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ORIGINAL PAPER

The rate of convergence of a generalization of Post–Widder operators and Rathore operators

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Abstract
In this paper, we study local approximation properties of certain gamma-type operators. They generalize the Post–Widder operators and the Rathore operators, and approximate locally integrable functions satisfying a certain growth condition on the infinite interval $[0, \infty)$. We derive the complete asymptotic expansion for these operators and prove a localization result. Also, we estimate the rate of convergence for functions of bounded variation.

Keywords Gamma type operators · Post–Widder operators · Generalized Rathore operators · Rate of convergence · Functions of bounded variation · Complete asymptotic expansion

Mathematics Subject Classification 41A25 · 41A36 · 41A60

1 Introduction

The Post–Widder operator plays a crucial role in the inversion of the Laplace transform. Let $f : [0, \infty) \to \mathbb{R}$ be locally integrable and let $\mathcal{L}f$ denote its Laplace transform

$$(\mathcal{L}f)(s) = \int_0^\infty e^{-sx} f(x)\, dx.$$

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If the integral converges for some $s > 0$, then the inversion formula

$$f(x) = \lim_{n\to\infty} L_{n,x}[\mathcal{L}f]$$

is valid, for all positive x in the Lebesgue set of f [18, Chapter 7, Theorem 6a]. The operator $L_{n,x}$ is defined by the equation

$$L_{n,x}[g] = (-1)^n g^{(n)}\left(\frac{n}{x}\right)\left(\frac{n}{x}\right)^{n+1},$$

for any real positive number x and any positive integer n [18, Chapter 7, Definition 6]. A simple calculation reveals that

$$L_{n,x}[\mathcal{L}f] = \frac{1}{n!}\left(\frac{n}{x}\right)^{n+1}\int_0^\infty e^{-nt/x}t^n f(t)\,\mathrm{d}t \quad (x > 0)$$

[18, Page 288]. For the sake of approximation, the Post–Widder operator P_n is defined in the slightly different form

$$(P_n f)(x) = \frac{(n/x)^n}{\Gamma(n)}\int_0^\infty e^{-nt/x}t^{n-1} f(t)\,\mathrm{d}t \quad (x > 0). \tag{1.1}$$

[2, Eq. (9.1.9)] (cf. [10, Eq. (3.5)]). The form (1.1) is an operator of exponential type and these operators preserve linear functions [8]. The connection is as follows. Fix $x > 0$. Define $f^{[x]}(t) = e^{-t/x} f(t)$. Then,

$$\begin{aligned}(P_{n+1}f)(x) &= \left(\frac{n+1}{n}\right)^{n+1}\frac{(n/x)^{n+1}}{n!}\int_0^\infty e^{-(n+1)t/x}t^n f(t)\,\mathrm{d}t \\ &= \left(\frac{n+1}{n}\right)^{n+1} L_{n,x}\left[\mathcal{L}f^{[x]}\right].\end{aligned}$$

Hence, in each Lebesgue point of f we have

$$\lim_{n\to\infty}(P_{n+1}f)(x) = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{n+1} L_{n,x}\left[\mathcal{L}f^{[x]}\right] = e\cdot e^{-x/x} f(x) = f(x).$$

The Post–Widder operators P_n were intensively studied by several authors [3, 4, 9]. In recent years, several authors defined and studied variants of the Post–Widder operator which preserve several test functions [5–7, 13, 16].

In order to include the similar operator by Rathore [12] (see below), we study in this paper a more general gamma type operator depending on a positive parameter, which includes both, the Post–Widder operators and the Rathore operators as special cases.

Let E be the class of all locally integrable functions of exponential type on $[0, +\infty)$ with the property $|f(t)| \le Me^{At}$ $(t \ge 0)$ for some finite constants $M, A > 0$. The gamma-type operators $P_{n,c}$ (cf. [10, Eq. (3.3)]) associate to each $f \in E$ the function

$$\left(P_{n,c}f\right)(x) = \frac{(nc)^{ncx}}{\Gamma(ncx)} \int_0^\infty e^{-nct} t^{ncx-1} f(t)\,\mathrm{d}t \quad (x > 0), \tag{1.2}$$

where c is a positive parameter. We emphasize the fact that c may depend on the variable x. Note that the integral exists if $nc > A$. The definition can be rewritten in the form

$$\left(P_{n,c}f\right)(x) = \int_0^\infty \phi_{n,c}(x,t) f(t)\,\mathrm{d}t$$

with the kernel function

$$\phi_{n,c}(x,t) = \frac{(nc)^{ncx}}{\Gamma(ncx)} e^{-nct} t^{ncx-1}. \tag{1.3}$$

In the special case $c = 1$ these operators reduce to the Rathore operators $R_n \equiv P_{n,1}$, given by [10, Eq. (3.6)]

$$(R_n f)(x) = \frac{n^{nx}}{\Gamma(nx)} \int_0^\infty e^{-nt} t^{nx-1} f(t)\,\mathrm{d}t \quad (x > 0).$$

If we substitute $c = 1/x$, we obtain the Post–Widder operators (1.1).

In this paper we derive the complete asymptotic expansion for the sequence of operators $P_{n,c}$ in the form

$$\left(P_{n,c}f\right)(x) \sim f(x) + \sum_{k=1}^{\infty} a_k(f,c,x)\; n^{-k} \quad (n \to \infty), \tag{1.4}$$

provided that f admits derivatives of sufficiently high order at $x > 0$. Formula (1.4) means that, for all $q = 0, 1, 2, \ldots$, there holds

$$\left(P_{n,c}f\right)(x) = \sum_{k=0}^{q} a_k(f,c,x)\; n^{-k} + o(n^{-q}) \quad (n \to \infty)$$

where $a_0(f,c,x) = f(x)$. The coefficients $a_k(f,c,x)$, which are independent of n, will be given in an explicit form. It turns out that associated Stirling numbers of the first kind play an important role. As a special case we obtain the complete asymptotic expansion for the Rathore operators R_n and for the Post–Widder operators P_n.

Secondly, we study the rate of convergence of the sequence $\left(P_{n,c}f\right)(x)$ as $n \to \infty$ for functions of bounded variation. More precisely, we present an estimate of the difference $\left(P_{n,c}f\right)(x) - \left(f(x+) + f(x-)\right)/2$.

2 Main results

For $q \in \mathbb{N}$ and $x \in (0, \infty)$, let $K[q; x]$ be the class of all functions $f \in E$ which are q times differentiable at x. The following theorem presents as our main result the complete asymptotic expansion for the operators $P_{n,c}$.

Theorem 2.1 *Let $q \in \mathbb{N}$ and $x \in (0, \infty)$. For each function $f \in K[2q; x]$, the operators $P_{n,c}$ possess the asymptotic expansion*

$$\left(P_{n,c} f\right)(x) = f(x) + \sum_{k=1}^{q} \frac{(-1)^k}{(nc)^k} \sum_{j=k}^{2k} s_2(j, j-k) \frac{f^{(j)}(x)}{j!} x^{j-k} + o\left(n^{-q}\right)$$

as $n \to \infty$, where $s_2(j, i)$ denote the associated Stirling numbers of the first kind.

The associated Stirling numbers of the first kind can be defined by their double generating function

$$\sum_{i,j=0}^{\infty} s_2(i, j) \frac{t^i}{i!} u^j = e^{-tu} (1+t)^u$$

(see [1, page 295, Ex. *20]).

For $q = 4$, we obtain

$$\begin{aligned} &\left(P_{n,c} f\right)(x) \\ &= f(x) + \frac{x f^{(2)}(x)}{2cn} + \frac{8x f^{(3)}(x) + 3x^2 f^{(4)}(x)}{24(cn)^2} \\ &+ \frac{12x f^{(4)}(x) + 8x^2 f^{(5)}(x) + x^3 f^{(6)}(x)}{48(cn)^3} \\ &+ \frac{1152x f^{(5)}(x) + 1040x^2 f^{(6)}(x) + 240x^3 f^{(7)}(x) + 15x^4 f^{(8)}(x)}{5760(cn)^4} \\ &+ o\left(n^{-4}\right) \end{aligned}$$

as $n \to \infty$. In particular, we obtain the Voronovskaja-type formula

$$\lim_{n \to \infty} n\left(\left(P_{n,c} f\right)(x) - f(x)\right) = \frac{x}{2c} f^{(2)}(x), \tag{2.1}$$

for $f \in K[2; x]$.

In the special case $c = 1$ we have the complete asymptotic expansion for the Rathore operators,

$$\left(R_n f\right)(x) \sim f(x) + \sum_{k=1}^{\infty} \frac{(-1)^k}{n^k} \sum_{j=k}^{2k} s_2(j, j-k) \frac{f^{(j)}(x)}{j!} x^{j-k}$$

as $n \to \infty$. In the special case $c = 1/x$ we have the complete asymptotic expansion for the Post–Widder operators

$$(P_n f)(x) \sim f(x) + \sum_{k=1}^{\infty} \frac{(-1)^k}{n^k} \sum_{j=k}^{2k} s_2(j, j-k) \frac{f^{(j)}(x)}{j!} x^j$$

as $n \to \infty$.

Our second main result is an estimate of the rate of convergence for functions $f \in E$, which are of bounded variation (BV) on each finite subinterval of $(0, \infty)$.

Theorem 2.2 *Let $f \in E$ be a function of bounded variation on each finite subinterval of $(0, \infty)$. Then, for each $x > 0$, we have the estimate*

$$\begin{aligned} &\left| \left(P_{n,c} f\right)(x) - \frac{f(x+) + f(x-)}{2} \right| \\ &\leq \left(\frac{1}{\sqrt{18\pi ncx}} + O\left(\frac{1}{n}\right) \right) |f(x+) - f(x-)| + \frac{cx+2}{ncx} \sum_{k=1}^{n} v_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(f_x) \\ &\quad + O(\exp(-\beta n)) \end{aligned}$$

as $n \to \infty$, where $\beta = (1 - \log 2)\, cx > 0$ and the function f_x is defined as

$$f_x(y) = \begin{cases} f(y) - f(x-), & 0 < y < x, \\ f(y) - f(x+), & x < y < \infty, \\ 0, & y = x. \end{cases}$$

For the proofs of Theorems 2.1 and 2.2 we need a localization result for the operators $P_{n,c}$. Since it is interesting in itself we state it as a theorem.

Theorem 2.3 *Let $x \geq \delta > 0$. If $f \in E$ vanishes in a neighborhood $(x - \delta, x + \delta)$ of x, then it exists a positive constant β such that*

$$\left(P_{n,c} f\right)(x) = O(\exp(-\beta cn)) \qquad (n \to \infty).$$

The constant β can be chosen to be

$$\beta = \delta - x \log\left(\frac{x+\delta}{x}\right) > 0.$$

Note that $\delta > x \log((x+\delta)/x)$ for $x, \delta > 0$.

3 Auxiliary results and proofs

Firstly, we study the moments of the operators $P_{n,c}$. Throughout the paper, let e_r denote the monomials, given by $e_r(x) = x^r$ $(r = 0, 1, 2, \ldots)$. Furthermore, define

$\psi_x = e_1 - xe_0$, for $x \in \mathbb{R}$. In the following, the quantities $\left[\begin{matrix} m \\ j \end{matrix}\right]$ denote the unsigned Stirling numbers of the first kind defined by

$$z^{\underline{m}} = \sum_{j=0}^{m} (-1)^{m-j} \left[\begin{matrix} m \\ j \end{matrix}\right] z^j \quad (m = 0, 1, 2, \ldots),$$

where $z^{\underline{0}} = 1, z^{\underline{m}} = z(z-1)\cdots(z-m+1), m \in \mathbb{N}$, are the falling factorials. Using $(-z)^{\underline{m}} = (-1)^m (z+m-1)^{\underline{m}}$ we obtain the relations

$$(z+m-1)^{\underline{m}} = \sum_{j=0}^{m} \left[\begin{matrix} m \\ j \end{matrix}\right] z^j \quad (m = 0, 1, 2, \ldots). \tag{3.1}$$

We recall some known facts about Stirling numbers which will be useful in the sequel. The Stirling numbers of the first kind possess the representation

$$\left[\begin{matrix} r \\ r-m \end{matrix}\right] = (-1)^m \sum_{i=m}^{2m} s_2(i, i-m) \binom{r}{i} = (-1)^m \sum_{i=0}^{m} s_2(i+m, i) \binom{r}{i+m}, \tag{3.2}$$

for $0 \le m \le r$, (see [1, page 226–227, Ex. 16]). The coefficients $s_2(i, i-m)$, called associated Stirling numbers of the first kind, are independent of r.

Lemma 3.1 *The moments of the operators $P_{n,c}$ are given by*

$$\left(P_{n,c}e_r\right)(x) = \sum_{k=0}^{r} \frac{1}{(nc)^k} \left[\begin{matrix} r \\ r-k \end{matrix}\right] x^{r-k} \qquad (r = 0, 1, 2, \ldots).$$

In particular, we have $P_{n,c}e_0 = e_0$, $P_{n,c}e_1 = e_1$ and

$$\begin{aligned}
\left(P_{n,c}e_2\right)(x) &= x^2 + \frac{x}{nc}, \\
\left(P_{n,c}e_3\right)(x) &= x^3 + \frac{3x^2}{nc} + \frac{2x}{(nc)^2}, \\
\left(P_{n,c}e_4\right)(x) &= x^4 + \frac{6x^3}{nc} + \frac{11x^2}{(nc)^2} + \frac{6x}{(nc)^3}.
\end{aligned}$$

Proof We have

$$\begin{aligned}
\left(P_{n,c}e_r\right)(x) &= \frac{(nc)^{ncx}}{\Gamma(ncx)} \int_0^\infty e^{-nct} t^{r+ncx-1} \mathrm{d}t = \frac{\Gamma(r+ncx)}{\Gamma(ncx)(nc)^r} \\
&= \frac{(ncx+r-1)^{\underline{r}}}{(nc)^r}.
\end{aligned}$$

Application of formula (3.1) yields

$$\left(P_{n,c}e_r\right)(x)=\frac{1}{(nc)^r}\sum_{j=0}^{r}\begin{bmatrix} r\\ j\end{bmatrix}(ncx)^j$$

and the index transform $j=r-k$ completes the proof. □

Lemma 3.2 *The central moments of the operators $P_{n,c}$ are given by*

$$\left(P_{n,c}\psi_x^j\right)(x)=\sum_{k=0}^{j}\frac{x^{j-k}}{(nc)^k}\sum_{r=0}^{j-k}(-1)^{j-k-r}\binom{j}{r+k}\begin{bmatrix} r+k\\ r\end{bmatrix}$$

$(j=0,1,2,\ldots)$.

In particular, we have $\left(P_{n,c}\psi_x^0\right)(x)=1$, $\left(P_{n,c}\psi_x^1\right)(x)=0$ and $\left(P_{n,c}\psi_x^2\right)(x)=x/(nc)$.

Proof Application of the binomial formula yields for the central moments

$$\begin{aligned}\left(P_{n,c}\psi_x^j\right)(x)&=\sum_{r=0}^{j}(-x)^{j-r}\binom{j}{r}\left(P_{n,c}e_r\right)(x)\\&=\sum_{k=0}^{j}\frac{x^{j-k}}{(nc)^k}\sum_{r=k}^{j}(-1)^{j-r}\binom{j}{r}\begin{bmatrix} r\\ r-k\end{bmatrix}\end{aligned}$$

and an index shift $r\to r+k$ yields the desired representation. □

Lemma 3.3 *For each $x>0$ and $j=0,1,2,\ldots$, the central moments of the operators $P_{n,c}$ satisfy the relation $\left(P_{n,c}\psi_x^j\right)(x)=O\left(n^{-\lfloor(j+1)/2\rfloor}\right)$ as $n\to\infty$. More precisely, they have the representation*

$$\left(P_{n,c}\psi_x^j\right)(x)=\sum_{k=\lfloor(j+1)/2\rfloor}^{j}(-1)^k\frac{x^{j-k}}{(nc)^k}s_2\left(j,j-k\right).$$

Proof Taking advantage of the formula (3.2) we obtain

$$\left(P_{n,c}\psi_x^j\right)(x)=\sum_{k=0}^{j}\frac{x^{j-k}}{(nc)^k}\sum_{r=0}^{j-k}(-1)^{j-r}\binom{j}{r+k}\sum_{i=0}^{k}s_2\left(i+k,i\right)\binom{r+k}{i+k}.$$

Note that $\binom{r+k}{i+k}=0$, for $i>r$. Using the binomial identity $\binom{j}{r+k}\binom{r+k}{i+k}=\binom{j}{i+k}\binom{j-i-k}{r-i}$, for $0\le i\le r$, we obtain

$$\left(P_{n,c}\psi_x^j\right)(x)=\sum_{k=0}^{j}\frac{x^{j-k}}{(nc)^k}\sum_{i=0}^{k}s_2\left(i+k,i\right)\binom{j}{i+k}\sum_{r=i}^{j-k}(-1)^{j-r}\binom{j-k-i}{r-i}.$$

The inner sum is to be read as zero if $i > j - k$. Since

$$\sum_{r=i}^{j-k} (-1)^{j-r} \binom{j-k-i}{r-i} = \sum_{r=0}^{j-k-i} (-1)^{j-r-i} \binom{j-k-i}{r} = \begin{cases} 0 \ (i < j-k), \\ 1 \ (i = j-k), \end{cases}$$

we conclude that

$$\left(P_{n,c}\psi_x^j\right)(x) = \sum_{k=\lfloor (j+1)/2 \rfloor}^{j} (-1)^k \frac{x^{j-k}}{(nc)^k} s_2(j, j-k),$$

which completes the proof. □

In order to derive Theorem 2.1, a general approximation theorem due to Sikkema [14, Theorem 3] (see also [15]) will be applied. For $j \in \mathbb{N}$ and $x > 0$, let $H^{(j)}(x)$ denote the class of all locally bounded real functions $f : [0, \infty) \to \mathbb{R}$, which are j times differentiable at x, and satisfy the additional condition $f(t) = O\left(t^{-j}\right)$ as $t \to +\infty$. An inspection of the proof of Sikkema's result reveals that it can be stated in the following form which is more appropriate for our purposes.

Lemma 3.4 *Let $q \in \mathbb{N}$ and let $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators, $L_n : H^{(2q)}(x) \to C[c, d]$, $x \in [c, d]$. Suppose that the operators L_n apply to ψ_x^{2q+1} and to ψ_x^{2q+2}. Then the condition*

$$\left(L_n\psi_x^j\right)(x) = O\left(n^{-\lfloor (j+1)/2 \rfloor}\right) \quad (n \to \infty), \quad \text{for } j = 0, 1, \ldots, 2q+2,$$

implies, for each function $f \in H^{(2q)}(x)$, the asymptotic relation

$$(L_n f)(x) = \sum_{j=0}^{2q} \frac{f^{(j)}(x)}{j!} \left(L_n\psi_x^j\right)(x) + o\left(n^{-q}\right) \quad (n \to \infty).$$

In the application used in the proof of Theorem 2.1, we restrict $H^{(j)}(x)$ to consist only of locally integrable functions. We proceed with the proof of the localization result (Theorem 2.3), which will be applied in the proofs of Theorems 2.1 and 2.2.

Proof of Theorem 2.3 Let $f \in E$. From $|f(t)| \leq Me^{At}$ $(t \geq 0)$ we obtain the estimate

$$\begin{aligned} \left|\left(P_{n,c}f\right)(x)\right| &\leq M\frac{(nc)^{ncx}}{\Gamma(ncx)} \left(\int_0^{x-\delta} + \int_{x+\delta}^{\infty}\right) e^{-(nc-A)t} t^{ncx-1} \mathrm{d}t \\ &= M\frac{s^{sx}}{\Gamma(sx)} (I_1 + I_2), \end{aligned}$$

say, where $s = nc > 0$ and

$$I_1 = \int_0^{x-\delta} e^{-(s-A)t} t^{sx-1} \mathrm{d}t = (s-A)^{-sx} \gamma(sx, (s-A)(x-\delta)),$$

$$I_2 = \int_{x+\delta}^{\infty} e^{-(s-A)t} t^{sx-1} \mathrm{d}t = (s-A)^{-sx} \, \Gamma\left(sx, (s-A)(x+\delta)\right),$$

where

$$\gamma(z, b) = \int_0^b e^{-t} t^{z-1} \mathrm{d}t, \quad \Gamma(z, b) = \int_b^{\infty} e^{-t} t^{z-1} \mathrm{d}t \quad (Rez > 0, \;\; b \geq 0)$$

denote the lower and the upper incomplete gamma function, respectively. We use the well-known asymptotic behaviour of the incomplete gamma function for large parameters z and b. It holds

$$\gamma(z, b) \sim \frac{b^z e^{-b}}{(1-\lambda) z}, \tag{3.3}$$

[17, Eq. (7.3.18)], as $z, b \to \infty$ such that the ratio $\lambda = b/z$ is bounded away from unity, i.e., $\lambda \leq \lambda_0 < 1$, where λ_0 is a fixed number in $(0, 1)$. In a similar kind it holds

$$\Gamma(z, b) \sim \frac{b^{z-1} e^{-b}}{1-\alpha}, \tag{3.4}$$

[17, Eq. (7.4.43)], as $z, b \to \infty$ such that the ratio $\alpha = z/b$ is bounded away from unity, i.e., $\alpha \leq \alpha_0 < 1$.
If $\delta = x$ the integral I_1 vanishes. Let us consider the case $\delta < x$. Since

$$\lambda = \frac{(s-A)(x-\delta)}{sx} \to \frac{x-\delta}{x} < 1 \quad (s \to \infty)$$

and $1 - \lambda = \delta/x + A(x-\delta)/(sx)$, Eq. (3.3) implies that

$$I_1 \sim (\delta s + A(x-\delta))^{-1} (x-\delta)^{sx} \, e^{-(s-A)(x-\delta)} \quad (s \to \infty).$$

Application of Stirling's formula,

$$\Gamma(z) \sim \sqrt{2\pi} e^{-z} z^{z-1/2} \quad (z \to +\infty),$$

leads to

$$M \frac{s^{sx}}{\Gamma(sx)} I_1 \sim \frac{M}{\sqrt{2\pi}} \frac{\sqrt{sx}}{\delta s + A(x-\delta)} \left(\frac{x-\delta}{x}\right)^{sx} e^{\delta s + A(x-\delta)} \quad (s \to \infty).$$

Since

$$\left(\frac{x-\delta}{x}\right)^x e^{\delta} < 1 \quad (0 < \delta < x),$$

we conclude that $M\frac{s^{sx}}{\Gamma(sx)}I_1 = O\left(\exp\left(-\beta_1 s\right)\right)$ as $s \to \infty$, where $\beta_1 = -\delta - x\log\left(\frac{x-\delta}{x}\right) > 0$. Now we turn to the estimate of I_2. Since

$$\alpha = \frac{sx}{(s-A)(x+\delta)} \to \frac{x}{x+\delta} < 1 \quad (s \to \infty)$$

and $1-\alpha = \frac{\delta s - A(x+\delta)}{(s-A)(x+\delta)}$, Eq. (3.4) implies that

$$I_2 \sim \frac{1}{\delta s - A(x+\delta)}(x+\delta)^{sx} e^{-(s-A)(x+\delta)} \quad (s \to \infty).$$

Application of Stirling's formula leads to

$$M\frac{s^{sx}}{\Gamma(sx)}I_2 \sim \frac{M}{\sqrt{2\pi}} \cdot \frac{\sqrt{sx}}{\delta s - A(x+\delta)}\left(\frac{x+\delta}{x}\right)^{sx} e^{-\delta s + A(x+\delta)} \quad (s \to \infty).$$

Since

$$\left(\frac{x+\delta}{x}\right)^{x} e^{-\delta} < 1 \quad (x, \delta > 0),$$

we conclude that $M\frac{s^{sx}}{\Gamma(sx)}I_2 = O\left(\exp\left(-\beta_2 s\right)\right)$ as $s \to \infty$, where $\beta_2 = \delta - x\log\left(\frac{x+\delta}{x}\right) > 0$. Observe that $\beta_1 \geq \beta_2$ because $\left(\frac{x-\delta}{x}\right)^{x} e^{\delta} \leq \left(\frac{x+\delta}{x}\right)^{x} e^{-\delta}$. The latter inequality is equivalent to the obvious inequality $2t \leq \log\left(\frac{1+t}{1-t}\right) = 2\left(t + t^3/3 + t^5/5 + t^7/7 + \cdots\right)$, for $t = \delta/x \in [0, 1)$. Combining the above results we obtain the desired estimate with the constant $\beta = \beta_2$. □

Proof of Theorem 2.1 Let $x > 0$ and put $U_r(x) = (x-r, x+r) \cap [0, +\infty)$, for $r > 0$. Let $\delta > 0$ be given. Suppose that $f^{(2q)}(x)$ exists. Choose a function $\varphi \in C^\infty([0, +\infty))$ with $\varphi(x) = 1$ on $U_\delta(x)$ and $\varphi(x) = 0$ on $[0, +\infty) \setminus U_{2\delta}(x)$. Put $\widetilde{f} = \varphi f$. Then we have $\widetilde{f} \equiv f$ on $U_\delta(x)$ which implies $\widetilde{f}^{(j)}(x) = f^{(j)}(x)$, for $j = 0, \ldots, 2q$, and $\widetilde{f} \equiv 0$ on $[0, +\infty) \setminus U_{2\delta}(x)$. By the localization theorem (Theorem 2.3), $\left(P_{n,c}\left(f - \widetilde{f}\right)\right)(x)$ decays exponentially fast as $n \to \infty$. Consequently, $\widetilde{f}$ and f possess the same asymptotic expansion of the form (1.4). Therefore, without loss of generality, we can assume that $f \equiv 0$ on $[0, +\infty) \setminus U_{2\delta}(x)$. By Lemma 3.3, we have $\left(P_{n,c}\psi_x^{2j}\right)(x) = O\left(n^{-j}\right)$ as $n \to \infty$. Under these conditions, Lemma 3.4 implies that

$$\left(P_{n,c}f\right)(x) = f(x) + \sum_{j=1}^{2q} \frac{f^{(j)}(x)}{j!}\left(P_{n,c}\psi_x^{j}\right)(x) + o\left(n^{-q}\right) \quad (n \to \infty).$$

By Lemma 3.3, we obtain

$$\sum_{j=1}^{2q} \frac{f^{(j)}(x)}{j!} \left(P_{n,c}\psi_x^j\right)(x) = \sum_{j=1}^{2q} \frac{f^{(j)}(x)}{j!} \sum_{k=\lfloor (j+1)/2 \rfloor}^{j} (-1)^k \frac{x^{j-k}}{(nc)^k} s_2(j, j-k).$$

Interchanging the order of summation, we obtain

$$\left(P_{n,c}f\right)(x) = f(x) + \sum_{k=1}^{2q} \frac{(-1)^k}{(nc)^k} \sum_{j=k}^{\min\{2k,2q\}} \frac{f^{(j)}(x)}{j!} x^{j-k} s_2(j, j-k) + o\left(n^{-q}\right)$$

as $n \to \infty$. Taking into account that

$$\sum_{k=q+1}^{2q} \frac{(-1)^k}{(nc)^k} \sum_{j=k}^{\min\{2k,2q\}} \frac{f^{(j)}(x)}{j!} x^{j-k} s_2(j, j-k) = o\left(n^{-q}\right) \quad (n \to \infty)$$

this implies the desired expansion (1.4) with the associated Stirling numbers of the first kind $s_2(i, j)$ as defined in Eq. (3.2). □

Now we turn to the estimate of the rate of convergence for BV functions. For the proof of Theorem 2.2 we apply the following properties of the kernel function $\phi_{n,c}(x, t)$ as defined in (1.3).

Lemma 3.5 *The kernel function $\phi_{n,c}(x, t)$ satisfies the following estimates:*

$$\int_0^y \phi_{n,c}(x, t)\, \mathrm{d}t \leq \frac{x}{nc(x-y)^2} \quad (0 < y < x)$$

and

$$\int_z^\infty \phi_{n,c}(x, t)\, \mathrm{d}t \leq \frac{x}{nc(z-x)^2} \quad (x < z < +\infty).$$

Proof Since $x - t \geq x - y > 0$, for $0 \leq t \leq y < x$, we have

$$\int_0^y \phi_{n,c}(x, t)\, \mathrm{d}t \leq \int_0^y \phi_{n,c}(x, t) \left(\frac{x-t}{x-y}\right)^2 \mathrm{d}t \leq \frac{1}{(x-y)^2} \left(P_{n,c}\psi_x^2\right)(x)$$
$$= \frac{x}{nc(x-y)^2}.$$

The second estimate

$$\int_z^\infty \phi_{n,c}(x, t)\, \mathrm{d}t \leq \int_z^\infty \phi_{n,c}(x, t) \left(\frac{t-x}{z-x}\right)^2 \mathrm{d}t \leq \frac{1}{(z-x)^2} \left(P_{n,c}\psi_x^2\right)(x)$$
$$= \frac{x}{nc(z-x)^2}$$

is obtained in an analogous manner. □

Lemma 3.6 *For fixed* $x > 0$,

$$\int_0^x \phi_{n,c}(x,t)\,\mathrm{d}t = \frac{1}{2} + \frac{1}{3\sqrt{2\pi ncx}} + O\left(\frac{1}{n}\right) \quad (n \to \infty).$$

Proof With $s = ncx$ we have

$$\int_0^x \phi_{n,c}(x,t)\,\mathrm{d}t = \frac{1}{\Gamma(ncx)} \int_0^{ncx} e^{-u} u^{ncx-1}\mathrm{d}u = \frac{1}{\Gamma(s)} \int_0^s e^{-u} u^{s-1}\mathrm{d}u$$
$$= \frac{\Gamma(s) - \Gamma(s,s)}{\Gamma(s)}.$$

Following [11, Eq. 8.11.12], the (upper) incomplete gamma function satisfies the asymptotic relation

$$\Gamma(s,s) = s^{s-1}e^{-s}\left(\sqrt{\frac{\pi s}{2}} - \frac{1}{3} + O\left(s^{-1/2}\right)\right) \quad (s \to +\infty).$$

Together with Stirling's formula

$$\Gamma(s) = s^s e^{-s}\sqrt{\frac{2\pi}{s}}\left(1 + \frac{1}{12s} + O\left(s^{-2}\right)\right) \quad (s \to +\infty)$$

we obtain

$$\begin{aligned}
\frac{\Gamma(s) - \Gamma(s,s)}{\Gamma(s)} &\sim 1 - \frac{s^{s-1}e^{-s}\left(\sqrt{\frac{\pi s}{2}} - \frac{1}{3} + O\left(s^{-1/2}\right)\right)}{s^s e^{-s}\sqrt{2\pi/s}\left(1 + O\left(s^{-1}\right)\right)} \\
&= 1 - \frac{\sqrt{\frac{\pi}{2}} - \frac{1}{3\sqrt{s}} + O\left(s^{-1}\right)}{\sqrt{2\pi} + O\left(s^{-1}\right)} \\
&= 1 - \frac{\frac{1}{2} - \frac{1}{3\sqrt{2\pi s}} + O\left(s^{-1}\right)}{1 + O\left(s^{-1}\right)} \\
&= 1 - \left(\frac{1}{2} - \frac{1}{3\sqrt{2\pi s}} + O\left(\frac{1}{s}\right)\right)\left(1 + O\left(\frac{1}{s}\right)\right) \\
&= \frac{1}{2} + \frac{1}{3\sqrt{2\pi s}} + O\left(\frac{1}{s}\right)
\end{aligned}$$

as $s \to +\infty$. Hence, for fixed $x > 0$,

$$\frac{1}{\Gamma(ncx)} \int_0^{ncx} e^{-u} u^{ncx-1}\mathrm{d}u = \frac{1}{2} + \frac{1}{3\sqrt{2\pi ncx}} + O\left(\frac{1}{n}\right) \quad (n \to \infty).$$

□

Proof of Theorem 2.2 Let $x \in (0, \infty)$. We start with the estimate

$$\begin{aligned}&\left|\left(P_{n,c} f\right)(x) - \frac{1}{2}\left(f\left(x+\right) + f\left(x-\right)\right)\right| \\ &\quad \leq \frac{1}{2}\left|f\left(x+\right) - f\left(x-\right)\right| \cdot \left|\left(P_{n,c}\mathrm{sign}\psi_x\right)(x)\right| + \left|\left(P_{n,c} f_x\right)(x)\right|.\end{aligned}$$

Due to the fact that $P_{n,c}$ preserve constant functions, we have

$$\left(P_{n,c}\mathrm{sign}\psi_x\right)(x) = \left(\int_x^\infty - \int_0^x\right)\phi_{n,c}(x,t)\,\mathrm{d}t = 2\left[\frac{1}{2} - \int_0^x \phi_{n,c}(x,t)\,\mathrm{d}t\right].$$

Thus, by Lemma 3.6, we have

$$\left(P_{n,c}\mathrm{sign}\psi_x\right)(x) = -\frac{2}{3\sqrt{2\pi ncx}} + O\left(\frac{1}{n}\right) \quad (n \to \infty).$$

Next we estimate $\left(P_{n,c} f_x\right)(x)$ as follows:

$$\begin{aligned}&\left(P_{n,c} f_x\right)(x) \\ &\quad = \int_0^\infty \phi_{n,c}(x,t)\, f_x(t)\,\mathrm{d}t \\ &\quad = \left(\int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} + \int_{x+x/\sqrt{n}}^{2x} + \int_{2x}^\infty\right)\phi_{n,c}(x,t)\, f_x(t)\,\mathrm{d}t \\ &\quad =: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Define

$$\eta_{n,c}(x,y) = \int_0^y \phi_{n,c}(x,t)\,\mathrm{d}t.$$

Integration by parts yields

$$\begin{aligned}I_1 &= \int_0^{x-x/\sqrt{n}} f_x(t)\, d_t\left(\eta_{n,c}(x,t)\right) \\ &= f_x\left(x - \frac{x}{\sqrt{n}}\right)\eta_{n,c}\left(x, x - \frac{x}{\sqrt{n}}\right) - \int_0^{x-x/\sqrt{n}} \eta_{n,c}(x,t)\, d_t\left(f_x(t)\right).\end{aligned}$$

Since $|f_x(t)| \leq v_t^x(f_x)$, we have

$$|I_1| \leq v_{x-x/\sqrt{n}}^x(f_x) \cdot \eta_{n,c}\left(x, x - \frac{x}{\sqrt{n}}\right) + \int_0^{x-x/\sqrt{n}} \eta_{n,c}(x,t)\, d_t\left(-v_t^x(f_x)\right).$$

Applying Lemma 3.5, and in the next step integrating by parts, we get

$$
\begin{aligned}
|I_1| &\leq \frac{1}{cx} v_{x-x/\sqrt{n}}^{x}(f_x) + \frac{x}{nc}\int_0^{x-x/\sqrt{n}} \frac{1}{(x-t)^2} d_t\left(-v_t^x(f_x)\right) \\
&= \frac{x}{nc}\left[\frac{1}{x^2} v_0^x(f_x) + 2\int_0^{x-x/\sqrt{n}} \frac{1}{(x-t)^3} v_t^x(f_x)\,\mathrm{d}t\right] \\
&\leq \frac{x}{nc}\left[\frac{1}{x^2} v_0^x(f_x) + \frac{1}{x^2}\sum_{k=1}^{n} v_{x-x/\sqrt{k}}^{x}(f_x)\right] \\
&\leq \frac{2}{ncx}\sum_{k=1}^{n} v_{x-x/\sqrt{k}}^{x}(f_x).
\end{aligned}
$$

Next for $t \in \left[x - x/\sqrt{n}, x + x/\sqrt{n}\right]$ and by fact $\int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} d_t\left(\eta_{n,c}(x,t)\right) \leq 1$, we conclude that

$$
|I_2| \leq \frac{1}{n}\sum_{k=1}^{n} v_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(f_x).
$$

Arguing analogously as in estimate of I_1, we have

$$
|I_3| \leq \frac{2}{ncx}\sum_{k=1}^{n} v_{x}^{x+x/\sqrt{k}}(f_x).
$$

Since $f_x \in E$, the localization result (Theorem 2.3) applied with $\delta = x$ implies that $I_4 = O\left(\exp\left(-\beta cn\right)\right)$ as $n \to \infty$ with the constant $\beta = (1 - \log 2)\, x > 0$. Collecting the estimates of I_1, I_2, I_3, I_4, we get the desired result. □

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