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Coupling constant dependence for the Schrödinger equation with an inverse-square potential

A. G. Smirnov[1] ⓘ

Abstract

We consider the one-dimensional Schrödinger equation $-f'' + q_\alpha f = Ef$ on the positive half-axis with the potential $q_\alpha(r) = (\alpha - 1/4)r^{-2}$. It is known that the value $\alpha = 0$ plays a special role in this problem: all self-adjoint realizations of the formal differential expression $-\partial_r^2 + q_\alpha(r)$ for the Hamiltonian have infinitely many eigenvalues for $\alpha < 0$ and at most one eigenvalue for $\alpha \geq 0$. We find a parametrization of self-adjoint boundary conditions and eigenfunction expansions that is analytic in α and, in particular, is not singular at $\alpha = 0$. Employing suitable singular Titchmarsh–Weyl m-functions, we explicitly find the spectral measures for all self-adjoint Hamiltonians and prove their smooth dependence on α and the boundary condition. Using the formulas for the spectral measures, we analyze in detail how the "phase transition" through the point $\alpha = 0$ occurs for both the eigenvalues and the continuous spectrum of the Hamiltonians.

1 Introduction

This paper is devoted to eigenfunction expansions connected with the one-dimensional Schrödinger equation[1]

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[1] The shift of the coupling constant by 1/4 in the potential term is a matter of technical convenience: it allows us to get simpler expressions for the solutions of (1.1).

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$$-\partial_r^2 f(r) + \frac{\alpha - 1/4}{r^2} f(r) = Ef(r), \quad r \in \mathbb{R}_+, \tag{1.1}$$

where α and E are real parameters, and $\mathbb{R}_+$ denotes the positive half-axis $(0, \infty)$.

There are two special values of the coupling constant α at which this problem undergoes a structural change. One of them is $\alpha = 1$. For $\alpha < 1$, all solutions of (1.1) are square-integrable on the interval $(0, a)$ for every $a > 0$. At the same time, only one solution (up to a constant factor) for each E possesses this property for $\alpha \geq 1$. In terms of the well-known Weyl alternative, this means that the differential expression

$$-\partial_r^2 + \frac{\alpha - 1/4}{r^2} \tag{1.2}$$

corresponds at $r = 0$ to the limit point case for $\alpha \geq 1$ and to the limit circle case for $\alpha < 1$. As a consequence, (1.2) has a unique self-adjoint realization in $L_2(\mathbb{R}_+)^2$ for $\alpha \geq 1$ and infinitely many self-adjoint realizations in $L_2(\mathbb{R}_+)$ for $\alpha < 1$. The latter correspond to various self-adjoint boundary conditions at $r = 0$.

Another special value of α is $\alpha = 0$. It has long been known [4, 17] that the spectrum of all self-adjoint realizations of (1.2) is not bounded from below and contains infinitely many negative eigenvalues for $\alpha < 0$. On the other hand, every self-adjoint realization of (1.2) has at most one eigenvalue for $\alpha \geq 0$ (the continuous spectrum is $[0, \infty)$ for all real α).

If $\kappa \in \mathbb{R}$ and $\alpha = \kappa^2$, then the function[3] $f(r) = \sqrt{r} J_\kappa(\sqrt{E}r)$, where J_κ is the Bessel function of the first kind of order κ, is a solution of (1.1) for every $E > 0$ (this follows immediately from the fact that J_κ satisfies the Bessel equation). These solutions can be used to expand square-integrable functions on $\mathbb{R}_+$. More precisely, given $\kappa \geq 0$ and a square-integrable complex function ψ on $\mathbb{R}_+$ that vanishes for large r, we can define the function $\hat{\psi}$ on $\mathbb{R}_+$ by setting:

$$\hat{\psi}(E) = \frac{1}{\sqrt{2}} \int_0^\infty \sqrt{r} J_\kappa(\sqrt{E}r) \psi(r) \, dr, \quad E > 0. \tag{1.3}$$

The map $\psi \to \hat{\psi}$ then coincides up to a change of variables with the well-known Hankel transformation [13] and induces a uniquely determined unitary operator in $L_2(\mathbb{R}_+)$. Since the development of a general theory of singular Sturm–Liouville operators by Weyl [27], this transformation has been used by many authors to illustrate various approaches to eigenfunction expansions for problems of this type [9, 10, 16, 18, 24, 28].

For $\alpha \geq 1$, transformation (1.3) with $\kappa = \sqrt{\alpha}$ provides an eigenfunction expansion (i.e., a diagonalizing unitary operator) for the unique self-adjoint realization of (1.2). If $0 \leq \alpha < 1$, then it is an eigenfunction expansion for one of infinitely many self-adjoint realizations of (1.2), namely, for the Friedrichs extension of the

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[2] Here and subsequently, we let $L_2(\mathbb{R}_+)$ denote the Hilbert space of (equivalence classes of) square-integrable complex functions on $\mathbb{R}_+$.

[3] In this paper, we use the symbol $\sqrt{x}$ only for nonnegative x; the notation $z^{1/2}$ will be used for a suitable branch of the square root in the complex plane.

minimal operator h_α associated with (1.2) (see [8]; the precise definition of h_α will be given later in this section). As we shall see, the latter is not bounded from below and, therefore, has no Friedrichs extension for $\alpha < 0$. Accordingly, the right-hand side of (1.3) as a function of α has a branch point at $\alpha = 0$ and cannot be analytically continued to the region $\alpha < 0$.

For $\alpha \geq 0$, eigenfunction expansions corresponding to all self-adjoint realizations of (1.2) were found in [24] (however, without explicitly using the language of operators in Hilbert space). In [17], all self-adjoint Hamiltonians associated with (1.2) and corresponding eigenfunction expansions were constructed for every real α using the theory of self-adjoint extensions of symmetric operators (a somewhat different treatment of this problem in the framework of self-adjoint extensions can be found in [11, 12]).

The generalized eigenfunctions used in [11, 12, 17, 24] had the same type of branch point singularity at $\alpha = 0$ as that appearing in Hankel transformation (1.3). As a result, the cases $0 < \alpha < 1$, $\alpha = 0$, and $\alpha < 0$ were treated separately and eigenfunction expansions for $\alpha = 0$ could not be obtained from those for $0 < \alpha < 1$ and $\alpha < 0$ by taking the limit $\alpha \to 0$. In [21], we considered problem (1.1) with $\alpha = \kappa^2$ and constructed a parametrization of self-adjoint realizations of (1.2) and corresponding eigenfunction expansions that is continuous in κ on the interval $(-1, 1)$ (and, in particular, at $\kappa = 0$). This work was motivated by our previous research [20] of the Aharonov–Bohm model, where zero and nonzero κ correspond to integer and noninteger values of the dimensionless magnetic flux through the solenoid. In terms of α, the results of [21] give a continuous transition from the region $0 < \alpha < 1$ to $\alpha = 0$.

In this paper, we extend the treatment in [21] to also cover the region $\alpha < 0$. We parametrize all eigenfunction expansions associated with (1.2) in such a way that the generalized eigenfunctions turn out to be analytic in α for $\alpha < 1$, while the corresponding spectral measures are infinitely differentiable in α on the same interval. Using explicit formulas for the spectral measures, we analyze in detail how the transition through the point $\alpha = 0$ occurs for both the eigenvalues and the continuous spectrum of self-adjoint realizations of (1.2) in this parametrization.

We now give a brief structural description of our results.

For every $\alpha \in \mathbb{C}$, we define the function q_α on $\mathbb{R}_+$ by setting:

$$q_\alpha(r) = \frac{\alpha - 1/4}{r^2}, \quad r > 0. \tag{1.4}$$

For real α, q_α is the potential term in (1.2).

Let λ_+ be the restriction to $\mathbb{R}_+$ of the Lebesgue measure λ on $\mathbb{R}$ and $\mathcal{D}$ be the space of all complex continuously differentiable functions on $\mathbb{R}_+$ whose derivative is absolutely continuous on $\mathbb{R}_+$ (i.e., absolutely continuous on every segment $[a, b]$ with $0 < a \leq b < \infty$). Given $\alpha, z \in \mathbb{C}$, we let $\mathscr{L}_{\alpha,z}$ denote the linear operator from $\mathcal{D}$ to the space of complex λ_+-equivalence classes, such that:

$$(\mathscr{L}_{\alpha,z}f)(r) = -f''(r) + q_\alpha(r)f(r) - zf(r) \tag{1.5}$$

for λ-a.e.[4] $r \in \mathbb{R}_+$ and set:

$$\mathscr{L}_\alpha = \mathscr{L}_{\alpha,0}. \tag{1.6}$$

Let $\alpha \in \mathbb{R}$. We define the linear subspace Δ_α of $\mathcal{D}$ by setting:

$$\Delta_\alpha = \{f \in \mathcal{D} : f \text{ and } \mathscr{L}_\alpha f \text{ are both square-integrable on } \mathbb{R}_+\}. \tag{1.7}$$

For every linear subspace Z of Δ_α, let $H_\alpha(Z)$ be the linear operator in $L_2(\mathbb{R}_+)$ defined by the relations[5]:

$$\begin{aligned}
D_{H_\alpha(Z)} &= \{[f] : f \in Z\}, \\
H_\alpha(Z)[f] &= \mathscr{L}_\alpha f, \quad f \in Z,
\end{aligned} \tag{1.8}$$

where $[f] = [f]_{\lambda_+}$ denotes the λ_+-equivalence class of f. We clearly have $C_0^\infty(\mathbb{R}_+) \subset \Delta_\alpha$, where $C_0^\infty(\mathbb{R}_+)$ is the space of all smooth functions on $\mathbb{R}_+$ with compact support. The operator

$$\check{h}_\alpha = H_\alpha(C_0^\infty(\mathbb{R}_+)) \tag{1.9}$$

is obviously symmetric and, hence, closable. The closure of $\check{h}_\alpha$ is denoted by h_α:

$$h_\alpha = \overline{\check{h}_\alpha}. \tag{1.10}$$

We shall see that the adjoint h_α^* of h_α is given by:

$$h_\alpha^* = H_\alpha(\Delta_\alpha). \tag{1.11}$$

If T is a symmetric extension of h_α, then h_α^* is an extension of T^* and hence of T. By (1.11), we conclude that T is of the form $H_\alpha(Z)$ for some subspace Z of Δ_α.

Self-adjoint operators of the form $H_\alpha(Z)$ can be naturally viewed as self-adjoint realizations of differential expression (1.2). If $H_\alpha(Z)$ is self-adjoint, then equality (1.11) and the closedness of h_α imply that $H_\alpha(Z)$ is an extension of h_α, because $H_\alpha(\Delta_\alpha)$ is an extension of $H_\alpha(Z)$. Therefore, the self-adjoint realizations of (1.2) are precisely the self-adjoint extensions of h_α (or, equivalently, of $\check{h}_\alpha$).

For every $\alpha, z \in \mathbb{C}$, we shall construct real-analytic functions $\mathcal{A}^\alpha(z)$ and $\mathcal{B}^\alpha(z)$ on $\mathbb{R}_+$, such that:

$$\mathscr{L}_{\alpha,z}\mathcal{A}^\alpha(z) = \mathscr{L}_{\alpha,z}\mathcal{B}^\alpha(z) = 0, \quad \alpha, z \in \mathbb{C}. \tag{1.12}$$

The functions $\mathcal{A}^\alpha(z)$ and $\mathcal{B}^\alpha(z)$ are real for real α and z. Moreover, the quantities[6] $\mathcal{A}^\alpha(z|r)$ and $\mathcal{B}^\alpha(z|r)$ are entire analytic in α and z for every fixed $r > 0$ and, in

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[4] Throughout the paper, a.e. means either "almost every" or "almost everywhere".

[5] Here and subsequently, we let D_F denote the domain of definition of a map F.

[6] Given a map F whose values are also maps, we let $F(x|y)$ denote the value of $F(x)$ at a point y: $F(x|y) = (F(x))(y)$.

particular, are not singular at $\alpha = 0$. If $\alpha < 1$ and $z \in \mathbb{C}$, then $\mathcal{A}^\alpha(z)$ and $\mathcal{B}^\alpha(z)$ are linear independent and are both square-integrable on the interval $(0, a)$ for every $a > 0$ (as mentioned above, we have the limit circle case for $\alpha < 1$).

Given $f, g \in \mathcal{D}$, we let $W_r(f, g)$ denote the Wronskian of f and g at a point $r > 0$:

$$W_r(f, g) = f(r)g'(r) - f'(r)g(r). \tag{1.13}$$

Clearly, $r \to W_r(f, g)$ is an absolutely continuous function on $\mathbb{R}_+$.

For every $\alpha, \vartheta, z \in \mathbb{C}$, we define the function $\mathcal{U}_\vartheta^\alpha(z)$ on $\mathbb{R}_+$ by the relation:

$$\mathcal{U}_\vartheta^\alpha(z) = \mathcal{A}^\alpha(z) \cos \vartheta + \mathcal{B}^\alpha(z) \sin \vartheta. \tag{1.14}$$

By (1.12), we obviously have:

$$\mathscr{L}_{\alpha, z} \mathcal{U}_\vartheta^\alpha(z) = 0, \quad \alpha, \vartheta, z \in \mathbb{C}. \tag{1.15}$$

The properties of $\mathcal{A}^\alpha(z)$ and $\mathcal{B}^\alpha(z)$ imply that $\mathcal{U}_\vartheta^\alpha(z)$ is real for real α, ϑ, and z and the quantity $\mathcal{U}_\vartheta^\alpha(z|r)$ is entire analytic in α, ϑ, and z for every fixed $r > 0$. If $\alpha < 1$, then $\mathcal{U}_\vartheta^\alpha(z)$ is nontrivial for every $\vartheta, z \in \mathbb{C}$. We shall show that $\lim_{r \downarrow 0} W_r(\mathcal{U}_\vartheta^\alpha(z), \mathcal{U}_\vartheta^\alpha(z')) = 0$ for all $\alpha < 1$, $\vartheta \in \mathbb{R}$, and $z, z' \in \mathbb{C}$. This condition means that $\mathcal{U}_\vartheta^\alpha(z|r)$ for various z have the same asymptotics as $r \downarrow 0$.

Let $f \in \mathcal{D}$ and $\alpha, \vartheta, z \in \mathbb{C}$. In view of (1.15), integration by parts yields:

$$\int_r^a (\mathscr{L}_{\alpha, z} f)(r') \mathcal{U}_\vartheta^\alpha(z|r') \, \mathrm{d}r' = W_r(\mathcal{U}_\vartheta^\alpha(z), f) - W_a(\mathcal{U}_\vartheta^\alpha(z), f)$$

for every $a, r > 0$. If $\alpha < 1$, then $\mathcal{U}_\vartheta^\alpha(z)$ is square-integrable on $(0, a)$ for every $a > 0$ and this equality implies that $W_r(\mathcal{U}_\vartheta^\alpha(z), f)$ has a limit as $r \downarrow 0$ for every $\vartheta, z \in \mathbb{C}$ and $f \in \Delta_\alpha$. Given $\alpha < 1$ and $\vartheta \in \mathbb{R}$, we define the operator $h_{\alpha, \vartheta}$ in $L_2(\mathbb{R}_+)$ by the relation:

$$h_{\alpha, \vartheta} = H_\alpha(Z_{\alpha, \vartheta}), \tag{1.16}$$

where the linear subspace $Z_{\alpha, \vartheta}$ of Δ_α is given by:

$$Z_{\alpha, \vartheta} = \{f \in \Delta_\alpha : \lim_{r \downarrow 0} W_r(\mathcal{U}_\vartheta^\alpha(0), f) = 0\}. \tag{1.17}$$

By (1.14) and the definition of $h_{\alpha, \vartheta}$, we have:

$$h_{\alpha, \vartheta + \pi} = h_{\alpha, \vartheta}, \quad \alpha < 1, \vartheta \in \mathbb{R}. \tag{1.18}$$

The next statement gives a complete description of the self-adjoint extensions of h_α for every $\alpha \in \mathbb{R}$.

Theorem 1.1 *For $\alpha \geq 1$, the operator h_α is self-adjoint. If $\alpha < 1$, then $h_{\alpha, \vartheta}$ is a self-adjoint extension of h_α for every $\vartheta \in \mathbb{R}$ and, conversely, every self-adjoint extension of h_α is equal to $h_{\alpha, \vartheta}$ for some $\vartheta \in \mathbb{R}$. Given $\vartheta, \vartheta' \in \mathbb{R}$, we have $h_{\alpha, \vartheta} = h_{\alpha, \vartheta'}$ if and only if $\vartheta - \vartheta' \in \pi \mathbb{Z}$.*

As will be obvious from the explicit definitions of $\mathcal{A}^\alpha(z)$ and $\mathcal{B}^\alpha(z)$ in Sect. 2, these functions are actually square-integrable on intervals $(0, a)$ with $a > 0$ for all α belonging to the domain:

$$\Pi = \{\alpha \in \mathbb{C} : \alpha = \kappa^2 \text{ for some } \kappa \in \mathbb{C} \text{ such that } |\operatorname{Re}\kappa| < 1\}.$$

and all $z \in \mathbb{C}$. Hence, the above definition of $h_{\alpha,\vartheta}$ can be naturally extended to all $\alpha \in \Pi$ and $\vartheta \in \mathbb{C}$. Moreover, it is possible to show that such an extended family of operators is holomorphic on $\Pi \times \mathbb{C}$ in the sense of Kato (see Ch. 7 in [14]) and, therefore, $h_{\alpha,\vartheta}$ with $\alpha < 1$ and $\vartheta \in \mathbb{R}$ constitute a real-analytic family of operators. This can be proved using a technique similar to that employed in [3] for the case of extensions of h_α homogeneous with respect to dilations of $\mathbb{R}_+$. The spectral analysis of a holomorphic family similar to $h_{\alpha,\vartheta}$ can be found in [6], where, however, κ rather than α was used as a parameter and the case $\alpha = 0$ was treated separately (see also [5] for an analogous treatment of the Coulomb potential; a possibility of removing the singularity at $\kappa = 0$ was indicated in Remark 2.5 in [6]). However, the analysis of $h_{\alpha,\vartheta}$ for complex α and ϑ is beyond the scope of this paper. In the sequel, we confine ourselves to the self-adjoint case $\alpha < 1$, $\vartheta \in \mathbb{R}$.

Given a positive Borel measure v on $\mathbb{R}$ and a v-measurable complex function g, we let $\mathcal{T}_g^v$ denote the operator of multiplication by g in the Hilbert space $L_2(\mathbb{R}, v)$ of v-square-integrable complex functions on $\mathbb{R}$.[7] If g is real, then $\mathcal{T}_g^v$ is self-adjoint. We let $L_2^c(\mathbb{R}_+)$ denote the subspace of $L_2(\mathbb{R}_+)$ consisting of all its elements vanishing λ-a.e. outside some compact subset of $\mathbb{R}_+$.

It turns out that the functions $\mathcal{U}_\vartheta^\alpha(E)$ with real E can be used as generalized eigenfunctions for constructing eigenfunction expansions for $h_{\alpha,\vartheta}$. More precisely, for every $\alpha < 1$ and $\vartheta \in \mathbb{R}$, we shall construct a positive Radon measure[8] $\mathcal{V}_{\alpha,\vartheta}$ on $\mathbb{R}$, such that:

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mathcal{V}_{\alpha,\vartheta}(E)}{E^2 + 1} < \infty, \tag{1.19}$$

and the following statement holds.

Theorem 1.2 *Let $\alpha < 1$ and $\vartheta \in \mathbb{R}$. Then, there is a unique unitary operator $U_{\alpha,\vartheta} : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}, \mathcal{V}_{\alpha,\vartheta})$, such that*:

$$(U_{\alpha,\vartheta}\psi)(E) = \int_0^\infty \mathcal{U}_\vartheta^\alpha(E|r)\psi(r)\,dr, \quad \psi \in L_2^c(\mathbb{R}_+),$$

for $\mathcal{V}_{\alpha,\vartheta}$-a.e. E, and we have:

$$h_{\alpha,\vartheta} = U_{\alpha,\vartheta}^{-1}\mathcal{T}_\iota^{\mathcal{V}_{\alpha,\vartheta}}U_{\alpha,\vartheta}, \tag{1.20}$$

where ι is the identity function on $\mathbb{R}$ (i.e., $\iota(E) = E$ for all $E \in \mathbb{R}$).

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[7] More precisely, $\mathcal{T}_g^v$ is the operator in $L_2(\mathbb{R}, v)$ whose graph consists of all pairs (φ_1, φ_2), such that $\varphi_1, \varphi_2 \in L_2(\mathbb{R}, v)$ and $\varphi_2(E) = g(E)\varphi_1(E)$ for v-a.e. E.

[8] We recall that a Borel measure v on $\mathbb{R}$ is called a Radon measure on $\mathbb{R}$ if $v(K) < \infty$ for every compact set $K \subset \mathbb{R}$.

Clearly, the measures $\mathcal{V}_{\alpha,\vartheta}$ (which will be referred to as the spectral measures) contain all information about the spectral properties of the operators $h_{\alpha,\vartheta}$. In particular, $E \in \mathbb{R}$ is an eigenvalue of $h_{\alpha,\vartheta}$ if and only if the measure $\mathcal{V}_{\alpha,\vartheta}$ of the one-point set $\{E\}$ is strictly positive. In agreement with (1.14) and (1.18), $\mathcal{V}_{\alpha,\vartheta}$ is π-periodic in ϑ:

$$\mathcal{V}_{\alpha,\vartheta+\pi} = \mathcal{V}_{\alpha,\vartheta}, \quad \alpha < 1, \ \vartheta \in \mathbb{R}. \tag{1.21}$$

Let ω be the π-periodic function on $\mathbb{R}$, such that:

$$\omega(\vartheta) = \left(1 - \frac{2|\vartheta|}{\pi}\right)^2, \quad -\frac{\pi}{2} \le \vartheta \le \frac{\pi}{2}. \tag{1.22}$$

Clearly, we have $0 \le \omega(\vartheta) \le 1$ for all $\vartheta \in \mathbb{R}$. We define the subsets Q_0, Q_1, and Q_∞ of $\mathbb{R}^2$ by the relations (see Fig. 1):

$$Q_0 = \{(\alpha,\vartheta) \in \mathbb{R}^2 : \omega(\vartheta) \le \alpha < 1\}, \tag{1.23}$$

$$Q_1 = \{(\alpha,\vartheta) \in \mathbb{R}^2 : 0 \le \alpha < \omega(\vartheta)\}, \tag{1.24}$$

$$Q_\infty = \{(\alpha,\vartheta) \in \mathbb{R}^2 : \alpha < 0\}. \tag{1.25}$$

The analysis of the measures $\mathcal{V}_{\alpha,\vartheta}$ shows that $h_{\alpha,\vartheta}$ has no eigenvalues for $(\alpha,\vartheta) \in Q_0$, one eigenvalue for $(\alpha,\vartheta) \in Q_1$, and infinitely many eigenvalues that are not bounded from below for $(\alpha,\vartheta) \in Q_\infty$. Using a parametrization of generalized eigenfunctions that is analytic on the entire domain:

$$Q = Q_0 \cup Q_1 \cup Q_\infty = \{(\alpha,\vartheta) \in \mathbb{R}^2 : \alpha < 1\} \tag{1.26}$$

allows us to understand in detail what happens to eigenvalues as we pass from Q_∞ to Q_1 through the line $\alpha = 0$. It turns out that there is one eigenvalue that crosses this line in an analytic manner, while the rest infinitely many eigenvalues tend either to $-\infty$ or to zero as $\alpha \uparrow 0$ and die away there. Moreover, the density of $\mathcal{V}_{\alpha,\vartheta}$ corresponding to the continuous part of the spectrum turns out to be real-analytic on Q.

If $\alpha < 0$, then the operator h_α is not bounded from below, because otherwise it would have self-adjoint extensions (e.g., its Friedrichs extension) that are bounded from below, in contradiction to Theorem 1.1 and the described properties of eigenvalues of $h_{\alpha,\vartheta}$ for $(\alpha,\vartheta) \in Q_\infty$. On the other hand, it is easy to see that h_α is positive for $\alpha \ge 0$. Indeed, let $f \in C_0^\infty(\mathbb{R}_+)$, $\psi = [f]$, and ϱ be a real number. Using the integration by parts, we easily derive from (1.9) that:

$$\langle \psi, \tilde{h}_\alpha \psi \rangle = \int_0^\infty r^{2\varrho}\left(|\partial_r \tilde{f}(r)|^2 + \frac{\alpha - 1/4 - \varrho^2 + \varrho}{r^2}|\tilde{f}(r)|^2\right) dr,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $L_2(\mathbb{R}_+)$ and the function $\tilde{f}$ on $\mathbb{R}_+$ is given by $\tilde{f}(r) = r^{-\varrho}f(r)$, $r > 0$. The maximum of $-\varrho^2 + \varrho$ is attained at $\varrho = 1/2$ and is equal to $1/4$. Substituting this value to the above equality, we deduce that $\langle \psi, \tilde{h}_\alpha \psi \rangle \ge 0$ for all $\alpha \ge 0$ and $\psi \in D_{\tilde{h}_\alpha}$. The positivity of h_α for $\alpha \ge 0$ now follows from (1.10).
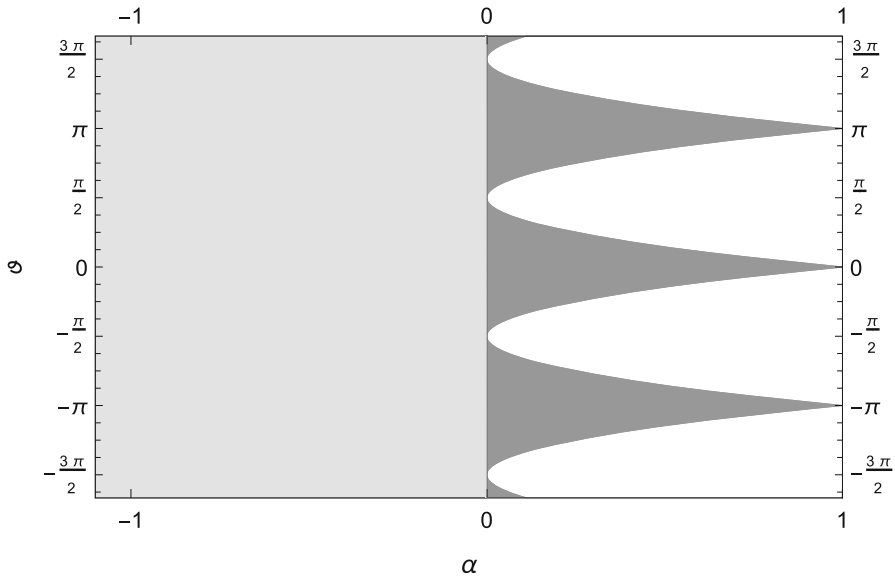
Fig. 1 The sets Q_0, Q_1, and Q_∞ are represented by white, dark gray, and light gray regions, respectively

It seems probable that the dependence of $\mathcal{V}_{\alpha,\vartheta}$ on α and ϑ is not analytic at the boundaries between the regions Q_0, Q_1, and Q_∞, where eigenvalues arise and disappear (however, we do not prove this claim in this paper). At the same time, we shall show that this dependence is smooth on Q in a suitable sense. To formulate this result precisely, we make use of the Schwartz space $\mathscr{S}$ of rapidly decreasing smooth functions. More specifically, $\mathscr{S}$ consists of all infinitely differentiable functions φ on $\mathbb{R}$, such that:

$$\sup_{E\in\mathbb{R},\, k\in\{0,\dots,n\}} |\varphi^{(k)}(E)|(1+|E|)^n < \infty$$

for every nonnegative integer n, where $\varphi^{(k)}$ stands for the kth derivative of φ. The space $\mathscr{S}$ is widely used in the theory of generalized functions as the test function space for tempered distributions. In view of (1.19), every $\varphi \in \mathscr{S}$ is $\mathcal{V}_{\alpha,\vartheta}$-integrable for all $\alpha < 1$ and $\vartheta \in \mathbb{R}$.

Theorem 1.3 *For every $\varphi \in \mathscr{S}$, the function $(\alpha, \vartheta) \to \int \varphi(E)\, d\mathcal{V}_{\alpha,\vartheta}(E)$ is infinitely differentiable on the domain Q given by (1.26).*

Thus, our construction of eigenfunction expansions is, as a whole, at least infinitely differentiable.

When considering Eq. (1.1), it is convenient to set $\alpha = \kappa^2$ and find its solutions as functions of κ (we have actually done so in the case of Hankel transformation (1.3)). To return to the initial variable α, it is then necessary to replace κ with the square root of α. As was discussed above, this may lead to the appearance of branch points and the loss of analyticity. This does not happen, however, if the solution in question is an even holomorphic function of κ. Indeed, suppose we have

an even holomorphic function g, which will be assumed for simplicity to be entire. Then, g has the power series expansion of the form $g(w) = \sum_{k=0}^{\infty} c_k w^{2k}$ for every $w \in \mathbb{C}$. If we define the entire analytic function G by the formula $G(\zeta) = \sum_{k=0}^{\infty} c_k \zeta^k$, $\zeta \in \mathbb{C}$, then we have:

$$G(w^2) = g(w) \tag{1.27}$$

for all $w \in \mathbb{C}$ and, hence, $G(\zeta)$ can be viewed as a result of "substituting the square root of ζ" in g. More generally, representations of type (1.27) can be obtained for even holomorphic functions on arbitrary reflection-symmetric domains and for the case of several complex variables (see Appendix A). Our construction of the solutions $\mathcal{A}^{\alpha}(z)$ and $\mathcal{B}^{\alpha}(z)$ is based on the described technique. We shall first find functions $\mathfrak{a}^{\kappa}(z)$ and $\mathfrak{b}^{\kappa}(z)$ that are even in κ and satisfy (1.1) with α and E replaced with κ^2 and z, respectively, and then define $\mathcal{A}^{\alpha}(z)$ and $\mathcal{B}^{\alpha}(z)$ by requiring that $\mathcal{A}^{\kappa^2}(z) = \mathfrak{a}^{\kappa}(z)$ and $\mathcal{B}^{\kappa^2}(z) = \mathfrak{b}^{\kappa}(z)$ for every $\kappa, z \in \mathbb{C}$.

Simple examples of representations of type (1.27), which will be important for us, are obtained if we choose g to be equal either to the cosine or the entire function sinc that is defined by the formula:

$$\operatorname{sinc} w = \begin{cases} w^{-1} \sin w, & w \in \mathbb{C} \setminus \{0\}, \\ 1, & w = 0. \end{cases}$$

Proceeding as above, we find that:

$$\cos w = \operatorname{Cos}(w^2), \quad \operatorname{sinc} w = \operatorname{Sinc}(w^2) \tag{1.28}$$

for every $w \in \mathbb{C}$, where the entire functions Cos and Sinc are given by:

$$\operatorname{Cos} \zeta = \sum_{k=0}^{\infty} \frac{(-\zeta)^k}{(2k)!}, \quad \operatorname{Sinc} \zeta = \sum_{k=0}^{\infty} \frac{(-\zeta)^k}{(2k+1)!}, \quad \zeta \in \mathbb{C}. \tag{1.29}$$

It follows from (1.28) that:

$$\operatorname{Cos}(-w^2) = \cos(iw) = \operatorname{ch} w,$$
$$w \operatorname{Sinc}(-w^2) = w \operatorname{sinc}(iw) = \operatorname{sh} w \tag{1.30}$$

for every $w \in \mathbb{C}$. In particular, we have:

$$\operatorname{Cos} \xi = \cos(\sqrt{\xi}), \quad \operatorname{Sinc} \xi = \operatorname{sinc}(\sqrt{\xi}), \quad \xi \geq 0, \tag{1.31}$$

$$\operatorname{Cos} \xi = \operatorname{ch}(\sqrt{|\xi|}), \quad \operatorname{Sinc} \xi = |\xi|^{-1/2} \operatorname{sh}(\sqrt{|\xi|}), \quad \xi < 0. \tag{1.32}$$

The graphs of $\operatorname{Cos} \xi$ and $\operatorname{Sinc} \xi$ are shown in Fig. 2.

Formulas (1.31) and (1.32) show that, in spite of being analytic, the functions Cos and Sinc are expressed in a piecewise way in terms of the standard trigonometric and hyperbolic functions. We shall see that various quantities related to the spectral measures (such as eigenvalues and the density of the absolutely
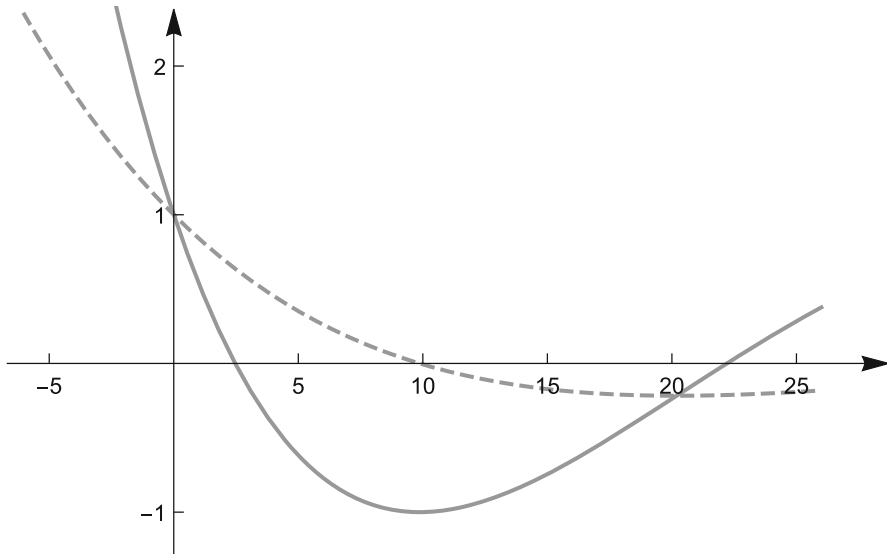
Fig. 2 Solid and dashed lines correspond to the functions Cos and Sinc, respectively

continuous part of $\mathcal{V}_{\alpha,\vartheta}$) can be conveniently expressed through Cos and Sinc. Accordingly, the formulas for these quantities in terms of the ordinary elementary functions are of a piecewise nature. This suggests that Cos and Sinc are more suitable as "elementary functions" for our problem. Some properties of these functions that will be necessary for us are summarized in Appendix B.

The paper is organized as follows. In Sect. 2, we define the solutions $\mathcal{A}^{\alpha}(z)$ and $\mathcal{B}^{\alpha}(z)$ and the spectral measures $\mathcal{V}_{\alpha,\vartheta}$, thus completing the formulation of our results. The definition of the measures $\mathcal{V}_{\alpha,\vartheta}$ in Sect. 2 is given via Herglotz representations (see Appendix C) of suitable holomorphic functions in the upper complex half-plane and is not quite explicit. In Sect. 3, we obtain concrete formulas for the point and absolutely continuous parts of $\mathcal{V}_{\alpha,\vartheta}$. In particular, this allows us to justify the "phase diagram" in Fig. 1 and analyze the dependence of eigenvalues of $h_{\alpha,\vartheta}$ on α and ϑ. In Sect. 4, we recall the general theory concerning self-adjoint extensions of one-dimensional Schrödinger operators and apply it to the proof of Theorem 1.1. Our treatment of eigenfunction expansions relies on the method of singular Titchmarsh–Weyl m-functions [16]. In Sect. 5, we briefly describe this method and then use it to prove Theorem 1.2. Section 6 is devoted to the proof of Theorem 1.3.

2 Definition of generalized eigenfunctions and spectral measures

2.1 Definition of $\mathcal{A}^{\alpha}(z)$ and $\mathcal{B}^{\alpha}(z)$

For any $z, \kappa \in \mathbb{C}$, we define the function $\mathfrak{u}^{\kappa}(z)$ on $\mathbb{R}_{+}$ by the relation:

$$\mathfrak{u}^\kappa(z|r) = r^{1/2+\kappa}\mathcal{X}_\kappa(r^2 z), \quad r > 0, \tag{2.1}$$

where the entire function $\mathcal{X}_\kappa$ is given by:

$$\mathcal{X}_\kappa(\zeta) = \frac{1}{2^\kappa}\sum_{n=0}^{\infty}\frac{(-1)^n\zeta^n}{\Gamma(\kappa+n+1)n!2^{2n}}, \quad \zeta \in \mathbb{C}. \tag{2.2}$$

The function $\mathcal{X}_\kappa$ is closely related to Bessel functions: for $\zeta \neq 0$, we have:

$$\mathcal{X}_\kappa(\zeta) = \zeta^{-\kappa/2}J_\kappa(\zeta^{1/2}). \tag{2.3}$$

Because J_κ satisfies the Bessel equation, it follows that:

$$-\partial_r^2\mathfrak{u}^{\pm\kappa}(z|r) + \frac{\kappa^2-1/4}{r^2}\mathfrak{u}^{\pm\kappa}(z|r) = z\mathfrak{u}^{\pm\kappa}(z|r), \quad r > 0, \tag{2.4}$$

for every $\kappa \in \mathbb{C}$ and $z \neq 0$. By continuity, this is also true for $z = 0$. We therefore have:

$$\mathscr{L}_{\kappa^2,z}\mathfrak{u}^{\pm\kappa}(z) = 0, \quad \kappa, z \in \mathbb{C}. \tag{2.5}$$

For every $\kappa, z \in \mathbb{C}$, we define the function $\mathfrak{a}^\kappa(z)$ on $\mathbb{R}_+$ by setting:

$$\mathfrak{a}^\kappa(z) = \frac{\mathfrak{u}^\kappa(z) - \mathfrak{u}^{-\kappa}(z)}{\kappa}\cos\vartheta_\kappa, \quad \kappa \in \mathbb{C} \setminus \{0\}, \tag{2.6}$$

and

$$\mathfrak{a}^0(z|r) = \lim_{\kappa \to 0}\mathfrak{a}^\kappa(z|r) = 2\left[\left(\ln\frac{r}{2} + \gamma\right)\mathfrak{u}^0(z|r) - \sqrt{r}\,\mathcal{Y}(r^2 z)\right], \quad r > 0, \tag{2.7}$$

where

$$\vartheta_\kappa = \frac{\pi\kappa}{2}, \tag{2.8}$$

the entire function $\mathcal{Y}$ is given by:

$$\mathcal{Y}(\zeta) = \sum_{n=1}^{\infty}\frac{(-1)^n c_n}{(n!)^2 2^{2n}}\zeta^n, \quad c_n = \sum_{j=1}^{n}\frac{1}{j},$$

and $\gamma = \lim_{n\to\infty}(c_n - \ln n) = 0,577\ldots$ is the Euler constant.[9]

Furthermore, for every $\kappa, z \in \mathbb{C}$, we define the function $\mathfrak{b}^\kappa(z)$ on $\mathbb{R}_+$ by the formula:

$$\mathfrak{b}^\kappa(z) = \frac{\pi}{2}\left(\mathfrak{u}^\kappa(z) + \mathfrak{u}^{-\kappa}(z)\right)\operatorname{sinc}\vartheta_\kappa, \tag{2.9}$$

where ϑ_κ is given by (2.8).

Given $\varphi \in \mathbb{R}$, we set $R_\varphi = \{z \in \mathbb{C} : z = re^{i\varphi} \text{ for some } r \geq 0\}$ and:

---

[9] To compute the limit of $\mathfrak{a}^\kappa(z|r)$ as $\kappa \to 0$, we can apply L'Hôpital's rule and use the equality $\Gamma'(1+n)/\Gamma(1+n) = c_n - \gamma$ (see formula (9) in Sect. 1.7.1 in [7]).

$$\mathbb{C}_\varphi = \mathbb{C} \backslash R_\varphi.$$

Hence, $\mathbb{C}_\varphi$ is the complex plane with a cut along the ray R_φ.

The next statement shows that, notwithstanding a piecewise definition of $\mathfrak{a}^\kappa(z)$, both quantities $\mathfrak{a}^\kappa(z|r)$ and $\mathfrak{b}^\kappa(z|r)$ are actually analytic in all their arguments.

Lemma 2.1 *There are unique holomorphic functions F_1 and F_2 on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_\pi$, such that*:

$$F_1(\kappa, z, r) = \mathfrak{a}^\kappa(z|r), \quad F_2(\kappa, z, r) = \mathfrak{b}^\kappa(z|r), \quad \kappa, z \in \mathbb{C}, \, r > 0. \tag{2.10}$$

The proof of Lemma 2.1 is given in Appendix D.

By (2.6) and (2.9), we have:

$$\mathfrak{a}^\kappa(z) = \mathfrak{a}^{-\kappa}(z), \quad \mathfrak{b}^\kappa(z) = \mathfrak{b}^{-\kappa}(z), \quad \kappa, z \in \mathbb{C}. \tag{2.11}$$

For every $\alpha, z \in \mathbb{C}$, we define the functions $\mathcal{A}^\alpha(z)$ and $\mathcal{B}^\alpha(z)$ on $\mathbb{R}_+$ by the relations $\mathcal{A}^\alpha(z) = \mathfrak{a}^\kappa(z)$ and $\mathcal{B}^\alpha(z) = \mathfrak{b}^\kappa(z)$, where $\kappa \in \mathbb{C}$ is such that $\kappa^2 = \alpha$ (by 2.11, this definition does not depend on the choice of κ). We therefore have:

$$\mathcal{A}^{\kappa^2}(z) = \mathfrak{a}^\kappa(z), \quad \mathcal{B}^{\kappa^2}(z) = \mathfrak{b}^\kappa(z), \quad \kappa, z \in \mathbb{C}. \tag{2.12}$$

Equalities (2.4) and (2.6) imply that:

$$-\partial_r^2 \mathfrak{a}^\kappa(z|r) + \frac{\kappa^2 - 1/4}{r^2} \mathfrak{a}^\kappa(z|r) = z \, \mathfrak{a}^\kappa(z|r), \quad r > 0, \tag{2.13}$$

for every $\kappa \in \mathbb{C} \backslash \{0\}$ and $z \in \mathbb{C}$. By Lemma 2.1, we can take the limit $\kappa \to 0$ and conclude that (2.13) also holds for $\kappa = 0$. We hence have $\mathscr{L}_{\kappa^2, z} \mathfrak{a}^\kappa(z) = 0$ for all $\kappa, z \in \mathbb{C}$. Since $\mathscr{L}_{\kappa^2, z} \mathfrak{b}^\kappa(z) = 0$ for every $\kappa, z \in \mathbb{C}$ by (2.5) and (2.9), it follows from (2.12) that (1.12) is valid for all $\alpha, z \in \mathbb{C}$.

We now use Lemma 2.1 to prove that the quantities $\mathcal{A}^\alpha(z|r)$ and $\mathcal{B}^\alpha(z|r)$ enjoy the same analyticity properties as $\mathfrak{a}^\kappa(z|r)$ and $\mathfrak{b}^\kappa(z|r)$.

Lemma 2.2 *There are unique holomorphic functions G_1 and G_2 on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_\pi$, such that $G_1(\alpha, z, r) = \mathcal{A}^\alpha(z|r)$ and $G_2(\alpha, z, r) = \mathcal{B}^\alpha(z|r)$ for every $\alpha, z \in \mathbb{C}$ and $r > 0$.*

Proof Let F_1 and F_2 be as in Lemma 2.1. It follows from (2.10), (2.11), and the uniqueness theorem for holomorphic functions that $F_{1,2}(\kappa, z, \zeta) = F_{1,2}(-\kappa, z, \zeta)$ for all $\kappa, z \in \mathbb{C}$ and $\zeta \in \mathbb{C}_\pi$. The existence of G_1 and G_2 with the required properties is therefore ensured by Lemma A.3, (2.10), and (2.12). The uniqueness of G_1 and G_2 follows from the uniqueness theorem for holomorphic functions. $\qquad \square$

It follows from (1.14), (2.6), (2.9), and (2.12) that:

$$\mathcal{U}_\vartheta^{\kappa^2}(z) = \frac{\mathfrak{u}^\kappa(z)\cos(\vartheta - \vartheta_\kappa) - \mathfrak{u}^{-\kappa}(z)\cos(\vartheta + \vartheta_\kappa)}{\kappa} \tag{2.14}$$

for every $z, \vartheta \in \mathbb{C}$ and $\kappa \in \mathbb{C}\backslash\{0\}$, where ϑ_κ is given by (2.8).

By (2.1) and (2.2), we have:

$$\overline{\mathfrak{u}^\kappa(z)} = \mathfrak{u}^{\bar\kappa}(\bar z), \quad z, \kappa \in \mathbb{C},$$

where the bar means complex conjugation. In view of (2.6), (2.7), and (2.9), this implies that $\mathfrak{a}^\kappa(z)$ and $\mathfrak{b}^\kappa(z)$ are real if z is real and κ is either real or purely imaginary. Since every $\alpha \in \mathbb{R}$ is equal to κ^2 for some κ that is either real or purely imaginary, it follows from (2.12) that $\mathcal{A}^\alpha(z)$ and $\mathcal{B}^\alpha(z)$ are real for every $\alpha, z \in \mathbb{R}$.

If $f, g \in \mathcal{D}$ are such that $r \to W_r(f, g)$ is a constant function on $\mathbb{R}_+$ (in particular, this is the case when f and g are solutions of $\mathscr{L}_{\alpha,z}f = \mathscr{L}_{\alpha,z}g = 0$ for some $\alpha, z \in \mathbb{C}$), then its value will be denoted by $W(f, g)$. Equality (2.5) implies that $W_r(\mathfrak{u}^\kappa(z), \mathfrak{u}^{-\kappa}(z))$ does not depend on r, and we derive from (2.1) and (2.2) that:

$$W(\mathfrak{u}^\kappa(z), \mathfrak{u}^{-\kappa}(z)) = \lim_{r\downarrow 0} W_r(\mathfrak{u}^\kappa(z), \mathfrak{u}^{-\kappa}(z)) = -\frac{2}{\pi}\sin\pi\kappa, \quad \kappa, z \in \mathbb{C}. \tag{2.15}$$

It follows from (2.6), (2.9), and (2.15) that $W(\mathfrak{a}^\kappa(z), \mathfrak{b}^\kappa(z)) = -2\pi\operatorname{sinc}^2\pi\kappa$ for all $\kappa \in \mathbb{C}\backslash\{0\}$ and $z \in \mathbb{C}$. By Lemma 2.1, $W(\mathfrak{a}^\kappa(z), \mathfrak{b}^\kappa(z))$ is continuous in κ at $\kappa = 0$ and, therefore, this equality holds for all $\kappa, z \in \mathbb{C}$. In view of (2.12), this yields:

$$W(\mathcal{A}^\alpha(z), \mathcal{B}^\alpha(z)) = -2\pi\operatorname{Sinc}^2(\pi^2\alpha), \quad \alpha, z \in \mathbb{C}. \tag{2.16}$$

Hence, $\mathcal{A}^\alpha(z)$ and $\mathcal{B}^\alpha(z)$ are linearly independent for all $\alpha, z \in \mathbb{C}$, such that α is not a square of a nonzero integer number and, in particular, for all $\alpha < 1$ and $z \in \mathbb{C}$.

2.2 Definition of $\mathcal{V}_{\alpha,\vartheta}$

We now turn to the definition of the spectral measures $\mathcal{V}_{\alpha,\vartheta}$. In what follows, we let $\ln$ denote the branch of the logarithm on $\mathbb{C}_{3\pi/2}$ satisfying the condition $\ln 1 = 0$ and set $z^\rho = e^{\rho\ln z}$ for all $z \in \mathbb{C}_{3\pi/2}$ and $\rho \in \mathbb{C}$.

Lemma 2.3 *There is a unique holomorphic function R on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_{3\pi/2}$, such that*:

$$R(\kappa^2, \vartheta, z) = \frac{z^{-\kappa/2}e^{i\pi\kappa/2}\cos(\vartheta - \vartheta_\kappa) - z^{\kappa/2}e^{-i\pi\kappa/2}\cos(\vartheta + \vartheta_\kappa)}{\kappa} \tag{2.17}$$

for every $\kappa \in \mathbb{C}\backslash\{0\}$, $\vartheta \in \mathbb{C}$ and $z \in \mathbb{C}_{3\pi/2}$, where ϑ_κ is given by (2.8). The function R satisfies the equality:

$$R(0, \vartheta, z) = (\pi i - \ln z)\cos\vartheta + \pi\sin\vartheta \tag{2.18}$$

for every $\vartheta \in \mathbb{C}$ and $z \in \mathbb{C}_{3\pi/2}$.

Proof Let the function $\tilde{R}$ on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_{3\pi/2}$ be such that $\tilde{R}(\kappa, \vartheta, z)$ is equal to the right-hand side of (2.17) for nonzero κ and $\tilde{R}(0, \vartheta, z)$ is equal to the right-hand side of (2.18). For every $\vartheta \in \mathbb{C}$ and $z \in \mathbb{C}_{3\pi/2}$, the function $\kappa \to \tilde{R}(\kappa, \vartheta, z)$ is holomorphic on $\mathbb{C}\backslash\{0\}$ and is continuous at $\kappa = 0$ (the calculation of the limit of the right-hand side of (2.17) shows that $\lim_{\kappa \to 0} \tilde{R}(\kappa, \vartheta, z) = \tilde{R}(0, \vartheta, z)$). This implies that the function $\kappa \to \tilde{R}(\kappa, \vartheta, z)$ is holomorphic on $\mathbb{C}$ for every $\vartheta \in \mathbb{C}$ and $z \in \mathbb{C}_{3\pi/2}$. On the other hand, the function $(\vartheta, z) \to \tilde{R}(\kappa, \vartheta, z)$ is obviously holomorphic on $\mathbb{C} \times \mathbb{C}_{3\pi/2}$ for every $\kappa \in \mathbb{C}$. By the Hartogs theorem, we conclude that $\tilde{R}$ is holomorphic on its domain. Moreover, we have $\tilde{R}(-\kappa, \vartheta, z) = \tilde{R}(\kappa, \vartheta, z)$ for every $\kappa, \vartheta \in \mathbb{C}$ and $z \in \mathbb{C}_{3\pi/2}$. Hence, the existence of R follows from Lemma A.3. The uniqueness of R is ensured by the uniqueness theorem for holomorphic functions. Formula (2.18) is obvious from the above. $\qquad \square$

It follows from (2.17) and (2.18) that:

$$R(\alpha, \vartheta + \pi, z) = -R(\alpha, \vartheta, z) \tag{2.19}$$

for every $\alpha, \vartheta \in \mathbb{C}$ and $z \in \mathbb{C}_{3\pi/2}$.

Given $z \in \mathbb{C}_{3\pi/2}$, there is a unique $\phi \in (-\pi/2, 3\pi/2)$, such that $z = |z|e^{i\phi}$. We shall denote this ϕ by ϕ_z.

Lemma 2.4 *Let R be as in Lemma 2.3. Then, we have*:

$$\text{Im}\left(R(\alpha, \vartheta + \pi/2, z)\overline{R(\alpha, \vartheta, z)}\right) = \pi(\phi_z - \pi)\,\text{Sinc}\left((\phi_z - \pi)^2\alpha\right)\text{Sinc}\left(\pi^2\alpha\right) \tag{2.20}$$

for every $\alpha, \vartheta \in \mathbb{R}$ and $z \in \mathbb{C}_{3\pi/2}$.

Proof By (2.17), we have:

$$\kappa^2 R(\kappa^2, \vartheta + \pi/2, Ee^{i\phi})\overline{R(\bar{\kappa}^2, \vartheta, Ee^{i\phi})}$$
$$= i\sin(\phi - \pi)\kappa\sin\pi\kappa$$
$$+ \cos(\phi - \pi)\kappa\sin 2\vartheta - E^\kappa\cos(\vartheta + \vartheta_\kappa)\sin(\vartheta + \vartheta_\kappa) - E^{-\kappa}\cos(\vartheta - \vartheta_\kappa)\sin(\vartheta - \vartheta_\kappa)$$

for all $\kappa \in \mathbb{C}\backslash\{0\}$, $\vartheta \in \mathbb{R}$, $E > 0$, and $-\pi/2 < \phi < 3\pi/2$. If κ is real or purely imaginary, then $\bar{\kappa}^2 = \kappa^2$ and the sum of the last three terms in the right-hand side is real. This implies (2.20) for nonzero α. By continuity, (2.20) remains valid for $\alpha = 0$. $\qquad \square$

Let R be as in Lemma 2.3. For every $\alpha, \vartheta \in \mathbb{C}$, we let $\mathcal{O}_{\alpha,\vartheta}$ denote the open subset of $\mathbb{C}_{3\pi/2}$, where the function $z \to R(\alpha, \vartheta, z)$ is nonzero:

$$\mathcal{O}_{\alpha,\vartheta} = \{z \in \mathbb{C}_{3\pi/2} : R(\alpha, \vartheta, z) \neq 0\}. \tag{2.21}$$

Suppose now that $\alpha < 1$ and $\vartheta \in \mathbb{R}$. Then, it follows from Lemma 2.4 that:

$$\mathbb{C}_+ \cup \mathbb{R}_+ \subset \mathcal{O}_{\alpha,\vartheta}, \tag{2.22}$$

where $\mathbb{C}_+$ denotes the open upper half-plane of the complex plane,

$\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Let the holomorphic function $\mathscr{M}_{\alpha,\vartheta}$ on $\mathcal{O}_{\alpha,\vartheta}$ be defined by the equality:

$$\mathscr{M}_{\alpha,\vartheta}(z) = -\frac{R(\alpha, \vartheta + \pi/2, z)}{2\pi^2 \operatorname{Sinc}^2(\pi^2\alpha)R(\alpha, \vartheta, z)}, \quad z \in \mathcal{O}_{\alpha,\vartheta}. \tag{2.23}$$

Lemma 2.4 implies that:

$$\operatorname{Im} \mathscr{M}_{\alpha,\vartheta}(z) = \frac{(\pi - \phi_z)\operatorname{Sinc}((\pi - \phi_z)^2\alpha)}{2\pi \operatorname{Sinc}(\pi^2\alpha)|R(\alpha, \vartheta, z)|^2}, \quad z \in \mathcal{O}_{\alpha,\vartheta}. \tag{2.24}$$

By Lemma B.1 and (2.22), we conclude that $\operatorname{Im} \mathscr{M}_{\alpha,\vartheta}(z) > 0$ for every $z \in \mathbb{C}_+$ and, hence, $\mathscr{M}_{\alpha,\vartheta}|_{\mathbb{C}_+}$ is a Herglotz function (see Appendix C). We now define $\mathcal{V}_{\alpha,\vartheta}$ as the Herglotz (and, hence, Radon) measure associated with the function $\pi\mathscr{M}_{\alpha,\vartheta}|_{\mathbb{C}_+}$. It follows from (C.2) and Lemma C.1 that (1.19) is valid and:

$$\int \varphi(E)\,\mathrm{d}\mathcal{V}_{\alpha,\vartheta}(E) = \lim_{\eta\downarrow0} \int \varphi(E)\operatorname{Im} \mathscr{M}_{\alpha,\vartheta}(E + i\eta)\,\mathrm{d}E \tag{2.25}$$

for every continuous complex function φ on $\mathbb{R}$ satisfying the bound $|\varphi(E)| \leq C(1 + E^2)^{-2}$, $E \in \mathbb{R}$, for some $C \geq 0$. In particular, (2.25) holds for every continuous function φ on $\mathbb{R}$ with compact support. In view of the Riesz representation theorem, this implies that $\mathcal{V}_{\alpha,\vartheta}$ is uniquely determined by equality (2.25). It follows from (2.19) and (2.23) that $\mathscr{M}_{\alpha,\vartheta+\pi} = \mathscr{M}_{\alpha,\vartheta}$ and, therefore, $\mathcal{V}_{\alpha,\vartheta}$ has π-periodicity property (1.21).

3 Explicit formulas for the spectral measures

In this section, we assume that Theorems 1.1 and 1.2 are valid and obtain explicit expressions for both the point and absolutely continuous parts of the spectral measures $\mathcal{V}_{\alpha,\vartheta}$. The proofs of Theorems 1.1 and 1.2 in Sects. 4 and 5 do not rely on the results of this section.

3.1 General structure of $\mathcal{V}_{\alpha,\vartheta}$

Given a positive Radon measure v on $\mathbb{R}$, we let $L_2^c(\mathbb{R}, v)$ denote the subspace of $L_2(\mathbb{R}, v)$ consisting of all its elements vanishing v-a.e. outside some compact subset of $\mathbb{R}$.

Lemma 3.1 *Let $\alpha < 1$, $\vartheta \in \mathbb{R}$, and $U_{\alpha,\vartheta}$ be as in Theorem 1.2. Then, we have:*

$$(U_{\alpha,\vartheta}^{-1}\varphi)(r) = \int \mathcal{U}_\vartheta^\alpha(E|r)\varphi(E)\,d\mathcal{V}_{\alpha,\vartheta}(E), \quad \varphi \in L_2^c(\mathbb{R}, \mathcal{V}_{\alpha,\vartheta}), \tag{3.1}$$

for λ-a.e. $r \in \mathbb{R}_+$. An $E \in \mathbb{R}$ is an eigenvalue of $h_{\alpha,\vartheta}$ if and only if $\mathcal{V}_{\alpha,\vartheta}(\{E\}) > 0$. For every eigenvalue E, the corresponding eigenspace is one-dimensional and is spanned by $[\mathcal{U}_\vartheta^\alpha(E)]$, and we have the equality $\|[\mathcal{U}_\vartheta^\alpha(E)]\| = \mathcal{V}_{\alpha,\vartheta}(\{E\})^{-1/2}$.

Proof For brevity, we set $h = h_{\alpha,\vartheta}$, $U = U_{\alpha,\vartheta}$, $\mathcal{V} = \mathcal{V}_{\alpha,\vartheta}$, and $\mathcal{U} = \mathcal{U}_\vartheta^\alpha$. Given $\varphi \in L_2^c(\mathbb{R}, \mathcal{V})$ and $r > 0$, let $\check{\varphi}(r)$ denote the right-hand side of (3.1). By the unitarity of U and the Fubini theorem, we have:

$$\langle \psi, U^{-1}\varphi \rangle_{L_2(\mathbb{R}_+)} = \langle U\psi, \varphi \rangle_{L_2(\mathbb{R}, \mathcal{V})}$$
$$= \int_{\mathbb{R}} d\mathcal{V}(E)\varphi(E) \int_{\mathbb{R}_+} \overline{\psi(r)}\mathcal{U}(E|r)\,\mathrm{d}r = \int_{\mathbb{R}_+} \overline{\psi(r)}\check{\varphi}(r)\,\mathrm{d}r$$

for any $\psi \in L_2^c(\mathbb{R}_+)$, where $\langle \cdot, \cdot \rangle_{L_2(\mathbb{R}_+)}$ and $\langle \cdot, \cdot \rangle_{L_2(\mathbb{R}, \mathcal{V})}$ are the scalar products in $L_2(\mathbb{R}_+)$ and $L_2(\mathbb{R}, \mathcal{V})$, respectively. This implies (3.1). Given $E \in \mathbb{R}$, let G_E be the subspace of $L_2(\mathbb{R}_+)$ composed of all ψ in the domain of h, such that $h\psi = E\psi$ and $\tilde{G}_E$ be the subspace of $L_2(\mathbb{R}, \mathcal{V})$ composed of all φ in the domain of $\mathcal{T}_\iota^\mathcal{V}$, such that $\mathcal{T}_\iota^\mathcal{V}\varphi = E\varphi$, where ι is the identity function on $\mathbb{R}$. By Theorem 1.2, U induces an isomorphism between G_E and $\tilde{G}_E$ for every $E \in \mathbb{R}$. This means, in particular, that the operators h and $\mathcal{T}_\iota^\mathcal{V}$ have the same eigenvalues. Hence, $E \in \mathbb{R}$ is an eigenvalue of h if and only if $\mathcal{V}(\{E\}) > 0$. If $\mathcal{V}(\{E\}) > 0$, then $\tilde{G}_E$ is one-dimensional and is spanned by $[\chi_E]_\mathcal{V}$, where χ_E is the characteristic function of the one-point set $\{E\}$. By (3.1), we have $U^{-1}[\chi_E]_\mathcal{V} = \mathcal{V}(\{E\})[\mathcal{U}_\vartheta^\alpha(E)]$. The space G_E is, therefore, one-dimensional and is spanned by $[\mathcal{U}_\vartheta^\alpha(E)]$. Since the norm of $[\chi_E]_\mathcal{V}$ in $L_2(\mathbb{R}, \mathcal{V})$ is equal to $\mathcal{V}(\{E\})^{1/2}$, the unitarity of U implies that $\|[\mathcal{U}_\vartheta^\alpha(E)]\| = \mathcal{V}_{\alpha,\vartheta}(\{E\})^{-1/2}$. $\qquad\square$

As in Sect. 1, let ω be the π-periodic function on $\mathbb{R}$ satisfying (1.22) and let Q_0, Q_1, Q_∞, and Q be defined by (1.23), (1.24), (1.25), and (1.26), respectively. We set:

$$Q_{\geq 1} = Q_1 \cup Q_\infty = \{(\alpha, \vartheta) \in \mathbb{R}^2 : \alpha < \omega(\vartheta)\}. \tag{3.2}$$

We obviously have:

$$Q_0 \cap Q_{\geq 1} = \varnothing, \quad Q_0 \cup Q_{\geq 1} = Q. \tag{3.3}$$

By (1.22), we have:

$$\mathrm{Cos}\,\frac{\pi^2\omega(\vartheta)}{4} = \cos\left(\frac{\pi}{2} - |\vartheta|\right) = |\sin\vartheta| \tag{3.4}$$

for every $\vartheta \in [-\pi/2, \pi/2]$. Since both sides of (3.4) are π-periodic, (3.4) remains valid for all $\vartheta \in \mathbb{R}$. By Lemma B.1, the function $\alpha \to \mathrm{Cos}\,(\pi^2\alpha/4)$ is nonnegative and strictly decreasing on $(-\infty, 1]$. Hence, for every $\alpha < 1$ and $\vartheta \in \mathbb{R}$, we have the chain of equivalent conditions:

$$\alpha < \omega(\vartheta) \Leftrightarrow \mathrm{Cos}\,\frac{\pi^2\omega(\vartheta)}{4} < \mathrm{Cos}\,\frac{\pi^2\alpha}{4} \Leftrightarrow |\sin\vartheta| < \mathrm{Cos}\,\frac{\pi^2\alpha}{4} \Leftrightarrow \sin^2\vartheta < \mathrm{Cos}^2\,\frac{\pi^2\alpha}{4}.$$

In view of (1.23) and (3.2), it follows that:

$$Q_0 = \{(\alpha, \vartheta) \in \mathbb{R}^2 : \alpha < 1 \text{ and } \sin^2\vartheta \geq \mathrm{Cos}^2(\pi^2\alpha/4)\}, \tag{3.5}$$

$$Q_{\geq 1} = \{(\alpha, \vartheta) \in \mathbb{R}^2 : \alpha < 1 \text{ and } \sin^2 \vartheta < \mathrm{Cos}^2(\pi^2 \alpha/4)\}. \tag{3.6}$$

Given $\alpha, \vartheta \in \mathbb{C}$, we set:

$$\Sigma_{\alpha,\vartheta} = \{E < 0 : R(\alpha, \vartheta, E) = 0\}, \tag{3.7}$$

where R is as in Lemma 2.3. By (2.17), we have:

$$E \in \Sigma_{\kappa^2, \vartheta} \iff E < 0 \text{ and } |E|^{-\kappa/2} \cos(\vartheta - \vartheta_\kappa) = |E|^{\kappa/2} \cos(\vartheta + \vartheta_\kappa) \tag{3.8}$$

for every $\kappa \in \mathbb{C} \setminus \{0\}$ and $\vartheta \in \mathbb{C}$.

In view of (2.21) and (2.22), we have $R(\alpha, \vartheta, E) \neq 0$ for every $\alpha < 1$, $\vartheta \in \mathbb{R}$ and $E > 0$. For every $\alpha < 1$ and $\vartheta \in \mathbb{R}$, we define the function $t_{\alpha,\vartheta}$ by the formula:

$$t_{\alpha,\vartheta}(E) = \begin{cases} \dfrac{1}{2|R(\alpha, \vartheta, E)|^2}, & E > 0, \\ 0, & E \leq 0. \end{cases} \tag{3.9}$$

Given a positive Radon measure v on $\mathbb{R}$, we set:

$$\mathcal{P}(v) = \{E \in \mathbb{R} : v(\{E\}) > 0\}.$$

Since v is σ-additive, the set $\mathcal{P}(v)$ is at most countable. We define the continuous part v^c and the point part v^p of v by the relations:

$$v^c = (1 - \chi_{\mathcal{P}(v)}) v, \quad v^p = \chi_{\mathcal{P}(v)} v, \tag{3.10}$$

where $\chi_{\mathcal{P}(v)}$ is the characteristic function of the set $\mathcal{P}(v)$ (i.e., it is equal to unity on $\mathcal{P}(v)$ and to zero on $\mathbb{R} \setminus \mathcal{P}(v)$). Clearly, $v^c(\{E\}) = 0$ for every $E \in \mathbb{R}$ and:

$$v = v^c + v^p.$$

A function φ on $\mathbb{R}$ is v^p-integrable if and only if $\{v(\{E\})\varphi(E)\}_{E \in \mathcal{P}(v)}$ is a summable family in $\mathbb{C}$, in which case we have:

$$\int \varphi(E) \, \mathrm{d}v^p(E) = \sum_{E \in \mathcal{P}(v)} v(\{E\})\varphi(E). \tag{3.11}$$

Thus, to completely describe a positive Radon measure v on $\mathbb{R}$, it suffices to find $\mathcal{P}(v)$ and v^c and specify $v(\{E\})$ for every $E \in \mathcal{P}(v)$. The next theorem gives such a description for the measures $\mathcal{V}_{\alpha,\vartheta}$. As in Sect. 1, we let λ denote the Lebesgue measure on $\mathbb{R}$.

Theorem 3.2 *For every $\alpha < 1$ and $\vartheta \in \mathbb{R}$, we have $\mathcal{P}(\mathcal{V}_{\alpha,\vartheta}) = \Sigma_{\alpha,\vartheta}$ and $\mathcal{V}_{\alpha,\vartheta}^c = t_{\alpha,\vartheta} \lambda$, where $t_{\alpha,\vartheta}$ is given by (3.9). If $(\alpha, \vartheta) \in Q_0$, then $\Sigma_{\alpha,\vartheta} = \varnothing$. If $(\alpha, \vartheta) \in Q_{\geq 1}$ and $E \in \Sigma_{\alpha,\vartheta}$, then*:

$$\mathcal{V}_{\alpha,\vartheta}(\{E\}) = \frac{|E|}{2 \, \mathrm{Sinc}\,(\pi^2 \alpha)(\mathrm{Cos}^2(\pi^2 \alpha/4) - \sin^2 \vartheta)}. \tag{3.12}$$

Corollary 3.3 *Let $\alpha < 1$ and $\vartheta \in \mathbb{R}$. Then, the set of eigenvalues of $h_{\alpha,\vartheta}$ is precisely $\Sigma_{\alpha,\vartheta}$. For every $E \in \Sigma_{\alpha,\vartheta}$, the corresponding eigenspace is one-dimensional and is spanned by $[\mathcal{U}_{\vartheta}^{\alpha}(E)]$, and we have:*

$$\int_0^{\infty} \mathcal{U}_{\vartheta}^{\alpha}(E|r)^2 \, dr = 2|E|^{-1} \operatorname{Sinc}(\pi^2 \alpha)(\operatorname{Cos}^2(\pi^2 \alpha/4) - \sin^2 \vartheta).$$

Proof The statement follows directly from Lemma 3.1 and Theorem 3.2. □

To prove Theorem 3.2, we shall need several auxiliary lemmas.

Lemma 3.4 *Let $\kappa \in \mathbb{C}$ be such that $-1 < |\operatorname{Re} \kappa| < 1$ and $f \in \mathcal{D}$ be a nontrivial solution of $\mathscr{L}_{\kappa^2} f = 0$. Then, f is not square-integrable on $\mathbb{R}_+$.*

Proof If $\mathscr{L}_0 f = 0$, then there exist $c_1, c_2 \in \mathbb{C}$, such that $f(r) = c_1 r^{1/2} + c_2 r^{1/2} \ln r$ for all $r > 0$. It is straightforward to verify that such a function is square-integrable on $\mathbb{R}_+$ if and only if $c_1 = c_2 = 0$. This proves our statement for $\kappa = 0$. If $\kappa = \kappa' + i\kappa''$ is nonzero and $\mathscr{L}_{\kappa^2} f = 0$, then there exist $c_1, c_2 \in \mathbb{C}$ such that $f(r) = c_1 r^{1/2+\kappa} + c_2 r^{1/2-\kappa}$ for all $r > 0$. Since $|\kappa'| < 1$, f is square-integrable on $(0, r]$ for every $r > 0$, and we have:

$$\int_0^r |f(r')|^2 dr' = \frac{g(r)}{2} + \operatorname{Re} \frac{c_1 \bar{c}_2 r^{2+2i\kappa''}}{1 + i\kappa''} \tag{3.13}$$

for every $r > 0$, where the function g on $\mathbb{R}_+$ is given by:

$$g(r) = \frac{|c_1|^2 r^{2+2\kappa'}}{1 + \kappa'} + \frac{|c_2|^2 r^{2-2\kappa'}}{1 - \kappa'}, \quad r > 0. \tag{3.14}$$

Applying the inequality $2ab \le a^2 + b^2$ to $g(r)$, we obtain:

$$\frac{g(r)}{2} \ge \frac{|c_1 c_2| r^2}{\sqrt{1 - \kappa'^2}} = \frac{1}{\sigma} \left| \frac{c_1 \bar{c}_2 r^{2+2i\kappa''}}{1 + i\kappa''} \right|, \quad r > 0,$$

where $\sigma = [(1 + \kappa''^2)^{-1}(1 - \kappa'^2)]^{1/2}$. In view of (3.13), we conclude that:

$$\int_0^r |f(r')|^2 dr' \ge \frac{1 - \sigma}{2} g(r) \tag{3.15}$$

for every $r > 0$. Since $\kappa \ne 0$, we have $\sigma < 1$. If f is nontrivial, then c_1 and c_2 are not both zero and it follows from (3.14) that $g(r) \to \infty$ as $r \to \infty$. By (3.15), this implies that $\int_0^r |f(r')|^2 dr' \to \infty$ as $r \to \infty$ and, hence, f is not square-integrable on $\mathbb{R}_+$. □

In what follows, we set $\mathbb{R}_- = (-\infty, 0)$.

Lemma 3.5 *Let $\alpha < 1$, $\vartheta \in \mathbb{R}$, and χ be the characteristic function of $\Sigma_{\alpha,\vartheta}$. Then, we have $(1 - \chi)\mathcal{V}_{\alpha,\vartheta} = t_{\alpha,\vartheta} \lambda$.*

Proof Let $v = (1 - \chi)\mathcal{V}_{\alpha,\vartheta}$ and $v' = t_{\alpha,\vartheta}\lambda$. Let O be the open subset of $\mathbb{R}$ defined by the relation $O = \mathbb{R}_+ \cup (\mathbb{R}_- \backslash \Sigma_{\alpha,\vartheta})$. As $\mathbb{R} = O \cup \Sigma_{\alpha,\vartheta} \cup \{0\}$, it suffices to show that v and v' have the same restrictions to each of the sets O, $\Sigma_{\alpha,\vartheta}$, and $\{0\}$. Since the functions $t_{\alpha,\vartheta}$ and $1 - \chi$ are locally bounded on O and $\mathcal{V}_{\alpha,\vartheta}$ and λ are Radon measures on $\mathbb{R}$, the restrictions of v and v' to O are Radon measures on O. By (2.21), (2.22), and (3.7), we have $O \subset \mathcal{O}_{\alpha,\vartheta}$, and it follows from (2.24) and (3.9) that $\operatorname{Im}\mathscr{M}_{\alpha,\vartheta}(E) = t_{\alpha,\vartheta}(E)$ for every $E \in O$. Since $1 - \chi$ is equal to unity on O, (2.25) and the dominated convergence theorem imply that:

$$\int \varphi(E)\,\mathrm{d}v(E) = \int \varphi(E)\,\mathrm{d}\mathcal{V}_{\alpha,\vartheta}(E) = \int t_{\alpha,\vartheta}(E)\varphi(E)\,\mathrm{d}E = \int \varphi(E)\,\mathrm{d}v'(E)$$

for every continuous function φ on $\mathbb{R}$, such that $\operatorname{supp}\varphi$ is a compact subset of O. By the Riesz representation theorem, we conclude that $v|_O = v'|_O$. Because $1 - \chi$ and $t_{\alpha,\vartheta}$ vanish on $\Sigma_{\alpha,\vartheta}$, both v and v' have zero restrictions to $\Sigma_{\alpha,\vartheta}$. We now note that $\mathcal{V}_{\alpha,\vartheta}(\{0\}) = 0$, because, otherwise, $\mathcal{U}_\vartheta^\alpha(0)$ would be a nontrivial square-integrable function on $\mathbb{R}_+$ by Lemma 3.1, in contradiction to (1.15) and Lemma 3.4. Since $v'(\{0\}) = 0$ and $v(\{0\}) = \mathcal{V}_{\alpha,\vartheta}(\{0\})$, we conclude that $v(\{0\}) = v'(\{0\})$. $\qquad\square$

Using elementary trigonometric transformations, we find that:

$$\cos(\vartheta - \vartheta_\kappa)\cos(\vartheta + \vartheta_\kappa) = \cos^2(\pi\kappa/2) - \sin^2\vartheta = \operatorname{Cos}^2(\pi^2\kappa^2/4) - \sin^2\vartheta \quad (3.16)$$

for all $\kappa, \vartheta \in \mathbb{C}$, where ϑ_κ is given by (2.8). In view of (3.6), this equality implies that:

$$\cos(\vartheta - \vartheta_\kappa)\cos(\vartheta + \vartheta_\kappa) > 0 \tag{3.17}$$

for all $\vartheta \in \mathbb{R}$ and $\kappa \in \mathbb{C}$, such that $(\kappa^2, \vartheta) \in Q_{\geq 1}$.

Lemma 3.6 $\Sigma_{\alpha,\vartheta} = \varnothing$ *for every* $(\alpha, \vartheta) \in Q_0$.

Proof Let $(\alpha, \vartheta) \in Q_0$. Suppose first that $\alpha = 0$. By (3.5), we have $\vartheta = \pi/2 + \pi k$ for some $k \in \mathbb{Z}$. Equality (2.18) therefore implies that $R(\alpha, \vartheta, z) = (-1)^k\pi$ for every $z \in \mathbb{C}_{3\pi/2}$. This means that $\Sigma_{\alpha,\vartheta} = \varnothing$. Now, let $\alpha \neq 0$. Since ω is nonnegative, it follows from (1.23) that $0 < \alpha < 1$ and, hence, $\alpha = \kappa^2$ for some $0 < \kappa < 1$. Suppose $\Sigma_{\alpha,\vartheta} \neq \varnothing$ and $E \in \Sigma_{\alpha,\vartheta}$. By (3.8) and (3.16), it follows that:

$$\operatorname{Cos}^2(\pi^2\alpha/4) - \sin^2\vartheta = \frac{1}{2}(|E|^\kappa\cos^2(\vartheta + \vartheta_\kappa) + |E|^{-\kappa}\cos^2(\vartheta - \vartheta_\kappa)). \tag{3.18}$$

Since κ is real, the right-hand side of (3.18) is nonnegative and can be zero only if $\cos(\vartheta - \vartheta_\kappa) = \cos(\vartheta + \vartheta_\kappa) = 0$ and, hence, only if $\kappa \in \mathbb{Z}$. The condition $0 < \kappa < 1$ therefore implies that the right-hand side of (3.18) is strictly positive. In view of (3.5), this contradicts the assumption that $(\alpha, \vartheta) \in Q_0$. Hence, $\Sigma_{\alpha,\vartheta} = \varnothing$. $\qquad\square$

Lemma 3.7 *Let* $(\alpha, \vartheta) \in Q_{\geq 1}$ *and* $E \in \Sigma_{\alpha,\vartheta}$*. Then, we have:*

$$\lim_{z \to E}(z - E)\mathcal{M}_{\alpha,\vartheta}(z) = -\frac{|E|}{2\pi \operatorname{Sinc}(\pi^2\alpha)(\operatorname{Cos}^2(\pi^2\alpha/4) - \sin^2\vartheta)}.$$

Proof We separately consider the cases $\alpha \neq 0$ and $\alpha = 0$.

1. Let $\alpha \neq 0$ and $\kappa \in \mathbb{C}$ be such that $\kappa^2 = \alpha$. By (2.17) and (3.8), we have:

$$\begin{aligned}
\partial_z R(\alpha, \vartheta, z)|_{z=E} &= \frac{|E|^{-\kappa/2}\cos(\vartheta - \vartheta_\kappa) + |E|^{\kappa/2}\cos(\vartheta + \vartheta_\kappa)}{2|E|} \\
&= |E|^{\kappa/2 - 1}\cos(\vartheta + \vartheta_\kappa).
\end{aligned} \tag{3.19}$$

It follows from (3.17) that $\cos(\vartheta + \vartheta_\kappa) \neq 0$ and, hence, $\partial_z R(\alpha, \vartheta, z)|_{z=E} \neq 0$. By (2.17), we have:

$$R(\alpha, \vartheta + \pi/2, E) = \frac{|E|^{\kappa/2}\sin(\vartheta + \vartheta_\kappa) - |E|^{-\kappa/2}\sin(\vartheta - \vartheta_\kappa)}{\kappa}.$$

Multiplying the numerator and denominator by $\cos(\vartheta - \vartheta_\kappa)$ (which is nonzero by (3.17)) and using (3.8), we obtain:

$$R(\alpha, \vartheta + \pi/2, E) = \frac{\pi|E|^{\kappa/2}\operatorname{sinc}\pi\kappa}{\cos(\vartheta - \vartheta_\kappa)}. \tag{3.20}$$

In view of (2.23), we have:

$$\lim_{z \to E}(z - E)\mathcal{M}_{\alpha,\vartheta}(z) = -\frac{R(\alpha, \vartheta + \pi/2, E)}{2\pi^2 \operatorname{Sinc}^2(\pi^2\alpha)\partial_z R(\alpha, \vartheta, z)|_{z=E}}.$$

Combining this formula with (3.16), (3.19), and (3.20), we arrive at the required equality.

2. Let $\alpha = 0$. Since $E \in \Sigma_{0,\vartheta}$, it follows from (2.18) that:

$$\ln|E|\cos\vartheta = \pi\sin\vartheta, \tag{3.21}$$

$$\partial_z R(0, \vartheta, z)|_{z=E} = \frac{\cos\vartheta}{|E|}. \tag{3.22}$$

Since $(0, \vartheta) \in Q_{\geq 1}$ and $\operatorname{Cos}(0) = 1$, (3.6) implies that $\cos\vartheta \neq 0$ and, hence, $\partial_z R(0, \vartheta, z)|_{z=E} \neq 0$. By (2.18) and (3.21), we obtain:

$$R(0, \vartheta + \pi/2, E) = \ln|E|\sin\vartheta + \pi\cos\vartheta = \frac{\pi}{\cos\vartheta}. \tag{3.23}$$

In view of (2.23) and the equality $\operatorname{Sinc}(0) = 1$, we have

$$\lim_{z \to E}(z - E)\mathcal{M}_{0,\vartheta}(z) = -\frac{R(0, \vartheta + \pi/2, E)}{2\pi^2\partial_z R(0, \vartheta, z)|_{z=E}}.$$

Combining this formula with (3.22) and (3.23) yields the required result. $\square$

Proof of Theorem 3.2 Let $(\alpha, \vartheta) \in Q_{\geq 1}$ and $E \in \Sigma_{\alpha,\vartheta}$. Let $O = \mathcal{O}_{\alpha,\vartheta} \cup \{E\}$, where $\mathcal{O}_{\alpha,\vartheta}$ is given by (2.21). Clearly, O is an open subset of $\mathbb{C}$ containing E. By Lemma 3.7, there exists a holomorphic function g on O, such that:

$$\mathcal{M}_{\alpha,\vartheta}(z) = -\frac{1}{\pi} \frac{A}{z - E} + g(z) \tag{3.24}$$

for all $z \in \mathcal{O}_{\alpha,\vartheta}$, where A denotes the right-hand side of (3.12). By (2.24), $\mathcal{M}_{\alpha,\vartheta}$ is real on $\mathcal{O}_{\alpha,\vartheta} \cap \mathbb{R}_-$ and, therefore, g is real on $O \cap \mathbb{R}_-$. By (2.25), (3.24), and the dominated convergence theorem, we conclude that $\int \varphi(E') \, d\mathcal{V}_{\alpha,\vartheta}(E')$ is equal to $A\varphi(E)$ for every continuous function φ on $\mathbb{R}$, such that supp φ is a compact subset of $O \cap \mathbb{R}_-$. Hence, $\mathcal{V}_{\alpha,\vartheta}(\{E\}) = A$. Thus, formula (3.12) holds for every $(\alpha, \vartheta) \in Q_{\geq 1}$ and $E \in \Sigma_{\alpha,\vartheta}$. This implies, in particular, that $\Sigma_{\alpha,\vartheta} \subset \mathcal{P}(\mathcal{V}_{\alpha,\vartheta})$ for all $(\alpha, \vartheta) \in Q_{\geq 1}$. By Lemma 3.6, we have $\Sigma_{\alpha,\vartheta} = \varnothing$ for all $(\alpha, \vartheta) \in Q_0$. It follows that $\Sigma_{\alpha,\vartheta} \subset \mathcal{P}(\mathcal{V}_{\alpha,\vartheta})$ for all $\alpha < 1$ and $\vartheta \in \mathbb{R}$. Since the opposite inclusion also holds by Lemma 3.5, we conclude that $\Sigma_{\alpha,\vartheta} = \mathcal{P}(\mathcal{V}_{\alpha,\vartheta})$ for all $\alpha < 1$ and $\vartheta \in \mathbb{R}$. The equality $\mathcal{V}_{\alpha,\vartheta}^c = t_{\alpha,\vartheta} \lambda$ now follows from (3.10) and Lemma 3.5. $\qquad \square$

Theorem 3.2 implies, in particular, that $t_{\alpha,\vartheta}$ is a locally integrable function on $\mathbb{R}$ for every $\alpha < 1$ and $\vartheta \in \mathbb{R}$. It is noteworthy that we established this property of $t_{\alpha,\vartheta}$ without explicitly estimating this function. Instead, we relied on the fact that $\mathcal{V}_{\alpha,\vartheta}$ is a Radon measure, which follows from its definition as a Herglotz measure. In Lemma 6.4, we shall obtain an explicit estimate for $|R(\alpha, \vartheta, E)|^{-1}$ that, when substituted in (3.9), immediately implies the local integrability of $t_{\alpha,\vartheta}$.

3.2 Eigenvalues of $h_{\alpha,\vartheta}$

We now turn to obtaining an explicit description of the set $\Sigma_{\alpha,\vartheta}$ of eigenvalues of $h_{\alpha,\vartheta}$ for every $\alpha < 1$ and $\vartheta \in \mathbb{R}$. To this end, it is convenient to use the logarithmic scale and pass from the set $\Sigma_{\alpha,\vartheta}$ to its inverse image $N_{\alpha,\vartheta}$ under the map $s \to -e^s$ from $\mathbb{R}$ to itself:

$$N_{\alpha,\vartheta} = \{s \in \mathbb{R} : -e^s \in \Sigma_{\alpha,\vartheta}\}. \tag{3.25}$$

We thus have $E \in \Sigma_{\alpha,\vartheta}$ if and only if $E < 0$ and $\ln |E| \in N_{\alpha,\vartheta}$. Furthermore, we define the open subsets W_0 and W of $\mathbb{R}^2$ by the relations:

$$W_0 = \{(\alpha, \vartheta) \in Q_1 : \alpha > 0 \text{ and } -\pi/2 < \vartheta < \pi/2\}, \tag{3.26}$$

$$W = \{(\alpha, \vartheta) \in Q_1 : -\pi/2 < \vartheta < \pi/2\} \cup Q_\infty. \tag{3.27}$$

Hence, W_0 is the interior of the central dark gray curvilinear triangular region in Fig. 1. Let $(\alpha, \vartheta) \in W_0$. Since $W_0 \subset Q_{\geq 1}$, inequality (3.17) for $\kappa = \sqrt{\alpha}$ implies that $\cos(\vartheta + \pi\sqrt{\alpha}/2) \neq 0$ and:

$$\frac{\cos(\vartheta - \pi\sqrt{\alpha}/2)}{\cos(\vartheta + \pi\sqrt{\alpha}/2)} > 0. \tag{3.28}$$

Hence, we can define a real-valued function S_0 on W_0 by the formula:

$$S_0(\alpha, \vartheta) = \frac{1}{\sqrt{\alpha}} \ln \frac{\cos(\vartheta - \pi\sqrt{\alpha}/2)}{\cos(\vartheta + \pi\sqrt{\alpha}/2)}. \tag{3.29}$$

Applying (3.8) to $\kappa = \sqrt{\alpha}$ and using (3.28), we conclude that $-\exp(S_0(\alpha, \vartheta))$ belongs to $\Sigma_{\alpha,\vartheta}$ for every $(\alpha, \vartheta) \in W_0$. In view of (3.25), this means that:

$$S_0(\alpha, \vartheta) \in N_{\alpha,\vartheta}, \quad (\alpha, \vartheta) \in W_0. \tag{3.30}$$

We shall see that the set $N_{\alpha,\vartheta}$ for every $(\alpha, \vartheta) \in Q_{\geq 1}$ can actually be completely described in terms of the analytic continuation of S_0 from W_0 to W. To construct such an analytic continuation, we calculate the derivative of $S_0(\alpha, \vartheta)$ with respect to ϑ. In view of (3.16), we find that:

$$\partial_\vartheta S_0(\alpha, \vartheta) = \frac{\pi \operatorname{Sinc}(\pi^2 \alpha)}{\operatorname{Cos}^2(\pi^2 \alpha/4) - \sin^2 \vartheta}, \quad (\alpha, \vartheta) \in W_0. \tag{3.31}$$

We now observe that the right-hand side of (3.31) is actually well-defined and real-analytic on the entire domain W. The real-analytic continuation of S_0 to W can therefore be obtained by integrating the right-hand side of (3.31). This argument is central to the proof of the next result.

Lemma 3.8 *Let W_0 and W be given by (3.26) and (3.27), respectively, and the function S_0 on W_0 be defined by (3.29). There is a unique real-analytic function S on W, such that $S|_{W_0} = S_0$. For every $\vartheta \in (-\pi/2, \pi/2)$, we have:*

$$S(0, \vartheta) = \pi \operatorname{tg} \vartheta, \tag{3.32}$$

$$S(\alpha, \vartheta) = \frac{2}{\sqrt{|\alpha|}} \operatorname{arctg}\left(\operatorname{tg} \vartheta \operatorname{th} \frac{\pi\sqrt{|\alpha|}}{2}\right), \quad \alpha < 0. \tag{3.33}$$

For all $\vartheta \in \mathbb{R}$ and $\alpha < 0$, we have:

$$S(\alpha, \vartheta + \pi) = S(\alpha, \vartheta) + \frac{2\pi}{\sqrt{|\alpha|}}. \tag{3.34}$$

The graph of the function S described by Lemma 3.8 is shown in Fig. 3.
The proof of Lemma 3.8 relies on the next auxiliary statement.

Lemma 3.9 *Let $x_0, \xi_0 \in \mathbb{R}$, $a, b > 0$, and f be a real-analytic function on the rectangle:*
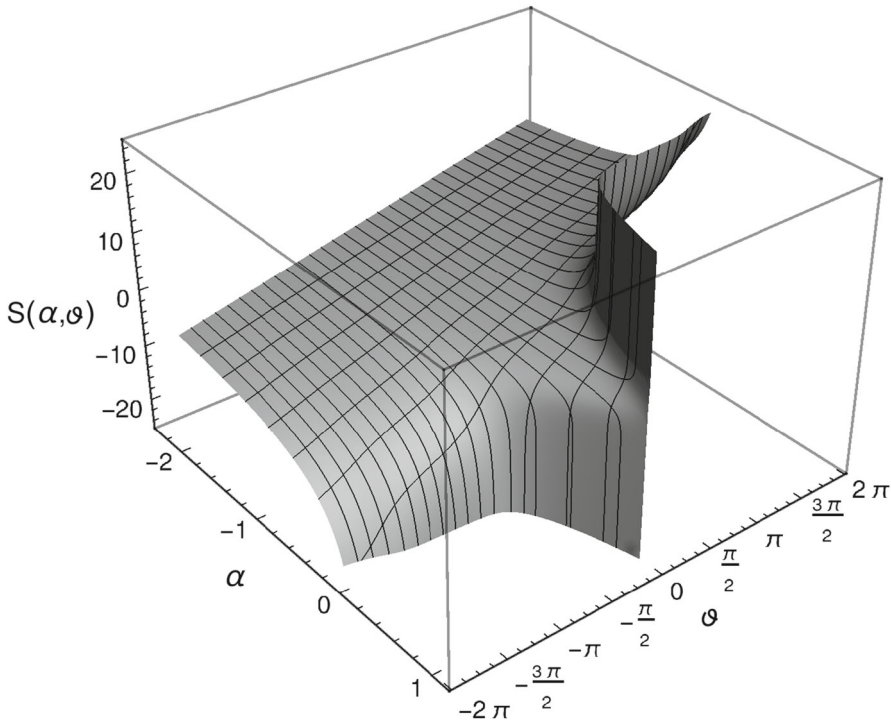
Fig. 3 The function S is plotted using formulas (3.29), (3.32), (3.33), and (3.34)

$$R_{a,b}(x_0, \xi_0) = \{(x, \xi) \in \mathbb{R}^2 : |x - x_0| < a \text{ and } |\xi - \xi_0| < b\}.$$

Then, $(x, \xi) \to \int_{\xi_0}^{\xi} f(x, \xi')\, d\xi'$ is a real-analytic function on $R_{a,b}(x_0, \xi_0)$.

Proof Since f is real-analytic on $R_{a,b}(x_0, \xi_0)$, there are an open set $O \subset \mathbb{C}^2$ and a holomorphic function $\tilde{f}$ on O, such that $R_{a,b}(x_0, \xi_0) \subset O$ and f is the restriction of $\tilde{f}$ to $R_{a,b}(x_0, \xi_0)$. Let F denote the function $(x, \xi) \to \int_{\xi_0}^{\xi} f(x, \xi')\, d\xi'$ on $R_{a,b}(x_0, \xi_0)$. Fix $0 < a' < a$ and $0 < b' < b$. There exist open subsets O' and O'' of $\mathbb{C}$, such that $R_{a',b'}(x_0, \xi_0) \subset O' \times O'' \subset O$. Moreover, we can assume that O'' is convex. We define the function $\tilde{F}$ on $O' \times O''$ by the formula:

$$\tilde{F}(z, \zeta) = \int_0^1 (\zeta - \xi_0)\tilde{f}(z, \xi_0 + (\zeta - \xi_0)t)\, dt, \quad z \in O', \zeta \in O''.$$

Clearly, $\tilde{F}$ is holomorphic on $O' \times O''$ and coincides with F on $R_{a',b'}(x_0, \xi_0)$. This means that F is real-analytic on $R_{a',b'}(x_0, \xi_0)$. Since a' and b' can be chosen arbitrarily close to a and b, we conclude that F is real-analytic on $R_{a,b}(x_0, \xi_0)$. □

Proof of Lemma 3.8 We define the function S on W by the formula:

$$S(\alpha, \vartheta) = \pi \operatorname{Sinc}(\pi^2 \alpha) \int_0^\vartheta \frac{d\vartheta'}{\operatorname{Cos}^2(\pi^2\alpha/4) - \sin^2 \vartheta'}, \quad (\alpha, \vartheta) \in W. \tag{3.35}$$

Since $S_0(\alpha, 0) = 0$ for all $0 < \alpha < 1$ by (3.29), it follows from (3.31) and (3.35) that $S_0 = S|_{W_0}$. For every $(\alpha, \vartheta) \in W$, there is $\varepsilon > 0$, such that W contains the rectangle:

$$\{(\alpha', \vartheta') \in \mathbb{R}^2 : |\alpha' - \alpha| < \varepsilon \text{ and } |\vartheta'| < |\vartheta| + \varepsilon\}.$$

It therefore follows from Lemma 3.9 and (3.35) that S is real-analytic in a neighborhood of every point of W. This means that S is real-analytic on W. Given $\alpha < 0$ and $-\pi/2 < \vartheta < \pi/2$, we let $A(\alpha, \vartheta)$ denote the right-hand side of (3.33) and set $\sigma_\alpha = \sqrt{|\alpha|}$. Using (1.32), we find for every $\alpha < 0$ that:

$$\partial_\vartheta A(\alpha, \vartheta) = \frac{\operatorname{sh}(\pi\sigma_\alpha)}{\sigma_\alpha(\operatorname{ch}^2(\pi\sigma_\alpha/2) - \sin^2 \vartheta)} = \frac{\pi \operatorname{Sinc}(\pi^2\alpha)}{\operatorname{Cos}^2(\pi^2\alpha/4) - \sin^2 \vartheta}, \quad |\vartheta| < \pi/2.$$

In view of (3.35) and the equality $A(\alpha, 0) = 0$, this implies that $S(\alpha, \vartheta) = A(\alpha, \vartheta)$ for all $\alpha < 0$ and $-\pi/2 < \vartheta < \pi/2$, i.e., (3.33) holds. Formula (3.32) follows immediately from (3.35) for $\alpha = 0$. Since $\vartheta \to (\operatorname{Cos}^2(\pi^2\alpha/4) - \sin^2 \vartheta)^{-1}$ is a continuous π-periodic function on $\mathbb{R}$ for every $\alpha < 0$, it follows from (3.35) that:

$$\begin{aligned} S(\alpha, \vartheta + \pi) &= S(\alpha, \vartheta) + \pi \operatorname{Sinc}(\pi^2 \alpha) \int_{-\pi/2}^{\pi/2} \frac{d\vartheta'}{\operatorname{Cos}^2(\pi^2\alpha/4) - \sin^2 \vartheta'} \\ &= S(\alpha, \vartheta) + S(\alpha, \pi/2) - S(\alpha, -\pi/2) \end{aligned}$$

for all $\alpha < 0$ and $\vartheta \in \mathbb{R}$. This implies (3.34), because $S(\alpha, \pm\pi/2) = \pm\pi/\sqrt{|\alpha|}$ by (3.33) and the continuity of S. The uniqueness of S follows from the uniqueness theorem for holomorphic functions. $\qquad\square$

Theorem 3.10 *Let the function S on W be as in Lemma 3.8. For every $\alpha < 1$ and $\vartheta \in \mathbb{R}$, $N_{\alpha, \vartheta}$ is equal to the set*:

$$\{s \in \mathbb{R} : s = S(\alpha, \vartheta + \pi k) \text{ for some } k \in \mathbb{Z} \text{ such that } (\alpha, \vartheta + \pi k) \in W\}. \tag{3.36}$$

Proof Given $\alpha < 1$ and $\vartheta \in \mathbb{R}$, we let $\tilde{N}_{\alpha, \vartheta}$ denote the set (3.36). We have to prove that:

$$N_{\alpha, \vartheta} = \tilde{N}_{\alpha, \vartheta} \tag{3.37}$$

for all $\alpha < 1$ and $\vartheta \in \mathbb{R}$. By (3.25) and (3.30), the set $\Sigma_{\alpha, \vartheta}$ contains $-e^{S_0(\alpha, \vartheta)}$ for every $(\alpha, \vartheta) \in W_0$. Since S coincides with S_0 on W_0, it follows from (3.7) that:

$$R(\alpha, \vartheta, -e^{S(\alpha, \vartheta)}) = 0 \tag{3.38}$$

for all $(\alpha, \vartheta) \in W_0$. By Lemma 3.8, the left-hand side of (3.38) is a real-analytic function of (α, ϑ) on W. In view of the uniqueness theorem for holomorphic functions, this implies that (3.38) remains valid for all $(\alpha, \vartheta) \in W$. Let $\alpha < 1$, $\vartheta \in \mathbb{R}$,

and $s \in \tilde{N}_{\alpha,\vartheta}$. Then, there is $k \in \mathbb{Z}$, such that $(\alpha, \vartheta + \pi k) \in W$ and $s = S(\alpha, \vartheta + \pi k)$. By (3.38), we have $R(\alpha, \vartheta + \pi k, -e^s) = 0$. By (2.19), it follows that $R(\alpha, \vartheta, -e^s) = 0$, i.e., $s \in N_{\alpha,\vartheta}$. We therefore have the inclusion:

$$\tilde{N}_{\alpha,\vartheta} \subset N_{\alpha,\vartheta}, \quad \alpha < 1, \ \vartheta \in \mathbb{R}. \tag{3.39}$$

If $(\alpha, \vartheta) \in Q_0$, then $N_{\alpha,\vartheta} = \varnothing$ by Lemma 3.6 and (3.25) and, therefore, (3.39) implies (3.37). By (3.3), it remains to prove (3.37) for $(\alpha, \vartheta) \in Q_{\geq 1}$. In this case, there is at least one $k \in \mathbb{Z}$, such that $(\alpha, \vartheta + \pi k) \in W$. We hence have:

$$\tilde{N}_{\alpha,\vartheta} \neq \varnothing, \quad (\alpha, \vartheta) \in Q_{\geq 1}. \tag{3.40}$$

We now prove (3.37) for $(\alpha, \vartheta) \in Q_{\geq 1}$ by separately considering the cases $\alpha > 0$, $\alpha = 0$, and $\alpha < 0$.

1. Let $\alpha > 0$, $\kappa = \sqrt{\alpha}$, and $E_1, E_2 \in \Sigma_{\alpha,\vartheta}$. By (3.17), we have $\cos(\vartheta \pm \vartheta_\kappa) \neq 0$, and it follows from (3.8) that $|E_1/E_2|^\kappa = 1$ and, hence, $E_1 = E_2$. This means that $\Sigma_{\alpha,\vartheta}$ and, consequently, $N_{\alpha,\vartheta}$ contain at most one element. In view of (3.39) and (3.40), this implies (3.37).

2. Let $\alpha = 0$ and $E_1, E_2 \in \Sigma_{0,\vartheta}$. By (2.18), we have:

$$-\ln |E_{1,2}| \cos \vartheta + \pi \sin \vartheta = 0.$$

   By (3.6), the condition $(0, \vartheta) \in Q_{\geq 1}$ ensures that $\cos \vartheta \neq 0$. It follows that $\ln |E_1/E_2| = 0$ and, hence, $E_1 = E_2$. This means that $\Sigma_{0,\vartheta}$ and, consequently, $N_{0,\vartheta}$ contain at most one element. In view of (3.39) and (3.40), this implies (3.37).

3. Let $\alpha < 0$ and $s \in N_{\alpha,\vartheta}$. Then, we have $(\alpha, \vartheta) \in W$ and, hence, $s' = S(\alpha, \vartheta)$ is an element of $\tilde{N}_{\alpha,\vartheta}$. Let $E = -e^s$ and $E' = -e^{s'}$. Since $s' \in N_{\alpha,\vartheta}$ by (3.39), we have $E, E' \in \Sigma_{\alpha,\vartheta}$. Let $\kappa = i\sqrt{|\alpha|}$. By (3.17), we have $\cos(\vartheta \pm \vartheta_\kappa) \neq 0$ and it follows from (3.8) that $|E/E'|^\kappa = 1$. This implies that $s = s' + 2\pi k/\sqrt{|\alpha|}$ for some $k \in \mathbb{Z}$. By Lemma 3.8, we conclude that $s = S(\alpha, \vartheta + \pi k)$ and, therefore, $s \in \tilde{N}_{\alpha,\vartheta}$. This means that $N_{\alpha,\vartheta} \subset \tilde{N}_{\alpha,\vartheta}$, whence (3.37) follows by (3.39). $\qquad\square$

By (1.23), (1.24), (1.25), (3.27), and Theorem 3.10, the set $N_{\alpha,\vartheta}$ is empty for $(\alpha, \vartheta) \in Q_0$, contains precisely one element for $(\alpha, \vartheta) \in Q_1$, and is countably infinite for $(\alpha, \vartheta) \in Q_\infty$. In view of (3.25), the same is true for $\Sigma_{\alpha,\vartheta}$ (the emptiness of $\Sigma_{\alpha,\vartheta}$ for $(\alpha, \vartheta) \in Q_0$ also follows from Theorem 3.2). Corollary 3.3 therefore implies that $h_{\alpha,\vartheta}$ has no eigenvalues for $(\alpha, \vartheta) \in Q_0$, has one eigenvalue for $(\alpha, \vartheta) \in Q_1$, and has infinitely many eigenvalues for $(\alpha, \vartheta) \in Q_\infty$, in agreement with what was claimed in Sect. 1.

We now obtain a graphical representation for the sets $N_{\alpha,\vartheta}$. Given $\vartheta \in \mathbb{R}$, we let N_ϑ denote the subset of (α, s)-plane whose sections by the lines $\alpha = \text{const}$ are precisely the sets $N_{\alpha,\vartheta}$:

$$N_\vartheta = \{(\alpha, s) \in \mathbb{R}^2 : \alpha < 1 \text{ and } s \in N_{\alpha, \vartheta}\}.$$

Furthermore, for every $\vartheta \in \mathbb{R}$, we let S_ϑ denote the function $\alpha \to S(\alpha, \vartheta)$ defined on the domain $D_{S_\vartheta} = \{\alpha \in \mathbb{R} : (\alpha, \vartheta) \in W\}$ and set:

$$G_\vartheta = \text{ graph of } S_\vartheta = \{(\alpha, s) \in \mathbb{R}^2 : (\alpha, \vartheta) \in W \text{ and } s = S(\alpha, \vartheta)\}.$$

By Theorem 3.10, we have:

$$N_\vartheta = \bigcup_{k \in \mathbb{Z}} G_{\vartheta + \pi k}. \tag{3.41}$$

It follows from (1.24), (1.25), and (3.27) that:

$$D_{S_\vartheta} = \begin{cases} (-\infty, \omega(\vartheta)), & |\vartheta| \leq \pi/2, \\ \mathbb{R}_-, & |\vartheta| > \pi/2, \end{cases}$$

where ω is given by (1.22). If $|\vartheta| < \pi/2$, then it follows from (3.29) and Lemma 3.8 that:

$$S_\vartheta(\alpha) = \begin{cases} \dfrac{1}{\sqrt{\alpha}} \ln \dfrac{\cos(\vartheta - \pi\sqrt{\alpha}/2)}{\cos(\vartheta + \pi\sqrt{\alpha}/2)}, & 0 < \alpha < \omega(\vartheta), \\[2mm] \pi \, \text{tg}\, \vartheta, & \alpha = 0, \\[2mm] \dfrac{2}{\sqrt{|\alpha|}} \, \text{arctg}\left(\text{tg}\, \vartheta \, \text{th} \, \dfrac{\pi\sqrt{|\alpha|}}{2}\right), & \alpha < 0. \end{cases} \tag{3.42}$$

Since S is continuous on W, we can calculate $S_{\pm\pi/2}(\alpha)$ for $\alpha < 0$ by passing to the limits $\vartheta \uparrow \pi/2$ and $\vartheta \downarrow -\pi/2$ in $S_\vartheta(\alpha)$. In view of (3.33), we obtain:

$$S_{\pm\pi/2}(\alpha) = \pm \frac{\pi}{\sqrt{|\alpha|}}, \quad \alpha < 0.$$

By Lemma 3.8, we have:

$$\begin{aligned} D_{S_{\vartheta + \pi k}} &= \mathbb{R}_-, \\ S_{\vartheta + \pi k}(\alpha) &= S_\vartheta(\alpha) + \frac{2\pi k}{\sqrt{|\alpha|}}, \quad \alpha < 0, \end{aligned} \tag{3.43}$$

for every $\vartheta \in [-\pi/2, \pi/2]$ and every nonzero $k \in \mathbb{Z}$. Formulas (3.41), (3.42), and (3.43) allow us to draw the set N_ϑ, for every $\vartheta \in \mathbb{R}$. In Fig. 4, this set, which represents the α-dependence of eigenvalues of $h_{\alpha, \vartheta}$ in the logarithmic scale, is shown for $\vartheta = \pm\pi/2, -\pi/3, 0$, and $\pi/6$. For $\vartheta \neq \pi/2 + \pi k$, where $k \in \mathbb{Z}$, there is precisely one eigenvalue that crosses the line $\alpha = 0$ in an analytic way, while all other eigenvalues die away at zero or minus infinity as $\alpha \uparrow 0$. If $\vartheta = \pi/2 + \pi k$ for some $k \in \mathbb{Z}$, then there are no eigenvalues for $\alpha \geq 0$.
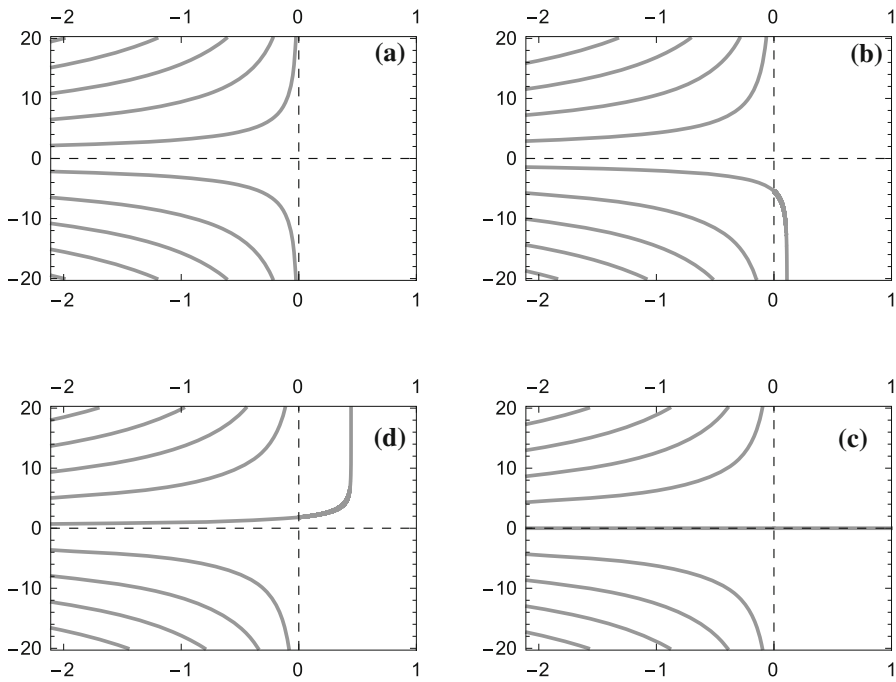
Fig. 4 Plots (a), (b), (c), and (d) represent the set N_ϑ for $\vartheta = \pm\pi/2$, $-\pi/3$, 0, and $\pi/6$, respectively. The horizontal and vertical axes correspond to the variables α and s

3.3 Continuous part of $\mathcal{V}_{\alpha,\vartheta}$

We now consider the absolutely continuous part of the spectral measure $\mathcal{V}_{\alpha,\vartheta}$. By Theorem 3.2, its density $t_{\alpha,\vartheta}$ is given by (3.9). Let the function T on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_{3\pi/2}$ be defined by the formula:

$$T(\alpha, \vartheta, z) = 2R(\alpha, \vartheta, z)\overline{R(\bar{\alpha}, \bar{\vartheta}, z)}, \quad \alpha, \vartheta \in \mathbb{C}, \ z \in \mathbb{C}_{3\pi/2}. \tag{3.44}$$

Clearly, $(\alpha, \vartheta) \to T(\alpha, \vartheta, z)$ is a holomorphic function on $\mathbb{C} \times \mathbb{C}$ for every $z \in \mathbb{C}_{3\pi/2}$. If α and ϑ are real, then we have:

$$T(\alpha, \vartheta, z) = 2|R(\alpha, \vartheta, z)|^2, \quad z \in \mathbb{C}_{3\pi/2}, \tag{3.45}$$

and it follows from (3.9) that:

$$t_{\alpha,\vartheta}(E) = T(\alpha, \vartheta, E)^{-1}, \quad E > 0, \tag{3.46}$$

for every $\alpha < 1$ and $\vartheta \in \mathbb{R}$. We shall explicitly express T in terms of the functions Sinc and Cos. In view of (3.46), this will also give us a formula for the density $t_{\alpha,\vartheta}$.

By (2.17) and (3.44), we have:

$$T(\kappa^2, \vartheta, Ee^{i\phi}) = \frac{2}{\kappa^2}\left(E^{-\kappa}\cos^2(\vartheta - \vartheta_\kappa)\right.$$
$$\left. -2\cos((\pi - \phi)\kappa)\cos(\vartheta - \vartheta_\kappa)\cos(\vartheta + \vartheta_\kappa) + E^\kappa\cos^2(\vartheta + \vartheta_\kappa)\right) \quad (3.47)$$

for all $\kappa \in \mathbb{C}\backslash\{0\}$, $\vartheta \in \mathbb{C}$, $E > 0$, and $-\pi/2 < \phi < 3\pi/2$. In view of the equality

$$\ln E \operatorname{Sinc}\left(-\kappa^2 \ln^2 E\right) = \frac{1}{\kappa}\operatorname{sh}\left(\kappa \ln E\right) = \frac{E^\kappa - E^{-\kappa}}{2\kappa}, \quad \kappa \in \mathbb{C}\backslash\{0\}, \; E > 0,$$

which follows from (1.30), and the trigonometric identities

$$2\cos^2(\vartheta \pm \vartheta_\kappa) = 1 + \cos 2\vartheta \cos \pi\kappa \mp \sin 2\vartheta \sin \pi\kappa,$$
$$2\cos(\vartheta + \vartheta_\kappa)\cos(\vartheta - \vartheta_\kappa) = \cos 2\vartheta + \cos \pi\kappa,$$

which hold for all $\vartheta, \kappa \in \mathbb{C}$, we derive from (3.47) that:

$$T(\kappa^2, \vartheta, Ee^{i\phi}) = \ln^2 E \operatorname{Sinc}^2\left(-\frac{\kappa^2}{4}\ln^2 E\right)(1 + \cos 2\vartheta \cos \pi\kappa)$$
$$-2\pi \ln E \operatorname{Sinc}\left(-\kappa^2 \ln^2 E\right)\operatorname{sinc} \pi\kappa \sin 2\vartheta + 2\frac{\cos \pi\kappa - \cos(\pi - \phi)\kappa}{\kappa^2}\cos 2\vartheta \quad (3.48)$$
$$+2\frac{1 - \cos \pi\kappa \cos(\pi - \phi)\kappa}{\kappa^2}$$

for every $\kappa \in \mathbb{C}\backslash\{0\}$, $\vartheta \in \mathbb{C}$, $E > 0$, and $-\pi/2 < \phi < 3\pi/2$. Let the functions τ and μ on $\mathbb{C} \times \mathbb{C}$ be defined by the formulas:

$$\tau(\alpha, \phi) = (\pi - \phi)^2 \operatorname{Sinc}^2\left(\frac{(\pi - \phi)^2 \alpha}{4}\right) - \pi^2 \operatorname{Sinc}^2\left(\frac{\pi^2 \alpha}{4}\right), \quad (3.49)$$

$$\mu(\alpha, \phi) = 2\pi^2 \operatorname{Sinc}^2(\pi^2 \alpha) + \operatorname{Cos}(\pi^2 \alpha)\tau(\alpha, \phi), \quad (3.50)$$

for every $\alpha, \phi \in \mathbb{C}$. Performing elementary trigonometric transformations, we obtain:

$$\tau(\kappa^2, \phi) = \frac{4}{\kappa^2}\left(\sin^2\frac{(\pi - \phi)\kappa}{2} - \sin^2\frac{\pi\kappa}{2}\right) = 2\frac{\cos \pi\kappa - \cos(\pi - \phi)\kappa}{\kappa^2}, \quad (3.51)$$

$$\mu(\kappa^2, \phi) = 2\frac{\sin^2 \pi\kappa}{\kappa^2} + \tau(\kappa^2, \phi)\cos \pi\kappa = 2\frac{1 - \cos \pi\kappa \cos(\pi - \phi)\kappa}{\kappa^2} \quad (3.52)$$

for all $\kappa \in \mathbb{C}\backslash\{0\}$ and $\phi \in \mathbb{C}$. In view of these formulas, (3.48) implies that:

$$T(\alpha, \vartheta, Ee^{i\phi}) = \ln^2 E \operatorname{Sinc}^2\left(-\frac{\alpha}{4}\ln^2 E\right)(1 + \cos 2\vartheta \operatorname{Cos}(\pi^2 \alpha))$$
$$-2\pi \ln E \operatorname{Sinc}\left(-\alpha \ln^2 E\right)\operatorname{Sinc}(\pi^2 \alpha)\sin 2\vartheta + \tau(\alpha, \phi)\cos 2\vartheta + \mu(\alpha, \phi) \quad (3.53)$$

for all $\alpha \in \mathbb{C}\backslash\{0\}$, $\vartheta \in \mathbb{C}$, $E > 0$, and $-\pi/2 < \phi < 3\pi/2$. By continuity, this equality remains valid for $\alpha = 0$. Thus, (3.53) holds for all $\alpha, \vartheta \in \mathbb{C}$, $E > 0$, and $-\pi/2 < \phi < 3\pi/2$.

Formulas (3.46) and (3.53) can be illustrated by drawing the graphs of the density $t_{\alpha,\vartheta}(E)$ as a function of α and E for various values of ϑ. For this, it is

convenient to use the logarithmic scale for the energy variable and multiply the density $t_{\alpha,\vartheta}$ by the factor $\mu(\alpha, 0) = 2\pi^2 \operatorname{Sinc}^2(\pi^2\alpha)$. More precisely, given $\vartheta \in \mathbb{R}$, we define the function t_ϑ on $(-\infty, 1) \times \mathbb{R}$ by setting:

$$t_\vartheta(\alpha, s) = 2\pi^2 \operatorname{Sinc}^2(\pi^2\alpha) t_{\alpha,\vartheta}(e^s), \quad s \in \mathbb{R}, \, \alpha < 1. \tag{3.54}$$

By (3.46), (3.50), and (3.53), we have:

$$t_\vartheta(\alpha, s)^{-1} = 1 + \frac{s^2 \operatorname{Sinc}^2(-\alpha s^2/4)}{2\pi^2 \operatorname{Sinc}^2(\pi^2\alpha)}(1 + \cos 2\vartheta \operatorname{Cos}(\pi^2\alpha)) - \frac{s \operatorname{Sinc}(-\alpha s^2)}{\pi \operatorname{Sinc}(\pi^2\alpha)} \sin 2\vartheta$$

for every $\alpha < 1$ and $\vartheta, s \in \mathbb{R}$. In Fig. 5, the function t_ϑ is plotted using this formula for $\vartheta = \pm\pi/2$, $-\pi/3$, 0, and $\pi/6$.

The comparison between Figs. 4 and 5 shows that the α-dependence of eigenvalues and of the density of the continuous part of $\mathcal{V}_{\alpha,\vartheta}$ follows the same pattern. This phenomenon can be easily understood if we recall that the point and continuous parts of $\mathcal{V}_{\alpha,\vartheta}$ are both determined by the imaginary part of $\mathscr{M}_{\alpha,\vartheta}$ via (2.25) and consider its behavior in the upper complex half-plane. As in the case of the density $t_{\alpha,\vartheta}$, to facilitate the visualization of $\operatorname{Im}\mathscr{M}_{\alpha,\vartheta}(z)$, we multiply it by $2\pi^2 \operatorname{Sinc}^2(\pi^2\alpha)$ and pass to the logarithmic scale for $|z|$. Therefore, for every $\alpha < 1$ and $\vartheta \in \mathbb{R}$, we introduce the function $J_{\alpha,\vartheta}$ on $\mathbb{R} \times [0, \pi)$ that is defined by the formula:

$$\begin{aligned}
J_{\alpha,\vartheta}(s, \phi) &= 2\pi^2 \operatorname{Sinc}^2(\pi^2\alpha) \operatorname{Im}\mathscr{M}_{\alpha,\vartheta}(e^{s+i\phi}) \\
&= 2\pi(\pi - \phi) \operatorname{Sinc}((\pi - \phi)^2\alpha) \operatorname{Sinc}(\pi^2\alpha) T(\alpha, \vartheta, e^{s+i\phi})^{-1}, \quad s \in \mathbb{R}, \, 0 \le \phi < \pi,
\end{aligned}$$

where the second equality follows from (2.22), (2.24), and (3.45). In view of (3.46) and (3.54), we have $J_{\alpha,\vartheta}(s, 0) = t_\vartheta(\alpha, s)$ for every $\alpha < 1$ and $\vartheta, s \in \mathbb{R}$. Using (3.53), we can explicitly express $J_{\alpha,\vartheta}$ in terms of Sinc and Cos. In Fig. 6, the function $J_{\alpha,\vartheta}$ is plotted for $\alpha = -1/2$ and $\vartheta = \pi/6$. We see that the graph of $J_{\alpha,\vartheta}$ contains equidistant vertical ridges that connect the points of $N_{\alpha,\vartheta}$ at their upper ends with the maxima of the function $s \to t_\vartheta(\alpha, s)$ at their lower ends. This means that the graph of $\operatorname{Im}\mathscr{M}_{\alpha,\vartheta}$ contains ridges along logarithmically equidistant semicircles in the upper half-plane that connect eigenvalues of $h_{\alpha,\vartheta}$ on the negative half-axis with the maxima of the density $t_{\alpha,\vartheta}$ on the positive half-axis. The values $\alpha = -1/2$ and $\vartheta = \pi/6$ chosen for Fig. 6 play no special role: the functions $J_{\alpha,\vartheta}$ and $\operatorname{Im}\mathscr{M}_{\alpha,\vartheta}$ behave in the same way for every $\alpha < 0$ and $\vartheta \in \mathbb{R}$.

4 Self-adjoint extensions

In this section, we recall basic facts concerning self-adjoint extensions of one-dimensional Schrödinger operators and then apply the general theory to proving Theorem 1.1. We refer the reader to [18, 23, 26] for a detailed treatment of one-dimensional Schrödinger operators.
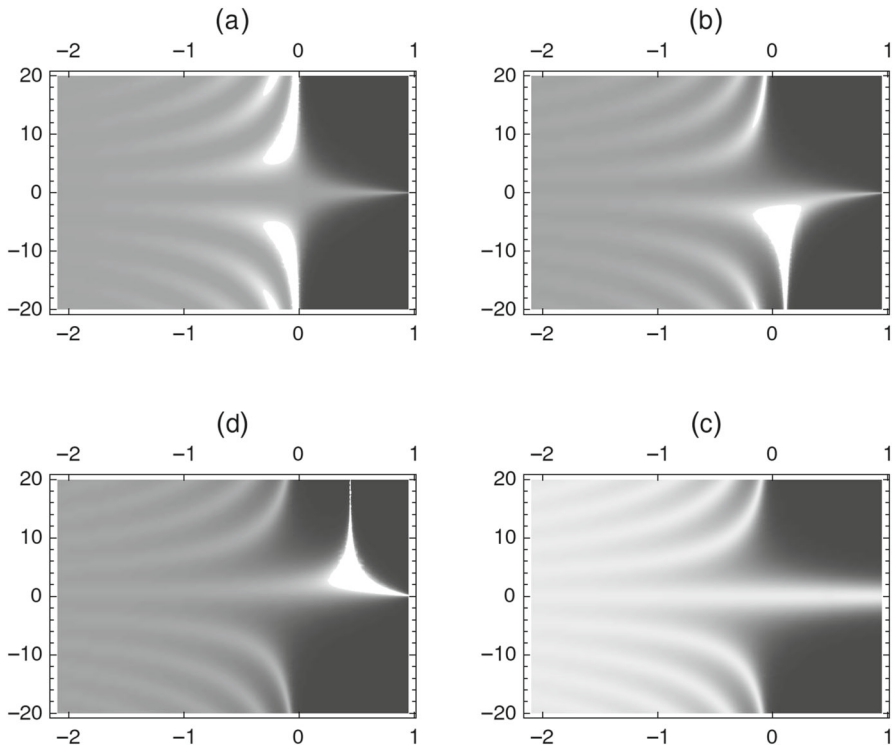
Fig. 5 Plots (a), (b), (c), and (d) represent the function $t_\vartheta(\alpha, s)$ for $\vartheta = \pm\pi/2$, $-\pi/3$, 0, and $\pi/6$, respectively. The horizontal and vertical axes correspond to the variables α and s. The value of t_ϑ is encoded in the brightness of the plot: brighter regions correspond to greater values of the function

4.1 General theory

As in Sect. 1, let λ_+ be the restriction to $\mathbb{R}_+$ of the Lebesgue measure λ on $\mathbb{R}$ and $\mathcal{D}$ be the space of all complex continuously differentiable functions on $\mathbb{R}_+$ whose derivative is absolutely continuous on $\mathbb{R}_+$. Let q be a complex locally integrable function on $\mathbb{R}_+$. Given $z \in \mathbb{C}$, we let $l_{q,z}$ denote the linear operator from $\mathcal{D}$ to the space of complex λ_+-equivalence classes, such that:

$$(l_{q,z}f)(r) = -f''(r) + q(r)f(r) - zf(r) \tag{4.1}$$

for λ-a.e. $r \in \mathbb{R}_+$ and set:

$$l_q = l_{q,0}. \tag{4.2}$$

For every $f \in \mathcal{D}$ and $z \in \mathbb{C}$, we have $l_{q,z}f = l_q f - z[f]$, where, as in Sect. 1, $[f] = [f]_{\lambda_+}$ denotes the λ_+-equivalence class of f. For every $a > 0$ and all complex numbers z, ζ_1, and ζ_2, there is a unique solution f of the equation $l_{q,z}f = 0$, such that $f(a) = \zeta_1$ and $f'(a) = \zeta_2$. This implies that solutions of $l_{q,z}f = 0$ constitute a two-
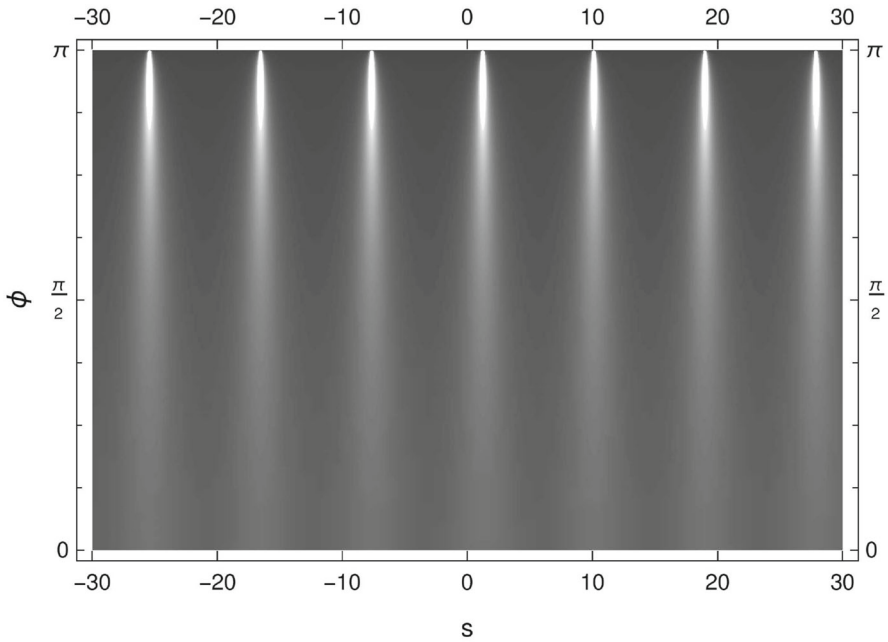
Fig. 6 The function $J_{\alpha,\vartheta}(s,\phi)$ is plotted for $\alpha = -1/2$ and $\vartheta = \pi/6$. As in Fig. 5, brighter regions correspond to greater values of the function

dimensional subspace of $\mathcal{D}$. If $f, g \in \mathcal{D}$ are such that $W_r(f, g)$ has a limit as $r \downarrow 0$, then we set:

$$W^{\downarrow}(f, g) = \lim_{r \downarrow 0} W_r(f, g). \tag{4.3}$$

Similarly, if $f, g \in \mathcal{D}$ are such that $W_r(f, g)$ has a limit as $r \uparrow \infty$, then we set:

$$W^{\uparrow}(f, g) = \lim_{r \uparrow \infty} W_r(f, g). \tag{4.4}$$

In the rest of this subsection, we assume that q is real. Let:

$$\mathcal{D}_q = \{f \in \mathcal{D} : f \text{ and } l_q f \text{ are both square-integrable on } \mathbb{R}_+\}. \tag{4.5}$$

A λ_+-measurable complex function f is said to be left or right square-integrable on $\mathbb{R}_+$ if, respectively, $\int_0^a |f(r)|^2 \, dr < \infty$ or $\int_a^\infty |f(r)|^2 \, dx < \infty$ for any $a > 0$. The subspace of $\mathcal{D}$ consisting of left or right square-integrable on $\mathbb{R}_+$ functions f, such that $l_q f$ is also, respectively, left or right square-integrable on $\mathbb{R}_+$ is denoted by $\mathcal{D}_q^{\downarrow}$ or $\mathcal{D}_q^{\uparrow}$. We obviously have $\mathcal{D}_q = \mathcal{D}_q^{\downarrow} \cap \mathcal{D}_q^{\uparrow}$. It follows from (4.1) by integrating by parts that:

$$\int_a^b ((l_{q,z}f)(r)g(r) - f(r)(l_{q,z}g)(r))\, \mathrm{d}r = W_b(f,g) - W_a(f,g)$$

for every $f, g \in \mathcal{D}$, $z \in \mathbb{C}$, and $a, b > 0$. This implies the existence of limits in the right-hand sides of (4.3) and (4.4), respectively, for every $f, g \in \mathcal{D}_q^\downarrow$ and $f, g \in \mathcal{D}_q^\uparrow$. Hence, $W^\downarrow(f, g)$ is well defined for every $f, g \in \mathcal{D}_q^\downarrow$ and $W^\uparrow(f, g)$ is well defined for every $f, g \in \mathcal{D}_q^\uparrow$. Moreover, it follows that:

$$\langle l_q f, [g] \rangle - \langle [f], l_q g \rangle = W^\uparrow(\bar{f}, g) - W^\downarrow(\bar{f}, g) \tag{4.6}$$

for any $f, g \in \mathcal{D}_q$, where $\langle \cdot, \cdot \rangle$ is the scalar product in $L_2(\mathbb{R}_+)$.

For any linear subspace Z of $\mathcal{D}_q$, let $L_q(Z)$ be the linear operator in $L_2(\mathbb{R}_+)$ defined by the relations:

$$\begin{aligned} D_{L_q(Z)} &= \{[f] : f \in Z\}, \\ L_q(Z)[f] &= l_q f, \quad f \in Z. \end{aligned} \tag{4.7}$$

We define the minimal operator L_q by setting:

$$L_q = L_q(\mathcal{D}_q^0), \tag{4.8}$$

where

$$\mathcal{D}_q^0 = \{f \in \mathcal{D}_q : W^\downarrow(f, g) = W^\uparrow(f, g) = 0 \text{ for every } g \in \mathcal{D}_q\}. \tag{4.9}$$

By (4.6), the operator $L_q(Z)$ is symmetric if and only if $W^\downarrow(\bar{f}, g) = W^\uparrow(\bar{f}, g)$ for any $f, g \in Z$. In particular, L_q is a symmetric operator. Moreover, L_q is closed and densely defined, and its adjoint L_q^* is given by:

$$L_q^* = L_q(\mathcal{D}_q) \tag{4.10}$$

(see Lemma 9.4 in [23]).

If $W^\downarrow(f, g) = 0$ for any $f, g \in \mathcal{D}_q^\downarrow$, then q is said to be in the limit point case (l.p.c.) on the left. Otherwise, q is said to be in the limit circle case (l.c.c.) on the left. Similarly, q is said to be in the l.p.c. on the right if $W^\uparrow(f, g) = 0$ for any $f, g \in \mathcal{D}_q^\uparrow$ and to be in the l.c.c. on the right otherwise. According to the well-known Weyl alternative (see, e.g., [23], Theorem 9.9), q is in the l.c.c. on the left if and only if all solutions of $l_q f = 0$ are left square-integrable on $\mathbb{R}_+$ (and, hence, belong to $\mathcal{D}_q^\downarrow$).

If q is in the l.p.c. both on the left and on the right, then (4.10) implies that L_q^* is symmetric and, therefore, L_q is self-adjoint.

If q is in the l.c.c. on the left and in the l.p.c. on the right, then L_q has deficiency indices (1, 1) and the self-adjoint extensions of L_q are precisely the operators (see [26], Theorem 5.8):

$$L_q^f = L_q(Z_q^f), \tag{4.11}$$

where f is a nontrivial real solution of $l_q f = 0$ and the subspace Z_q^f of $\mathcal{D}_q$ is given by:

$$Z_q^f = \{g \in \mathcal{D}_q : W^{\downarrow}(f, g) = 0\}. \tag{4.12}$$

The operator L_q^f determines f uniquely up to a nonzero real coefficient.

If q is locally square-integrable on $\mathbb{R}_+$, then formulas (4.8) and (4.9) imply that $C_0^\infty(\mathbb{R}_+)$ is contained in $\mathcal{D}_q^0$ and L_q is an extension of $L_q(C_0^\infty(\mathbb{R}_+))$.

Lemma 4.1 *Let q be a real locally square-integrable function on $\mathbb{R}_+$. Then, L_q is the closure of $L_q(C_0^\infty(\mathbb{R}_+))$.*

Proof See Lemma 17 in [21]. □

4.2 The case of the inverse-square potential

By (1.5) and (4.1), we have:

$$\mathscr{L}_{\alpha,z} = l_{q_\alpha,z}, \quad \alpha, z \in \mathbb{C}, \tag{4.13}$$

where the function q_α on $\mathbb{R}_+$ is given by (1.4). In view of (1.6) and (4.2), this implies that:

$$\mathscr{L}_\alpha = l_{q_\alpha}, \quad \alpha \in \mathbb{C}. \tag{4.14}$$

By (1.15) and (4.13), we obtain:

$$l_{q_\alpha,z} \mathcal{U}_\vartheta^\alpha(z) = 0, \quad \alpha, \vartheta, z \in \mathbb{C}. \tag{4.15}$$

If α is real, then q_α is real. It follows from (1.7), (1.8), (4.5), (4.7), and (4.14) that:

$$\Delta_\alpha = \mathcal{D}_{q_\alpha}, \tag{4.16}$$

$$H_\alpha(Z) = L_{q_\alpha}(Z) \tag{4.17}$$

for every $\alpha \in \mathbb{R}$ and every linear subspace Z of Δ_α. Hence, (1.9), (1.10), and Lemma 4.1 imply that:

$$h_\alpha = L_{q_\alpha}, \quad \alpha \in \mathbb{R}. \tag{4.18}$$

By (4.10), (4.16), (4.17), and (4.18), equality (1.11) holds for all real α.

If $\alpha = \kappa^2$ for $\kappa \in \mathbb{C}$, then the equation $l_{q_\alpha} f = 0$ has linearly independent solutions $r^{1/2 \pm \kappa}$ for $\kappa \neq 0$ and $r^{1/2}$ and $r^{1/2} \ln r$ for $\kappa = 0$. It follows that:

(i) q_α is in the l.p.c. both on the left and on the right for $\alpha \geq 1$ and
(ii) q_α is in the l.p.c. on the right and in the l.c.c. on the left for $\alpha < 1$.

In view of (4.2) and (4.15), we have $l_{q_\alpha} \mathcal{U}_\vartheta^\alpha(0) = 0$ for every $\alpha, \vartheta \in \mathbb{C}$. For $\alpha < 1$ and $\vartheta \in \mathbb{R}$, the function $\mathcal{U}_\vartheta^\alpha(0)$ is real and nontrivial, and it follows from (1.17), (4.12), and (4.16) that $Z_{\alpha,\vartheta} = Z_{q_\alpha}^{\mathcal{U}_\vartheta^\alpha(0)}$. By (1.16), (4.11), and (4.17), we conclude that:

$$h_{\alpha,\vartheta} = L_{q_\alpha}^{\mathcal{U}_\vartheta^\alpha(0)}, \quad \alpha < 1, \ \vartheta \in \mathbb{R}. \tag{4.19}$$

Proof of Theorem 1.1 In view of (4.18) and condition (i), the operator h_α is self-adjoint for $\alpha \geq 1$. Let $\alpha < 1$. By (4.18) and (4.19), $h_{\alpha,\vartheta}$ a self-adjoint extension of h_α for every $\vartheta \in \mathbb{R}$. Conversely, let H be a self-adjoint extension of h_α. Then, $H = L_{q_\alpha}^f$ for some nontrivial real $f \in \mathcal{D}$ satisfying $l_{q_\alpha} f = 0$. By (1.12), (4.2), and (4.13), we have $l_{q_\alpha} \mathcal{A}^\alpha(0) = l_{q_\alpha} \mathcal{B}^\alpha(0) = 0$. Since $\mathcal{A}^\alpha(0)$ and $\mathcal{B}^\alpha(0)$ are real and linearly independent, it follows from (1.14) that $f = c \mathcal{U}_\vartheta^\alpha(0)$ for some $c, \vartheta \in \mathbb{R}$ such that $c \neq 0$. In view of (4.19), this means that $H = h_{\alpha,\vartheta}$.

Suppose now that $\vartheta, \vartheta' \in \mathbb{R}$ and $h_{\alpha,\vartheta} = h_{\alpha,\vartheta'}$. By (4.19), we have $\mathcal{U}_\vartheta^\alpha(0) = c \mathcal{U}_{\vartheta'}^\alpha(0)$ for some nonzero real c. In view of (1.14), this implies that $\sin \vartheta = c \sin \vartheta'$ and $\cos \vartheta = c \cos \vartheta'$ and, therefore, $e^{i\vartheta} = c e^{i\vartheta'}$. It follows that $e^{i(\vartheta - \vartheta')} = c$, whence $c = \pm 1$ and $\vartheta - \vartheta' \in \pi\mathbb{Z}$. $\qquad\square$

5 Eigenfunction expansions

This section consists of two subsections. In the first one, we briefly describe the construction of eigenfunction expansions of one-dimensional Schrödinger operators developed in [10, 16]. This construction, which is adapted to the case of operators with a simple spectrum and relies on the so-called singular Titchmarsh–Weyl m-functions, can be viewed as a variant of Kodaira's general approach [15] based on matrix-valued measures (see Remark 16 in [21]). In the second subsection, we prove Theorem 1.2 by applying the general theory to the case of the inverse-square potential.

5.1 General theory

Let q be a real locally integrable function on $\mathbb{R}_+$. We assume that q is in the l.c.c. on the left and in the l.p.c. on the right.

Let $O \subset \mathbb{C}$ be an open set. We say that a map $u : O \to \mathcal{D}$ is a q-solution on O if $l_{q,z} u(z) = 0$ for every $z \in O$. A q-solution u on O is said to be holomorphic if the functions $z \to u(z|r)$ and $z \to \partial_r u(z|r)$ are holomorphic on O for any $r \in \mathbb{R}_+$. A q-solution u on O is said to be nonvanishing if $u(z) \neq 0$ for every $z \in O$. A q-solution in $\mathbb{C}$ is said to be real-entire if it is holomorphic on $\mathbb{C}$ and $u(E)$ is real for every $E \in \mathbb{R}$.

Let u be a real-entire q-solution. Since q is in l.c.c. on the left, we have $u(z) \in \mathcal{D}_q^\downarrow$ for every $z \in \mathbb{C}$. Suppose that u is nonvanishing and:

$$W^{\downarrow}(u(z), u(z')) = 0, \quad z, z' \in \mathbb{C}. \tag{5.1}$$

Let v be a nonvanishing holomorphic q-solution on $\mathbb{C}_+$, such that $v(z)$ is right square-integrable for every $z \in \mathbb{C}_+$ (such a v always exists; see Lemma 9.8 in [23]). If $W(u(z), v(z)) = 0$ for some $z \in \mathbb{C}_+$,[10] then $u(z)$ is proportional to $v(z)$ and, hence, $u(z) \in \mathcal{D}_q$. In view of (4.12) and (5.1), this means that $u(z) \in Z_q^{u(0)}$ and, therefore, $[u(z)]$ is an eigenvector of the self-adjoint operator $L_q^{u(0)}$ with the eigenvalue z. However, this cannot be the case, because all eigenvalues of $L_q^{u(0)}$ must be real. It follows that $W(u(z), v(z)) \neq 0$ for all $z \in \mathbb{C}_+$.

Given a nonvanishing real-entire q-solution u, one can always find another real-entire q-solution $\tilde{u}$, such that $W(u(z), \tilde{u}(z)) \neq 0$ for all $z \in \mathbb{C}$ (see Lemma 2.4 in [16]).

Let u and $\tilde{u}$ be real-entire q-solutions, such that (5.1) is satisfied and $W(u(z), \tilde{u}(z)) \neq 0$ for all $z \in \mathbb{C}$. We define the holomorphic function $\mathcal{M}_{u,\tilde{u}}^q$ on $\mathbb{C}_+$ by setting:

$$\mathcal{M}_{u,\tilde{u}}^q(z) = \frac{1}{\pi} \frac{W(v(z), \tilde{u}(z))}{W(v(z), u(z))W(u(z), \tilde{u}(z))}, \tag{5.2}$$

where v is a nonvanishing holomorphic q-solution on $\mathbb{C}_+$, such that $v(z)$ is right square-integrable for all $z \in \mathbb{C}_+$ (since q is in the l.p.c. on the right, this definition is independent of the choice of v). Following [16], we call such functions singular Titchmarsh–Weyl m-functions.

The proof of the next statement can be found in [16].

Proposition 5.1 *Let a locally integrable real function q on $\mathbb{R}_+$ be in the l.c.c. on the left and in the l.p.c. on the right. Let u be a nonvanishing real-entire q solution, such that (5.1) holds for all $z \in \mathbb{C}$. Then, the following statements hold:*

1. *There exists a unique positive Radon measure v on $\mathbb{R}$ (called the spectral measure for q and u), such that:*

$$\int \varphi(E) \operatorname{Im} \mathcal{M}_{u,\tilde{u}}^q(E + i\eta)\, dE \to \int \varphi(E)\, dv(E) \quad (\eta \downarrow 0)$$

   for every continuous function φ on $\mathbb{R}$ with compact support and every real-entire q-solution $\tilde{u}$, such that $W(u(z), \tilde{u}(z)) \neq 0$ for every $z \in \mathbb{C}$.
2. *Let v be the spectral measure for q and u. There is a unique unitary operator $U : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}, v)$ (called the spectral transformation for q and u), such that:*

$$(U\psi)(E) = \int_{\mathbb{R}_+} u(E|r)\psi(r)\, dr, \quad \psi \in L_2^c(\mathbb{R}_+),$$

   for v-a.e. E.

---

[10] We recall that $W(f, g)$ denotes the value of the function $r \to W_r(f, g)$ if $f, g \in \mathcal{D}$ are such that this function is constant (in particular, if $l_{q,z}f = l_{q,z}g = 0$ for some $z \in \mathbb{C}$).

3. *Let v and U be the spectral measure and transformation for q and u. Then, we have $L_q^{u(0)} = U^{-1}\mathcal{T}_\iota^v U$, where ι is the identity function on $\mathbb{R}$.*

In the next subsection, we shall verify that $\mathcal{V}_{\alpha,\vartheta}$ is actually the spectral measure for q_α and $\mathcal{U}_\vartheta^\alpha$. This justifies using the same term "spectral measure" for $\mathcal{V}_{\alpha,\vartheta}$ and for the measures described by Proposition 5.1.

5.2 The case of the inverse-square potential

For any $\kappa \in \mathbb{C}$, we define the map $\mathfrak{v}^\kappa : \mathbb{C}_{3\pi/2} \to \mathcal{D}$ by the relation:

$$\mathfrak{v}^\kappa(z|r) = \frac{i\pi}{2}e^{i\pi\kappa/2}r^{1/2}H_\kappa^{(1)}(rz^{1/2}), \quad r \in \mathbb{R}_+, z \in \mathbb{C}_{3\pi/2}, \tag{5.3}$$

where $H_\kappa^{(1)}$ is the first Hankel function of order κ. Because $H_\kappa^{(1)}$ is a solution of the Bessel equation, we have:

$$\mathscr{L}_{\kappa^2,z}\mathfrak{v}^\kappa(z) = 0, \quad \kappa \in \mathbb{C}, z \in \mathbb{C}_{3\pi/2}. \tag{5.4}$$

It follows from the relation $H_{-\kappa}^{(1)} = e^{i\pi\kappa}H_\kappa^{(1)}$ (formula (9) in Sec. 7.2.1 in [7]) that:

$$\mathfrak{v}^{-\kappa}(z) = \mathfrak{v}^\kappa(z), \quad \kappa \in \mathbb{C}, z \in \mathbb{C}_{3\pi/2}. \tag{5.5}$$

In view of (2.5) and (5.4), the Wronskian $W_r(\mathfrak{v}^\kappa(z), \mathfrak{u}^{\pm\kappa}(z))$ does not depend on r. To find it explicitly, we can use the expression for the Wronskian of Bessel functions (formula (29) in Sec. 7.11 in [7]):

$$W_z(J_\kappa, H_\kappa^{(1)}) = \frac{2i}{\pi z}. \tag{5.6}$$

Taking (5.5) into account and combining (5.6) with (2.1), (2.3), and (5.3), we derive that:

$$W(\mathfrak{v}^\kappa(z), \mathfrak{u}^\kappa(z)) = z^{-\kappa/2}e^{i\pi\kappa/2}, \quad W(\mathfrak{v}^\kappa(z), \mathfrak{u}^{-\kappa}(z)) = z^{\kappa/2}e^{-i\pi\kappa/2} \tag{5.7}$$

for any $\kappa \in \mathbb{C}$ and $z \in \mathbb{C}_{3\pi/2}$.

For $\alpha \in \mathbb{C}$ and $z \in \mathbb{C}_{3\pi/2}$, let the function $\mathscr{V}^\alpha(z)$ on $\mathbb{R}_+$ be defined by the relation $\mathscr{V}^\alpha(z) = \mathfrak{v}^\kappa(z)$, where $\kappa \in \mathbb{C}$ is such that $\kappa^2 = \alpha$ (by (5.5), this definition does not depend on the choice of κ). We therefore have:

$$\mathscr{V}^{\kappa^2}(z) = \mathfrak{v}^\kappa(z) \tag{5.8}$$

for every $\kappa \in \mathbb{C}$ and $z \in \mathbb{C}_{3\pi/2}$. By (5.4) and (5.8), we obtain:

$$\mathscr{L}_{\alpha,z}\mathscr{V}^\alpha(z) = 0, \quad z \in \mathbb{C}_{3\pi/2}, \alpha \in \mathbb{C}. \tag{5.9}$$

Using the well-known asymptotic form of $H_\kappa^{(1)}(\zeta)$ for $\zeta \to \infty$ (see formula (1) in Sec. 7.13.1 in [7]), we find that:

$$\mathfrak{v}^\kappa(z|r) \sim 2^{-1}\sqrt{\pi}(i+1)z^{-1/4}e^{iz^{1/2}r}, \quad r \to \infty,$$

for every $\kappa \in \mathbb{C}$ and $z \in \mathbb{C}_{3\pi/2}$ and, hence, $\mathfrak{v}^\kappa(z)$ is right square-integrable for all $\kappa \in \mathbb{C}$ and $z \in \mathbb{C}_+$. In view of (5.8), this implies that $\mathscr{V}^\alpha(z)$ is right square-integrable for all $\alpha \in \mathbb{C}$ and $z \in \mathbb{C}_+$.

Lemma 5.2 *There is a unique holomorphic function F on $\mathbb{C} \times \mathbb{C}_{3\pi/2} \times \mathbb{C}_\pi$, such that $F(\alpha, z, r) = \mathscr{V}^\alpha(z|r)$ for every $\alpha \in \mathbb{C}$, $z \in \mathbb{C}_{3\pi/2}$, and $r > 0$.*

Proof By (5.3), there is a holomorphic function G on $\mathbb{C} \times \mathbb{C}_{3\pi/2} \times \mathbb{C}_\pi$, such that $G(\kappa, z, r) = \mathfrak{v}^\kappa(z|r)$ for every $\kappa \in \mathbb{C}$, $z \in \mathbb{C}_{3\pi/2}$, and $r > 0$. It follows from (5.5) and the uniqueness theorem for holomorphic functions that $G(\kappa, z, \zeta) = G(-\kappa, z, \zeta)$ for all $\kappa \in \mathbb{C}$, $z \in \mathbb{C}_{3\pi/2}$, and $\zeta \in \mathbb{C}_\pi$. The existence of F with the required properties is now ensured by Lemma A.3 and (5.8). The uniqueness of F follows from the uniqueness theorem for holomorphic functions. $\square$

It follows immediately from (1.13) that the identity

$$W_r(f_1 f_2, f_3 f_4) = f_1(r)f_3(r)W_r(f_2, f_4) + W_r(f_1, f_3)f_2(r)f_4(r) \tag{5.10}$$

holds for every $f_1, f_2, f_3, f_4 \in \mathcal{D}$ and $r > 0$.

Lemma 5.3 *Let $\kappa \in \mathbb{C}$ be such that $|\mathrm{Re}\,\kappa| < 1$. Then, we have:*

$$W^\downarrow(\mathfrak{u}^\kappa(z), \mathfrak{u}^\kappa(z')) = 0, \quad W^\downarrow(\mathfrak{u}^\kappa(z), \mathfrak{u}^{-\kappa}(z')) = -\frac{2}{\pi}\sin\pi\kappa \tag{5.11}$$

for every $z, z' \in \mathbb{C}$.

Proof By (2.1) and (5.10), we have:

$$W_r(\mathfrak{u}^\kappa(z), \mathfrak{u}^\kappa(z')) = 2r^{2+2\kappa}(z'\mathcal{X}_\kappa(r^2z)\mathcal{X}'_\kappa(r^2z') - z\mathcal{X}'_\kappa(r^2z)\mathcal{X}_\kappa(r^2z')), \tag{5.12}$$

$$\begin{aligned} W_r(\mathfrak{u}^\kappa(z), \mathfrak{u}^{-\kappa}(z')) = &- 2\kappa\mathcal{X}_\kappa(r^2z)\mathcal{X}_{-\kappa}(r^2z') \\ &+ 2r^2(z'\mathcal{X}_\kappa(r^2z)\mathcal{X}'_{-\kappa}(r^2z') - z\mathcal{X}'_\kappa(r^2z)\mathcal{X}_{-\kappa}(r^2z')) \end{aligned} \tag{5.13}$$

for all $z, z' \in \mathbb{C}$ and $r > 0$. Since $|\mathrm{Re}\,\kappa| < 1$, the left equality in (5.11) follows from (5.12) for every $z, z' \in \mathbb{C}$. Formula (2.2) implies that:

$$\mathcal{X}_\kappa(0)\mathcal{X}_{-\kappa}(0) = (\Gamma(1+\kappa)\Gamma(1-\kappa))^{-1} = \mathrm{sinc}\,\pi\kappa.$$

By (5.13), we conclude that the right equality in (5.11) holds for all $z, z' \in \mathbb{C}$. $\square$

Lemma 5.4 *Let $\alpha < 1$ and $\vartheta \in \mathbb{R}$. Then, we have:*

$$W^\downarrow(\mathcal{U}^\alpha_\vartheta(z), \mathcal{U}^\alpha_\vartheta(z')) = 0, \quad z, z' \in \mathbb{C}. \tag{5.14}$$

Proof Let $\alpha \neq 0$ and $\kappa \in \mathbb{C}$ be such that $\kappa^2 = \alpha$. By (2.14), we have:

$$
\begin{aligned}
\alpha W_r(\mathcal{U}_\vartheta^\alpha(z), \mathcal{U}_\vartheta^\alpha(z')) = {} & W_r(u^\kappa(z), u^\kappa(z')) \cos^2(\vartheta - \vartheta_\kappa) \\
& + W_r(u^{-\kappa}(z), u^{-\kappa}(z')) \cos^2(\vartheta + \vartheta_\kappa) + (W_r(u^\kappa(z'), u^{-\kappa}(z)) \\
& - W_r(u^\kappa(z), u^{-\kappa}(z'))) \cos(\vartheta - \vartheta_\kappa) \cos(\vartheta + \vartheta_\kappa)
\end{aligned} \tag{5.15}
$$

for all $z, z' \in \mathbb{C}$ and $r > 0$. Since $\alpha < 1$, we have $|\operatorname{Re} \kappa| < 1$ and, therefore, (5.14) follows from (5.15) and Lemma 5.3. It remains to consider the case $\alpha = 0$. For every $z \in \mathbb{C}$, we define the smooth functions $c(z)$ and $d(z)$ on $\mathbb{R}$ by the relations:

$$
c(z|r) = \mathcal{X}_0(r^2 z), \quad d(z|r) = (\gamma - \ln 2) c(z|r) - \mathcal{Y}(r^2 z), \quad r \in \mathbb{R}.
$$

By (2.1), (2.7), (2.9), and (2.12), we have:

$$
\mathcal{A}^0(z|r) = 2r^{1/2}(c(z|r) \ln r + d(z|r)), \quad \mathcal{B}^0(z|r) = \pi r^{1/2} c(z|r), \quad z \in \mathbb{C}, r > 0.
$$

Using (5.10), we obtain:

$$
\begin{aligned}
\frac{1}{4} W_r(\mathcal{A}^0(z), \mathcal{A}^0(z')) = {} & r \ln^2 r W_r(c(z), c(z')) + r W_r(d(z), d(z')) \\
& + r \ln r (W_r(c(z), d(z')) + W_r(d(z), c(z'))) \\
\frac{1}{2\pi} W_r(\mathcal{A}^0(z), \mathcal{B}^0(z')) = {} & r W_r(d(z), c(z')) + r \ln r W_r(c(z), c(z')) \\
& - c(z|r) c(z'|r), \\
W_r(\mathcal{B}^0(z), \mathcal{B}^0(z')) = {} & \pi^2 r W_r(c(z), c(z'))
\end{aligned}
$$

for every $r > 0$ and $z, z' \in \mathbb{C}$. Since $c(z|0) = 1$ and $d(z|0) = \gamma - \ln 2$ for all $z \in \mathbb{C}$, we find that:

$$
W^\downarrow(\mathcal{A}^0(z), \mathcal{B}^0(z')) = -2\pi, \quad W^\downarrow(\mathcal{A}^0(z), \mathcal{A}^0(z')) = W^\downarrow(\mathcal{B}^0(z), \mathcal{B}^0(z')) = 0
$$

for all $z, z' \in \mathbb{C}$. In view of (1.14), this implies (5.14) for $\alpha = 0$. □

It follows from (1.15) and (5.9) that $W_r(\mathscr{V}^\alpha(z), \mathcal{U}_\vartheta^\alpha(z))$ does not depend on r for every $\alpha, \vartheta \in \mathbb{C}$ and $z \in \mathbb{C}_{3\pi/2}$. It is easy to see that:

$$
W(\mathscr{V}^\alpha(z), \mathcal{U}_\vartheta^\alpha(z)) = R(\alpha, \vartheta, z), \quad \alpha, \vartheta \in \mathbb{C}, z \in \mathbb{C}_{3\pi/2}, \tag{5.16}
$$

where R is the function defined in Lemma 2.3. Indeed, it follows from (2.14), (5.7), and (5.8) that $W(\mathscr{V}^{\kappa^2}(z), \mathcal{U}_\vartheta^{\kappa^2}(z))$ is equal to the right-hand side of formula (2.17) for every $\kappa \in \mathbb{C} \backslash \{0\}$, $\vartheta \in \mathbb{C}$, and $z \in \mathbb{C}_{3\pi/2}$. Equality (5.16), therefore, holds for all $\alpha \in \mathbb{C} \backslash \{0\}$, $\vartheta \in \mathbb{C}$, and $z \in \mathbb{C}_{3\pi/2}$. By (1.14) and Lemmas 2.2, 2.3, and 5.2, both sides of (5.16) are holomorphic with respect to (α, ϑ, z) on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_{3\pi/2}$. Hence, (5.16) remains valid for $\alpha = 0$.

By (1.14) and (2.16), we have:

$$
W(\mathcal{U}_\vartheta^\alpha(z), \mathcal{U}_{\vartheta+\pi/2}^\alpha(z)) = -2\pi \operatorname{Sinc}^2(\pi^2 \alpha), \quad \alpha, \vartheta, z \in \mathbb{C}. \tag{5.17}
$$

Proof of Theorem 1.2 It follows immediately from (1.14) and the definition of $\mathcal{A}^\alpha(z)$ and $\mathcal{B}^\alpha(z)$ that the functions $z \to \mathcal{U}_\vartheta^\alpha(z|r)$ and $z \to \partial_r \mathcal{U}_\vartheta^\alpha(z|r)$ are holomorphic

on $\mathbb{C}$ for every $r > 0$ and $\alpha, \vartheta \in \mathbb{C}$. Equality (4.15), therefore, implies that $\mathcal{U}_\vartheta^\alpha$ is a holomorphic q_α-solution on $\mathbb{C}$ for every $\alpha, \vartheta \in \mathbb{C}$. If α and ϑ are real, then $\mathcal{U}_\vartheta^\alpha(E)$ is real for real E and, hence, $\mathcal{U}_\vartheta^\alpha$ is a real-entire q_α-solution. By (5.3) and (5.8), the functions $z \to \mathscr{V}^\alpha(z|r)$ and $z \to \partial_r \mathscr{V}^\alpha(z|r)$ are holomorphic on $\mathbb{C}_{3\pi/2}$ for every $r > 0$ and $\alpha \in \mathbb{C}$. In view of (4.13) and (5.9), it follows that $\mathscr{V}^\alpha$ is a holomorphic q_α-solution on $\mathbb{C}_{3\pi/2}$ for every $\alpha \in \mathbb{C}$. Moreover, $\mathscr{V}^\alpha$ is nonvanishing for every $\alpha \in \mathbb{C}$ by (5.7) and (5.8).

We now fix $\alpha < 1$ and $\vartheta \in \mathbb{R}$, set $q = q_\alpha$, and let u and $\tilde{u}$ denote the real-entire q-solutions $\mathcal{U}_\vartheta^\alpha$ and $\mathcal{U}_{\vartheta+\pi/2}^\alpha$ respectively. Because $\alpha < 1$, (5.17) implies that:

$$W(u(z), \tilde{u}(z)) \neq 0, \quad z \in \mathbb{C}, \tag{5.18}$$

and, hence, u is nonvanishing. By Lemma 5.4, we conclude that q and u satisfy all conditions of Proposition 5.1. Moreover, it follows from (5.18) that there exists a well-defined Titchmarsh–Weyl m-function $\mathcal{M}_{u,\tilde{u}}^q$. Since $\mathscr{V}^\alpha(z)$ is right square-integrable for every $z \in \mathbb{C}_+$, the latter can be found by substituting $v = \mathscr{V}^\alpha|_{\mathbb{C}_+}$ in the right-hand side of (5.2). Using (2.23), (5.16), and (5.17), we obtain $\mathcal{M}_{u,\tilde{u}}^q(z) = \mathscr{M}_{\alpha,\vartheta}(z)$ for every $z \in \mathbb{C}_+$. In view of (5.18), statement 1 of Proposition 5.1 and formula (2.25) imply that $\mathcal{V}_{\alpha,\vartheta}$ is precisely the spectral measure for q and u. The existence and uniqueness of $U_{\alpha,\vartheta}$ and equality (1.20) are, therefore, ensured by statements 2 and 3 of Proposition 5.1 and formula (4.19). $\qquad \square$

6 Smoothness properties of the spectral measure

In this section, we shall establish Theorem 1.3.

Before passing to the proof, we note that the smoothness of $\mathcal{V}_{\alpha,\vartheta}$ with respect to α and ϑ is suggested by explicit formulas obtained in Sect. 3. For example, let us set $\vartheta = \pi/2$ and examine the α-dependence of the point part $\mathcal{V}_{\alpha,\pi/2}^p$ of the measure $\mathcal{V}_{\alpha,\pi/2}$. It follows from (3.25), (3.27), Lemma 3.8, and Theorem 3.10 that $\Sigma_{\alpha,\pi/2} = \varnothing$ for $\alpha \geq 0$ and:

$$\Sigma_{\alpha,\pi/2} = \left\{ E \in \mathbb{R} : E = -\exp\left((2k+1)\pi|\alpha|^{-1/2}\right) \text{ for some } k \in \mathbb{Z} \right\}$$

for $\alpha < 0$. Let $\varphi \in \mathscr{S}$. In view of (3.11) and (B.2), Theorem 3.2 implies that:

$$\int \varphi(E) \, d\mathcal{V}_{\alpha,\pi/2}^p(E) = \frac{2}{\pi^2 \operatorname{Sinc}(\pi^2\alpha) \operatorname{Sinc}^2(\pi^2\alpha/4)} \sum_{k \in \mathbb{Z}} \varphi_{(2k+1)\pi}(\alpha) \tag{6.1}$$

for every $\alpha < 1$, where φ_c, $c \in \mathbb{R}$, denotes the function on $\mathbb{R}$ that is identically zero on $[0, \infty)$ and is given by:

$$\varphi_c(\alpha) = \frac{1}{|\alpha|} \exp\left(c|\alpha|^{-1/2}\right) \varphi\left(-\exp\left(c|\alpha|^{-1/2}\right)\right)$$

for $\alpha < 0$. It follows immediately from the definition of the space $\mathscr{S}$ that φ_c is an infinitely differentiable function on $\mathbb{R}$ for every nonzero real c (however, in general, φ_c is not real-analytic at $\alpha = 0$ even for real-analytic φ). It is also possible to verify directly that $\sum_{k\in\mathbb{Z}} \varphi_{(2k+1)\pi}$ is infinitely differentiable on $\mathbb{R}$ and, therefore, the left-hand side of (6.1) is infinitely differentiable with respect to α on $(-\infty, 1)$.

It seems, however, that a complete proof of Theorem 1.3 (including the continuous part of $\mathcal{V}_{\alpha,\vartheta}$ and the dependence on both α and ϑ) based on explicit formulas for the spectral measures would be extremely cumbersome. We shall adopt a different approach based on representation (2.25) of $\mathcal{V}_{\alpha,\vartheta}$ as a boundary value of the holomorphic function $\mathscr{M}_{\alpha,\vartheta}$. The idea is to derive the infinite differentiability of $\mathcal{V}_{\alpha,\vartheta}$ with respect to α and ϑ from that of $\mathscr{M}_{\alpha,\vartheta}$. Lemma 6.1 below gives a condition under which the differentiability of a holomorphic function on $\mathbb{C}_+$ with respect to some parameters implies the same property for its boundary value. This condition involves certain uniform estimates on the derivatives of this function with respect to the parameters in question. In the case of the function $\mathscr{M}_{\alpha,\vartheta}$, estimates of this type, which are the most nontrivial part of the proof of Theorem 1.3, are provided by Proposition 6.2 below. Combining Lemma 6.1 and Proposition 6.2, we shall obtain the infinite differentiability of $\mathcal{V}_{\alpha,\vartheta}$.

We now give a formal exposition.

For every $\varrho, \sigma \geq 0$, we let $\mathcal{H}_{\varrho,\sigma}$ denote the Banach space consisting of all holomorphic functions on $\mathbb{C}_+$ with the finite norm:

$$\|f\|_{\varrho,\sigma} = \sup_{z\in\mathbb{C}_+} |f(z)| \mathcal{N}_{\varrho,\sigma}(z),$$

where the function $\mathcal{N}_{\varrho,\sigma}$ on $\mathbb{C}_+$ is given by:

$$\mathcal{N}_{\varrho,\sigma}(z) = \frac{1}{(1+|z|)^\varrho} \left(\frac{\operatorname{Im} z}{1+|z|}\right)^\sigma.$$

If $\varrho' \geq \varrho \geq 0$ and $\sigma' \geq \sigma \geq 0$, then $\mathcal{H}_{\varrho,\sigma} \subset \mathcal{H}_{\varrho',\sigma'}$ and the inclusion map is continuous. We define the linear space $\mathcal{H}$ by setting $\mathcal{H} = \bigcup_{\varrho,\sigma \geq 0} \mathcal{H}_{\varrho,\sigma}$. It is well known (see, e.g., [25], Ch. 5, Sec. 26.3) that $\int_{\mathbb{R}} f(E + i\eta)\varphi(E)\,dE$ has a limit as $\eta \downarrow 0$ for every $f \in \mathcal{H}$ and $\varphi \in \mathscr{S}$. Given $\varphi \in \mathscr{S}$, we let B_φ denote the map $f \to \lim_{\eta\downarrow 0} \int_{\mathbb{R}} f(E + i\eta)\varphi(E)\,dE$ on $\mathcal{H}$. The definitions of $\mathscr{S}$ and $\mathcal{H}_{\varrho,\sigma}$ imply that $f \to \int_{\mathbb{R}} f(E + i\eta)\varphi(E)\,dE$ is a continuous linear functional on $\mathcal{H}_{\varrho,\sigma}$ for every $\eta > 0$, $\varphi \in \mathscr{S}$, and $\varrho, \sigma \geq 0$. By the Banach–Steinhaus theorem (see Theorem III.9 in [19]), it follows that the restriction of B_φ to $\mathcal{H}_{\varrho,\sigma}$ is a continuous linear functional on $\mathcal{H}_{\varrho,\sigma}$ for every $\varphi \in \mathscr{S}$ and $\varrho, \sigma \geq 0$.

Lemma 6.1 *Let $\varrho, \sigma \geq 0$, $n = 1, 2, \ldots$, and $\varphi \in \mathscr{S}$. Let O be an open subset of $\mathbb{R}^n$ and M be a map from O to $\mathcal{H}_{\varrho,\sigma}$, such that $x \to M(x|z)$ is a continuously differentiable function on O for every $z \in \mathbb{C}_+$. For every $j = 1, \ldots, n$ and $x \in O$, let*

the function $M_j(x)$ on $\mathbb{C}_+$ be defined by the equality $M_j(x|z) = \partial_{x_j} M(x|z)$. Suppose there is $C > 0$, such that:

$$|M_j(x|z)| \leq C \mathcal{N}_{\varrho,\sigma}(z)^{-1}, \quad j = 1,\ldots,n, \ x \in O, \ z \in \mathbb{C}_+. \tag{6.2}$$

Then, the function $x \to B_\varphi(M(x))$ on O is continuously differentiable and we have $M_j(x) \in \mathcal{H}_{\varrho,\sigma}$ and $\partial_{x_j} B_\varphi(M(x)) = B_\varphi(M_j(x))$ for all $j = 1,\ldots,n$ and $x \in O$.

Proof The proof relies on the following convergence property for sequences of holomorphic functions, which easily follows from the Montel theorem (see, e.g., Theorem 12 in Sec. 5.4.4 in [1]).

(C) Let $V \subset \mathbb{C}$ be an open set, $f_1, f_2, \ldots$ be holomorphic functions on V, and f be a complex function on V. Suppose the functions f_k are uniformly bounded on compact subsets of V and $f_k(z) \to f(z)$ as $k \to \infty$ for every $z \in V$. Then, f is holomorphic on V and $f_k \to f$ as $k \to \infty$ uniformly on compact subsets of V.

Let $e_1, \ldots, e_n$ be the standard basis in $\mathbb{R}^n$. Given $x \in O$ and $j = 1, \ldots, n$, we choose $\delta > 0$, such that $x + te_j \in O$ for every $t \in [0, \delta]$. We define the holomorphic functions $h_1, h_2, \ldots$ on $\mathbb{C}_+$ by setting $h_k = t_k^{-1}(M(x + t_k e_j) - M(x))$, where $t_k = \delta/k$. By (6.2) and the mean value theorem, we conclude that $|h_k(z)| \leq C \mathcal{N}_{\varrho,\sigma}(z)^{-1}$ for every $z \in \mathbb{C}_+$ and $k = 1, 2, \ldots$ and, therefore, the functions h_k are uniformly bounded on compact subsets of $\mathbb{C}_+$. Since $\lim_{k\to\infty} h_k(z) = M_j(x|z)$ for every $z \in \mathbb{C}_+$, property (C) implies that $M_j(x)$ is holomorphic on $\mathbb{C}_+$. In view of (6.2), it follows that $M_j(x) \in \mathcal{H}_{\varrho,\sigma}$ and $\|M_j(x)\|_{\varrho,\sigma} \leq C$ for every $x \in O$ and $j = 1, \ldots, n$.

We now choose $\varrho' > \varrho$, $\sigma' > \sigma$, and let $x \in O$ and $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$ be such that $x + th \in O$ for all $t \in [0, 1]$. By the mean value theorem, we have:

$$\left| M(x+h|z) - M(x|z) - \sum_j M_j(x|z) h_j \right|$$
$$\leq |h| \sup_{0 < t < 1} \sum_j |M_j(x + th|z) - M_j(x|z)|$$

for every $z \in \mathbb{C}_+$, where $|h| = \max_{1 \leq j \leq n} |h_j|$. This implies that:

$$\left\| M(x+h) - M(x) - \sum_j M_j(x) h_j \right\|_{\varrho',\sigma'}$$
$$\leq |h| \sup_{0 < t < 1} \sum_j \|M_j(x + th) - M_j(x)\|_{\varrho',\sigma'}.$$

Since B_φ is continuous on $\mathcal{H}_{\varrho',\sigma'}$, our statement will be proved if we show that M_j is a continuous map from O to $\mathcal{H}_{\varrho',\sigma'}$ for every $j = 1, \ldots, n$. To this end, we fix $\varepsilon > 0$ and choose a compact subset K of $\mathbb{C}_+$, such that:

$$\sup_{z \in \mathbb{C}_+ \backslash K} \mathcal{N}_{\varrho'-\varrho,\sigma'-\sigma}(z) < \frac{\varepsilon}{2C}.$$

Then, we have:

$$\sup_{z\in\mathbb{C}_+\setminus K}|M_j(x'|z)-M_j(x|z)|\mathcal{N}_{\varrho',\sigma'}(z)\leq\frac{\varepsilon}{2C}\|M_j(x')-M_j(x)\|_{\varrho,\sigma}<\varepsilon$$

for all $x, x' \in O$ and $j = 1,\ldots,n$. On the other hand, property (C) and the continuity of the functions $y \to M_j(y|z)$ on O for every $z \in \mathbb{C}_+$ imply that every $x \in O$ has a neighborhood $O_x \subset O$, such that:

$$\sup_{z\in K}|M_j(x'|z)-M_j(x|z)|\mathcal{N}_{\varrho',\sigma'}(z)<\varepsilon,\quad x'\in O_x, j=1,\ldots,n.$$

Hence, $\|M_j(x')-M_j(x)\|_{\varrho',\sigma'}<\varepsilon$ for all $x'\in O_x$ and $j=1,\ldots,n$, i.e., M_j is a continuous map from O to $\mathcal{H}_{\varrho',\sigma'}$ for every $j=1,\ldots,n$. □

Let R be as in Lemma 2.3. We set:

$$\mathcal{O}=\{(\alpha,\vartheta,z)\in\mathbb{C}\times\mathbb{C}\times\mathbb{C}_{3\pi/2}:R(\alpha,\vartheta,z)\neq0\},$$

and define the holomorphic function $\mathscr{F}$ on $\mathcal{O}$ by the equality:

$$\mathscr{F}(\alpha,\vartheta,z)=\frac{R(\alpha,\vartheta+\pi/2,z)}{R(\alpha,\vartheta,z)},\quad(\alpha,\vartheta,z)\in\mathcal{O}. \tag{6.3}$$

It follows from (2.21), (2.22), and (2.23) that $(\alpha,\vartheta,z)\in\mathcal{O}$ and:

$$\mathscr{M}_{\alpha,\vartheta}(z)=-\frac{\mathscr{F}(\alpha,\vartheta,z)}{2\pi^2\operatorname{Sinc}^2(\pi^2\alpha)} \tag{6.4}$$

for every $\alpha<1$, $\vartheta\in\mathbb{R}$, and $z\in\mathbb{C}_+$.

In the sequel, we let $\mathbb{Z}_+$ denote the set of all nonnegative integer numbers.

Proposition 6.2 *Let $0\leq a<1$, $b\in\mathbb{R}$, and $k,l\in\mathbb{Z}_+$. Then, we have:*

$$\left|\partial_\alpha^k\partial_\vartheta^l\mathscr{F}(\alpha,\vartheta,z)\right|\leq P_{a,b}(k,l)(1+\ln^2|z|)^{2k+l+1}\left(\frac{(1+|z|)^{1+a}}{\operatorname{Im}z}\right)^{k+l+1}$$

for every $\alpha\in[-b^2,a^2]$, $\vartheta\in\mathbb{R}$, and $z\in\mathbb{C}_+$, where:

$$P_{a,b}(k,l)=\frac{\pi^{2k}}{2}\left(\frac{24\pi\operatorname{ch}(\pi b)}{\operatorname{sinc}^2(\pi a)}\right)^{k+l+1}(k+l)!.$$

Before proceeding with the proof of Proposition 6.2, we shall derive Theorem 1.3 from Lemma 6.1 and Proposition 6.2.

Proof of Theorem 1.3 For every $k,l\in\mathbb{Z}_+$ and $(\alpha,\vartheta)\in Q$, we define the holomorphic function $M_{k,l}(\alpha,\vartheta)$ on $\mathbb{C}_+$ by the equality:

$$M_{k,l}(\alpha,\vartheta|z)=\partial_\alpha^k\partial_\vartheta^l\mathscr{F}(\alpha,\vartheta,z),\quad z\in\mathbb{C}_+.$$

Clearly, the function $(\alpha,\vartheta)\to M_{k,l}(\alpha,\vartheta|z)$ on Q is infinitely differentiable for every $z\in\mathbb{C}_+$ and we have:

$$\partial_\alpha M_{k,l}(\alpha, \vartheta | z) = M_{k+1,l}(\alpha, \vartheta | z), \quad \partial_\vartheta M_{k,l}(\alpha, \vartheta | z) = M_{k,l+1}(\alpha, \vartheta | z) \tag{6.5}$$

for every $(\alpha, \vartheta) \in Q$, $z \in \mathbb{C}_+$, and $k, l \in \mathbb{Z}_+$. Given $0 \leq a < 1$ and $b \in \mathbb{R}$, we set $O_{a,b} = \{(\alpha, \vartheta) \in Q : -b^2 < \alpha < a^2\}$. By Proposition 6.2, we have:

$$|M_{k,l}(\alpha, \vartheta | z)| \leq P_{a,b}(k,l) \sup_{E > 0} \frac{E(1 + \ln^2 E)^{2k+l+1}}{(1 + E)^2} \mathcal{N}_{k+l+2,k+l+2}(z)^{-1} \tag{6.6}$$

for every $(\alpha, \vartheta) \in O_{a,b}$, $z \in \mathbb{C}_+$, and $k, l \in \mathbb{Z}_+$. Since

$$Q = \bigcup_{a \in [0,1), b \in \mathbb{R}} O_{a,b}, \tag{6.7}$$

this implies that $M_{k,l}(\alpha, \vartheta) \in \mathcal{H}$ for every $(\alpha, \vartheta) \in Q$ and $k, l \in \mathbb{Z}_+$. We now fix $\varphi \in \mathscr{S}$. Given $k, l \in \mathbb{Z}_+$, we let $F_{k,l}$ denote the function $(\alpha, \vartheta) \to B_\varphi(M_{k,l}(\alpha, \vartheta))$ on Q. Let $0 \leq a < 1$, $b \in \mathbb{R}$, and $k, l \in \mathbb{Z}_+$. In view of (6.5) and (6.6), we can apply Lemma 6.1 to $O = O_{a,b}$, $M = M_{k,l}|_O$, and $\varrho = \sigma = k + l + 3$. As a result, we conclude that $F_{k,l}$ is continuously differentiable on $O_{a,b}$ and:

$$\partial_\alpha F_{k,l}(\alpha, \vartheta) = F_{k+1,l}(\alpha, \vartheta), \quad \partial_\vartheta F_{k,l}(\alpha, \vartheta) = F_{k,l+1}(\alpha, \vartheta) \tag{6.8}$$

for all $(\alpha, \vartheta) \in O_{a,b}$. By (6.7), it follows that $F_{k,l}$ is continuously differentiable on Q and equalities (6.8) hold for all $(\alpha, \vartheta) \in Q$ and $k, l \in \mathbb{Z}_+$. We now use induction on n to prove the following statement:

(S_n)　$F_{0,0}$ is n times differentiable on Q and $\partial_\alpha^k \partial_\vartheta^l F_{0,0}(\alpha, \vartheta) = F_{k,l}(\alpha, \vartheta)$ for every $(\alpha, \vartheta) \in Q$ and $k, l \in \mathbb{Z}_+$, such that $k + l = n$.

The statement (S_0) trivially holds, because every function on Q is 0 times differentiable. We now suppose $n \geq 1$ and derive (S_n) from (S_{n-1}). Let $k, l \in \mathbb{Z}_+$ be such that $k + l = n - 1$. Since $F_{k,l}$ is differentiable, (S_{n-1}) implies that the function $(\alpha, \vartheta) \to \partial_\alpha^k \partial_\vartheta^l F_{0,0}(\alpha, \vartheta)$ on Q is differentiable. This means that $F_{0,0}$ is n times differentiable. Suppose now $k, l \in \mathbb{Z}_+$ are such that $k + l = n$. Then, we have either $k > 0$ or $l > 0$. Hence, we can represent $\partial_\alpha^k \partial_\vartheta^l F_{0,0}(\alpha, \vartheta)$ either as $\partial_\alpha(\partial_\alpha^{k-1} \partial_\vartheta^l F_{0,0}(\alpha, \vartheta))$ or as $\partial_\vartheta(\partial_\alpha^k \partial_\vartheta^{l-1} F_{0,0}(\alpha, \vartheta))$. In both cases, it follows from (S_{n-1}) and (6.8) that $\partial_\alpha^k \partial_\vartheta^l F_{0,0}(\alpha, \vartheta) = F_{k,l}(\alpha, \vartheta)$ for all $(\alpha, \vartheta) \in Q$. This completes the derivation of (S_n) from (S_{n-1}). By induction, we conclude that (S_n) holds for all $n \in \mathbb{Z}_+$ and, therefore, the function $F_{0,0}$ is infinitely differentiable. Given $(\alpha, \vartheta) \in Q$, we set $G_{\alpha, \vartheta} = \mathscr{M}_{\alpha, \vartheta}|_{\mathbb{C}_+}$. By (6.4), we have $M_{0,0}(\alpha, \vartheta) = -2\pi^2 \operatorname{Sinc}(\pi^2 \alpha) G_{\alpha, \vartheta}$ for every $(\alpha, \vartheta) \in Q$. Since $F_{0,0}(\alpha, \vartheta) = B_\varphi(M_{0,0}(\alpha, \vartheta))$, we conclude that the function $(\alpha, \vartheta) \to B_\varphi(G_{\alpha, \vartheta})$ on Q is infinitely differentiable for every $\varphi \in \mathscr{S}$. To complete the proof, it remains to note that:

$$\int \varphi(E) \, d\mathcal{V}_{\alpha, \vartheta}(E) = \frac{B_\varphi(G_{\alpha, \vartheta}) - \overline{B_{\bar\varphi}(G_{\alpha, \vartheta})}}{2i}$$

for every $(\alpha, \vartheta) \in Q$ and $\varphi \in \mathscr{S}$ by (2.25). $\qquad\square$

The rest of this section is devoted to the proof of Proposition 6.2.

It follows from (3.51) and (3.52) that:

$$\mu(\kappa^2, \phi)^2 - \tau(\kappa^2, \phi)^2 = 4\pi^2(\pi - \phi)^2 \operatorname{sinc}^2 \pi\kappa \operatorname{sinc}^2(\pi - \phi)\kappa$$

for every $\kappa \in \mathbb{C}\backslash\{0\}$ and $\phi \in \mathbb{C}$. By continuity, this equality remains valid for $\kappa = 0$. Hence, we have:

$$\mu(\alpha, \phi)^2 - \tau(\alpha, \phi)^2 = 4\pi^2(\pi - \phi)^2 \operatorname{Sinc}^2(\pi^2\alpha) \operatorname{Sinc}^2((\pi - \phi)^2\alpha), \quad \alpha, \phi \in \mathbb{C}. \quad (6.9)$$

Let the function Φ on $\mathbb{C} \times \mathbb{R}_+$ be defined by the relation:

$$\Phi(\alpha, E) = \ln E \operatorname{Sinc}\left(-\frac{\alpha}{4}\ln^2 E\right), \quad \alpha \in \mathbb{C}, E > 0. \quad (6.10)$$

In view of (B.1), rewriting equality (3.53) in terms of Φ yields:

$$\begin{aligned}
T(\alpha, \vartheta, Ee^{i\phi}) =& \mu(\alpha, \phi) + \Phi(\alpha, E)^2 \\
&+ \left(\Phi(\alpha, E)^2 \operatorname{Cos}(\pi^2\alpha) + \tau(\alpha, \phi)\right) \cos 2\vartheta \\
&- 2\pi\Phi(\alpha, E) \operatorname{Cos}\left(-\frac{\alpha}{4}\ln^2 E\right) \operatorname{Sinc}(\pi^2\alpha) \sin 2\vartheta
\end{aligned} \quad (6.11)$$

for all $\alpha, \vartheta \in \mathbb{C}$, $E > 0$, and $-\pi/2 < \phi < 3\pi/2$.

By (3.49), $\tau(\alpha, \phi)$ is real for real α and ϕ. We also observe that:

$$\mu(\alpha, \phi) \geq 0, \quad \alpha, \phi \in \mathbb{R}. \quad (6.12)$$

Indeed, (3.52) implies that this is true for all $\alpha \in \mathbb{R} \setminus \{0\}$ and $\phi \in \mathbb{R}$. By continuity, (6.12) remains valid for $\alpha = 0$.

The next lemma is the key part of the proof of Proposition 6.2.

Lemma 6.3 *Let $\alpha < 1$, $\vartheta \in \mathbb{R}$, $E > 0$, and $0 \leq \phi < \pi$. Then, we have*:

$$\frac{1}{|R(\alpha, \vartheta, Ee^{i\phi})|} \leq \frac{(\Phi(\alpha, E)^2 + \mu(\alpha, \phi))^{1/2}}{\pi(\pi - \phi) \operatorname{Sinc}(\pi^2\alpha) \operatorname{Sinc}((\pi - \phi)^2\alpha)}.$$

Proof Let

$$G = \Phi(\alpha, E)^2 + \mu(\alpha, \phi), \quad (6.13)$$

$$H = \sqrt{\tau(\alpha, \phi)^2 + \Phi(\alpha, E)^4 + 2\mu(\alpha, \phi)\Phi(\alpha, E)^2}. \quad (6.14)$$

By (6.12), H is well defined, and both G and H are nonnegative. Using (3.50) and the identity:

$$\operatorname{Cos}^2\left(-\frac{\alpha}{4}\ln^2 E\right) = 1 + \frac{\alpha}{4}\Phi(\alpha, E)^2,$$

which follows from (6.10) and (B.2), we find that

$$H = \sqrt{c^2 + d^2}, \tag{6.15}$$

where

$$c = \Phi(\alpha, E)^2 \operatorname{Cos}(\pi^2 \alpha) + \tau(\alpha, \phi),$$

$$d = -2\pi \Phi(\alpha, E) \operatorname{Cos}\left(-\frac{\alpha}{4}\ln^2 E\right) \operatorname{Sinc}(\pi^2 \alpha).$$

Since $|c \cos 2\vartheta + d \sin 2\vartheta| \leq \sqrt{c^2 + d^2}$ by the Cauchy–Bunyakovsky inequality, (6.11) and (6.15) imply that:

$$T(\alpha, \vartheta, E e^{i\phi}) = G + c \cos 2\vartheta + d \sin 2\vartheta \geq G - H. \tag{6.16}$$

By (6.13) and (6.14), we have:

$$G^2 - H^2 = \mu(\alpha, \phi)^2 - \tau(\alpha, \phi)^2. \tag{6.17}$$

In view of (6.9), it follows that the left-hand side of (6.17) is nonnegative. We therefore have $H \leq G$, whence $2G \geq G + H$. Multiplying this inequality with (6.16) and using (6.9) and (6.17), we obtain:

$$2GT(\alpha, \vartheta, E e^{i\phi}) \geq 4\pi^2 (\pi - \phi)^2 \operatorname{Sinc}^2(\pi^2 \alpha) \operatorname{Sinc}^2((\pi - \phi)^2 \alpha). \tag{6.18}$$

Since $\alpha < 1$ and $0 \leq \phi < \pi$, the right-hand side of (6.18) is strictly positive. Hence, $T(\alpha, \vartheta, E e^{i\phi}) > 0$, and (6.13) and (6.18) imply that:

$$\frac{1}{T(\alpha, \vartheta, E e^{i\phi})} \leq \frac{\Phi(\alpha, E)^2 + \mu(\alpha, \phi)}{2\pi^2 (\pi - \phi)^2 \operatorname{Sinc}^2(\pi^2 \alpha) \operatorname{Sinc}^2((\pi - \phi)^2 \alpha)}.$$

The required estimate is now ensured by (3.45), because both $\operatorname{Sinc}(\pi^2 \alpha)$ and $\operatorname{Sinc}((\pi - \phi)^2 \alpha)$ are strictly positive by Lemma B.1. $\qquad\square$

For every $\alpha \in \mathbb{R}$ and $\phi \in [0, \pi]$, we have:

$$\mu(\alpha, \phi) \leq 2\pi^2 \operatorname{Sinc}^2(\pi^2 \alpha) + \pi^2. \tag{6.19}$$

Indeed, let $\phi \in [0, \pi]$. If $\alpha < 0$, then $\pi^2 \alpha \leq (\pi - \phi)^2 \alpha$ and, therefore, $0 < \operatorname{Sinc}((\pi - \phi)^2 \alpha / 4) \leq \operatorname{Sinc}(\pi^2 \alpha / 4)$ by Lemma B.1. In view of (3.49), this implies that $\tau(\alpha, \phi) \leq 0$ and, hence, (6.19) follows from (1.32) and (3.50). If $\alpha \geq 0$, then (3.49) ensures that $|\tau(\alpha, \phi)| \leq \pi^2$ (note that $|\operatorname{Sinc}\xi| \leq 1$ for $\xi \geq 0$ by Lemma B.1). Since $|\operatorname{Cos}(\pi^2 \alpha)| \leq 1$ by (1.31), it follows from (3.50) that (6.19) is again satisfied. This completes the proof of (6.19).

Lemma 6.4 *Let* $-1 < a < 1$. *Then, we have*:

$$\frac{1}{|R(\alpha, \vartheta, E e^{i\phi})|} \leq \frac{(|\ln E| + 3\pi)(E^{a/2} + E^{-a/2})}{2\pi(\pi - \phi)\operatorname{sinc}^2(\pi a)}$$

for all $\alpha \leq a^2$, $\vartheta \in \mathbb{R}$, $E > 0$, *and* $0 \leq \phi < \pi$.

Proof It follows from (6.10) and Lemma B.1 that:

$$|\Phi(\alpha, E)| \leq |\ln E| \operatorname{ch}\left(\frac{a}{2}\ln E\right) = \frac{|\ln E|}{2}\left(E^{a/2} + E^{-a/2}\right)$$

for all $\alpha \leq a^2$ and $E > 0$. By Lemma B.1, we have $\operatorname{Sinc}(\pi^2\alpha) \geq \operatorname{Sinc}(\pi^2 a^2) = \operatorname{sinc}(\pi a)$ for every $\alpha \leq a^2$. Since $0 < \operatorname{sinc}(\pi a) \leq 1$, inequality (6.19) and the above estimate imply that:

$$\frac{(\Phi(\alpha, E)^2 + \mu(\alpha, \phi))^{1/2}}{\operatorname{Sinc}(\pi^2\alpha)} \leq \frac{\sqrt{2}\pi \operatorname{Sinc}(\pi^2\alpha) + \pi + |\Phi(\alpha, E)|}{\operatorname{Sinc}(\pi^2\alpha)}$$

$$= \sqrt{2}\pi + \frac{|\Phi(\alpha, E)| + \pi}{\operatorname{Sinc}(\pi^2\alpha)} \leq \frac{3\pi + |\Phi(\alpha, E)|}{\operatorname{sinc}(\pi a)}$$

$$\leq \frac{3\pi + |\ln E|}{2\operatorname{sinc}(\pi a)}\left(E^{a/2} + E^{-a/2}\right)$$

for every $\alpha \leq a^2$, $E > 0$, and $0 \leq \phi \leq \pi$. The required inequality now follows from Lemma 6.3, because $\operatorname{Sinc}((\pi - \phi)^2\alpha) \geq \operatorname{sinc}(\pi a)$ for all $\alpha \leq a^2$ and $0 \leq \phi \leq 2\pi$ by Lemma B.1.   □

Lemma 6.5 *Let* a, b *be real numbers*, $A, B, C \geq 0$, *and* $n, k_1, k_2, k_3 \in \mathbb{Z}_+$. *Then*:

$$\left|\partial_\alpha^n\left(\operatorname{Cos}^{(k_1)}(A^2\alpha)\operatorname{Cos}^{(k_2)}(B^2\alpha)\operatorname{Cos}^{(k_3)}(-C^2\alpha)\right)\right|$$

$$\leq \frac{n!(A + B + C)^{2n}}{(2n)!2^{k_1+k_2+k_3}}\operatorname{ch}(Ab)\operatorname{ch}(Bb)\operatorname{ch}(Ca) \tag{6.20}$$

for every $\alpha \in [-b^2, a^2]$.

Proof For every $u \in \mathbb{R}$ and $n \in \mathbb{Z}_+$, we have the inequality (see formula (12) in [22])

$$|\operatorname{Cos}^{(n)}(\xi)| \leq \frac{n!}{(2n)!}\operatorname{ch} u, \quad \xi \geq -u^2.$$

Using this estimate and the standard formula for the nth derivative of a product of functions, we find that the left-hand side of (6.20) is bounded by the product of $\operatorname{ch}(Ab)\operatorname{ch}(Bb)\operatorname{ch}(Ca)$ and:

$$\sum \frac{n!}{n_1!n_2!n_3!}\frac{(n_1 + k_1)!A^{2n_1}}{(2n_1 + 2k_1)!}\frac{(n_2 + k_2)!B^{2n_2}}{(2n_2 + 2k_2)!}\frac{(n_3 + k_3)!C^{2n_3}}{(2n_3 + 2k_3)!}$$

for every $\alpha \in [-b^2, a^2]$, where the sum is taken over all $n_1, n_2, n_3 \in \mathbb{Z}_+$, such that $n_1 + n_2 + n_3 = n$. Since

$$\frac{(n + k)!}{(2n + 2k)!} \leq \frac{1}{2^k}\frac{n!}{(2n)!}$$

for all $n, k \in \mathbb{Z}_+$, the sum in the above expression does not exceed:

$$\frac{1}{2^{k_1+k_2+k_3}} \frac{n!}{(2n)!} \sum_{n_1+n_2+n_3=n} \frac{(2n)!}{(2n_1)!(2n_2)!(2n_3)!} A^{2n_1} B^{2n_2} C^{2n_3},$$

whence the required estimate follows immediately. $\qquad\square$

Lemma 6.6 *Let $a, b \in \mathbb{R}$. Then, we have*:

$$|\partial_\alpha^n R(\alpha, \vartheta, Ee^{i\phi})| \le \frac{n!}{(2n)!} (\pi + |\ln E|)^{2n+1} \operatorname{ch}(\pi b)(E^{a/2} + E^{-a/2})$$

for all $\alpha \in [-b^2, a^2]$, $\vartheta \in \mathbb{R}$, $E > 0$, $\phi \in [0, \pi]$, and $n \in \mathbb{Z}_+$.

Proof By elementary trigonometric transformations, we derive from (2.17) that:

$$R(\kappa^2, \vartheta, Ee^{i\phi})$$
$$= \left(ie^{-i\vartheta} \frac{\sin \vartheta_\kappa}{\kappa} \cos \frac{(\pi - \phi)\kappa}{2} - \frac{i}{\kappa} \cos \vartheta \sin \frac{\phi\kappa}{2} \right)(E^{\kappa/2} + E^{-\kappa/2})$$
$$- \left(e^{-i\vartheta} \cos \vartheta_\kappa \cos \frac{(\pi - \phi)\kappa}{2} + i \sin \vartheta \cos \frac{\phi\kappa}{2} \right) \frac{E^{\kappa/2} - E^{-\kappa/2}}{\kappa}$$

for every $\kappa \in \mathbb{C}\backslash\{0\}$, $\vartheta \in \mathbb{C}$, $E > 0$, and $-\pi/2 < \phi < 3\pi/2$. We now fix $E > 0$ and $\phi \in [0, \pi]$, and set $A = \pi/2$, $\tilde{A} = \phi/2$, $B = (\pi - \phi)/2$, and $C = |\ln E|/2$. In view of the equalities:

$$\frac{E^{\kappa/2} - E^{-\kappa/2}}{\kappa} = \ln E \operatorname{Sinc}\left(-\frac{\kappa^2}{4} \ln^2 E\right),$$
$$\frac{E^{\kappa/2} + E^{-\kappa/2}}{2} = \operatorname{Cos}\left(-\frac{\kappa^2}{4} \ln^2 E\right),$$

which hold for all $\kappa \in \mathbb{C}\backslash\{0\}$ by (1.30), we conclude that:

$$R(\alpha, \vartheta, Ee^{i\phi}) = i\left(\pi e^{-i\vartheta} \operatorname{Sinc}(A^2\alpha) \operatorname{Cos}(B^2\alpha) - \phi \cos\vartheta \operatorname{Sinc}(\tilde{A}^2\alpha)\right) \operatorname{Cos}(-C^2\alpha)$$
$$- \ln E\left(e^{-i\vartheta} \operatorname{Cos}(A^2\alpha) \operatorname{Cos}(B^2\alpha) + i\sin\vartheta \operatorname{Cos}(\tilde{A}^2\alpha)\right) \operatorname{Sinc}(-C^2\alpha)$$

for all $\alpha \in \mathbb{C}\backslash\{0\}$ and $\vartheta \in \mathbb{C}$. By continuity, this equality remains valid for $\alpha = 0$. By Lemma 6.5 and (B.3), it follows that:

$$|\partial_\alpha^n R(\alpha, \vartheta, Ee^{i\phi})|$$
$$\le \frac{n!}{(2n)!} ((A + B + C)^{2n} \operatorname{ch}(Ab) \operatorname{ch}(Bb) + (\tilde{A} + C)^{2n} \operatorname{ch}(\tilde{A}b))(|\ln E| + \pi) \operatorname{ch}(Ca)$$

for every $\alpha \in [-b^2, a^2]$ and $\vartheta \in \mathbb{R}$. This implies the required estimate, because A, $\tilde{A}$, and B do not exceed $\pi/2$ by the condition $\phi \in [0, \pi]$ and $\operatorname{ch}^2(\pi b/2) \le \operatorname{ch}(\pi b)$. $\square$

Given $k \in \mathbb{Z}_+$, let $\Lambda_k = \mathbb{Z}_+^{\{0,\dots,k\}}$, i.e., Λ_k is the set of all maps from $\{0, \dots, k\}$ to $\mathbb{Z}_+$. For every $k, l \in \mathbb{Z}_+$, we set:

$$\mathcal{Q}_{k,l} = \left\{ (s,t) \in \Lambda_k \times \Lambda_k : \sum_{j=0}^{k} j(s(j) + t(j)) = k, \right.$$

$$\left. \sum_{j=0}^{k} (s(j) + t(j)) = k + l + 1 \right\}. \tag{6.21}$$

Lemma 6.7 *For every $k, l \in \mathbb{Z}_+$, there is a map $C : \mathcal{Q}_{k,l} \to \mathbb{Z}$, such that*:

$$\sum_{(s,t) \in \mathcal{Q}_{k,l}} |C(s,t)| \le 2^{k+l}(k+l)! \tag{6.22}$$

and

$$\partial_\alpha^k \partial_\vartheta^l \mathscr{F}(\alpha, \vartheta, z) = R(\alpha, \vartheta, z)^{-k-l-1}$$

$$\times \sum_{(s,t) \in \mathcal{Q}_{k,l}} C(s,t) \prod_{j=0}^{k} (\partial_\alpha^j R(\alpha, \vartheta, z))^{s(j)} (\partial_\alpha^j R(\alpha, \vartheta + \pi/2, z))^{t(j)} \tag{6.23}$$

for every $(\alpha, \vartheta, z) \in \mathcal{O}$ (we assume that $\zeta^0 = 1$ for every $\zeta \in \mathbb{C}$).

The proof of Lemma 6.7, which is of purely combinatorial nature, is elementary but rather lengthy. It is given in Appendix E.

Proof of Proposition 6.2 We fix $z \in \mathbb{C}_+$ and set $E = |z|$. Then, there is a unique $\phi \in (0, \pi)$, such that $z = Ee^{i\phi}$. Let the map $C : \mathcal{Q}_{k,l} \to \mathbb{Z}$ be as in Lemma 6.7. In view of Lemma 6.6, the sum in the right-hand side of (6.23) is bounded above by:

$$\sum_{(s,t) \in \mathcal{Q}_{k,l}} |C(s,t)| \prod_{j=0}^{k} \left(\frac{j!}{(2j)!} (\pi + |\ln E|)^{2j+1} \operatorname{ch}(\pi b)(E^{a/2} + E^{-a/2}) \right)^{s(j)+t(j)}$$

for every $\alpha \in [-b^2, a^2]$ and $\vartheta \in \mathbb{R}$. As $j!/(2j)! \le 2^{-j}$ for every $j \in \mathbb{Z}_+$, it follows from (6.21) and (6.22) that this expression does not exceed:

$$2^l (k+l)! (\pi + |\ln E|)^{3k+l+1} \left(\operatorname{ch}(\pi b)(E^{a/2} + E^{-a/2}) \right)^{k+l+1}.$$

Since

$$E(E^{a/2} + E^{-a/2})^2 \le 4(1+E)^{1+a}, \quad (1 + |\ln E|)^2 \le 2(1 + \ln^2 E),$$

Lemma 6.4 and (6.23) therefore imply that:

$$\left| \partial_\alpha^k \partial_\vartheta^l \mathscr{F}(\alpha, \vartheta, z) \right| \le P_{a,b}(k,l)(1 + \ln^2 E)^{2k+l+1} \left(\frac{(1+E)^{1+a}}{E(\pi - \phi)} \right)^{k+l+1}$$

for every $\alpha \in [-b^2, a^2]$ and $\vartheta \in \mathbb{R}$. Hence, the required estimate follows, because $E(\pi - \phi) \ge E \sin(\pi - \phi) = E \sin \phi = \operatorname{Im} z$. $\qquad\square$

Appendix A. Even holomorphic functions

Given $r > 0$, we let $\mathbb{D}_r$ denote the disc of radius r in the complex plane centred at the origin: $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$.

Lemma A.1

1. *The map $z \to z^2$ from $\mathbb{C}$ to itself is open.*
2. *Let X be a topological space. The map $(z, x) \to (z^2, x)$ from $\mathbb{C} \times X$ to itself is open.*

Proof 1. Let O be an open subset of $\mathbb{C}$ and $\tilde{O}$ be its image under the map $z \to z^2$. Let $\zeta \in \tilde{O} \backslash \{0\}$. Then, there is a holomorphic function g defined on a neighborhood of ζ and such that $g(\zeta) \in O$ and $g(\zeta')^2 = \zeta'$ for every $\zeta' \in D_g$ (i.e., g is a holomorphic branch of the square root in a neighborhood of ζ whose value at ζ belongs to O). By continuity of g, there is a neighborhood V of ζ, such that $g(\zeta') \in O$ for all $\zeta' \in V$ and, hence, $V \subset \tilde{O}$. This means that ζ is an interior point of $\tilde{O}$ for every nonzero $\zeta \in \tilde{O}$. If $0 \in \tilde{O}$, then $0 \in O$ and $\mathbb{D}_r \subset O$ for some $r > 0$. It follows that $\mathbb{D}_{r^2} \subset \tilde{O}$ and, therefore, 0 is an interior point of $\tilde{O}$. This implies that $\tilde{O}$ is open.

2. The assertion follows immediately from statement 1 and the definition of product topology. $\qquad\square$

Lemma A.2 *Let $O \subset \mathbb{C}$ be such that $-z \in O$ for every $z \in O$. Let $\tilde{O} = \{\zeta \in \mathbb{C} : \zeta = z^2$ for some $z \in O\}$ and f be a map on O, such that $f(-z) = f(z)$ for every $z \in O$. Then, there is a unique map $\tilde{f}$ on $\tilde{O}$, such that $f(z) = \tilde{f}(z^2)$ for all $z \in O$. If O is open and f is a holomorphic function on O, then $\tilde{O}$ is open and $\tilde{f}$ is a holomorphic function on $\tilde{O}$.*

Proof The uniqueness of $\tilde{f}$ is obvious. To prove the existence, we choose $w_\zeta \in \mathbb{C}$, such that $\zeta = w_\zeta^2$ for every $\zeta \in \mathbb{C}$. Clearly, $w_\zeta \in O$ for every $\zeta \in \tilde{O}$, and we can define $\tilde{f}$ as the map on $\tilde{O}$ taking $\zeta \in \tilde{O}$ to $f(w_\zeta)$. Then, we have $f(z) = f(w_{z^2}) = \tilde{f}(z^2)$ for every $z \in O$, because $z = \pm w_{z^2}$.

Let O be open and f be holomorphic. By statement 1 of Lemma A.1, $\tilde{O}$ is open. Let $\zeta \in \tilde{O} \backslash \{0\}$. As in the proof of statement 1 of Lemma A.1, we choose a holomorphic branch of the square root g and a neighborhood V of ζ, such that $V \subset D_g$ and $g(V) \subset O$. Then, $V \subset \tilde{O}$ and $f(g(\zeta')) = \tilde{f}(g(\zeta')^2) = \tilde{f}(\zeta')$ for every $\zeta' \in V$. This means that $\tilde{f}$ coincides on V with the composition of holomorphic functions f and g and, hence, is holomorphic on V. This implies that $\tilde{f}$ is holomorphic on $\tilde{O} \backslash \{0\}$. If $0 \in \tilde{O}$, then $0 \in O$ and there is $r > 0$, such that $\mathbb{D}_r \subset O$. For $k = 0, 1, \ldots$, let $a_k = f^{(k)}(0)/k!$ be the Taylor coefficients of $f(z)$ at $z = 0$. Since f is even, we have $a_k = 0$ for odd k and, hence, $f(z) = \sum_{n=0}^{\infty} a_{2n} z^{2n}$ for all $z \in \mathbb{D}_r$. It follows that the series $\sum_{n=0}^{\infty} a_{2n} \zeta^n$ converges to some $h(\zeta)$ for every $\zeta \in \mathbb{D}_{r^2}$. Clearly, h is holomorphic on $\mathbb{D}_{r^2}$ and we have $h(z^2) = f(z) = \tilde{f}(z^2)$ for every $z \in \mathbb{D}_r$. This means that $\tilde{f}$ coincides with h on $\mathbb{D}_{r^2}$ and, therefore, $\tilde{f}$ is holomorphic on $\tilde{O}$. $\qquad\square$

Lemma A.3 *Let $n = 1, 2, \ldots$ and $O \subset \mathbb{C} \times \mathbb{C}^n$ be such that $(-z, u) \in O$ for every $(z, u) \in O$. Let $\tilde{O}$ be the image of O under the map $(z, u) \to (z^2, u)$ from $\mathbb{C} \times \mathbb{C}^n$ to itself and f be a map on O, such that $f(-z, u) = f(z, u)$ for every $(z, u) \in O$. Then, there is a unique map $\tilde{f}$ on $\tilde{O}$, such that $f(z, u) = \tilde{f}(z^2, u)$ for all $(z, u) \in O$. If O is open and f is a holomorphic function on O, then $\tilde{O}$ is open and $\tilde{f}$ is a holomorphic function on $\tilde{O}$.*

Proof The uniqueness of $\tilde{f}$ is obvious. To prove the existence, we choose $w_\zeta \in \mathbb{C}$, such that $\zeta = w_\zeta^2$ for every $\zeta \in \mathbb{C}$. Clearly, $(w_\zeta, u) \in O$ for every $(\zeta, u) \in \tilde{O}$, and we can define $\tilde{f}$ as the map on $\tilde{O}$ taking $(\zeta, u) \in \tilde{O}$ to $f(w_\zeta, u)$. Then, we have $f(z, u) = f(w_{z^2}, u) = \tilde{f}(z^2, u)$ for every $(z, u) \in O$, because $z = \pm w_{z^2}$.

Let O be open and f be holomorphic. By statement 2 of Lemma A.1, $\tilde{O}$ is open. For every $u \in \mathbb{C}^n$, we let $j_1(u)$ denote the holomorphic map $z \to (z, u)$ from $\mathbb{C}$ to $\mathbb{C} \times \mathbb{C}^n$. For every $z \in \mathbb{C}$, we let $j_2(z)$ denote the holomorphic map $u \to (z, u)$ from $\mathbb{C}^n$ to $\mathbb{C} \times \mathbb{C}^n$. In view of the Hartogs theorem, the holomorphy of $\tilde{f}$ will be proved if we show that $\tilde{f} \circ j_1(u)$ and $\tilde{f} \circ j_2(z)$ are holomorphic functions for every $u \in \mathbb{C}^n$ and $z \in \mathbb{C}$. Let s be the map $z \to z^2$ from $\mathbb{C}$ to itself and t be the map $(z, u) \to (z^2, u)$ from $\mathbb{C} \times \mathbb{C}^n$ to itself. Since $f = \tilde{f} \circ t$ and $t \circ j_1(u) = j_1(u) \circ s$, we have $f \circ j_1(u) = (\tilde{f} \circ j_1(u)) \circ s$ for every $u \in \mathbb{C}^n$. Because $f \circ j_1(u)$ is a holomorphic function, Lemma A.2 implies that $\tilde{f} \circ j_1(u)$ is a holomorphic function for every $u \in \mathbb{C}^n$. Since $t \circ j_2(z) = j_2(z^2)$, we have $\tilde{f} \circ j_2(z) = f \circ j_2(z^2)$ and, hence, $\tilde{f} \circ j_2(z)$ is a holomorphic function for every $z \in \mathbb{C}$. $\qquad\square$

Appendix B. Some properties of the functions commaCoscomma and commaSinccomma

Dividing the identity $\sin 2w = 2 \sin w \cos w$ by $2w$, we find that $\operatorname{sinc} 2w = \operatorname{sinc} w \cos w$ for every $w \in \mathbb{C} \backslash \{0\}$. By continuity, this equality remains valid for $w = 0$ and it follows from (1.28) that:

$$\operatorname{Sinc} 4\zeta = \operatorname{Sinc} \zeta \operatorname{Cos} \zeta, \quad \zeta \in \mathbb{C}. \tag{B.1}$$

Substituting $\sin w = w \operatorname{sinc} w$ in the identity $\sin^2 w + \cos^2 w = 1$, we obtain $w^2 \operatorname{sinc}^2 w + \cos^2 w = 1$ for every $w \in \mathbb{C}$. In view of (1.28), this implies that:

$$\zeta \operatorname{Sinc}^2 \zeta + \operatorname{Cos}^2 \zeta = 1, \quad \zeta \in \mathbb{C}. \tag{B.2}$$

Differentiating the left equality in (1.28) yields $\sin w = -2w \operatorname{Cos}'(w^2)$ for every $w \in \mathbb{C}$. Dividing this identity by w, we obtain $\operatorname{sinc} w = -2 \operatorname{Cos}'(w^2)$ for all $w \in \mathbb{C} \backslash \{0\}$. By continuity, this formula remains valid for $w = 0$ and it follows from the right equality in (1.28) that:

$$\operatorname{Sinc} \zeta = -2 \operatorname{Cos}' \zeta, \quad \zeta \in \mathbb{C}. \tag{B.3}$$

Lemma B.1 *The functions* Cos *and* Sinc *are strictly decreasing on the interval* $(-\infty, \pi^2]$. *For every* $\xi < \pi^2$, *we have* Sinc $\xi > 0$. *If* $a \in \mathbb{R}$ *and* $\xi \geq -a^2$, *then* $|\text{Sinc}\,\xi| \leq \text{ch}\,a$.

Proof Let the function f on $\mathbb{R}$ be defined by the equality $f(x) = x \cos x - \sin x$. Then, we have $\text{sinc}'x = f(x)/x^2$ for every $x > 0$. Since $f'(x) = -x \sin x$, we have $f'(x) < 0$ for $x \in (0, \pi)$. As $f(0) = 0$, we have $f(x) < 0$ for $x \in (0, \pi)$ and, therefore, sinc strictly decreases on $[0, \pi]$. In view of (1.31), this implies that Sinc strictly decreases on $[0, \pi^2]$. Since Sinc is strictly decreasing on $(-\infty, 0]$ by (1.29), we conclude that Sinc strictly decreases on $(-\infty, \pi^2]$. In view of the equality Sinc $(\pi^2) = 0$, it follows that Sinc $\xi > 0$ for $\xi < \pi^2$. By (B.3), we have $\text{Cos}'\xi = -2^{-1}\,\text{Sinc}\,\xi < 0$ for $\xi < \pi^2$ and, hence, Cos is strictly decreasing on $(-\infty, \pi^2]$. Let $a \in \mathbb{R}$. In view of (1.29), (1.30), and the monotonicity of Sinc established above, we have:

$$\text{Sinc}\,\xi \leq \text{Sinc}\,(-a^2) \leq \text{Cos}\,(-a^2) = \text{ch}\,a, \quad \xi \in [-a^2, \pi^2].$$

If $\xi \geq \pi^2$, then we have $|\text{Sinc}\,\xi| = |\text{sinc}\,\sqrt{\xi}| \leq 1/\pi < \text{ch}\,a$. Hence, $|\text{Sinc}\,\xi| \leq \text{ch}\,a$ for all $\xi \geq -a^2$. $\qquad\square$

Appendix C: Herglotz functions

A holomorphic function f on $\mathbb{C}_+$ is said to be a Herglotz function if $\text{Im}\,f(z) > 0$ for every $z \in \mathbb{C}_+$. It is well known (see [2], Ch. 6, Sec. 69) that every Herglotz function f admits a unique representation of the form:

$$f(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{t^2 + 1}\right) dv(t), \quad z \in \mathbb{C}_+, \tag{C.1}$$

where $a \in \mathbb{R}$, $b \geq 0$, and v is a positive Radon measure on $\mathbb{R}$, such that:

$$\int_{\mathbb{R}} \frac{dv(t)}{t^2 + 1} \leq \infty. \tag{C.2}$$

We call v the Herglotz measure associated with f.

Lemma C.1 *Let f be a Herglotz function and φ be a continuous complex function on* $\mathbb{R}$ *satisfying the bound* $|\varphi(E)| \leq C(1 + E^2)^{-2}$, $E \in \mathbb{R}$, *for some* $C \geq 0$. *Then, the function* $E \to \varphi(E)\text{Im}\,f(E + i\eta)$ *on* $\mathbb{R}$ *is integrable for every* $\eta > 0$ *and we have* $\int_{\mathbb{R}} \varphi(E)\text{Im}\,f(E + i\eta)\,dE \to \pi \int_{\mathbb{R}} \varphi(E)\,dv(E)$ *as* $\eta \downarrow 0$, *where* v *is the Herglotz measure associated with f.*

Proof Since v is the Herglotz measure associated with f, representation (C.1) holds for some $a \in \mathbb{R}$ and $b \geq 0$. We, therefore, have:

$$\mathrm{Im} f(E + i\eta) = b\eta + \int_{\mathbb{R}} \frac{\eta \, \mathrm{d}v(t)}{(t - E)^2 + \eta^2}, \quad E \in \mathbb{R}, \, \eta > 0.$$

Since

$$\int_{\mathbb{R}} \frac{|\varphi(E)|\eta \, \mathrm{d}E}{(t - E)^2 + \eta^2} \leq C \int_{\mathbb{R}} \frac{\eta \, \mathrm{d}E}{(E^2 + 1)((t - E)^2 + \eta^2)} = \frac{C\pi(\eta + 1)}{t^2 + (\eta + 1)^2}, \qquad \text{(C.3)}$$

the function $(E, t) \to \eta((t - E)^2 + \eta^2)^{-1}\varphi(E)$ on $\mathbb{R} \times \mathbb{R}$ is integrable with respect to the measure $\lambda \times v$, where λ is the Lebesgue measure on $\mathbb{R}$. By the Fubini theorem, we conclude that the function $E \to \varphi(E)\mathrm{Im} f(E + i\eta)$ on $\mathbb{R}$ is integrable for every $\eta > 0$ and:

$$\int_{\mathbb{R}} \varphi(E)\mathrm{Im} f(E + i\eta) \, \mathrm{d}E = b\eta \int_{\mathbb{R}} \varphi(E) \, \mathrm{d}E + \int_{\mathbb{R}} g_\eta(t) \, \mathrm{d}v(t), \qquad \text{(C.4)}$$

where the function g_η on $\mathbb{R}$ is given by:

$$g_\eta(t) = \int_{\mathbb{R}} \frac{\varphi(E)\eta \, \mathrm{d}E}{(t - E)^2 + \eta^2} = \int_{\mathbb{R}} \frac{\varphi(t + \eta E) \, \mathrm{d}E}{E^2 + 1}.$$

By the dominated convergence theorem, $g_\eta(t) \to \pi\varphi(t)$ as $\eta \downarrow 0$ for every $t \in \mathbb{R}$. Since $|g_\eta(t)| \leq 2\pi C(t^2 + 1)^{-1}$ for $0 < \eta < 1$ by (C.3) and v satisfies (C.2), we can apply the dominated convergence theorem again and conclude that $\int_{\mathbb{R}} g_\eta(t) \, \mathrm{d}v(t) \to \pi \int_{\mathbb{R}} \varphi(t) \, \mathrm{d}v(t)$ as $\eta \downarrow 0$. In view of (C.4), it follows that $\int_{\mathbb{R}} \varphi(E)\mathrm{Im} f(E + i\eta) \, \mathrm{d}E \to \pi \int_{\mathbb{R}} \varphi(E) \, \mathrm{d}v(E)$ as $\eta \downarrow 0$. □

Appendix D: Proof of Lemma 2.1

The proof given below is similar to that of Lemma 2 in [21].

Let Ln be the branch of the logarithm on $\mathbb{C}_\pi$ satisfying $\mathrm{Ln}\, 1 = 0$ and p be the holomorphic function on $\mathbb{C} \times \mathbb{C}_\pi$ defined by the relation $p(\kappa, \zeta) = e^{\kappa \mathrm{Ln}\, \zeta}$ (hence, $p(\kappa, r) = r^\kappa$ for $r > 0$). Let h be the holomorphic function on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_\pi$, such that $h(\kappa, z, \zeta) = p(1/2 + \kappa, \zeta)\mathcal{X}_\kappa(\zeta^2 z)$ for all $\kappa, z \in \mathbb{C}$ and $\zeta \in \mathbb{C}_\pi$. By (2.1), we have:

$$h(\kappa, z, r) = \mathfrak{u}^\kappa(z|r), \quad \kappa, z \in \mathbb{C}, \, r > 0. \qquad \text{(D.1)}$$

We define the holomorphic function F_2 on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_\pi$ by the formula:

$$F_2(\kappa, z, \zeta) = \frac{\pi}{2}(h(\kappa, z, \zeta) + h(-\kappa, z, \zeta)) \, \mathrm{sinc}\, \vartheta_\kappa, \quad \kappa, z \in \mathbb{C}, \, \zeta \in \mathbb{C}_\pi.$$

In view of (2.9) and (D.1), the equality $F_2(\kappa, z, r) = \mathfrak{b}^\kappa(z|r)$ holds for every $\kappa, z \in \mathbb{C}$ and $r > 0$. Furthermore, we define the function F_1 on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_\pi$ by setting:

$$F_1(\kappa, z, \zeta) = \frac{h(\kappa, z, \zeta) - h(-\kappa, z, \zeta)}{\kappa} \cos \vartheta_\kappa, \quad \kappa \in \mathbb{C} \backslash \{0\},$$

$$F_1(0, z, \zeta) = 2\left[\left(\mathrm{Ln}\,\frac{\zeta}{2} + \gamma\right) h(0, z, \zeta) - p(1/2, \zeta)\,\mathcal{Y}(\zeta^2 z)\right]$$

for every $z \in \mathbb{C}$ and $\zeta \in \mathbb{C}_\pi$. It follows immediately from (2.6), (2.7), (D.1), and the definition of F_1 that $F_1(\kappa, z, r) = \mathfrak{a}^\kappa(z|r)$ for every $\kappa, z \in \mathbb{C}$ and $r > 0$. The function $(z, \zeta) \to F_1(\kappa, z, \zeta)$ is obviously holomorphic on $\mathbb{C} \times \mathbb{C}_\pi$ for every fixed $\kappa \in \mathbb{C}$. The function $\kappa \to F_1(\kappa, z, \zeta)$ is holomorphic on $\mathbb{C} \backslash \{0\}$ and continuous at $\kappa = 0$ (this is ensured by the same calculation as the one used to find the limit in (2.7)) and is, therefore, holomorphic on $\mathbb{C}$ for every fixed $z \in \mathbb{C}$ and $\zeta \in \mathbb{C}_\pi$. Hence, F_1 is holomorphic on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_\pi$ by the Hartogs theorem. The uniqueness of F_1 and F_2 follows from the uniqueness theorem for holomorphic functions.

Appendix E: Proof of Lemma 6.7

Let Ξ denote the set of all maps from $\mathbb{Z}_+$ to $\mathbb{Z}$ that vanish outside a finite subset of $\mathbb{Z}_+$. We let $\mathbf{0}$ denote the element of Ξ which is identically zero on $\mathbb{Z}_+$: $\mathbf{0}(j) = 0$ for all $j \in \mathbb{Z}_+$. For $s \in \Xi$, let $\mathcal{K}_s = \{j \in \mathbb{Z}_+ : s(j) \neq 0\}$. By the definition of Ξ, the set $\mathcal{K}_s$ is finite for every $s \in \Xi$. Let Ξ_+ be the set of all $s \in \Xi$, such that $s(j) \geq 0$ for every $j \in \mathbb{Z}_+$. Given $s, t \in \Xi$, we let $s + t$ denote the pointwise sum of s and t: $(s + t)(j) = s(j) + t(j)$ for every $j \in \mathbb{Z}_+$. Endowed with this addition, Ξ becomes an Abelian group with zero element $\mathbf{0}$. If $s, t \in \Xi_+$, then $s + t \in \Xi_+$. Given $s \in \Xi_+$, we define the function $e(s)$ on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_{3\pi/2}$ by the relation:

$$e(s|\alpha, \vartheta, z) = \prod_{j \in I} (\partial_\alpha^j R(\alpha, \vartheta, z))^{s(j)}, \quad \alpha, \vartheta \in \mathbb{C},\ z \in \mathbb{C}_{3\pi/2},$$

where I is any finite subset of $\mathbb{Z}_+$, such that $\mathcal{K}_s \subset I$ (clearly, the definition of $e(s)$ does not depend on the choice of I). It follows immediately from the above definition that:

$$e(s + t) = e(s)e(t), \quad s, t \in \Xi_+. \tag{E.1}$$

For every $j \in \mathbb{Z}_+$, we define $\delta_j \in \Xi_+$ by the formula:

$$\delta_j(k) = \begin{cases} 1, & k = j, \\ 0, & k \in \mathbb{Z}_+ \backslash \{j\}. \end{cases}$$

If $j \in \mathbb{Z}_+$ and $s \in \Xi$, then we set:

$$[s]_j = s - \delta_j + \delta_{j+1}.$$

If $s \in \Xi_+$ and $j \in \mathcal{K}_s$, then $[s]_j \in \Xi_+$. We now show that:

$$\partial_\alpha e(s|\alpha, \vartheta, z) = \sum_{j \in \mathcal{K}_s} s(j) e([s]_j|\alpha, \vartheta, z), \tag{E.2}$$

$$\partial_\vartheta e(s|\alpha, \vartheta, z) = \sum_{j \in \mathcal{K}_s} s(j) e(s - \delta_j|\alpha, \vartheta, z) \partial_\alpha^j R(\alpha, \vartheta + \pi/2, z) \tag{E.3}$$

for every $s \in \Xi_+$, $\alpha, \vartheta \in \mathbb{C}$, and $z \in \mathbb{C}_{3\pi/2}$. The proof is by induction on the cardinality Card $\mathcal{K}_s$ of $\mathcal{K}_s$. Let Card $\mathcal{K}_s = 1$ and $j \in \mathbb{Z}_+$ be such that $\mathcal{K}_s = \{j\}$. Then, we have $e(s|\alpha, \vartheta, z) = (\partial_\alpha^j R(\alpha, \vartheta, z))^{s(j)}$. This implies (E.2) and (E.3), because $s(j) > 0$ and:

$$\partial_\vartheta R(\alpha, \vartheta, z) = R(\alpha, \vartheta + \pi/2, z) \tag{E.4}$$

by (2.17) and (2.18). Suppose now that Card $\mathcal{K}_s > 1$. Then, we can find $s', s'' \in \Xi_+$, such that $s = s' + s''$, the sets $\mathcal{K}_{s'}$ and $\mathcal{K}_{s''}$ are both nonempty, and $\mathcal{K}_{s'} \cap \mathcal{K}_{s''} = \varnothing$. Since Card $\mathcal{K}_{s'} <$ Card $\mathcal{K}_s$ and Card $\mathcal{K}_{s''} <$ Card $\mathcal{K}_s$, it follows from the Leibniz rule, the induction hypothesis, and formula (E.1) that:

$$\partial_\alpha e(s|\alpha, \vartheta, z) = e(s''|\alpha, \vartheta, z) \sum_{j \in \mathcal{K}_{s'}} s'(j) e([s']_j|\alpha, \vartheta, z) +$$
$$+ e(s'|\alpha, \vartheta, z) \sum_{j \in \mathcal{K}_{s''}} s''(j) e([s'']_j|\alpha, \vartheta, z).$$

Applying (E.1) again and observing that $[s]_j = [s']_j + s'' = s' + [s'']_j$ for every $j \in \mathbb{Z}_+$, we obtain (E.2). In the same way, (E.3) follows from the Leibniz rule, the induction hypothesis, and formula (E.1). This completes the proof of (E.2) and (E.3) for all $s \in \Xi_+$ that are not identically zero. It remains to note that these formulas obviously hold for $s = \mathbf{0}$.[11]

Given $s, t \in \Xi_+$, we define the function $\mathcal{E}(s, t)$ on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_{3\pi/2}$ by the relation:

$$\mathcal{E}(s, t|\alpha, \vartheta, z) = e(s|\alpha, \vartheta, z) e(t|\alpha, \vartheta + \pi/2, z), \quad \alpha, \vartheta \in \mathbb{C}, \, z \in \mathbb{C}_{3\pi/2}.$$

Since $\partial_\alpha^j R(\alpha, \vartheta, z) = e(\delta_j|\alpha, \vartheta, z)$ for every $j \in \mathbb{Z}_+$, it follows from (2.19), (E.2), and (E.3) that:

$$\partial_\alpha \mathcal{E}(s, t|\alpha, \vartheta, z) = \sum_{j \in \mathcal{K}_s} s(j) \mathcal{E}([s]_j, t|\alpha, \vartheta, z) + \sum_{j \in \mathcal{K}_t} t(j) \mathcal{E}(s, [t]_j|\alpha, \vartheta, z),$$
$$\partial_\vartheta \mathcal{E}(s, t|\alpha, \vartheta, z) = \sum_{j \in \mathcal{K}_s} s(j) \mathcal{E}(s - \delta_j, t + \delta_j|\alpha, \vartheta, z)$$
$$- \sum_{j \in \mathcal{K}_t} t(j) \mathcal{E}(s + \delta_j, t - \delta_j|\alpha, \vartheta, z).$$

In view of (E.4), this implies that:

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[11] As usual, we assume that sum and product of an empty family are equal to zero and unity respectively. In particular, $e(\mathbf{0})$ is identically unity on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_{3\pi/2}$.

$$\partial_\alpha \frac{\mathcal{E}(s,t|\alpha,\vartheta,z)}{R(\alpha,\vartheta,z)^m} = \frac{G_m(s,t|\alpha,\vartheta,z)}{R(\alpha,\vartheta,z)^{m+1}},$$

$$\partial_\vartheta \frac{\mathcal{E}(s,t|\alpha,\vartheta,z)}{R(\alpha,\vartheta,z)^m} = \frac{H_m(s,t|\alpha,\vartheta,z)}{R(\alpha,\vartheta,z)^{m+1}}$$

(E.5)

for every $s,t \in \Xi_+$, $m = 1, 2, \ldots$, and $(\alpha,\vartheta,z) \in \mathcal{O}$, where the functions $G_m(s,t)$ and $H_m(s,t)$ on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_{3\pi/2}$ are given by:

$$G_m(s,t) = \sum_{j \in \mathcal{K}_s} s(j)\mathcal{E}([s]_j + \delta_0, t)$$
$$+ \sum_{j \in \mathcal{K}_t} t(j)\mathcal{E}(s + \delta_0, [t]_j) - m\mathcal{E}(s + \delta_1, t),$$

(E.6)

$$H_m(s,t) = \sum_{j \in \mathcal{K}_s} s(j)\mathcal{E}(s - \delta_j + \delta_0, t + \delta_j)$$
$$- \sum_{j \in \mathcal{K}_t} t(j)\mathcal{E}(s + \delta_j + \delta_0, t - \delta_j) - m\mathcal{E}(s, t + \delta_0).$$

(E.7)

For $k, l \in \mathbb{Z}_+$, we set $\Xi_{k+} = \{s \in \Xi_+ : s(j) = 0 \text{ for } j > k\}$ and:

$$\mathcal{S}_{k,l} = \{(s,t) \in \Xi_{k+} \times \Xi_{k+} : (s|_{\{0,\ldots,k\}}, t|_{\{0,\ldots,k\}}) \in \mathcal{Q}_{k,l}\},$$

where the set $\mathcal{Q}_{k,l}$ is given by (6.21).

Let $k, l \in \mathbb{Z}_+$, $a \geq 0$, and f be a function on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_{3\pi/2}$. We say that f is a function of type (k, l, a) if there exists a map $c : \mathcal{S}_{k,l} \to \mathbb{Z}$, such that $\sum_{(s,t) \in \mathcal{S}_{k,l}} |c(s,t)| \leq a$ and:

$$f(\alpha,\vartheta,z) = \sum_{(s,t) \in \mathcal{S}_{k,l}} c(s,t)\mathcal{E}(s,t|\alpha,\vartheta,z)$$

(E.8)

for every $\alpha, \vartheta \in \mathbb{C}$ and $z \in \mathbb{C}_{3\pi/2}$. In particular, $n\mathcal{E}(s,t)$ is a function of type $(k, l, |n|)$ for every $(s,t) \in \mathcal{S}_{k,l}$ and $n \in \mathbb{Z}$. Clearly, the following properties hold:

(1) If f is a function of type (k, l, a), then nf is a function of type $(k, l, |n|a)$ for every $n \in \mathbb{Z}$.
(2) If f_1 and f_2 are functions of types (k, l, a_1) and (k, l, a_2), then $f_1 + f_2$ is a function of type $(k, l, a_1 + a_2)$.
(3) Let I be a finite set and F and A be maps on I, such that $F(\iota)$ is a function of type $(k, l, A(\iota))$ for every $\iota \in I$. Then, $\sum_{\iota \in I} F(\iota)$ is a function of type $\left(k, l, \sum_{\iota \in I} A(\iota)\right)$.

Given $k, l \in \mathbb{Z}_+$, we can define a bijection p between $\mathcal{S}_{k,l}$ and $\mathcal{Q}_{k,l}$ by setting $p(s,t) = (s|_{\{0,\ldots,k\}}, t|_{\{0,\ldots,k\}})$ for every $(s,t) \in \mathcal{S}_{k,l}$. The right-hand side of (E.8) then coincides with the sum in the right-hand side of (6.23) for $C = c \circ p^{-1}$. Hence, Lemma 6.7 is ensured by the following statement.

Lemma E.1 *For every $k, l \in \mathbb{Z}_+$, there is a function f of type $(k, l, 2^{k+l}(k+l)!)$, such that:*

$$\partial_\alpha^k \partial_\vartheta^l \mathscr{F}(\alpha, \vartheta, z) = \frac{f(\alpha, \vartheta, z)}{R(\alpha, \vartheta, z)^{k+l+1}}$$

for every $(\alpha, \vartheta, z) \in \mathcal{O}$.

Proof We note that $(\mathbf{0}, \delta_0) \in \mathcal{S}_{0,0}$ and $\mathcal{E}(\mathbf{0}, \delta_0 | \alpha, \vartheta, z) = R(\alpha, \vartheta + \pi/2, z)$ for every $\alpha, \vartheta \in \mathbb{C}$ and $z \in \mathbb{C}_{3\pi/2}$. It therefore follows from (6.3) that the statement holds for $k = l = 0$ with $f = \mathcal{E}(\mathbf{0}, \delta_0)$. Suppose the statement is true for $k, l \in \mathbb{Z}_+$ and the map $c : \mathcal{S}_{k,l} \to \mathbb{Z}$ is such that:

$$\sum_{(s,t) \in \mathcal{S}_{k,l}} |c(s,t)| \leq 2^{k+l}(k+l)! \tag{E.9}$$

and

$$\partial_\alpha^k \partial_\vartheta^l \mathscr{F}(\alpha, \vartheta, z) = \frac{1}{R(\alpha, \vartheta, z)^{k+l+1}} \sum_{(s,t) \in \mathcal{S}_{k,l}} c(s,t) \mathcal{E}(s,t | \alpha, \vartheta, z)$$

for every $(\alpha, \vartheta, z) \in \mathcal{O}$. In view of (E.5), it follows that:

$$\partial_\alpha^{k+1} \partial_\vartheta^l \mathscr{F}(\alpha, \vartheta, z) = \frac{g(\alpha, \vartheta, z)}{R(\alpha, \vartheta, z)^{k+l+2}}, \quad \partial_\alpha^k \partial_\vartheta^{l+1} \mathscr{F}(\alpha, \vartheta, z) = \frac{h(\alpha, \vartheta, z)}{R(\alpha, \vartheta, z)^{k+l+2}}$$

for every $(\alpha, \vartheta, z) \in \mathcal{O}$, where the functions g and h on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_{3\pi/2}$ are given by:

$$g = \sum_{(s,t) \in \mathcal{S}_{k,l}} c(s,t) G_{k+l+1}(s,t), \quad h = \sum_{(s,t) \in \mathcal{S}_{k,l}} c(s,t) H_{k+l+1}(s,t). \tag{E.10}$$

Our statement will be proved by induction on k and l if we show that g and h are functions of type $(k+1, l, a)$ and $(k, l+1, a)$ respectively, where $a = 2^{k+l+1}(k+l+1)!$. If $(s,t) \in \mathcal{S}_{k,l}$, then $([s]_j + \delta_0, t) \in \mathcal{S}_{k+1,l}$ for every $j \in \mathcal{K}_s$. Choosing $I = \mathcal{K}_s$, setting $F(j) = s(j) \mathcal{E}([s]_j + \delta_0, t)$ and $A(j) = s(j)$ for every $j \in I$, and applying property (3), we conclude that $\sum_{j \in \mathcal{K}_s} s(j) \mathcal{E}([s]_j + \delta_0, t)$ is a function of type $(k+1, l, \sum_{j \in \mathcal{K}_s} s(j))$. Furthermore, if $(s,t) \in \mathcal{S}_{k,l}$, then $(s + \delta_0, [t]_j) \in \mathcal{S}_{k+1,l}$ for every $j \in \mathcal{K}_t$. Hence, choosing $I = \mathcal{K}_t$, setting $F(j) = t(j) \mathcal{E}(s + \delta_0, [t]_j)$ and $A(j) = t(j)$ for every $j \in I$, and applying property (3), we deduce that $\sum_{j \in \mathcal{K}_t} t(j) \mathcal{E}(s + \delta_0, [t]_j)$ is a function of type $(k+1, l, \sum_{j \in \mathcal{K}_t} t(j))$. Since $(s + \delta_1, t) \in \mathcal{S}_{k+1,l}$ and $\sum_{j=0}^k (s(j) + t(j)) = k + l + 1$ for every $(s,t) \in \mathcal{S}_{k,l}$, it follows from (E.6) and property (2) that $G_{k+l+1}(s,t)$ is a function of type $(k+1, l, 2(k+l+1))$ for every $(s,t) \in \mathcal{S}_{k,l}$. In a similar way, (E.7) and properties (2) and (3) imply that $H_{k+l+1}(s,t)$ is a function of type $(k, l+1, 2(k+l+1))$ for every $(s,t) \in \mathcal{S}_{k,l}$. Our claim therefore follows from (E.9), (E.10), and properties (1) and (3). $\qquad \square$

References

1. Ahlfors, L.V.: Complex Analysis: An Introduction of the Theory of Analytic Functions of One Complex Variable, 2nd edn. McGraw-Hill Book Co., New York (1966)
2. Akhiezer, N.I., Glazman, I.M.: Theory of Linear Operators in Hilbert Space. Monographs and Studies in Mathematics. Pitman, Boston (1981)
3. Bruneau, L., Dereziński, J., Georgescu, V.: Homogeneous Schrödinger operators on half-line. Ann. Henri Poincaré **12**(3), 547–590 (2011)
4. Case, K.M.: Singular potentials. Phys. Rev. **80**(5), 797–806 (1950)
5. Dereziński, J., Faupin, J., Nguyen, Q.N., Richard, S.: On radial Schrödinger operators with a Coulomb potential: general boundary conditions. Adv. Oper. Theory **5**(3), 1132–1192 (2020)
6. Dereziński, J., Richard, S.: On Schrödinger operators with inverse square potentials on the half-line. Ann. Henri Poincaré **18**(3), 869–928 (2017)
7. Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F.G.: Higher Transcendental Functions, vol. 2. McGraw-Hill Book Company Inc., New York (1953)
8. Everitt, W.N., Kalf, H.: The Bessel differential equation and the Hankel transform. J. Comput. Appl. Math. **208**(1), 3–19 (2007)
9. Fulton, C.: Titchmarsh–Weyl *m*-functions for second-order Sturm–Liouville problems with two singular endpoints. Math. Nachr. **281**(10), 1418–1475 (2008)
10. Gesztesy, F., Zinchenko, M.: On spectral theory for Schrödinger operators with strongly singular potentials. Math. Nachr. **279**(9–10), 1041–1082 (2006)
11. Gitman, D.M., Tyutin, I.V., Voronov, B.L.: Self-adjoint extensions and spectral analysis in the Calogero problem. J. Phys. A **43**(14), 145205 (2010). (34pp)
12. Gitman, D.M., Tyutin, I.V., Voronov, B.L.: Self-Adjoint Extensions in Quantum Mechanics. Progress in Mathematical Physics, vol. 62. Springer, New York (2012)
13. Hankel, H.: Die Fourier'schen Reihen und Integrale für Cylinderfunctionen. Math. Ann. **8**(4), 471–494 (1875)
14. Kato, T.: Perturbation Theory for Linear Operators. Die Grundlehren der Mathematischen Wissenschaften, vol. 132. Springer, New York (1966)
15. Kodaira, K.: The eigenvalue problem for ordinary differential equations of the second order and Heisenberg's theory of *S*-matrices. Am. J. Math. **71**, 921–945 (1949)
16. Kostenko, A., Sakhnovich, A., Teschl, G.: Weyl–Titchmarsh theory for Schrödinger operators with strongly singular potentials. Int. Math. Res. Not. **2012**, 1699–1747 (2012)
17. Meetz, K.: Singular potentials in nonrelativistic quantum mechanics. Nuovo Cimento **34**(3), 690–708 (1964)
18. Naĭmark, M.A.: Linear Differential Operators. Part II: Linear Differential Operators in Hilbert Space. Frederick Ungar Publishing Co., New York (1968)
19. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. I. Functional Analysis. Academic Press, New York (1972)
20. Smirnov, A.G.: Reduction by symmetries in singular quantum-mechanical problems: general scheme and application to Aharonov–Bohm model. J. Math. Phys. **56**, 122101 (2015)
21. Smirnov, A.G.: Eigenfunction expansions for the Schrödinger equation with inverse-square potential. Theor. Math. Phys. **187**(2), 762–781 (2016)
22. Smirnov, A.G.: A Gronwall-type trigonometric inequality. Am. Math. Mon. **125**(5), 453–456 (2018)
23. Teschl, G.: Mathematical Methods in Quantum Mechanics. Graduate Studies in Mathematics, vol. 99. American Mathematical Society, Providence (2009)
24. Titchmarsh, E.C.: Eigenfunction Expansions Associated with Second-Order Differential Equations. Part I, 2nd edn. Clarendon Press, Oxford (1962)
25. Vladimirov, V.S.: Methods of the Theory of Functions of Many Complex Variables. MIT Press, Cambridge, Massachusetts (1966)
26. Weidmann, J.: Spectral Theory of Ordinary Differential Operators. Lecture Notes in Mathematics, vol. 1258. Springer, Berlin (1987)
27. Weyl, H.: Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen. Math. Ann. **68**(2), 220–269 (1910)
28. Weyl, H.: Über gewöhnliche lineare Differentialgleichungen mit singulären Stellen und ihre Eigenfunktionen (2 note). Gött. Nachr. **2**, 442–467 (1910)