=0$ is a solution to the Eq. (4.57) up to $O\left(\dfrac{j^3}{q^2}\right)$. Consider $X_2^{(1)}$ and $X_3^{(1)}$. By analogy with the estimates (4.28) and (4.26), we obtain that in the formula (4.55) all factors in front of u_1 are limited, and since $u_1=u_{1,j}^*+O\left(\dfrac{j^3}{q^2}\right)$

then

$$X_2^{(1)} = 0 + O\left(\frac{j^3}{q^2}\right). \tag{4.63}$$

Also using a similar estimate as in the formula (4.26) and equality (4.23) we obtain

$$|X_3^{(1)}| \leq 2\frac{|C_1(d_{1,j})\gamma'(d_{1,j})|}{|\cos^2\left(q\gamma(d_{1,j}) + \gamma'(d_{1,j})\right)|} = 2\frac{1}{|\cos^2\left(q\gamma(d_{1,j}) + \gamma'(d_{1,j})\right)|} \leq 0.61 < 1. \tag{4.64}$$

Then for $u_{2,j}^* = 0$, we get that $|u_{2,j}^* - \tilde{u}_{2,j}^*| = O\left(\frac{j^3}{q^2}\right)$. Taking into account the remark 4.1 we get

$$\varphi_{2j-1} = d_{1,j} + O\left(\frac{j^3}{q^3}\right). \tag{4.65}$$

From similar reasoning, taking into account the fact that $qd_{2,j} = \pi j + \frac{\pi}{2}$

$$\begin{aligned} X_1^{(2)} &= \frac{-2\sin\left(qd_{2,j} + w_1\right)}{-\cos\left(qd_{2,j} + w_1\right) + \cosh\left((qd_{2,j} + w_1)\sqrt{3}\right)} + O\left(\frac{j^3}{q^2}\right) \\ &= \frac{2(-1)^{j+1}\cos(w_1)}{(-1)^j\sin(w_1) + \cosh\left((qd_{2,j} + w_1)\sqrt{3}\right)} + O\left(\frac{j^3}{q^2}\right) \end{aligned} \tag{4.66}$$

Let us assume that $w_{1,j}^*$ is a solution to Eq. (4.59), taking into account the equality (4.66). Since the expressions $C_1(d_{2,j})\gamma''(d_{2,j})$, $C_2(d_{2,j})\beta''(d_{2,j})$, $C_1'(d_{2,j})$, $C_2'(d_{2,j})$ are $O\left(\frac{j}{q}\right)$ and in the expression for $X_2^{(2)}$ they are multiplied by limited functions, we get that $X_2^{(2)} = O\left(\frac{j}{q}\right)$ and as a consequence $w_2^* = 0 + O\left(\frac{j}{q}\right)$. Then

$$\varphi_{2j} = d_{2,j} + \frac{w_1^*}{q} + O\left(\frac{j^3}{q^3}\right).$$

□

Proof of the Theorem 2.5 If $j > \dfrac{2\ln(n+3)}{\pi} + 1$ then

$$\begin{aligned} \lambda_{2j-1}^{(n)} = g(\varphi_{2j-1}^{(n)}) &= g\left(d_{1,j} + \frac{2u_{1,j}^*}{n+3} + \frac{4u_{2,j}^*}{(n+3)^2} + O\left(\frac{1}{q^3}\right)\right) \\ &= g(d_{1,j}) + g'(d_{1,j})\left(\frac{2u_{1,j}^*}{n+3} + \frac{4u_{2,j}^*}{(n+3)^2} + O\left(\frac{1}{q^3}\right)\right) \end{aligned}$$

$$+\frac{1}{2}g''(d_{1,j})\left(\frac{2u^*_{1,j}}{n+3}+\frac{4u^*_{2,j}}{(n+3)^2}+O\left(\frac{1}{q^3}\right)\right)^2+O\left(\frac{1}{q^3}\right).$$

Expanding the brackets and leaving the terms of order no more than $O\left(\frac{1}{q^3}\right)$, we obtain a statement of the theorem for $\lambda^{(n)}_{2j-1}$. The rest of the cases are obtained similarly. □

Proof of Theorem 2.6 Consider the case when $j > \dfrac{2\ln(n+3)}{\pi}+1$. We know that $\lambda^{(n)}_m = g(\varphi^{(n)}_m)$ for all m and n. Given that $q=\frac{n+3}{2}$ and $\varphi^{(n)}_{2j-1} = d_{i,j} + \frac{u^*_{1,j}}{q} + \frac{u^*_{2,j}}{q^2} + O\left(\frac{1}{q^3}\right)$ we get

$$\lambda^{(n)}_{2j-1} = -2^6\sin^6\left(\frac{d_{i,j}}{2}+\frac{u^*_{1,j}}{2q}+\frac{u^*_{2,j}}{2q^2}+O\left(\frac{1}{q^3}\right)\right).$$

Taking Lemma 4.1 and equalities (2.2) into account, it is easy to obtain that

$$-i(C_1\left(d_{1,j}\right)+C_2\left(d_{1,j}\right)) = \frac{\sqrt{3}}{8}d^2_{1,j}+O(d^4_{1,j}).$$

Then from equalities (2.15) and (2.16) we get that

$$u_{1,j} = \frac{\sqrt{3}}{8}d^2_{1,j}+O(d^4_{1,j}),\quad u_{2,j}=O(d^3_{1,j}). \tag{4.67}$$

And as a consequence

$$\lambda^{(n)}_{2j-1} = -2^6\sin^6\left(\frac{d_{1,j}}{2}+\frac{\sqrt{3}}{16}\frac{d^2_{1,j}}{q}+O\left(\frac{d^4}{q}\right)+O\left(\frac{1}{q^3}\right)\right).$$

Since $\sin x = x-\frac{1}{6}x^3+O(x^5)$, $x\to 0$, a simple calculation shows that

$$\begin{aligned}\lambda^{(n)}_{2j-1} &= -2^6\left(\frac{d_{1,j}}{2}+\frac{\sqrt{3}}{16}\frac{d^2_{1,j}}{q}-\frac{1}{48}d^3_{1,j}+O\left(d^5_{1,j}\right)+O\left(\frac{1}{q^3}\right)\right)^6\\ &= -d^6_{1,j}+\frac{1}{4}d^8_{1,j}-\frac{3\sqrt{3}}{4}\frac{d^7_{1,j}}{q}\\ &\quad +O\left(d^{10}_{1,j}\right)+O\left(\frac{d^5_{1,j}}{q^3}\right).\end{aligned}$$

Since $j=o(n^{2/3})$ then for sufficiently large n the residual terms in the resulting asymptotic expansion are significantly smaller than those taken into account. The rest of the cases are obtained similarly. □

Proof of Remark 2.2 The proof is similar to the proof of the Theorem 2.5. □

5 Numerical experiments

All numerical experiments were carried out in the Maple mathematical package. In all calculations, all values were set with 50-decimal approximation. The exact value of the eigenvalues means the eigenvalues calculated using the predefined function of Maple.

In this paper, finding the eigenvalues was reduced to solving two sets of equations (depending on the parity of the eigenvalues). Each of the equations is solved with respect to the parameter φ and has a single root φ_m, and each such root corresponds to a single eigenvalue, which can be found by a simple substitution $\lambda_m = g(\varphi_m)$. Here m is the number of eigenvalues that are ordered in ascending order of the module.

Theorem 2.2 makes it possible to calculate φ_m using the Fix Point Method. At the k -th iteration, the approximate value of the root depending on the parity is found by the formulas (2.3) and (2.4).

$$\varphi_{2j-1}^{(k)} = \frac{2}{n+3}\left[\pi j + \arctan f(\varphi_{2j-1}^{(k-1)}, n)\right]$$
$$\varphi_{2j}^{(k)} = \frac{2}{n+3}\left[\pi j + \frac{\pi}{2} - \arctan h(\varphi_{2j}^{(k-1)}, n)\right]$$

The Table 1 shows the dependence of the error $\Delta\varphi_m = |\varphi_m^{(k)} - \varphi_m|$ on the iteration number k. And also an error is given for the corresponding eigenvalue $|\Delta\lambda_m = \lambda_m^{(k)} - \lambda_m|$, where $\lambda_m^{(k)} = g(\varphi_m^{(k)})$. In the experiment, the matrix size is $n = 200$. This dependence was considered for three eigenvalues $m = 1, 100, 200$.

The paper also presents asymptotics formulas (2.14), (2.17), (2.18) and (2.20) for the roots of φ_m. Denote

$$\varphi_{2j-1}^{*} = d_{1,j} + \frac{2u_{1,j}^{*}}{n+3} + \frac{4u_{2,j}^{*}}{(n+3)^2}, \tag{5.1}$$

$$\varphi_{2j}^{*} = d_{2,j} + \frac{2w_{1,j}^{*}}{n+3} + \frac{4w_{2,j}^{*}}{(n+3)^2}, \tag{5.2}$$

where $d_{1,j} := \frac{2\pi j}{n+3}$, $d_{2,j} := \frac{\pi(2j+1)}{n+3}$, and values $u_{1,j}^{*}, u_{2,j}^{*}, w_{1,j}^{*}, w_{2,j}^{*}$ are calculated by different formulas depending on the number of the corresponding eigenvalue. If $j > \dfrac{2\ln(n+3)}{\pi} + 1$ then

$$u_{1,j}^{*} = \arctan\left(-i(C_1\left(d_{1,j}\right) + C_2\left(d_{1,j}\right))\right), \tag{5.3}$$

$$u_{2,j}^{*} = -i\frac{C_1'(d_{1,j}) + C_2'(d_{1,j})}{1 + \left(-iC_1\left(d_{1,j}\right) - iC_2\left(d_{1,j}\right)\right)^2}, \tag{5.4}$$

$$w_{1,j}^{*} = \arctan\left(-i(C_1\left(d_{2,j}\right) + C_2\left(d_{2,j}\right))\right), \tag{5.5}$$

Table 1 Dependence of the error on the number of iterations

	k	1	2	3	4	5
m = 1	$\Delta\varphi$	3.6×10^{-8}	6.3×10^{-10}	1.1×10^{-11}	1.9×10^{-13}	3.3×10^{-15}
	$\Delta\lambda$	6.2×10^{-15}	1.1×10^{-16}	1.9×10^{-18}	3.2×10^{-20}	5.6×10^{-22}
m = 100	$\Delta\varphi$	2.8×10^{-5}	1.6×10^{-7}	8.8×10^{-10}	5×10^{-12}	2.8×10^{-14}
	$\Delta\lambda$	6.6×10^{-4}	3.7×10^{-6}	2.1×10^{-8}	1.2×10^{-10}	6.8×10^{-13}
m = 200	$\Delta\varphi$	1.1×10^{-4}	8×10^{-7}	5.8×10^{-9}	4.2×10^{-11}	3×10^{-13}
	$\Delta\lambda$	1.7×10^{-4}	1.2×10^{-6}	8.7×10^{-9}	6.3×10^{-11}	4.5×10^{-13}
	k	6	7	8	9	10
m = 1	$\Delta\varphi$	5.7×10^{-17}	1×10^{-18}	1.7×10^{-20}	2.3×10^{-22}	5.1×10^{-24}
	$\Delta\lambda$	9.7×10^{-24}	1.7×10^{-25}	2.9×10^{-27}	5.1×10^{-29}	8.8×10^{-31}
m = 100	$\Delta\varphi$	1.6×10^{-16}	9.1×10^{-19}	5.2×10^{-21}	2.9×10^{-23}	1.7×10^{-25}
	$\Delta\lambda$	3.8×10^{-15}	2.2×10^{-17}	1.2×10^{-19}	7×10^{-22}	4×10^{-24}
m = 200	$\Delta\varphi$	2.2×10^{-15}	1.6×10^{-17}	1.1×10^{-19}	8.3×10^{-22}	6×10^{-24}
	$\Delta\lambda$	3.3×10^{-15}	2.4×10^{-17}	1.7×10^{-19}	1.2×10^{-21}	9×10^{-24}

$$w^*_{2,j} = -i\frac{C_1'(d_{2,j}) + C_2'(d_{2,j})}{1+\left(-iC_1\left(d_{2,j}\right) - iC_2\left(d_{2,j}\right)\right)^2}, \tag{5.6}$$

and $|\varphi_m - \varphi^*_m| = O(\frac{1}{n^3})$.
If $j \le \dfrac{2\ln(n+3)}{\pi} + 1$ then $u^*_{1,j} = 0$, $u^*_{2,j} = 0$, $w^*_{2,j} = 0$ and $w^*_{1,j}$-solution of the equation

$$w_1 = \arctan\left(\frac{2(-1)^{j+1}\cos(w_1)}{(-1)^j\sin(w_1) + \cosh\left((qd_{2,j} + w_1)\sqrt{3}\right)}\right), \tag{5.7}$$

and $|\varphi_m - \varphi^*_m| = O(\frac{j^3}{n^3})$. The Table 2 shows the dependence of the maximum error $\Delta\varphi^* = \max\limits_{\left[\frac{m+1}{2}\right] > \tilde{j}_n} |\varphi_m - \varphi^*_m|$ and the maximum relative error $\Delta_r\varphi^* = \max\limits_{\left[\frac{m+1}{2}\right] > \tilde{j}_n} \left|\frac{\varphi_m - \varphi^*_m}{\varphi_m}\right|$ depending on the size of the matrix n. Where $\tilde{j}_n = \dfrac{2\ln(n+3)}{\pi} + 1$. Similarly for the corresponding eigenvalues, where $\lambda^*_m = g(\varphi^*_m)$, $\Delta\lambda^* = \max\limits_{\left[\frac{m+1}{2}\right] > \tilde{j}_n} |\lambda_m - \lambda^*_m|$ and $\Delta_r\lambda^* = \max\limits_{\left[\frac{m+1}{2}\right] > \tilde{j}_n} \left|\frac{\lambda_m - \lambda^*_m}{\lambda_m}\right|$.

A similar dependence is given in the Table 3, only in this case the maximum is found over all numbers m for which $m \le \tilde{j}_n$

Table 2 Maximum error when using the formulas (5.1), (5.2) when $\left[\frac{m+1}{2}\right] > \frac{2\ln(n+3)}{\pi} + 1$

n	32	64	128	256	512	1024
$\Delta\varphi^*$	1.5×10^{-4}	2.2×10^{-5}	3×10^{-6}	3.9×10^{-7}	5×10^{-8}	6.2×10^{-9}
$\Delta_r\varphi^*$	5×10^{-5}	7.2×10^{-6}	9.6×10^{-7}	1.2×10^{-7}	1.6×10^{-8}	2×10^{-9}
$\Delta\lambda^*$	5.2×10^{-3}	7.5×10^{-4}	1×10^{-4}	1.3×10^{-5}	1.6×10^{-6}	2.1×10^{-7}
$\Delta_r\lambda^*$	1.3×10^{-4}	1.9×10^{-5}	2.6×10^{-6}	3.3×10^{-7}	4.2×10^{-8}	5.3×10^{-9}

Table 3 Maximum error when using the formulas (5.1), (5.2) when $\left[\frac{m+1}{2}\right] \leq \frac{2\ln(n+3)}{\pi} + 1$

n	32	64	128	256	512	1024
$\Delta\varphi^*$	0.5×10^{-3}	9.1×10^{-4}	1.5×10^{-4}	2.5×10^{-5}	3.1×10^{-6}	4.8×10^{-8}
$\Delta_r\varphi^*$	7.8×10^{-3}	2.4×10^{-3}	7.1×10^{-4}	2×10^{-4}	5.1×10^{-5}	1.4×10^{-5}
$\Delta\lambda^*$	2.6×10^{-3}	3.9×10^{-5}	4.3×10^{-7}	3.9×10^{-9}	1.6×10^{-11}	1.2×10^{-13}
$\Delta_r\lambda^*$	4.4×10^{-2}	1.4×10^{-2}	4.2×10^{-3}	1.2×10^{-3}	3×10^{-4}	8.5×10^{-3}

Table 4 Maximum error when using the formulas (5.8) and (5.9)

n	32	64	128	256	512	1024
$\Delta\hat{\lambda}$	3×10^{-2}	4.6×10^{-3}	6.2×10^{-4}	8×10^{-5}	1×10^{-5}	1.3×10^{-6}
$\Delta_r\hat{\lambda}$	4.4×10^{-2}	1.4×10^{-2}	4.3×10^{-3}	1.2×10^{-3}	3×10^{-4}	8.6×10^{-5}

Theorem 2.5 presents formulas (2.21) and (2.22) for eigenvalues. Let 's put

$$\hat{\lambda}_{2j-1} = g(d_{1,j}) + g'(d_{1,j})\frac{2u^*_{1,j}}{n+3} + \frac{4u^*_{2,j}g'(d_{1,j}) + 2(u^*_{1,j})^2 g''(d_{1,j})}{(n+3)^2}, \quad (5.8)$$

$$\hat{\lambda}_{2j} = g(d_{2,j}) + g'(d_{2,j})\frac{2w^*_{1,j}}{n+3} + \frac{4w^*_{2,j}g'(d_{2,j}) + 2(w^*_{1,j})^2 g''(d_{2,j})}{(n+3)^2}, \quad (5.9)$$

where, $u^*_{1,j}$, $u^*_{2,j}$, $w^*_{1,j}$, $w^*_{2,j}$, are from the formulas (5.3)–(5.7) described above in this section.

The Table 4 shows the dependence of the maximum error $\Delta\hat{\lambda} = \max\limits_{1\leq j\leq n} |\hat{\lambda}_m - \lambda_m|$ of eigenvalues and the maximum relative error $\Delta_r\hat{\lambda} = \max\limits_{1\leq j\leq n} \left|\frac{\hat{\lambda}_m - \lambda_m}{\lambda_m}\right|$ of eigenvalues depending on the size of the matrix n. Where $\hat{\lambda}_m$ are calculated using the formulas (5.8) and (5.9). In this case, the maximum is taken for all eigenvalues.

The Theorem 2.6 presents asymptotic formulas for the eigenvalues of λ_m, provided that $m = o(n^{\frac{2}{3}})$. Let's check Theorem 2.6 if the eigenvalue with the number m depends on n as follows: $m = [3\ln(n+3)]$, where $[x]$ is the rounding of the number x to an integer value. In this case, the eigenvalue can be found up to $O(\frac{\ln^5(n+3)}{n^8})$ by the

Table 5 Error when using the formulas (5.10) and (5.11)

n	32	64	128	256	512	1024
m	11	13	15	17	19	21
$\Delta\bar{\lambda}$	1.7×10^{-2}	1.6×10^{-4}	8.9×10^{-7}	3.6×10^{-9}	1.2×10^{-11}	3.2×10^{-14}
$\Delta_r\bar{\lambda}$	1.4×10^{-2}	2.3×10^{-3}	2.9×10^{-4}	3.3×10^{-5}	3.5×10^{-6}	3.5×10^{-7}

formulas

$$\bar{\lambda}_{2j-1} = -d_{1,j}^6 + \frac{1}{4}d_{1,j}^8 - \frac{3\sqrt{3}}{2}\frac{d_{1,j}^7}{n+3}, \tag{5.10}$$

$$\bar{\lambda}_{2j} = -d_{2,j}^6 + \frac{1}{4}d_{2,j}^8 - \frac{3\sqrt{3}}{2}\frac{d_{2,j}^7}{n+3}. \tag{5.11}$$

The Table 5 shows the dependence of the error $\Delta\bar{\lambda_m} = |\bar{\lambda}_m - \lambda_m|$ of eigenvalues and the relative error $\Delta_r\bar{\lambda_m} = \left|\frac{\hat{\lambda}_m - \lambda_m}{\lambda_m}\right|$ of eigenvalues depending on the size of the matrix n. Where $\bar{\lambda}_m$ are calculated using the formulas (5.10) and (5.11).

By the Theorem 2.6, if m is a constant, the eigenvalues can be cleared using the formulas (2.27) and (2.28). Let 's put

$$\lambda_{2j-1}^{(n)} = -\frac{(2\pi j)^6}{(n+3)^6}, \tag{5.12}$$

$$\lambda_{2j}^{(n)} = -\frac{((2j+1)\pi + 2w_{1,j}^*)^6}{(n+3)^6}, \tag{5.13}$$

where $w_{1,j}^*$ - solution of the Eq. (5.7).
Let us compare our results with the results of the well-known works of Seymour Parter devoted to the asymptotics of the first eigenvalues in the case when the symbol of the Toeplitz matrix has a singularity of power order (see [16, 17]). Consider the class of functions g satisfying: In [16] the author considered the class of functions g satisfying:

(a) g is real, continuous, and periodic with period 2π; $\min g = g(0) = m^*$ and $\varphi = 0$ is the only value of φ $(mod\ 2\pi)$ for which this minimum is attained.
(b) If g satisfies (a), then it has continuous derivatives of order $2k$ $(k \in \mathbb{N})$ in some neighborhood of $\varphi = 0$ and $g^{(2k)}(0) = \sigma^2 > 0$ is the first non-vanishing derivative of g at $\varphi = 0$.

Theorem 5.1 ([17, Theorem 1]) *Let g be a function which satisfies Conditions (a) and (b). Let $\lambda_{\nu,n}(\nu = 1, 2, \ldots n)$ be the eigenvalues of $T_n(a)$ arranged in nondecreasing order. For fixed ν, as $n \to \infty$ we have*

$$\lambda_{\nu,n} = m^* + \frac{\sigma^2}{(2\alpha)!}\Lambda_\nu\left(\frac{1}{n^{2\alpha}}\right) + o\left(\frac{1}{n^{2\alpha}}\right),$$

Table 6 Error of the eigenvalue when using the formulas (5.12)–(5.13) and (5.15)–(5.16)

	n	32	64	128	256	512	1024
$m = 1$	$\Delta\tilde{\lambda}_m$	1.9×10^{-07}	1×10^{-09}	4.8×10^{-12}	2.1×10^{-14}	8.5×10^{-17}	3.4×10^{-19}
	$\Delta\lambda_m^{(p)}$	2.4×10^{-05}	2.1×10^{-07}	1.8×10^{-09}	1.5×10^{-11}	1.2×10^{-13}	9.3×10^{-16}
	$\Delta_r\tilde{\lambda}_m$	5.6×10^{-03}	1.5×10^{-03}	4×10^{-04}	1×10^{-04}	2.5×10^{-05}	6.5×10^{-06}
	$\Delta_r\lambda_m^{(p)}$	7×10^{-01}	3.1×10^{-01}	1.5×10^{-01}	7.2×10^{-02}	3.6×10^{-02}	1.8×10^{-02}
$m = 2$	$\Delta\tilde{\lambda}_m$	7×10^{-07}	4×10^{-09}	1.9×10^{-11}	8×10^{-14}	3.3×10^{-16}	1.3×10^{-18}
	$\Delta\lambda_m^{(p)}$	2.7×10^{-04}	2.5×10^{-06}	2.1×10^{-08}	2×10^{-10}	1.3×10^{-12}	1.1×10^{-14}
	$\Delta_r\tilde{\lambda}_m$	1.8×10^{-03}	5.1×10^{-04}	1.3×10^{-04}	3.5×10^{-05}	8.7×10^{-06}	2.2×10^{-06}
	$\Delta_r\lambda_m^{(p)}$	7.1×10^{-01}	3.2×10^{-01}	1.5×10^{-01}	7.2×10^{-02}	3.6×10^{-02}	1.8×10^{-02}
$m = 3$	$\Delta\tilde{\lambda}_m$	1.2×10^{-05}	6.7×10^{-08}	3.1×10^{-10}	1.3×10^{-12}	5.4×10^{-15}	2.2×10^{-17}
	$\Delta\lambda_m^{(p)}$	1.5×10^{-03}	1.4×10^{-05}	1.2×10^{-07}	9×10^{-10}	7.5×10^{-12}	5.9×10^{-14}
	$\Delta_r\tilde{\lambda}_m$	5.8×10^{-03}	1.5×10^{-03}	4×10^{-04}	1×10^{-04}	2.6×10^{-05}	6.5×10^{-06}
	$\Delta_r\lambda_m^{(p)}$	7.2×10^{-01}	3.2×10^{-01}	1.5×10^{-01}	7.3×10^{-02}	3.6×10^{-02}	1.8×10^{-02}
$m = 6$	$\Delta\tilde{\lambda}_m$	3.2×10^{-03}	1.8×10^{-05}	8.3×10^{-08}	3.6×10^{-10}	1.6×10^{-12}	5.8×10^{-15}
	$\Delta\lambda_m^{(p)}$	4.7×10^{-02}	4.1×10^{-04}	3.4×10^{-06}	2.8×10^{-08}	2.2×10^{-10}	1.7×10^{-12}
	$\Delta_r\tilde{\lambda}_m$	5.5×10^{-02}	1.4×10^{-02}	3.7×10^{-03}	9.5×10^{-04}	2.4×10^{-04}	6×10^{-05}
	$\Delta_r\lambda_m^{(p)}$	8.1×10^{-01}	3.4×10^{-01}	1.5×10^{-01}	7.3×10^{-02}	3.6×10^{-02}	1.8×10^{-02}

where the numbers Λ_ν are the eigenvalues arranged in nondecreasing order of

$$\left[-\left(\frac{d}{dx}\right)^2\right]^\alpha U - \Lambda U = 0, \quad 0 \le x \le 1$$

with boundary conditions

$$\left(\frac{d}{dx}\right)^i U(0) = \left(\frac{d}{dx}\right)^i U(1), \quad i = 0, 1, \ldots, \alpha - 1.$$

Let $g_1(\varphi) = -g(\varphi) = (2\sin\frac{\varphi}{2})^6$. Notice that g_1 satisfies Conditions (a) and (b) with $m^* = 0, \alpha = 3$ and $g_1^{(6)}(0) = 720 > 0$. Therefore, from Theorem 5.1 in our case we get

$$\lambda_m = -\Lambda_m\left(\frac{1}{n^6}\right) + o\left(\frac{1}{n^6}\right) \quad (n \to \infty). \tag{5.14}$$

To find Λ_m it is also convenient to consider two cases when m is even and odd. After finding Λ_m, the formula (5.14) will split into two cases and take the form

$$\lambda_{2j-1} = -\frac{(2\pi j)^6}{n^6} + +o\left(\frac{1}{n^6}\right), \tag{5.15}$$

$$\lambda_{2j} = -\frac{((2j+1)\pi + 2w^*_{1,j})^6}{n^6} + o\left(\frac{1}{n^6}\right), \tag{5.16}$$

where $w^*_{1,j}$ - solution of the Eq. (5.7). Note that formulas (5.15), (5.16) differ from formulas (2.27) and (2.28) in the denominator of the main part, and have a greater error. In addition, we proved in Theorem 2.6 that the remainder term (2.27)–(2.28) has a uniform estimate respect to n if quantity $m = o(n^{\frac{2}{3}})$, while it was shown in work [17] that formula (5.14) is valid for a fixed number m. Next, we will compare the error of the formulas (5.12) and (5.13) with the formulas s (5.15), (5.16). The Table 6 shows the dependence of the error $\Delta\tilde{\lambda_m} = \left|\tilde{\lambda}_m - \lambda_m\right|$, relative error $\Delta_r\tilde{\lambda_m} = \left|\frac{\tilde{\lambda}_m - \lambda_m}{\lambda_m}\right|$, where $\tilde{\lambda}_m$ calculated by the formulas (2.27) and (2.28), and the error $\Delta\lambda_m^{(p)} = \left|\lambda_m^{(p)} - \lambda_m\right|$, relative error $\Delta_r\lambda_m^{(p)} = \left|\frac{\lambda_m^{(p)} - \lambda_m}{\lambda_m}\right|$ when using formulas (5.15), (5.16). In this case, the number of the eigenvalue m was fixed. The dependence is based on the size of the matrix n. The experiment was carried out at $m = 1, 2, 3, 6$.

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