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ORIGINAL PAPER

Estimates of the best approximations of the functions of the Nikol'skii–Besov class in the generalized space of Lorentz

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Abstract
In this paper, we consider the generalized Lorentz space of periodic functions of several variables and the Nikol'skii–Besov space of functions. The article establishes a sufficient condition for a function to belong from one generalized Lorentz space to another space in terms of the difference of the partial sums of the Fourier series of a given function. Exact in order estimates of the best approximation by trigonometric polynomials of functions of the Nikol'skii–Besov class are obtained.

Keywords Lorentz space · Besov class · Best approximation · Logarithmic smoothness

Mathematics Subject Classification 41A10 · 41A25 · 42A05 · 42A10

1 Introduction

Let $\mathbf{R}^m$ be a m – dimensional Euclidean space of points $\bar{x} = (x_1, \ldots, x_m)$ with real coordinates; $\mathbf{I}^m = \{\bar{x} \in \mathbf{R}^m; 0 \leq x_j \leq 1;\ j = 1, \ldots, m\}$ — m – dimensional cube.

Definition 1.1 (see [21, Chapter 2, Sect. 2]). Two nonnegative Lebesgue measurable functions $f,\ g$ are called equimeasurable if

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$$\mu\{\bar{x} \in \mathbf{I}^m : f(\bar{x}) > \lambda\} = \mu\{\bar{x} \in \mathbf{I}^m : g(\bar{x}) > \lambda\}, \quad \lambda > 0,$$

where μe — Lebesgue measure of the set $e \subset \mathbf{I}^m$.

Let X be a Banach space of Lebesgue measurable functions on $\mathbf{I}^m$ of functions f with the norm $\|f\|_X$. The space X is called symmetric

(1) if $|f(\bar{x})| \leq |g(\bar{x})|$ almost everywhere on $\mathbf{I}^m$ and $g \in X$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$;
(2) if $f \in X$ and $|f|$, $|g|$ are equimeasurable, then $g \in X$ and $\|f\|_X = \|g\|_X$ (see [21, Chapter 2, Sect. 4]).

The norm $\|\chi_e\|_X$ of the characteristic function $\chi_e(t)$ of the measurable set $e \subset \mathbf{I}^m$ is called is the fundamental function of space X and is denoted by $\varphi(\mu e) = \|\chi_e\|_X$. Further, the symmetric space X with the fundamental function φ will be denoted by $X(\varphi)$.

It is known that the fundamental function of the symmetric space X is the function $\varphi(t) = \|\chi_{[0,t]}\|_X$ defined on the interval [0, 1]. She is a concave, non-decreasing, continuous function on [0, 1], and $\varphi(0) = 0$ (see [21, p. 70, 137, 164]). Such functions are called Φ - functions.

For this function $\varphi(t)$, $t \in [0,1]$, put $\alpha_\varphi = \underline{\lim}_{t\to 0} \frac{\varphi(2t)}{\varphi(t)}$, $\quad \beta_\varphi = \overline{\lim}_{t\to 0} \frac{\varphi(2t)}{\varphi(t)}$. It is known that for any symmetric space $X(\varphi)$ we have inequalities $1 \leq \alpha_\varphi \leq \beta_\varphi \leq 2$ (see [26]).

One example of a symmetric space is $L_q(\mathbf{T}^m)$ — Lebesgue space 2π periodic for each variable of the function f with norm (see [24, Chapter 1, Sect. 1.1])

$$\|f\|_q = \left(\int\limits_{\mathbf{I}^m} |f(2\pi\bar{x})|^q \mathrm{d}\bar{x} \right)^{1/q}, \quad 1 \leq q < \infty.$$

Here and in after, $\mathbf{T}^m = [0, 2\pi]^m$.

The space $C(\mathbf{T}^m)$ consists of continuous functions f with the norm $\|f\|_\infty = \max\limits_{\bar{x}\in\mathbf{I}^m} |f(2\pi\bar{x})|$.

Let the function ψ be continuous, non-decreasing, concave by [0, 1], $\psi(0) = 0$ and $0 < \tau < \infty$. A generalized Lorentz space $L_{\psi,\tau}(\mathbf{T}^m)$ is the set of measurable on $\mathbf{T}^m = [0, 2\pi]^m$ having 2π-period for each variable $x_j, j = 1, \ldots, m$, of functions $f(\bar{x}) = f(x_1, \ldots, x_m)$, for which (see [27])

$$\|f\|_{\psi,\tau} = \left(\int\limits_0^1 f^{*^\tau}(t) \psi^\tau(t) \frac{\mathrm{d}t}{t} \right)^{1/\tau} < \infty,$$

where f^* denotes the nonincreasing rearrangement of the function $|f(2\pi\bar{x})|$, $\bar{x} \in \mathbf{I}^m$ (see e.g. [21, 27]). It is known that under the conditions $1 < \alpha_\psi, \beta_\psi < 2$, the space

$L_{\psi,\tau}(\mathbf{T}^m)$ will be a symmetric space with the fundamental function ψ and the functional $\|f\|_{\psi,\tau}$ will be equivalent to the norm

$$\|f\|^*_{\psi,\tau} = \left(\int_0^1 \left(\frac{1}{t} \int_0^t f^*(y)\mathrm{d}y \right)^\tau \psi^\tau(t) \frac{\mathrm{d}t}{t} \right)^{1/\tau}$$

spaces $L_{\psi,\tau}(\mathbf{T}^m)$ [27, Lemma 3.1].

Note that for $\psi(t) = t^{1/q}$ the space $L_{\psi,\tau}(\mathbf{T}^m)$ coincides with the Lorentz space denoted by $L_{q,\tau}(\mathbf{T}^m)$, $1 < q, \tau < \infty$ (see [32, p. 228]).

For a given positive integer M, consider the set $\Delta_M = \{\bar{k} = (k_1, \ldots, k_m) \in \mathbf{Z}^m \ : |k_j| < M, \ j = 1, \ldots, m\}$. We will consider the multiple Dirichlet kernel

$$D_{\Delta_M}(2\pi\bar{x}) = \sum_{\bar{k}\in\Delta_M} e^{i\langle \bar{k}, 2\pi\bar{x}\rangle}, \ \bar{x} \in \mathbf{I}^m$$

and the convolution of a function $f \in L_{\psi,\tau}(\mathbf{T}^m)$

$$\sigma_s(f, 2\pi\bar{x}) = \int_{\mathbf{I}^m} f(2\pi\bar{y})(D_{\Delta_{2^s}}(2\pi\bar{x} - 2\pi\bar{y}) - D_{\Delta_{2^{s-1}}}(2\pi\bar{x} - 2\pi\bar{y}))\mathrm{d}\bar{y},$$

where $s \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$, $\mathbf{N}$ is the set of natural numbers.

Let $E_M(f)_{\psi,\tau} \equiv E_{M,\ldots,M}(f)_{\psi,\tau} = \inf_{T\in\Gamma_{\Delta_M}} \|f - T\|_{\psi,\tau}$ is the best approximation of the function $f \in L_{\psi,\tau}(\mathbf{T}^m)$ by the set Γ_{Δ_M} of trigonometric polynomials of order at most $M - 1$ in each variable. For a given class $F \subset L_{\psi,\tau}(\mathbf{T}^m)$ we put $E_M(F)_{\psi,\tau} = \sup_{f\in F} E_M(f)_{\psi,\tau}$.

Let $0 < \theta \le \infty$ and a number $r > 0$. We consider the space of all functions $f \in L_{\psi,\tau}(\mathbf{T}^m)$ for which

$$\sum_{s=0}^{\infty} 2^{sr\theta} \|\sigma_s(f)\|^\theta_{\psi,\tau} < \infty$$

for $0 < \theta < \infty$ and

$$\sup_{s\in\mathbf{N}_0} 2^{sr} \|\sigma_s(f)\|_{\psi,\tau} < \infty$$

for $\theta = \infty$.

This space is denoted by the symbol $B^r_{\psi,\tau,\theta}$ and is called the Nikol'skii–Besov space. In this space, we consider a unit ball

$$\mathbf{B}^r_{\psi,\tau,\theta} = \{f \in B^r_{\psi,\tau,\theta} \ : \|f\|_{B^r_{\psi,\tau,\theta}} \le 1\},$$

where

$$||f||_{B^r_{\psi,\tau,\theta}} = \|f\|_{\psi,\tau} + \left\{\sum_{s=0}^{\infty} 2^{sr\theta}||\sigma_s(f)||^{\theta}_{\psi,\tau}\right\}^{\frac{1}{\theta}},$$

for $0<\theta<\infty$ and

$$||f||_{B^r_{\psi,\tau,\theta}} = \|f\|_{\psi,\tau} + \sup_{s\in\mathbf{N}_0} 2^{sr}||\sigma_s(f)||_{\psi,\tau}$$

for $\theta=\infty$.

In the case $\psi(t)=t^{1/p}$ and $\tau=p$, the space $B^r_{\psi,\tau,\theta}$ is defined in [7, 24] and is denoted by $B^r_{p,\theta}$.

Note that the generalized Nikol'skii–Besov space in the Lebesgue space is defined and investigated in [8, 15, 16].

One of the generalizations of the Nikol'skii–Besov space is the Nikol'skii–Besov space with logarithmic smoothness, defined as a subset of $L_p(\mathbf{T}^m)$ (see [9–11, 13]). Dominguez O., Tikhonov S. [13] established characterizations and embeddings of Besov functional spaces with logarithmic smoothness.

In the space of continuous functions $C(\mathbf{T}^1)$ S.B.Kashin and V.N.Temlyakov [18] defined the following class:

$$LG^r = \left\{f\in C(\mathbf{T}^1):\ \ \|\sigma_s(f)\|_{\infty} \le (s+1)^{-r}, s=0,1,\ldots\right\},\ \ r>0.$$

Now we define a similar Nikol'skii–Besov space with logarithmic smoothness in the generalized Lorentz space.

Let $0<\theta\le\infty$ and a number $\alpha>0$. Consider the space of all functions $f\in L_{\psi,\tau}(\mathbf{T}^m)$ for which

$$\sum_{s=0}^{\infty}(s+1)^{\alpha\theta}\|\sigma_s(f)\|^{\theta}_{\psi,\tau}<\infty,$$

for $0<\theta<\infty$ and

$$\sup_{s\in\mathbf{N}_0}(s+1)^{\alpha}\|\sigma_s(f)\|_{\psi,\tau}<\infty,$$

for $\theta=\infty$.

This space is denoted by $B^{0,\alpha}_{\psi,\tau,\theta}$ and is called the Nikol'skii–Besov space of logarithmic smoothness in the generalized Lorentz space.

In this space, we consider a unit ball

$$\mathbf{B}^{0,\alpha}_{\psi,\tau,\theta} = \{f\in B^{0,\alpha}_{\psi,\tau,\theta}\ :\|f\|_{B^{0,\alpha}_{\psi,\tau,\theta}}\le 1\},$$

where

$$\|f\|_{B^{0,\alpha}_{\psi,\tau,\theta}} = \|f\|_{\psi,\tau} + \left\{ \sum_{s=0}^{\infty} (s+1)^{\alpha\theta} \|\sigma_s(f)\|^{\theta}_{\psi,\tau} \right\}^{\frac{1}{\theta}},$$

for $0 < \theta < \infty$ and

$$\|f\|_{B^{0,\alpha}_{\psi,\tau,\infty}} = \|f\|_{\psi,\tau} + \sup_{s \in \mathbf{N_0}} (s+1)^{\alpha} \|\sigma_s(f)\|_{\psi,\tau}.$$

In the case $\psi(t) = t^{1/p}$ and $\tau = p$, the space $B^{0,\alpha}_{\psi,\tau,\theta} = B^{0,\alpha}_{p,\theta}$ is defined in [30, 31]. Other generalizations ofthe Nikol'skii–Besov space are given in [8, 15, 16].

In the case of $\tau_1 = p$, $\tau_2 = q$ for the Nikol'skii–Besov class, $\mathbf{B}^{r}_{p,\theta}$ in order exact estimates of the best approximation in the space $L_q(T^m)$ received A.S. Romanyuk [25]. In the case $\tau = p$, estimates of the approximative characteristics of the class $\mathbf{B}^{0,\alpha}_{p,\tau,\theta}$ got S.A. Stasyuk [30, 31]. In [4], estimates of the best approximations of functions of the class $\mathbf{B}^{0,\alpha}_{\psi,\tau_1,\theta}$ in the Lorentz space $L_{\psi,\tau_2}(\mathbf{T}^m)$ in the case of $\psi(t) = t^{1/p}$. A survey of results on the theory of approximation of functions of many classes of Sobolev, Nikol'skii and Besov is given in [14], also see the bibliography in [34, 35].

It is known that $L_{\psi,\tau_2}(\mathbf{T}^m) \subset L_{\psi,\tau_1}(\mathbf{T}^m)$ for $0 < \tau_2 < \tau_1 < \infty$ and the fundamental functions of these spaces are equivalent to the function ψ.

In [5], the following statement was proved.

Lemma 1.1 *Let $1 < \tau_2 < \tau_1 < \infty$ and the functions ψ_1, ψ_2 satisfy the conditions $\alpha_{\psi_1} = \alpha_{\psi_2}$, $\beta_{\psi_1} = \beta_{\psi_2}$ and*

$$C_0 = \sup_{t \in (0,1]} \frac{\psi_1(t)}{\psi_2(t)} < \infty.$$

Then $L_{\psi_2,\tau_2}(\mathbf{T}^m) \subset L_{\psi_1,\tau_1}(\mathbf{T}^m)$ and $\|f\|_{\psi_1,\tau_1} \leq C \|f\|_{\psi_2,\tau_2}$.

Therefore, the main goal of this article is to find the exact order

$$E_M(\mathbf{B}^{0,\alpha}_{\psi_1,\tau_1,\theta})_{\psi_2,\tau_2}$$

in various relations between the parameters $p, \tau_1, \tau_2, \theta$.

The article consists of three sections. In the Sect. 2, several statements are proved necessary to prove the main results. In the Sect. 3, estimates for the value $E_M(\mathbf{B}^{r}_{\psi_1,\tau_1,\theta})_{\psi_2,\tau_2}$.

In the Sect. 4, we establish estimates of $E_M(\mathbf{B}^{0,\alpha}_{\psi_1,\tau_1,\theta})_{\psi_2,\tau_2}$. The main result of this section is Theorems 4.1, 4.2.

For theorems, lemmas, formulas, double numbering is used. Further, $a_+ = \max\{a, 0\}$ and the record $A(y) \asymp B(y)$ means that there are positive numbers C_1 C_2 independent of y such that $C_1 A(y) \leq B(y) \leq C_2 A(y)$. For brevity, in the case of the inequalities $B \geq C_1 A$ or $B \leq C_2 A$, we often write $B > > A$ or $B < < A$, respectively.

For a function g defined on [0, 1], the notation $g \uparrow$ (respectively $g \downarrow$) means that the function g is non-decreasing (respectively non-increasing) by [0, 1].

2 Auxiliary results

Theorem 2.1 *(see [23]). Let $1<p<\infty$. Then for any function $f \in L_p(\mathbf{T}^m)$ the following relation holds*

$$\|f\|_p \asymp \Big\| \Big(\sum_{s=0}^{\infty} |\sigma_s(f)|^2 \Big)^{\frac{1}{2}} \Big\|_p .$$

Theorem 2.2 *Let $1<\tau<\infty$ and give $\Phi-$ function ψ, $1<\alpha_\psi, \beta_\psi<2$. Then for any function $f \in L_{\psi,\tau}(\mathbf{T}^m)$ the relation*

$$\|f\|_{\psi,\tau} \asymp \Big\| \Big(\sum_{s=0}^{\infty} |\sigma_s(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{\psi,\tau} .$$

Proof Let $f \in L_{\psi,\tau}(\mathbf{T}^m)$. We consider the operator P:

$$P(f, 2\pi\bar{x}) = \Big(\sum_{s=0}^{\infty} |\sigma_s(f, 2\pi\bar{x})|^2 \Big)^{\frac{1}{2}}, \ \bar{x} \in \mathbf{I}^m .$$

P is known to be a sublinear operator. By Theorem 2.1, this operator acts boundedly in the space $L_p(\mathbf{T}^m)$, $1<p<\infty$. Therefore, by the Janson interpolation theorem [17], this operator is bounded in the space $L_{\psi,\tau}(\mathbf{T}^m)$ i.e. $\|P(f)\|_{\psi,\tau} \le C_2(p,\tau)\|f\|_{\psi,\tau}$ for any function $f \in L_{\psi,\tau}(\mathbf{T}^m)$.

The converse inequality follows from the duality principle. Let $f \in L_{\psi,\tau}(\mathbf{T}^m)$, $g \in L_{\bar{\psi},\tau'}(\mathbf{T}^m)$, $1<\tau<\infty$, $\frac{1}{\tau}+\frac{1}{\tau'}=1$. Here and in the sequel $\bar{\psi}(t) = t/\psi(t)$ for $t \in (0,1]$ and $\bar{\psi}(0)=0$. Then, due to the orthogonality of the functions $\sigma_s(f, 2\pi\bar{x})$, we have

$$\int_{\mathbf{I}^m} f(2\pi\bar{x})g(2\pi\bar{x})\mathrm{d}\bar{x} = \int_{\mathbf{I}^m} \sum_{s=0}^{\infty} \sigma_s(f, 2\pi\bar{x})\sigma_s(g, 2\pi\bar{x})\mathrm{d}\bar{x}.$$

Further, applying the Hölder inequalities for the sum and the integral, we obtain

$$\Big| \int_{\mathbf{I}^m} f(2\pi\bar{x})g(2\pi\bar{x})\mathrm{d}\bar{x} \Big| \le \Big\| \Big(\sum_{s=0}^{\infty} |\sigma_s(f)|^2 \Big)^{\frac{1}{2}} \Big\|_{\psi,\tau} \Big\| \Big(\sum_{s=0}^{\infty} |\sigma_s(g)|^2 \Big)^{\frac{1}{2}} \Big\|_{\bar{\psi},\tau'}$$

for any function $g \in L_{\bar{\psi},\tau'}(\mathbf{T}^m)$. Therefore, taking into account the well-known relation (see [27])

$$\|f\|_{\psi,\tau} \asymp \sup_{\|g\|_{\bar{\psi},\tau'} \le 1} \Big| \int_{\mathbf{I}^m} f(2\pi\bar{x})g(2\pi\bar{x})\mathrm{d}\bar{x} \Big| \tag{2.1}$$

and the boundedness of the operator P, we have

$$\|f\|_{\psi,\tau} << \left\| \left(\sum_{s=0}^{\infty} |\sigma_s(f)|^2 \right)^{\frac{1}{2}} \right\|_{\psi,\tau}.$$

□

Lemma 2.1 *Let $\Phi-$ function ψ satisfy the condition $1 < \alpha_\psi, \beta_\psi < 2^{1/\tau}$ and $1 < \tau \leq 2$. Then for an arbitrary system of functions $\{\varphi_j\}_{j=1}^n \subset L_{\psi,\tau}(\mathbf{T}^m)$ the inequality hold*

$$\left\| \left(\sum_{j=1}^{n} |\varphi_j|^2 \right)^{\frac{1}{2}} \right\|_{\psi,\tau} \leq C \left(\sum_{j=1}^{n} \|\varphi_j\|_{\psi,\tau}^{\tau} \right)^{\frac{1}{\tau}},$$

where the constant C is independent of φ_j and n.

Proof It is known that $(f^*)^\theta = (|f|^\theta)^*$ for the number $\theta > 0$. Therefore,

$$\begin{aligned} I = & \left\| \left(\sum_{j=1}^{n} |\varphi_j|^2 \right)^{\frac{1}{2}} \right\|_{\psi,\tau} = \left[\int_0^1 \left(\left(\sum_{j=1}^{n} |\varphi_j|^2 \right)^{1/2} \right)^{*^\tau} (t) \psi^\tau(t) \frac{\mathrm{d}t}{t} \right]^{\frac{1}{\tau}} \\ & = \left[\int_0^1 \left(\left(\sum_{j=1}^{n} |\varphi_j|^2 \right)^{\frac{\tau}{2}} \right)^{*} (t) \psi^\tau(t) \frac{\mathrm{d}t}{t} \right]^{\frac{1}{\tau}}. \end{aligned} \tag{2.2}$$

Now, using Jensen's inequality (see [24, Lemma 3.3.3]) and taking into account that the function f^* is non-increasing from (2.2), we obtain

$$I \leq \left[\int_0^1 \left(\sum_{j=1}^{n} |\varphi_j|^\tau \right)^{*} (t) \psi^\tau(t) \frac{\mathrm{d}t}{t} \right]^{\frac{1}{\tau}} \leq \left[\int_0^1 \left[\frac{1}{t} \int_0^t \left(\sum_{j=1}^{n} |\varphi_j|^\tau \right)^{*} (u) \mathrm{d}u \right] \psi^\tau(t) \frac{\mathrm{d}t}{t} \right]^{\frac{1}{\tau}}. \tag{2.3}$$

Applying the formula (see [21, p.89])

$$\int_0^t f^*(u)\mathrm{d}u = \sup_{E \subset \mathbf{I}^m, \mu E = t} \int_E |f(\bar{x})| \mathrm{d}\bar{x}, \tag{2.4}$$

where μE is the Lebesgue measure of the set E and the properties of the integral we have

$$\int_0^t \Big(\sum_{j=1}^n |\varphi_j|^\tau\Big)^*(u)\mathrm{d}u = \sup_{E\subset \mathbf{I}^m, \mu E = t} \int\limits_E \sum_{j=1}^n |\varphi_j(\bar{x})|^\tau \mathrm{d}\bar{x}$$
$$= \sup_{E\subset \mathbf{I}^m, \mu E = t} \sum_{j=1}^n \int\limits_E |\varphi_j(\bar{x})|^\tau \mathrm{d}\bar{x} = \sum_{j=1}^n \sup_{E\subset \mathbf{I}^m, \mu E = t} \int\limits_E |\varphi_j|^\tau(\bar{x})\mathrm{d}\bar{x}$$
$$= \sum_{j=1}^n \int_0^t \Big(|\varphi_j|^\tau\Big)^*(u)\mathrm{d}u. \tag{2.5}$$

Now it follows from inequalities (2.3) and (2.5) that

$$I \le \left[\int\limits_0^1 \left[\sum_{j=1}^n \frac{1}{t}\int_0^t \Big(|\varphi_j|^\tau\Big)^*(u)\mathrm{d}u\right]\psi^\tau(t)\frac{\mathrm{d}t}{t}\right]^{\frac{1}{\tau}}$$
$$= \left[\sum_{j=1}^n \int\limits_0^1 \left[\frac{1}{t}\int_0^t \Big(|\varphi_j|^\tau\Big)^*(u)\mathrm{d}u\right]\psi^\tau(t)\frac{\mathrm{d}t}{t}\right]^{\frac{1}{\tau}}. \tag{2.6}$$

Changing the order of integration, we have

$$\int\limits_0^1 \left[\frac{1}{t}\int_0^t \Big(|\varphi_j|^\tau\Big)^*(u)\mathrm{d}u\right]\psi^\tau(t)\frac{\mathrm{d}t}{t} = \int\limits_0^1 \Big(|\varphi_j|^\tau\Big)^*(u)\int_u^1 \psi^\tau(t)\frac{\mathrm{d}t}{t^2}\mathrm{d}u. \tag{2.7}$$

We will consider the function $\varphi(t) = t^{1/\tau}$. By the assumption of the lemma, $\beta_\psi < 2^{1/\tau}$ i.e. $\alpha_\varphi = 2^{1/\tau} > \beta_\psi$. Therefore, by [22, Lemma 4] there exists a $\Phi-$ function $g(t)$ such that $\varphi(t)/\psi(t) \asymp g(t)$ and $\alpha_g > 1$. Therefore, the [28, Lemma 3] holds the estimate

$$\int_u^1 \psi^\tau(t)\frac{\mathrm{d}t}{t^2} = \int_u^1 \left(\frac{\psi(t)}{t^{1/\tau}}\right)^\tau \frac{\mathrm{d}t}{t} = \int_u^1 \left(\frac{1}{g(t)}\right)^\tau \frac{\mathrm{d}t}{t} \le C\frac{\psi^\tau(u)}{u}.$$

Now, according to this estimate, from equality (2.7) we obtain

$$\int\limits_0^1 \left[\frac{1}{t}\int_0^t \Big(|\varphi_j|^\tau\Big)^*(u)\mathrm{d}u\right]\psi^\tau(t)\frac{\mathrm{d}t}{t} \le C\int\limits_0^1 \Big(|\varphi_j|^\tau\Big)^*(u)\frac{\psi^\tau(u)}{u}\mathrm{d}u. \tag{2.8}$$

It follows from inequalities (2.6) and (2.8) that

$$I \leq \left[\sum_{j=1}^{n} \int_{0}^{1} \left(|\varphi_j|^{\tau} \right)^{*}(u) \frac{\psi^{\tau}(u)}{u} \mathrm{d}u \right]^{\frac{1}{\tau}}$$

$$= \left[\sum_{j=1}^{n} \int_{0}^{1} \left(\varphi_j^{*}(u) \right)^{\tau} \frac{\psi^{\tau}(u)}{u} \mathrm{d}u \right]^{\frac{1}{\tau}} \leq C \left(\sum_{j=1}^{n} \|\varphi_j\|_{\psi,\tau}^{\tau} \right)^{\frac{1}{\tau}}.$$

□

Lemma 2.2 *Let $2 < \tau < \infty$ and give a $\Phi-$ function ψ and $1 < \alpha_\psi, \beta_\psi < 2^{1/2}$. Then for an arbitrary system of functions $\{\varphi_j\}_{j=1}^{n} \subset L_{\psi,\tau}(\mathbf{T}^m)$ the inequality hold*

$$\left\| \left(\sum_{j=1}^{n} |\varphi_j|^2 \right)^{\frac{1}{2}} \right\|_{\psi,\tau} \leq C \left(\sum_{j=1}^{n} \|\varphi_j\|_{\psi,\tau}^{2} \right)^{\frac{1}{2}},$$

where the constant C is independent of φ_j and n.

Proof By the property of nonincreasing rearrangment of the function, we have (see (2.2))

$$I = \left\| \left(\sum_{j=1}^{n} |\varphi_j|^2 \right)^{\frac{1}{2}} \right\|_{\psi,\tau} \leq \left[\int_{0}^{1} \left[\frac{1}{t} \int_{0}^{t} \left(\sum_{j=1}^{n} |\varphi_j|^2 \right)^{*}(u) \mathrm{d}u \right]^{\tau/2} \psi^{\tau}(t) \frac{\mathrm{d}t}{t} \right]^{\frac{1}{\tau}}. \qquad (2.9)$$

We will consider the function $\varphi(t) = t^{1/2}$, $t \in (0, 1]$ and $\varphi(0) = 0$. By the assumption of the lemma, $\beta_\psi < 2^{1/2}$ i.e. $\alpha_\varphi = 2^{1/2} > \beta_\psi$. Therefore, by [22, Lemma 4] there exists a $\Phi-$ function $g(t)$ such that $\varphi(t)/\psi(t) \asymp g(t)$ and $\alpha_g > 1$. Therefore,

$$\psi^{\tau}(t) = \left(\frac{\psi(t)}{t^{1/2}} \right)^{\tau/2} \left(\psi(t) t^{1/2} \right)^{\tau/2} \asymp \left(\frac{1}{g(t)} \right)^{\tau/2} \left(\psi(t) t^{1/2} \right)^{\tau/2}.$$

The functions $\psi(t)$ and $\varphi(t) = t^{1/2}$ are concave, so their product is a concave function. Therefore, $L_{\psi\varphi,\tau/2}(\mathbf{T}^m)$ is a generalized Lorentz space and, moreover, $\tau/2 > 1$. Now, taking into account that $\frac{1}{g(t)}$ decreasing on (0, 1] and applying the triangle inequality, we obtain

$$
\begin{aligned}
&\left[\int\limits_0^1\left[\frac{1}{t}\int_0^t\Big(\sum_{j=1}^n|\varphi_j|^2\Big)^*(u)\mathrm{d}u\right]^{\tau/2}\psi^\tau(t)\frac{\mathrm{d}t}{t}\right]^{\frac{1}{\tau}}\\
&\le C\left[\int\limits_0^1\left[\frac{1}{t}\int_0^t\Big(\sum_{j=1}^n|\varphi_j|^2\Big)^*(u)\mathrm{d}u\right]^{\tau/2}\Big(\frac{1}{g(t)}\Big)^{\tau/2}(t^{1/2}\psi(t))^{\tau/2}\frac{\mathrm{d}t}{t}\right]^{\frac{1}{\tau}}\\
&\le C\left[\int\limits_0^1\left[\frac{1}{t}\int_0^t\Big(\sum_{j=1}^n|\varphi_j|^2\Big)^*(u)\frac{1}{g(u)}\mathrm{d}u\right]^{\tau/2}(t^{1/2}\psi(t))^{\tau/2}\frac{\mathrm{d}t}{t}\right]^{\frac{1}{\tau}}\\
&\le C\left\{\sum_{j=1}^n\left[\int\limits_0^1\left[\frac{1}{t}\int_0^t\Big(|\varphi_j|^2\Big)^*(u)\frac{1}{g(u)}\mathrm{d}u\right]^{\tau/2}(t^{1/2}\psi(t))^{\tau/2}\frac{\mathrm{d}t}{t}\right]^{\frac{2}{\tau}}\right\}^{1/2}.
\end{aligned}
\tag{2.10}
$$

Since, by the assumption of the lemma, $\beta_\psi < 2^{1/2}$, then

$$\overline{\lim}_{t\to 0}\frac{(2t)^{1/2}\psi(2t)}{t^{1/2}\psi(t)} = 2^{1/2}\beta_\psi < 2.$$

Therefore, according to Hardy's inequality in the generalized Lorentz space , we have

$$
\begin{aligned}
&\left\{\sum_{j=1}^n\left[\int\limits_0^1\left[\frac{1}{t}\int_0^t\Big(|\varphi_j|^2\Big)^*(u)\frac{1}{g(u)}\mathrm{d}u\right]^{\tau/2}(t^{1/2}\psi(t))^{\tau/2}\frac{\mathrm{d}t}{t}\right]^{\frac{2}{\tau}}\right\}^{1/2}\\
&\le\left\{\sum_{j=1}^n\left[\int\limits_0^1\left[\Big(|\varphi_j|^2\Big)^*(t)\frac{1}{g(t)}\right]^{\tau/2}(t^{1/2}\psi(t))^{\tau/2}\frac{\mathrm{d}t}{t}\right]^{\frac{2}{\tau}}\right\}^{1/2}\\
&=\left\{\sum_{j=1}^n\left[\int\limits_0^1\left[\Big(\varphi_j^*(t)\Big)^2\frac{1}{g(t)}\right]^{\tau/2}(t^{1/2}\psi(t))^{\tau/2}\frac{\mathrm{d}t}{t}\right]^{\frac{2}{\tau}}\right\}^{1/2}\\
&\le C\left\{\sum_{j=1}^n\left[\int\limits_0^1\left[\Big(\varphi_j^*(t)\Big)^2\frac{\psi(t)}{t^{1/2}}\right]^{\tau/2}(t^{1/2}\psi(t))^{\tau/2}\frac{\mathrm{d}t}{t}\right]^{\frac{2}{\tau}}\right\}^{1/2}\\
&\le C\left\{\sum_{j=1}^n\left[\int\limits_0^1\Big(\varphi_j^*(t)\Big)^\tau\psi^\tau(t)\frac{\mathrm{d}t}{t}\right]^{\frac{2}{\tau}}\right\}^{1/2} = C\left\{\sum_{j=1}^n\|\varphi_j\|_{\psi,\tau}^2\right\}^{1/2}.
\end{aligned}
\tag{2.11}
$$

Now, inequalities (2.9)–(2.11) imply the assertion of Lemma 2.2. □

Remark 2.1 These lemmas in the one-dimensional case in the Lorentz weighted space were proved by Kokilashvili and Yildirir [20].

Lemma 2.3 *Let ψ a given Φ be a function. If $1<\alpha_{\psi},\beta_{\psi}<2^{1/\tau}$ and $1<\tau\leq 2$ or $1<\alpha_{\psi},\beta_{\psi}<2^{1/2}$ and $2\leq\tau<\infty$, then for any function $f\in L_{\psi,\tau}(\mathbf{T}^m)$ the inequality hold*

$$\|f\|_{\psi,\tau}<<\Big(\sum_{s=0}^{\infty}\|\sigma_s(f)\|_{\psi,\tau}^{\tau_0}\Big)^{\frac{1}{\tau_0}},$$

where $\tau_0=\min\{\tau,2\}$.

Proof Let $f\in L_{\psi,\tau}(\mathbf{T}^m)$. Then by Theorem 2.2 we have

$$\Big\|\sum_{s=0}^{n}\sigma_s(f)\Big\|_{\psi,\tau}<<\Big\|\Big(\sum_{s=0}^{n}|\sigma_s(f)|^2\Big)^{\frac{1}{2}}\Big\|_{\psi,\tau}.$$

From this inequality, according to the Lemma 2.1 and the Lemma 2.2, we obtain

$$\Big\|\sum_{s=0}^{n}\sigma_s(f)\Big\|_{\psi,\tau}<<\Big(\sum_{s=0}^{\infty}\|\sigma_s(f)\|_{\psi,\tau}^{\tau_0}\Big)^{\frac{1}{\tau_0}},\ \forall n\in\mathbf{N}. \tag{2.12}$$

It is known that the Fourier series of the function $f\in L_{\psi,\tau}(\mathbf{T}^m)$ converges to it in the norm of the space $L_{\psi,\tau}(\mathbf{T}^m)$. Therefore, in inequality (2.12), passing to the limit for $n\to\infty$, we obtain the assertion of Lemma. □

Lemma 2.4 *Let Φ - the function ψ satisfy the condition $1<\alpha_{\psi},\beta_{\psi}<2^{1/2}$ and $2\leq\tau<\infty$. Then for an arbitrary system of functions $\{\varphi_j\}_{j=1}^{n}\subset L_{\psi,\tau}(\mathbf{T}^m)$ the inequality hold*

$$\Big(\sum_{j=1}^{n}\|\varphi_j\|_{\psi,\tau}^{\tau}\Big)^{\frac{1}{\tau}}<<\Big\|\Big(\sum_{j=1}^{n}|\varphi_j|^2\Big)^{\frac{1}{2}}\Big\|_{\psi,\tau}.$$

Proof It is known that $(f^*)^{\theta}=(|f|^{\theta})^*$ for the number $\theta>0$. Therefore,

$$\Big\|\Big(\sum_{j=1}^{n}|\varphi_j|^2\Big)^{\frac{1}{2}}\Big\|_{\psi,\tau}=\left[\int_0^1\Big(\Big(\sum_{j=1}^{n}|\varphi_j|^2\Big)^*\Big)^{\frac{\tau}{2}}(t)\psi^{\tau}(t)\frac{dt}{t}\right]^{\frac{1}{\tau}}. \tag{2.13}$$

We consider the function $\varphi(t)=t^{1/2}$, $t\in[0,1]$. This function is increasing, continuous, concave, and $\alpha_{\varphi}=\beta_{\varphi}=2^{1/2}$.

By the assumption of Lemma 2.4, $\beta_{\psi}<2^{1/2}$ i.e. $\alpha_{\varphi}=2^{1/2}>\beta_{\psi}$. Therefore, by [22, Lemma 4] there exists a Φ–function $g(t)$ equivalent to the function φ/ψ and $\alpha_g>1$ (also see the proof of Lemma 2.2). Then $\psi(t)=\frac{\psi(t)}{t^{1/2}}t^{1/2}\asymp\frac{1}{g(t)}t^{1/2}$. Therefore,

$$\psi^{\tau}(t) = \Big(\frac{\psi(t)}{t^{1/2}}\Big)^{\tau/2}\Big(\psi(t)t^{1/2}\Big)^{\tau/2} \asymp \Big(\frac{1}{g(t)}\Big)^{\tau/2}\Big(\psi(t)t^{1/2}\Big)^{\tau/2}.$$

The functions $\psi(t)$ and $\varphi(t) = t^{1/2}$ are concave, so their product is a concave function. Therefore, $L_{\psi\varphi,\tau/2}$ is a generalized Lorentz space and, moreover, $\tau/2 > 1$. Take into account these considerations, we have

$$\begin{aligned}
&\Bigg[\int_0^1 \Big(\Big(\sum_{j=1}^{n}|\varphi_j|^2\Big)^*\Big)^{\frac{\tau}{2}}(t)\psi^{\tau}(t)\frac{dt}{t}\Bigg]^{\frac{1}{\tau}} \\
&= \Bigg[\int_0^1 \Big(\Big(\sum_{j=1}^{n}|\varphi_j|^2\Big)^*\Big)^{\frac{\tau}{2}}(t)\Big(\frac{\psi(t)}{t^{1/2}}\Big)^{\tau/2}\Big(\psi(t)t^{1/2}\Big)^{\tau/2}\frac{dt}{t}\Bigg]^{\frac{1}{\tau}} \\
&\geq C\Bigg[\int_0^1 \Big(\Big(\sum_{j=1}^{n}|\varphi_j|^2\Big)^*(t)\Big)^{\frac{\tau}{2}}\Big(\frac{1}{g(t)}\Big)^{\tau/2}\Big(\psi(t)t^{1/2}\Big)^{\tau/2}\frac{dt}{t}\Bigg]^{\frac{1}{\tau}} \\
&= C\Bigg[\int_0^1 \Big(\frac{1}{g(t)}\Big(\sum_{j=1}^{n}|\varphi_j|^2\Big)^*(t)\Big)^{\frac{\tau}{2}}\Big(\psi(t)t^{1/2}\Big)^{\tau/2}\frac{dt}{t}\Bigg]^{\frac{1}{\tau}}.
\end{aligned} \tag{2.14}$$

Now in the space $L_{\varphi\psi,\frac{\tau}{2}}(\mathbf{T}^m)$ applying Hardy's inequality (see [27]) from (2.14) we get

$$\begin{aligned}
&\Bigg[\int_0^1 \Big(\Big(\sum_{j=1}^{n}|\varphi_j|^2\Big)^*\Big)^{\frac{\tau}{2}}(t)\psi^{\tau}(t)\frac{dt}{t}\Bigg]^{\frac{1}{\tau}} \\
&\geq C\Bigg\{\int_0^1\Bigg[\frac{1}{t}\int_0^t \Big(\frac{1}{g(u)}\Big(\sum_{j=1}^{n}|\varphi_j|^2\Big)^*(u)du\Bigg]^{\frac{\tau}{2}}\Big(\psi(t)t^{1/2}\Big)^{\tau/2}\frac{dt}{t}\Bigg\}^{\frac{1}{\tau}}.
\end{aligned} \tag{2.15}$$

Now, taking into account that the function $\frac{1}{g(u)}$ decreasing from the inequalities (2.13) and (2.15) we get

$$
\begin{aligned}
&\Big\|\Big(\sum_{j=1}^{n}|\varphi_j|^2\Big)^{\frac{1}{2}}\Big\|_{\psi,\tau}\\
&\geq C\left\{\int_0^1\left[\frac{1}{t}\int_0^t\Big(\frac{1}{g(u)}\Big(\sum_{j=1}^{n}|\varphi_j|^2\Big)^*(u)\mathrm{d}u\right]^{\frac{\tau}{2}}\Big(\psi(t)t^{1/2}\Big)^{\tau/2}\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{\tau}}\\
&\geq C\left\{\int_0^1\left[\frac{1}{t}\int_0^t\Big(\sum_{j=1}^{n}|\varphi_j|^2\Big)^*(u)\mathrm{d}u\right]^{\frac{\tau}{2}}\Big(\frac{1}{g(t)}\Big)^{\frac{\tau}{2}}\Big(\psi(t)t^{1/2}\Big)^{\tau/2}\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{\tau}}\\
&\geq C\left\{\int_0^1\left[\frac{1}{t}\int_0^t\Big(\sum_{j=1}^{n}|\varphi_j|^2\Big)^*(u)\mathrm{d}u\right]^{\frac{\tau}{2}}\Big(\frac{\psi(t)}{t^{1/2}}\Big)^{\frac{\tau}{2}}\Big(\psi(t)t^{1/2}\Big)^{\tau/2}\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{\tau}}\\
&= C\left\{\int_0^1\left[\frac{1}{t}\int_0^t\Big(\sum_{j=1}^{n}|\varphi_j|^2\Big)^*(u)\mathrm{d}u\right]^{\frac{\tau}{2}}\psi^\tau(t)\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{\tau}}.
\end{aligned}
\tag{2.16}
$$

Further, using equality (2.5), Jensen's inequality (since $\frac{2}{\tau}\leq 1$) (see [24, Lemma 3.3.3]) from (2.16) we get

$$
\begin{aligned}
\Big\|\Big(\sum_{j=1}^{n}|\varphi_j|^2\Big)^{\frac{1}{2}}\Big\|_{\psi,\tau} &\geq C\left\{\int_0^1\left[\sum_{j=1}^{n}\frac{1}{t}\int_0^t\Big(|\varphi_j|^2\Big)^*(u)\mathrm{d}u\right]^{\frac{\tau}{2}}\psi^\tau(t)\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{\tau}}\\
&\geq C\left\{\int_0^1\sum_{j=1}^{n}\left[\frac{1}{t}\int_0^t\Big(\varphi_j^2(u)\Big)^*\mathrm{d}u\right]^{\frac{\tau}{2}}\psi^\tau(t)\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{\tau}}\\
&\geq C\left\{\sum_{j=1}^{n}\int_0^1\Big(\varphi_j^*(t)\Big)^{2\frac{\tau}{2}}\psi^\tau(t)\frac{\mathrm{d}t}{t}\right\}^{\frac{1}{\tau}}\geq C\Big(\sum_{j=1}^{n}\|\varphi_j\|_{\psi,\tau}^{\tau}\Big)^{\frac{1}{\tau}}.
\end{aligned}
$$

□

Lemma 2.5 *Let Φ-the function ψ satisfy the condition* $2^{1/2}<\alpha_\psi,\beta_\psi<2$ *and* $1<\tau\leq 2$. *Then for the function* $f\in L_{\psi,\tau}(\mathbf{T}^m)$ *the inequality hold*

$$\Big\|\sum_{s=0}^{n}\sigma_s(f)\Big\|_{\psi,\tau}\leq C\Big(\sum_{j=1}^{n}\|\sigma_s(f)\|_{\psi,\tau}^{\tau}\Big)^{\frac{1}{\tau}},\ n\in\mathbf{N}.$$

Proof Let $f\in L_{\psi,\tau}(\mathbf{T}^m),\ g\in L_{\bar{\psi},\tau'}(\mathbf{T}^m),\ 1<\tau<\infty,\ \frac{1}{\tau}+\frac{1}{\tau'}=1$. Then, taking into account the orthogonality of the function $\sigma_s(f,\bar{x})$, we have

$$\int\limits_{\mathbf{I}^m} \sum_{s=0}^{n} \sigma_s(f,\bar{x}) g(\bar{x}) \mathrm{d}\bar{x} = \int\limits_{\mathbf{I}^m} \sum_{s=0}^{n} \sigma_s(f,\bar{x}) \sigma_s(g,\bar{x}) \mathrm{d}\bar{x}. \tag{2.17}$$

Here and in the sequel $\bar{\psi}(t) = t/\psi(t)$ for $t \in (0,1]$ and $\bar{\psi}(0) = 0$. Further, applying the Hölder inequalities for the sum and the integral, we obtain

$$\begin{aligned} \Big| \int\limits_{\mathbf{I}^m} \sum_{s=0}^{n} \sigma_s(f,\bar{x}) g(\bar{x}) \mathrm{d}\bar{x} \Big| &\leq \sum_{s=0}^{n} \|\sigma_s(f)\|_{\psi,\tau} \|\sigma_s(g)\|_{\bar{\psi},\tau'} \\ &\leq \Big(\sum_{s=0}^{n} \|\sigma_s(g)\|_{\psi,\tau}^{\tau} \Big)^{\frac{1}{\tau}} \Big(\sum_{s=0}^{n} \|\sigma_s(g)\|_{\bar{\psi},\tau'}^{\tau'} \Big)^{\frac{1}{\tau'}} \end{aligned} \tag{2.18}$$

for any function $g \in L_{\bar{\psi},\tau'}(\mathbf{T}^m)$. Now, taking into account the relation (2.1) from the inequalities (2.17), (2.18) we have

$$\Big\| \sum_{s=0}^{n} \sigma_s(f) \Big\|_{\psi,\tau} \leq \Big(\sum_{s=0}^{n} \|\sigma_s(f)\|_{\psi,\tau}^{\tau} \Big)^{\frac{1}{\tau}} \Big(\sum_{s=0}^{n} \|\sigma_s(g)\|_{\bar{\psi},\tau'}^{\tau'} \Big)^{\frac{1}{\tau'}}. \tag{2.19}$$

Since $2^{1/2} < \alpha_\psi, \beta_\psi < 2$ and $1 < \tau \leq 2$, then $\beta_{\bar{\psi}} < 2^{1/2}$ and $2 < \tau' < \infty$. Therefore, by applying Lemma 2.4, from the inequality (2.19) we obtain the assertion of Lemma 2.5. □

Lemma 2.6 *Let the Φ - function ψ satisfy the condition $2^{1/2} < \alpha_\psi, \beta_\psi < 2$ and $1 < \tau \leq 2$. Then for the function $f \in L_{\psi,\tau}(\mathbf{T}^m)$ the inequality hold*

$$\Big(\sum_{s=0}^{n} \|\sigma_s(f)\|_{\psi,\tau}^{2} \Big)^{\frac{1}{2}} \leq \Big\| \sum_{s=0}^{n} \sigma_s(f) \Big\|_{\psi,\tau}, \ n \in \mathbf{N}.$$

Proof To prove this lemma, we use the method applied by V.N. Temlyakov (see [34, p.28-29] and [35, p.98]) . From the formulas (2.17), (2.1) we get

$$\Big\| \sum_{s=0}^{n} \sigma_s(f) \Big\|_{\psi,\tau} \geq C \sup_{\|g\|_{\bar{\psi},\tau'} \leq 1} \int\limits_{\mathbf{I}^m} \sum_{s=0}^{n} \sigma_s(f,\bar{x}) \sigma_s(g,\bar{x}) \mathrm{d}\bar{x} \tag{2.20}$$

Consider the set

$$G_{\bar{\psi},\tau'}(\varepsilon) = \Big\{ g \in L_{\bar{\psi},\tau'}(\mathbf{T}^m) : \|\sigma_s(g)\|_{\bar{\psi},\tau'} \leq \varepsilon_s, \ s \in \mathbf{N}_0 \Big\},$$

where $\frac{1}{\tau} + \frac{1}{\tau'} = 1$ and the number sequence $\{\varepsilon_s\}$ satisfies the condition

$$\Big(\sum_{s=0}^{\infty} \varepsilon_s^2 \Big)^{\frac{1}{2}} \leq 1.$$

The set of such sequences $\{\varepsilon_s\}$ is denoted by Λ_2.

Since $2^{1/2} < \alpha_\psi, \beta_\psi < 2$ and $1 < \tau \le 2$, then $1 < \alpha_{\bar{\psi}}, \beta_{\bar{\psi}} < 2^{1/2}$ and $2 \le \tau' < \infty$. Therefore, according to Lemma 2.2, we have

$$\|g\|_{\bar{\psi},\tau'} \le \Big(\sum_{s=1}^{n} \|\sigma_s(g)\|_{\bar{\psi},\tau'}^2\Big)^{\frac{1}{2}} \le C\Big(\sum_{s=0}^{\infty} \varepsilon_s^2\Big)^{\frac{1}{2}} \le C.$$

Therefore, from the inequality (2.20) we obtain

$$\Big\|\sum_{s=0}^{n} \sigma_s(f)\Big\|_{\psi,\tau} \ge C \sup_{\{\varepsilon_s\}\in\Lambda_2} \sum_{s=0}^{n} \sup_{g\in G_{\bar{\psi},\tau'}(\varepsilon)} \int_{\mathbf{I}^m} \sigma_s(f,\bar{x})\sigma_s(g,\bar{x})\mathrm{d}\bar{x}. \tag{2.21}$$

As in the article by V.N. Temlyakov (see [34, p.28-29] and [35, p.98]) we can prove that

$$\sup_{g\in G_{\bar{\psi},\tau'}(\varepsilon)} \int_{\mathbf{I}^m} \sigma_s(f,\bar{x})\sigma_s(g,\bar{x})\mathrm{d}\bar{x} = \varepsilon_s \|\sigma_s(f)\|_{\psi,\tau}, \ s \in \mathbf{N}.$$

Therefore, from the inequality (2.21) and taking into account the properties of the norm in the space l_2, we obtain

$$\Big\|\sum_{s=0}^{n} \sigma_s(f)\Big\|_{\psi,\tau} \ge C \sup_{\{\varepsilon_s\}\in\Lambda_2} \sum_{s=0}^{n} \varepsilon_s \|\sigma_s(f)\|_{\psi,\tau} = C\Big\{\sum_{s=0}^{n} \|\sigma_s(f)\|_{\psi,\tau}^2\Big\}^{\frac{1}{2}}.$$

□

Theorem 2.3 *Let Φ - functions ψ_1, ψ_2 be given such that*

$$\sup_{0<t\le 1} \frac{\psi_1(t)}{\psi_2(t)} < \infty,$$

$1 < \alpha_{\psi_1} = \alpha_{\psi_2} \le \beta_{\psi_1} = \beta_{\psi_2} < 2^{1/\tau_2}$ *and* $1 < \tau_2 \le 2$. *If* $\tau_2 < \tau_1$ *and the function* $f \in L_{\psi_1,\tau_1}(\mathbf{T}^m)$ *satisfies the condition*

$$\sum_{s=0}^{\infty} \Big[\int_{(2^s+1)^{-m}}^{1} \Big(\frac{\psi_2(t)}{\psi_1(t)}\Big)^{\frac{\tau_1\tau_2}{\tau_1-\tau_2}} \frac{\mathrm{d}t}{t}\Big]^{\frac{\tau_1-\tau_2}{\tau_1}} \|\sigma_s(f)\|_{\psi_1,\tau_1}^{\tau_2} < \infty,$$

then $f \in L_{\psi_2,\tau_2}(\mathbf{T}^m)$ *and the inequality hold*

$$\|f\|_{\psi_2,\tau_2} \le C\Big(\sum_{s=0}^{\infty} \Big[\int_{(2^s+1)^{-m}}^{1} \Big(\frac{\psi_2(t)}{\psi_1(t)}\Big)^{\frac{\tau_1\tau_2}{\tau_1-\tau_2}} \frac{\mathrm{d}t}{t}\Big]^{\frac{\tau_1-\tau_2}{\tau_1}} \|\sigma_s(f)\|_{\psi_1,\tau_1}^{\tau_2}\Big)^{1/\tau_2}.$$

Proof Let the function $f \in L_{\psi_1,\tau_1}(\mathbf{T}^m)$ and the conditions of the theorem be satisfied. Then using the inequality different for trigonometric polynomials (see [5, Theorem 1]) we have

$$\sum_{s=0}^{\infty} \|\sigma_s(f)\|_{\psi_2,\tau_2}^{\tau_2} \leq \sum_{s=0}^{\infty} \left[\int\limits_{(2^s+1)^{-m}}^{1} \left(\frac{\psi_2(t)}{\psi_1(t)}\right)^{\frac{\tau_1\tau_2}{\tau_1-\tau_2}} \frac{\mathrm{d}t}{t} \right]^{\frac{\tau_1-\tau_2}{\tau_1}} \|\sigma_s(f)\|_{\psi_1,\tau_1}^{\tau_2}.$$

Therefore, taking into account that $1 < \tau_2 \leq 2$, by Lemma 2.3 we obtain the assertions of the theorem. □

Definition 2.1 (see [29, 33]). We denote by SVL the set of all non-negative functions on [0, 1] of $\psi(t)$ for which $(\log 2/t)^{\varepsilon}\psi(t) \uparrow +\infty$ and $(\log 2/t)^{-\varepsilon}\psi(t) \downarrow 0$ for $t \downarrow 0$.

Here and below, the notation $\log x$ means the logarithm with base 2 of the number $x > 0$.

Corollary 2.1 *Let Φ - functions ψ_1, ψ_2 satisfy the conditions of Theorem 2.3 and $\frac{\psi_2}{\psi_1} \in SVL$. If $\tau_2 < \tau_1$ and the function $f \in L_{\psi_1,\tau_1}(\mathbf{T}^m)$ satisfies the condition*

$$\sum_{s=0}^{\infty} \left(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\right)^{\tau_2} (s+1)^{\tau_2(\frac{1}{\tau_2}-\frac{1}{\tau_1})} \|\sigma_s(f)\|_{\psi_1,\tau_1}^{\tau_2} < \infty,$$

then $f \in L_{\psi_2,\tau_2}(\mathbf{T}^m)$ and the inequality hold

$$\|f\|_{\psi_2,\tau_2} \leq C\left(\sum_{s=0}^{\infty} \left(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\right)^{\tau_2} (s+1)^{\tau_2(\frac{1}{\tau_2}-\frac{1}{\tau_1})} \|\sigma_s(f)\|_{\psi_1,\tau_1}^{\tau_2}\right)^{1/\tau_2}.$$

Proof By condition, the function $\frac{\psi_2}{\psi_1} \in SVL$. Let $0 < t_1 < t_2 \leq 1$ i.e. $1/t_1 > 1/t_2 > 0$. Consequently

$$\left(\log\frac{2}{t_1}\right)^{\varepsilon} \frac{\psi_2(t_1)}{\psi_1(t_1)} > \left(\log\frac{2}{t_2}\right)^{\varepsilon} \frac{\psi_2(t_2)}{\psi_1(t_2)}$$

for the number $\varepsilon > 0$. Put $t_1 = \prod_{j=1}^{m}(n_j+1)^{-1}$ and $t_2 = t$. Then

$$\int_{\prod_{j=1}^{m}(n_j+1)^{-1}}^{1} \left(\frac{\psi_2(t)}{\psi_1(t)}\right)^{\frac{\tau_1\tau_2}{\tau_1-\tau_2}} \frac{\mathrm{d}t}{t}$$

$$\leq \left(\frac{\psi_2(\prod_{j=1}^{m}(n_j+1)^{-1})}{\psi_1(\prod_{j=1}^{m}(n_j+1)^{-1})} (\log 2 \prod_{j=1}^{m}(n_j+1))^{\varepsilon} \right)^{\frac{\tau_1\tau_2}{\tau_1-\tau_2}} \tag{2.22}$$

$$\times \int_{\prod_{j=1}^{m}(n_j+1)^{-1}}^{1} \left(\log\frac{2}{t}\right)^{-\varepsilon\frac{\tau_1\tau_2}{\tau_1-\tau_2}} \frac{\mathrm{d}t}{t}.$$

Put $\varepsilon = \frac{1}{2}(\frac{1}{\tau_2} - \frac{1}{\tau_1})$. Then

$$\int_{\prod_{j=1}^{m}(n_j+1)^{-1}}^{1} \left(\log\frac{2}{t}\right)^{-\varepsilon\frac{\tau_1\tau_2}{\tau_1-\tau_2}} \frac{\mathrm{d}t}{t} = - \int_{\prod_{j=1}^{m}(n_j+1)^{-1}}^{1} \left(\log\frac{2}{t}\right)^{-1/2} d(\log\frac{2}{t})$$

$$= -2\left[(\log 2)^{1/2} - (\log 2\prod_{j=1}^{m}(n_j+1))^{1/2}\right] \leq 2(\log 2\prod_{j=1}^{m}(n_j+1))^{1/2}.$$

Therefore, for $\varepsilon = \frac{1}{2}(\frac{1}{\tau_2} - \frac{1}{\tau_1})$ from inequality (2.22) we obtain

$$\int_{\prod_{j=1}^{m}(n_j+1)^{-1}}^{1} \left(\frac{\psi_2(t)}{\psi_1(t)}\right)^{\frac{\tau_1\tau_2}{\tau_1-\tau_2}} \frac{dt}{t} \leq 2 \left(\frac{\psi_2(\prod_{j=1}^{m}(n_j+1)^{-1})}{\psi_1(\prod_{j=1}^{m}(n_j+1)^{-1})} \right)^{\frac{\tau_1\tau_2}{\tau_1-\tau_2}} \log 2\prod_{j=1}^{m}(n_j+1)$$

Now, using this inequality and Theorem 2.3 we obtain the corollary. □

Theorem 2.4 *Let Φ–functions ψ_1, ψ_2 satisfy the conditions $1 < \alpha_{\psi_2} \leq \beta_{\psi_2} < \alpha_{\psi_1} \leq \beta_{\psi_1} < 2$ and $1 < \tau_1, \tau_2 < \infty$. If function $f \in L_{\psi_1,\tau_1}(\mathbf{T}^m)$ and*

$$\sum_{s=0}^{\infty} \left(\frac{\psi_2(2^{-sm})}{\psi_1(2^{-sm})}\right)^{\tau_2} \|\sigma_s(f)\|_{\psi_1,\tau_1}^{\tau_2} < \infty, \tag{2.23}$$

then $f \in L_{\psi_2,\tau_2}(\mathbf{T}^m)$ and the inequality

$$\|f\|_{\psi_2,\tau_2} \leq C\left\{ \sum_{s=0}^{\infty} \left(\frac{\psi_2(2^{-sm})}{\psi_1(2^{-sm})}\right)^{\tau_2} \|\sigma_s(f)\|_{\psi_1,\tau_1}^{\tau_2} \right\}^{1/\tau_2}.$$

Proof Using the formula (2.4) and [3, Lemma 5], we can prove that

$$\frac{1}{t}\int_0^t f^*(u)\mathrm{d}u \le C\left\{\sum_{s=0}^{n}\frac{1}{\psi_1(2^{-sm})}\|\sigma_s(f)\|_{\psi_1,\tau_1}+\sum_{s=n}^{\infty}\|\sigma_s(f)\|_{\psi_1,\tau_1}\right\}, 2^{-(n+1)^m}<t\le 2^{-nm}.$$

Further, using this inequality due to the boundedness of the Hardy operator in the generalized Lorentz space $L_{\psi_2,\tau_2}(\mathbf{T}^m)$ we can verify that the statement of the theorem is true. □

Corollary 2.2 *Let the functions ψ_1, ψ_2 satisfies the conditions of the Theorem 2.4. If function $f \in L_{\psi_1,\tau_1}(\mathbf{T}^m)$ and*

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\psi_2(n^{-m})}{\psi_1(n^{-m})}\right)^{\tau_2}E_n(f)_{\psi_1,\tau_1}^{\tau_2}<\infty, \tag{2.24}$$

then $f \in L_{\psi_2,\tau_2}(\mathbf{T}^m)$.

Proof We consider the Fourier sum

$$S_{\Delta_M}(f, 2\pi\bar{x}) = \sum_{\bar{k}\in\Delta_M} a_{\bar{k}}(f)e^{i\langle\bar{k},2\pi\bar{x}\rangle},\ \bar{x}\in\mathbf{I}^m,$$

where $a_{\bar{k}}(f)$ as usual denote the Fourier coefficients of the function f with respect to the system $\{e^{i\langle\bar{k},2\pi\bar{x}\rangle}\}$. Then, by the property of the norm and the best approximation of the function, the following inequalities hold:

$$\begin{aligned}\|\sigma_s(f)\|_{\psi_1,\tau_1} &\le \|f - S_{\Delta_{2^{s-1}}}(f\|_{\psi_1,\tau_1} + \|f - S_{\Delta_{2^s}}(f\|_{\psi_1,\tau_1}\\ &\le CE_{2^{s-1}}(f)_{\psi_1,\tau_1} \le C\sum_{l=s}^{\infty}\|\sigma_l(f)\|_{\psi_1,\tau_1}.\end{aligned} \tag{2.25}$$

Further, from the properties of the functions ψ_1, ψ_2, the best approximation of the function, and [2, Lemma 2] it follows that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\psi_2(n^{-m})}{\psi_1(n^{-m})}\right)^{\tau_2}E_n(f)_{\psi_1,\tau_1}^{\tau_2} &<< \sum_{s=1}^{\infty}\left(\frac{\psi_2(2^{-sm})}{\psi_1(2^{-sm})}\right)^{\tau_2}E_{2^{s-1}}(f)_{\psi_1,\tau_1}^{\tau_2}\\ &<< \sum_{s=1}^{\infty}\left(\frac{\psi_2(2^{-sm})}{\psi_1(2^{-sm})}\right)^{\tau_2}\left(\sum_{l=s}^{\infty}\|\sigma_l(f)\|_{\psi_1,\tau_1}\right)^{\tau_2}\\ &\le C\sum_{s=1}^{\infty}\left(\frac{\psi_2(2^{-sm})}{\psi_1(2^{-sm})}\right)^{\tau_2}\|\sigma_l(f)\|_{\psi_1,\tau_1}^{\tau_2}.\end{aligned} \tag{2.26}$$

Now it follows from inequalities (2.25) and (2.26) that conditions (2.23) and (2.24) are equivalent. Therefore, the assertion of Corollary 2.2 follows from Theorem 2.4. □

Remark 2.2 In case $\psi_1(t) = t^{1/p}$, $\psi_2(t) = t^{1/q}$ for $1 < \tau_1 = p < \tau_2 = q$ Corollary 2.2 is proved in [12, Theorem 2.3], and for $1 < \tau_1 = p < q < \infty$ and $0 < \tau_2 < \infty$ in [1, Theorem 1].

3 On orders of approximation of functions of Nikol'skii–Besov classes

In this section, we prove estimates of the best approximations of a function from the class $\mathbf{B}^r_{\psi,\tau,\theta}$.

Theorem 3.1 *Let Φ - the function ψ satisfy the conditions $1 < \alpha_\psi \le \beta_\psi < 2$ and $1 \le \tau < \infty$, $0 < \theta < \infty$. Then for the number $r > 0$ the relation hold*

$$E_n(\mathbf{B}^r_{\psi,\tau,\theta})_{\psi,\tau} \asymp n^{-r}, n \in \mathbf{N}.$$

Proof Let $f \in \mathbf{B}^r_{\psi,\tau,\theta}$ and a positive integer l such that $2^{l-1} \le n < 2^l$. Then by the property of best approximation and norm we have

$$E_n(f)_{\psi,\tau} \le E_{2^{l-1}}(f)_{\psi,\tau} \le \|f - \sum_{s=0}^{l-1} \sigma_s(f)\|_{\psi,\tau} << \sum_{s=l}^{\infty} \|\sigma_s(f)\|_{\psi,\tau}. \tag{3.1}$$

If $1 < \theta < \infty$, then applying the Hölder's inequality $(\frac{1}{\theta} + \frac{1}{\theta'} = 1)$ from (3.1) we obtain

$$\begin{aligned} E_n(f)_{\psi,\tau} &\le \left(\sum_{s=l}^{\infty} 2^{sr\theta} \|\sigma_s(f)\|^{\theta}_{\psi,\tau}\right)^{\frac{1}{\theta}} \left(\sum_{s=l}^{\infty} 2^{-sr\theta'}\right)^{\frac{1}{\theta'}} \\ &<< 2^{-lr} \left(\sum_{s=0}^{\infty} 2^{sr\theta} \|\sigma_s(f)\|^{\theta}_{\psi,\tau}\right)^{\frac{1}{\theta}}. \end{aligned}$$

If $0 < \theta \le 1$, then applying Jensen's inequality (see [24, Lemma 3.3.3]) from (3.1) we obtain

$$E_n(f)_{\psi,\tau} \le \left(\sum_{s=l}^{\infty} \|\sigma_s(f)\|^{\theta}_{\psi,\tau}\right)^{\frac{1}{\theta}} \le 2^{-lr} \left(\sum_{s=l}^{\infty} 2^{sr\theta} \|\sigma_s(f)\|^{\theta}_{\psi,\tau}\right)^{\frac{1}{\theta}}.$$

Thus, $E_n(f)_{\psi,\tau} \le C n^{-r}$ for any function $f \in \mathbf{B}^r_{\psi,\tau,\theta}$, $0 < \theta < \infty$. The upper bound is proved.

Let us prove the lower bound for $E_n(\mathbf{B}^r_{\psi,\tau,\theta})_{\psi,\tau}$. Let a natural number l be such that $2^{l-1} \le n < 2^l$. We will consider the function

$$f_0(2\pi\bar{x}) = 2^{-lr} \frac{2^{-lm}}{\psi(2^{-lm})} \sum_{\bar{k} \in \Delta_{2^{l+1}} \setminus \Delta_{2^l}} e^{i\langle \bar{k}, 2\pi\bar{x}\rangle}, \ \bar{x} \in \mathbf{I}^m, \ n \in \mathbf{N}_0.$$

According to the estimate of the norm of the Dirichlet kernel in the generalized Lorentz space, we have (see [3, p.67])

$$\Big\| \sum_{\bar{k}\in \Delta_{2^s}\setminus \Delta_{2^{s-1}}} e^{i\langle \bar{k}, 2\pi\vec{x}\rangle} \Big\|_{\psi,\tau} \asymp 2^{sm}\psi(2^{-sm}), \tag{3.2}$$

for $1<\tau<\infty,\ 1<\alpha_\psi \le \beta_\psi <2$. Therefore,

$$\Big(\sum_{s=0}^{\infty} 2^{sr\theta}\|\sigma_s(f_0)\|_{\psi,\tau}^{\theta}\Big)^{\frac{1}{\theta}} = 2^{lr}\|\sigma_l(f_0)\|_{\psi,\tau}$$
$$= 2^{lr}2^{-lr}\frac{2^{-lm}}{\psi(2^{-lm})}\Big\| \sum_{\bar{k}\in \Delta_{2^l}\setminus \Delta_{2^{l-1}}} e^{i\langle \bar{k}, 2\pi\vec{x}\rangle} \Big\|_{\psi,\tau} \le C_0.$$

Therefore, the function $F_0 = C_0^{-1}f_0 \in \mathbf{B}^r_{\psi,\tau,\theta}$. Now, by the best approximation property and relation (3.2), we have

$$E_n(F_0)_{\psi,\tau} \ge E_{2^l}(F_0)_{\psi,\tau} = C_0^{-1}\|\sigma_s(f_0)\|_{\psi,\tau} \ge C2^{-lr} \ge Cn^{-r}.$$

Consequently $E_n(\mathbf{B}^r_{\psi,\tau,\theta})_{\psi,\tau} \ge Cn^{-r}, n \in \mathbf{N}$. □

Remark 3.1 In the case $\psi(t) = t^{1/p}$ and $\tau = p,\ 1 \le \theta < \infty$, Theorem 3.1 was proved in [25, Theorem 1].

Theorem 3.2 *Let Φ - functions ψ_1, ψ_2 satisfy the conditions of Theorem 2.3 and $\frac{\psi_2}{\psi_1} \in SVL$, $0<\theta\le\infty$. If $1<\tau_2<\tau_1<\infty$, $r>0$, then the relation hold*

$$E_n(\mathbf{B}^r_{\psi_1,\tau_1,\theta})_{\psi_2,\tau_2} \asymp n^{-r}\frac{\psi_2(1/n)}{\psi_1(1/n)}(\log(n+1))^{\frac{1}{\tau_2}-\frac{1}{\tau_1}}, n \in \mathbf{N}. \tag{3.3}$$

Proof Let $f \in \mathbf{B}^r_{\psi_1,\tau_1,\theta}$. If $\tau_2<\theta$, then put $q = \frac{\theta}{\tau_2} > 1, \frac{1}{q}+\frac{1}{q'} = 1$. Applying the Hölder's inequality, we obtain

$$\begin{aligned}&\Big(\sum_{s=0}^{\infty}\Big(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\Big)^{\tau_2}(s+1)^{\tau_2(\frac{1}{\tau_2}-\frac{1}{\tau_1})}\|\sigma_s(f)\|_{\psi_1,\tau_1}^{\tau_2}\Big)^{\frac{1}{\tau_2}}\\ &\le \Big(\sum_{s=0}^{\infty}2^{sr\theta}\|\sigma_s(f)\|_{\psi_1,\tau_1}^{\theta}\Big)^{\frac{1}{\theta}}\Big(\sum_{s=0}^{\infty}2^{-sr\tau_2 q'}\Big(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\Big)^{\tau_2 q'}(s+1)^{\tau_2(\frac{1}{\tau_2}-\frac{1}{\tau_1})q'}\Big)^{\frac{1}{\tau_2 q'}}.\end{aligned} \tag{3.4}$$

Since the function $\frac{\psi_2}{\psi_1} \in SVL$, then

$$\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})} \le \frac{\psi_2(1)}{\psi_1(1)}(\log 2)^{-\varepsilon}(\log 2^{s+1})^{\varepsilon} \tag{3.5}$$

for $\varepsilon > 0$, $s = 0, 1, 2, \ldots$. Therefore,

$$\sum_{s=0}^{\infty} 2^{-sr\tau_2 q'} \left(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\right)^{\tau_2 q'} (s+1)^{\tau_2(\frac{1}{\tau_2}-\frac{1}{\tau_1})q'} \\ \leq \left(\frac{\psi_2(1)}{\psi_1(1)}\right)^{\tau_2 q'} \sum_{s=0}^{\infty} 2^{-sr\tau_2 q'} (s+1)^{(\varepsilon+\frac{1}{\tau_2}-\frac{1}{\tau_1})\tau_2 q'} < \infty. \tag{3.6}$$

Therefore, it follows from (3.4) that the series

$$\sum_{s=0}^{\infty} \left(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\right)^{\tau_2} (s+1)^{\tau_2(\frac{1}{\tau_2}-\frac{1}{\tau_1})} \|\sigma_s(f)\|_{\psi_1,\tau_1}^{\tau_2}$$

converges for any function $f \in \mathbf{B}^r_{\psi_1,\tau_1,\theta}$ for $\tau_2 < \theta$. Therefore, according to Corollary 2.1, the inclusion $\mathbf{B}^r_{\psi_1,\tau_1,\theta} \subset L_{\psi_2,\tau_2}(\mathbf{T}^m)$ for $\tau_2 < \theta$. If $\theta \leq \tau_2$, then applying Jensen's inequality (see [24, Lemma 3.3.3]) and taking into account (3.5) we obtain

$$\left(\sum_{s=0}^{\infty} \left(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\right)^{\tau_2} (s+1)^{\tau_2(\frac{1}{\tau_2}-\frac{1}{\tau_1})} \|\sigma_s(f)\|_{\psi_1,\tau_1}^{\tau_2}\right)^{\frac{1}{\tau_2}} \\ \leq \left(\sum_{s=0}^{\infty} \left(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\right)^{\theta} (s+1)^{\theta(\frac{1}{\tau_2}-\frac{1}{\tau_1})} \|\sigma_s(f)\|_{\psi_1,\tau_1}^{\theta}\right)^{\frac{1}{\theta}} \\ \leq \frac{\psi_2(1)}{\psi_1(1)} (\log 2)^{-\varepsilon} \left(\sum_{s=0}^{\infty} 2^{sr\theta} \|\sigma_s(f)\|_{\psi_1,\tau_1}^{\theta} 2^{-sr\theta} (s+1)^{(\varepsilon+\frac{1}{\tau_2}-\frac{1}{\tau_1})\theta}\right)^{\frac{1}{\theta}} \\ \leq \frac{\psi_2(1)}{\psi_1(1)} \left(\sum_{s=0}^{\infty} 2^{sr\theta} \|\sigma_s(f)\|_{\psi_1,\tau_1}^{\theta}\right)^{\frac{1}{\theta}}.$$

Therefore, it follows from 3.4 that

$$\sum_{s=0}^{\infty} \left(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\right)^{\tau_2} (s+1)^{\tau_2(\frac{1}{\tau_2}-\frac{1}{\tau_1})} \|\sigma_s(f)\|_{\psi_1,\tau_1}^{\tau_2} < \infty.$$

Therefore, again according to Corollary 2.1, we can state that

$$\mathbf{B}^r_{\psi_1,\tau_1,\theta} \subset L_{\psi_2,\tau_2}(\mathbf{T}^m)$$

for $\theta \leq \tau_2$.

Now we prove relation (3.3). Let us prove an upper bound for the quantity $E_n(\mathbf{B}^r_{\psi_1,\tau_1,\theta})_{\psi_2,\tau_2}$. Let a natural number l be such that $2^{l-1} \leq n < 2^l$. For the function $f \in \mathbf{B}^r_{\psi_1,\tau_1,\theta}$ by Corollary 2.1 we have

$$E_n(f)_{\psi_2,\tau_2} \leq E_{2^{l-1}}(f)_{\psi_2,\tau_2} \leq \|f - \sum_{s=0}^{l-1} \sigma_s(f)\|_{\psi_2,\tau_2} \\ << \left(\sum_{s=l}^{\infty} \left(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\right)^{\tau_2} (s+1)^{\tau_2(\frac{1}{\tau_2}-\frac{1}{\tau_1})} \|\sigma_s(f)\|_{\psi_1,\tau_1}^{\tau_2}\right)^{\frac{1}{\tau_2}}. \tag{3.7}$$

If $\tau_2 < \theta < \infty$, then put $q = \frac{\theta}{\tau_2} > 1, \frac{1}{q} + \frac{1}{q'} = 1$. Applying the Hölder's inequality and taking into account the condition $\frac{\psi_2}{\psi_1} \in SVL$ (see the proof of (3.6) from (3.7) we obtain

$$\begin{aligned} E_n(f)_{\psi_2,\tau_2} &\leq \Big(\sum_{s=l}^{\infty} 2^{-sr\tau_2 q'} \Big(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\Big)^{\tau_2 q'} (s+1)^{\tau_2(\frac{1}{\tau_2}-\frac{1}{\tau_1})q'}\Big)^{\frac{1}{\tau_2 q'}} \\ &\quad \times \Big(\sum_{s=l}^{\infty} 2^{sr\theta}\|\sigma_s(f)\|_{\psi_1,\tau_1}^{\theta}\Big)^{\frac{1}{\theta}} << \frac{\psi_2(2^{-l})}{\psi_1(2^{-l})}(\log 2^{l+1})^{-\varepsilon} \\ &\quad \times \Big(\sum_{s=l}^{\infty} 2^{-sr\tau_2 q'}(s+1)^{(\varepsilon+\frac{1}{\tau_2}-\frac{1}{\tau_1})\tau_2 q'}\Big)^{\frac{1}{\tau_2 q'}} \Big(\sum_{s=0}^{\infty} 2^{sr\theta}\|\sigma_s(f)\|_{\psi_1,\tau_1}^{\theta}\Big)^{\frac{1}{\theta}} \\ &<< \frac{\psi_2(2^{-l})}{\psi_1(2^{-l})}(\log 2^{l+1})^{-\varepsilon} 2^{-lr}(l+1)^{(\varepsilon+\frac{1}{\tau_2}-\frac{1}{\tau_1})} \Big(\sum_{s=0}^{\infty} 2^{sr\theta}\|\sigma_s(f)\|_{\psi_1,\tau_1}^{\theta}\Big)^{\frac{1}{\theta}} \\ &= C\frac{\psi_2(2^{-l})}{\psi_1(2^{-l})} 2^{-lr}(l+1)^{\frac{1}{\tau_2}-\frac{1}{\tau_1}} \Big(\sum_{s=0}^{\infty} 2^{sr\theta}\|\sigma_s(f)\|_{\psi_1,\tau_1}^{\theta}\Big)^{\frac{1}{\theta}} \end{aligned} \tag{3.8}$$

for any function $f \in \mathbf{B}^r_{\psi_1,\tau_1,\theta}$ in the case of $\tau_2 < \theta < \infty$.

If $\theta = \infty$, then from inequalities (3.5) and (3.7) we obtain

$$\begin{aligned} E_n(f)_{\psi_2,\tau_2} &<< \sup_{s\in\mathbf{N}_0} 2^{sr}\|\sigma_s(f)\|_{\psi_1,\tau_1} \Big(\sum_{s=l}^{\infty}\Big(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\Big)^{\tau_2}(s+1)^{\tau_2(\frac{1}{\tau_2}-\frac{1}{\tau_1})}\Big)^{\frac{1}{\tau_2}} \\ &<< \frac{\psi_2(2^{-l})}{\psi_1(2^{-l})} 2^{-lr}(l+1)^{\frac{1}{\tau_2}-\frac{1}{\tau_1}} \end{aligned}$$

for any function $f \in \mathbf{B}^r_{\psi_1,\tau_1,\infty}$.

If $\theta \leq \tau_2$, then applying Jensen's inequality (see [24, Lemma 3.3.3]) and taking into account the condition $\frac{\psi_2}{\psi_1} \in SVL$ from (3.7) we get

$$\begin{aligned} E_n(f)_{\psi_2,\tau_2} &\leq \Big(\sum_{s=l}^{\infty}\Big(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\Big)^{\theta}(s+1)^{\theta(\frac{1}{\tau_2}-\frac{1}{\tau_1})}\|\sigma_s(f)\|_{\psi_1,\tau_1}^{\theta}\Big)^{\frac{1}{\theta}} \\ &\leq \frac{\psi_2(2^{-l})}{\psi_1(2^{-l})}(\log 2^{l+1})^{-\varepsilon}\Big(\sum_{s=l}^{\infty} 2^{sr\theta}\|\sigma_s(f)\|_{\psi_1,\tau_1}^{\theta} 2^{-sr\theta}(s+1)^{(\varepsilon+\frac{1}{\tau_2}-\frac{1}{\tau_1})\theta}\Big)^{\frac{1}{\theta}} \\ &\leq \frac{\psi_2(2^{-l})}{\psi_1(2^{-l})}(\log 2^{l+1})^{-\varepsilon}2^{-lr}(l+1)^{(\varepsilon+\frac{1}{\tau_2}-\frac{1}{\tau_1})}\Big(\sum_{s=0}^{\infty} 2^{sr\theta}\|\sigma_s(f)\|_{\psi_1,\tau_1}^{\theta}\Big)^{\frac{1}{\theta}} \\ &= \frac{\psi_2(2^{-l})}{\psi_1(2^{-l})} 2^{-lr}(l+1)^{(\frac{1}{\tau_2}-\frac{1}{\tau_1})}\Big(\sum_{s=0}^{\infty} 2^{sr\theta}\|\sigma_s(f)\|_{\psi_1,\tau_1}^{\theta}\Big)^{\frac{1}{\theta}} \end{aligned} \tag{3.9}$$

for any function $f \in \mathbf{B}^r_{\psi_1,\tau_1,\theta}$ in the case of $\theta \leq \tau_2$. Now it follows from inequalities (3.8) and (3.9) that

$$E_n(\mathbf{B}^r_{\psi_1,\tau_1,\theta})_{\psi_2,\tau_2} \leq \frac{\psi_2(n^{-1})}{\psi_1(n^{-1})} n^{-r} (\log(n+1))^{\frac{1}{\tau_2}-\frac{1}{\tau_1}}.$$

The upper bound is proved.

Let us prove the lower bound for the quantity $E_n(\mathbf{B}^r_{\psi_1,\tau_1,\theta})_{\psi_2,\tau_2}$. Let a natural number l be such that $2^{l-1} \leq n < 2^l$. We will consider the function

$$f_1(2\pi\bar{x}) = 2^{-lr}(l+1)^{-\frac{1}{\tau_1}} \prod_{j=2}^{m} e^{i2^l 2\pi x_j} \sum_{k_1=2^l}^{2^{l+1}-1} \frac{\cos(2\pi k_1 x_1)}{(k_1 - 2^l + 1)\psi_1(1/(k_1 - 2^l + 1))},$$

for $\bar{x} \in \mathbf{I}^m, \ l \in \mathbf{N}_0$.

Since $|\prod_{j=2}^{m} e^{i2^l 2\pi x_j}| = 1, \ x_j \in [0,1], \ j = 2, ..., m$, then

$$|f_1(2\pi\bar{x})| = 2^{-lr}(l+1)^{-\frac{1}{\tau_1}} \Big| \sum_{k_1=2^l}^{2^{l+1}-1} \frac{\cos(2\pi k_1 x_1)}{(k_1 - 2^l + 1)\psi_1(1/(k_1 - 2^l + 1))} \Big|.$$

Therefore, non-increasing rearrangement of these functions are equal. Hence,

$$\big\|f_1\big\|_{\psi_1,\tau_1} = \frac{(l+1)^{-\frac{1}{\tau_1}}}{2^{lr}} \Big\| \sum_{k_1=2^l}^{2^{l+1}-1} \frac{\cos(2\pi k_1 x_1)}{(k_1 - 2^l + 1)\psi_1(1/(k_1 - 2^l + 1))} \Big\|_{\psi_1,\tau_1}. \tag{3.10}$$

By the norm property and taking into account the boundedness of the conjugate function operator in the space $L_{\psi_1,\tau_1}(\mathbf{T}^m)$ we have

$$\begin{aligned}
&\Big\| \sum_{k_1=2^l}^{2^{l+1}-1} \frac{\cos(2\pi k_1 x_1)}{(k_1 - 2^l + 1)\psi_1(1/(k_1 - 2^l + 1))} \Big\|_{\psi_1,\tau_1} \\
&\quad = \Big\| \sum_{v=1}^{2^l} \frac{\cos((v + 2^l - 1)2\pi x_1)}{v\psi_1(1/v)} \Big\|_{\psi_1,\tau_1} \\
&\quad = \Big\| \sum_{v=1}^{2^l} \frac{\cos(v2\pi x_1)}{v\psi_1(1/v)} \cos((2^l - 1)2\pi x_1) - \sum_{v=1}^{2^l} \frac{\sin(v2\pi x_1)}{v\psi_1(1/v)} \sin((2^l - 1)2\pi x_1) \Big\|_{\psi_1,\tau_1} \\
&\quad \leq \Big\| \sum_{v=1}^{2^l} \frac{\sin(v2\pi x_1)}{v\psi_1(1/v)} \sin((2^l - 1)2\pi x_1) \Big\|_{\psi_1,\tau_1} \\
&\qquad + \Big\| \sum_{v=1}^{2^l} \frac{\cos(v2\pi x_1)}{v\psi_1(1/v)} \Big\|_{\psi_1,\tau_1} \leq C \Big\| \sum_{v=1}^{2^l} \frac{\cos(v2\pi x_1)}{v\psi_1(1/v)} \Big\|_{\psi_1,\tau_1}.
\end{aligned} \tag{3.11}$$

In the article [6] it was proved that

$$\Big\| \sum_{v=1}^{2^l} \frac{\cos(v2\pi x_1)}{v\psi_1(1/v)} \Big\|_{\psi_1,\tau_1} \leq C(\log(2^l + 1))^{1/\tau_1}.$$

Therefore, from the estimate (3.11) it follows that

$$\Big\|\sum_{k_1=2^l}^{2^{l+1}-1}\frac{\cos(2\pi k_1 x_1)}{(k_1-2^l+1)\psi_1(1/(k_1-2^l+1))}\Big\|_{\psi_1,\tau_1}\leq Cl^{1/\tau_1}. \tag{3.12}$$

Therefore, from equality (3.10) we obtain $\big\|f_1\big\|_{\psi_1,\tau_1}\leq C2^{-lr},\ l=1,2,...$ Then

$$\Big(\sum_{s=0}^{\infty}2^{sr\theta}\|\sigma_s(f_1)\|_{\psi_1,\tau_1}^{\theta}\Big)^{\frac{1}{\theta}}=2^{(l+1)r}\|\sigma_{l+1}(f_1)\|_{\psi_1,\tau_1}=2^{(l+1)r}\|f_1\|_{\psi_1,\tau_1}\leq C_1.$$

Therefore, the function $F_1=C_1^{-1}f_1\in\mathbf{B}^r_{\psi_1,\tau_1,\theta}$. In the article [6] it was proved that

$$\Big\|\sum_{\nu=1}^{2^l}\frac{\cos(\nu 2\pi x_1)}{\nu\psi_1(1/\nu)}\Big\|_{\psi_2,\tau_2}\geq C\frac{\psi_2(2^{-l})}{\psi_1(2^{-l})}(\log 2^l)^{1/\tau_2}.$$

on condition $\frac{\psi_2}{\psi_1}\in SVL$. Using this inequality, we can verify that

$$\Big\|\sum_{k_1=2^l}^{2^{l+1}-1}\frac{\cos(2\pi k_1 x_1)}{(k_1-2^l+1)\psi_1(1/(k_1-2^l+1))}\Big\|_{\psi_1,\tau_1}\geq C\frac{\psi_2(2^{-l})}{\psi_1(2^{-l})}l^{1/\tau_2}. \tag{3.13}$$

Now, by the property of the best approximation of the function and inequality (3.13), we obtain

$$\begin{aligned}E_n(F_1)_{\psi_2,\tau_2}\geq E_{2^l}(F_1)_{\psi_2,\tau_2}&=\|F_1\|_{\psi_2,\tau_2}=C_1^{-1}\|f_1\|_{\psi_2,\tau_2}\\&=2^{-lr}(l+1)^{-\frac{1}{\tau_1}}\Big\|\sum_{k_1=2^l}^{2^{l+1}-1}\frac{\cos(2\pi k_1 x_1)}{(k_1-2^l+1)\psi_1(1/(k_1-2^l+1))}\Big\|_{\psi_2,\tau_2}\\&\geq C2^{-lr}\frac{\psi_2(2^{-l})}{\psi_1(2^{-l})}l^{1/\tau_2-1/\tau_1}\geq Cn^{-r}\frac{\psi_2(n^{-1})}{\psi_1(n^{-1})}(\log(n+1))^{1/\tau_2-1/\tau_1}.\end{aligned}$$

Hence,

$$E_n(\mathbf{B}^r_{\psi_1,\tau_1,\theta})_{\psi_2,\tau_2}\geq C\frac{\psi_2(n^{-1})}{\psi_1(n^{-1})}n^{-r}(\log(n+1))^{(\frac{1}{\tau_2}-\frac{1}{\tau_1})}.$$

□

4 Estimates of the best approximations of logarithmic smoothness functions in a generalized Lorentz space

In this section, we prove estimates of the best approximations of functions from the class $\mathbf{B}^{0,\alpha}_{\psi_1,\tau_1,\theta}$.

Theorem 4.1 *Let Φ - functions ψ_1,ψ_2 satisfy the conditions of Theorem 2.3 and $\frac{\psi_2}{\psi_1}\in SVL,\ 0<\theta\leq\infty$. If $1<\tau_2\leq 2,\tau_2<\tau_1<\infty$ and $\alpha>(\frac{1}{\tau_2}-\frac{1}{\tau_1})+(\frac{1}{\tau_2}-\frac{1}{\theta})_+$, then the relation hold*

$$E_n(\mathbf{B}^{0,\alpha}_{\psi_1,\tau_1,\theta})_{\psi_2,\tau_2} << \frac{\psi_2(1/n)}{\psi_1(1/n)}(\log(n+1))^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})+(\frac{1}{\tau_2}-\frac{1}{\theta})_+}, n \in \mathbf{N},$$

where $a_+ = \min\{0, a\}$. *In the case of* $\theta \le \tau_2$, *this estimate is sharp in order.*

In case $1 < \tau_2 < \theta \le \infty$, *if* $2^{1/2} < \alpha_{\psi_2}, \beta_{\psi_2} < 2$ *and* $1 < \tau_2 \le 2$ *or* $1 < \alpha_{\psi_2}, \beta_{\psi_2} < 2^{1/2}$ *and* $2 < \tau_2 < \infty$, *then*

$$E_n(\mathbf{B}^{0,\alpha}_{\psi_1,\tau_1,\theta})_{\psi_2,\tau_2} >> \frac{\psi_2(1/n)}{\psi_1(1/n)}(\log(n+1))^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})+\frac{1}{\gamma}-\frac{1}{\theta}}, n \in \mathbf{N},$$

where $\gamma = \max\{2, \tau_2\}$.

Proof Let $f \in \mathbf{B}^{0,\alpha}_{\psi_1,\tau_1,\theta}$. If $\tau_2 < \theta$, then for $q = \frac{\theta}{\tau_2}, \frac{1}{q} + \frac{1}{q'} = 1)$ applying the Hölder's inequality we get

$$\begin{aligned}
&\Big(\sum_{s=l}^{\infty} s^{(\frac{1}{\tau_2}-\frac{1}{\tau_1})\tau_2}\Big(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\Big)^{\tau_2}\Big\|\sigma_s(f)\Big\|^{\tau_2}_{\psi_1,\tau_1}\Big)^{\frac{1}{\tau_2}} \\
&\le \Big(\sum_{s=l}^{\infty}(s+1)^{\alpha\theta}\Big\|\sigma_s(f)\Big\|^{\theta}_{\psi_1,\tau_1}\Big)^{\frac{1}{\theta}} \\
&\quad\times \Big(\sum_{s=l}^{\infty}\Big(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\Big)^{\tau_2 q'}(s+1)^{-(\alpha-(\frac{1}{\tau_2}-\frac{1}{\tau_1}))\tau_2 q'}\Big)^{\frac{1}{\tau_2 q'}}.
\end{aligned} \tag{4.1}$$

Since the function $\frac{\psi_2}{\psi_1} \in SVL$, then

$$\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})} \le \frac{\psi_2(2^{-l})}{\psi_1(2^{-l})}(\log 2^{l+1})^{-\varepsilon}(\log 2^{s+1})^{\varepsilon} \tag{4.2}$$

for $\varepsilon > 0$, $s = l, l+1, l+2, \ldots$. Therefore,

$$\begin{aligned}
&\sum_{s=l}^{\infty}\Big(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\Big)^{\tau_2 q'}(s+1)^{-(\alpha-(\frac{1}{\tau_2}-\frac{1}{\tau_1}))\tau_2 q'} \\
&\le \Big(\frac{\psi_2(2^{-l})}{\psi_1(2^{-l})}\Big)^{\tau_2 q'}(l+1)^{-\varepsilon\tau_2 q'}\sum_{s=l}^{\infty}(s+1)^{-(\alpha-(\frac{1}{\tau_2}-\frac{1}{\tau_1}))\tau_2 q'}(s+1)^{\varepsilon\tau_2 q'}
\end{aligned} \tag{4.3}$$

for $l = 0, 1, 2, \ldots$.

Since $\tau_2 < \theta$, then $\alpha > (\frac{1}{\tau_2} - \frac{1}{\tau_1}) + (\frac{1}{\tau_2} - \frac{1}{\theta})$. Therefore, you can choose a number ε such that $0 < \varepsilon < \alpha + (\frac{1}{\tau_1} - \frac{1}{\tau_2}) - (\frac{1}{\tau_2} - \frac{1}{\theta})$. Then the series

$$\sum_{s=1}^{\infty}(s+1)^{-(\alpha-(\frac{1}{\tau_2}-\frac{1}{\tau_1}))\tau_2 q'}(s+1)^{\varepsilon\tau_2 q'}$$

converges and

$$\Big(\sum_{s=l}^{\infty}(s+1)^{-(\alpha-(\frac{1}{\tau_2}-\frac{1}{\tau_1}))\tau_2 q'}(s+1)^{\varepsilon\tau_2 q'}\Big)^{\frac{1}{\tau_2 q'}} << (l+1)^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2}-\varepsilon)+\frac{1}{\tau_2}-\frac{1}{\theta}}. \tag{4.4}$$

Now it follows from inequalities (4.3) and (4.4) that

$$\begin{aligned}\Big(\sum_{s=l}^{\infty}\Big(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\Big)^{\tau_2 q'}(s+1)^{-(\alpha-(\frac{1}{\tau_2}-\frac{1}{\tau_1}))\tau_2 q'}\Big)^{\frac{1}{\tau_2 q'}} &<< \frac{\psi_2(2^{-l})}{\psi_1(2^{-l})}(l+1)^{-\varepsilon}(l+1)^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2}-\varepsilon)+\frac{1}{\tau_2}-\frac{1}{\theta}} \\ &= C\frac{\psi_2(2^{-l})}{\psi_1(2^{-l})}(l+1)^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})+\frac{1}{\tau_2}-\frac{1}{\theta}}.\end{aligned} \tag{4.5}$$

for $l=0,1,2,\ldots$. From inequalities (4.1) and (4.5) we obtain

$$\begin{aligned}&\Big(\sum_{s=l}^{\infty}(s+1)^{(\frac{1}{\tau_2}-\frac{1}{\tau_1})\tau_2}\Big(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\Big)^{\tau_2}\Big\|\sigma_s(f)\Big\|_{\psi_1,\tau_1}^{\tau_2}\Big)^{\frac{1}{\tau_2}} \\ &<< \Big(\sum_{s=0}^{\infty}(s+1)^{\alpha\theta}\Big\|\sigma_s(f)\Big\|_{\psi_1,\tau_1}^{\theta}\Big)^{\frac{1}{\theta}}\frac{\psi_2(2^{-l})}{\psi_1(2^{-l})}(l+1)^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})+\frac{1}{\tau_2}-\frac{1}{\theta}}\end{aligned} \tag{4.6}$$

in the case of $\tau_2<\theta<\infty$, for $l=0,1,2,\ldots$.

If $\theta=\infty$, then

$$\begin{aligned}&\Big(\sum_{s=l}^{\infty}s^{(\frac{1}{\tau_2}-\frac{1}{\tau_1})\tau_2}\Big(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\Big)^{\tau_2}\Big\|\sigma_s(f)\Big\|_{\psi_1,\tau_1}^{\tau_2}\Big)^{\frac{1}{\tau_2}} \\ &\Big(\sum_{s=l}^{\infty}s^{(\frac{1}{\tau_2}-\frac{1}{\tau_1}-\alpha)\tau_2}\Big(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\Big)^{\tau_2}\Big)^{\frac{1}{\tau_2}}\sup_{s\in\mathbf{N}_0}(s+1)^{\alpha}\Big\|\sigma_s(f)\Big\|_{\psi_1,\tau_1} \\ &<< \frac{\psi_2(2^{-l})}{\psi_1(2^{-l})}(l+1)^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})+\frac{1}{\tau_2}}\sup_{s\in\mathbf{N}_0}(s+1)^{\alpha}\Big\|\sigma_s(f)\Big\|_{\psi_1,\tau_1}\end{aligned}$$

for any function $f\in\mathbf{B}^{0,\alpha}_{\psi,\tau_1,\infty}$.

If $\theta\le\tau_2$, then applying Jensen's inequality (see [24, Lemma 3.3.3]) we have

$$\begin{aligned}&\Big(\sum_{s=l}^{\infty}\Big(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\Big)^{\tau_2}(s+1)^{(\frac{1}{\tau_2}-\frac{1}{\tau_1})\tau_2}\Big\|\sigma_s(f)\Big\|_{\psi_1,\tau_1}^{\tau_2}\Big)^{\frac{1}{\tau_2}} \\ &\le\Big(\sum_{s=l}^{\infty}\Big(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\Big)^{\theta}(s+1)^{(\frac{1}{\tau_2}-\frac{1}{\tau_1})\theta}\Big\|\sigma_s(f)\Big\|_{\psi_1,\tau_1}^{\theta}\Big)^{\frac{1}{\theta}} \\ &=\Big(\sum_{s=l}^{\infty}(s+1)^{\alpha\theta}\Big\|\sigma_s(f)\Big\|_{\psi_1,\tau_1}^{\theta}\Big(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\Big)^{\theta}(s+1)^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})\theta}\Big)^{\frac{1}{\theta}}.\end{aligned} \tag{4.7}$$

Since $\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2}>0$ and $\frac{\psi_2}{\psi_1}\in SVL$, then using inequality (4.2) for $\varepsilon=\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2}$ from the formula (4.7) we get

$$\Big(\sum_{s=l}^{\infty}\Big(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\Big)^{\tau_2}(s+1)^{(\frac{1}{\tau_2}-\frac{1}{\tau_1})\tau_2}\Big\|\sigma_s(f)\Big\|_{\psi_1,\tau_1}^{\tau_2}\Big)^{\frac{1}{\tau_2}} \\ \leq \Big(\sum_{s=0}^{\infty}(s+1)^{\alpha\theta}\Big\|\sigma_s(f)\Big\|_{\psi_1,\tau_1}^{\theta}\Big)^{\frac{1}{\theta}}\frac{\psi_2(2^{-l})}{\psi_1(2^{-l})}(l+1)^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})} \tag{4.8}$$

in the case $\theta \leq \tau_2$ for $l = 0, 1, 2, \ldots$.

In particular, for $l = 0$ it follows from estimates (4.6), (4.8) that

$$\sum_{s=0}^{\infty}\Big(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\Big)^{\tau_2}(s+1)^{(\frac{1}{\tau_2}-\frac{1}{\tau_1})\tau_2}\Big\|\sigma_s(f)\Big\|_{\psi_1,\tau_1}^{\tau_2} < \infty$$

for any function $f \in \mathbf{B}^{0,\alpha}_{\psi_1,\tau_1,\theta}$. Therefore, according to Corollary 2.1, the inclusion $\mathbf{B}^{0,\alpha}_{\psi_1,\tau_1,\theta} \subset L_{\psi_2,\tau_2}(\mathbf{T}^m)$ is true.

Now we estimate the value $E_n(\mathbf{B}^{0,\alpha}_{\psi_1,\tau_1,\theta})_{\psi_2,\tau_2}$. Let a natural number l be such that $2^{l-1} \leq n < 2^l$. Using the properties of the best approximation function and Corollary 2.1, we have

$$E_n(f)_{\psi_2,\tau_2} \leq E_{2^{l-1}}(f)_{\psi_2,\tau_2} \leq \Big\|f - \sum_{s=0}^{l-1}\sigma_s(f)\Big\|_{\psi_1,\tau_1} \\ \leq \Big(\sum_{s=l}^{\infty}\Big(\frac{\psi_2(2^{-s})}{\psi_1(2^{-s})}\Big)^{\tau_2}(s+1)^{(\frac{1}{\tau_2}-\frac{1}{\tau_1})\tau_2}\Big\|\sigma_s(f)\Big\|_{\psi_1,\tau_1}^{\tau_2}\Big)^{\frac{1}{\tau_2}}.$$

Further, using inequalities (4.6), (4.8) and the properties of the functions ψ_1, ψ_2, we obtain

$$E_n(f)_{\psi_2,\tau_2} \leq C\frac{\psi_2(2^{-l})}{\psi_1(2^{-l})}(l+1)^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})+(\frac{1}{\tau_2}-\frac{1}{\theta})_+} \\ \times \Big(\sum_{s=0}^{\infty}(s+1)^{\alpha\theta}\Big\|\sigma_s(f)\Big\|_{\psi_1,\tau_1}^{\theta}\Big)^{\frac{1}{\theta}} \leq C\frac{\psi_2(n^{-1})}{\psi_1(n^{-1})}(\log(n+1))^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})+(\frac{1}{\tau_2}-\frac{1}{\theta})_+}$$

for any function $f \in \mathbf{B}^{0,\alpha}_{\psi_1,\tau_1,\theta}$. Thus,

$$E_n(\mathbf{B}^{0,\alpha}_{\psi_1,\tau_1,\theta})_{\psi_2,\tau_2} << \frac{\psi_2(1/n)}{\psi_1(1/n)}(\log(n+1))^{-(\alpha+\frac{1}{\tau_2}-\frac{1}{\tau_1})+(\frac{1}{\tau_2}-\frac{1}{\theta})_+}.$$

This proves the upper bound.

Now we prove the lower bounds. We will consider the function

$$f_2(2\pi\bar{x}) = (l+1)^{-(\alpha+\frac{1}{\tau_1})}\prod_{j=2}^{m}e^{i2^l 2\pi x_j}\sum_{k_1=2^l}^{2^{l+1}-1}\frac{\cos(2\pi k_1 x_1)}{(k_1-2^l+1)\psi_1(1/(k_1-2^l+1))},$$

for $\bar{x} \in \mathbf{I}^m$, $l \in \mathbf{N}_0$. By continuity, the function $f_2 \in L_{\psi_1,\tau_1}(\mathbf{T}^m)$. Using estimate (3.12) we have

$$\left\|\sigma_{l+1}(f_2)\right\|_{\psi_1,\tau_1} = \left\|f_2\right\|_{\psi_1,\tau_1} < < (l+1)^{-\alpha}.$$

If $s \neq l+1$, then $\left\|\sigma_s(f_2)\right\|_{\psi_1,\tau_1} = 0$. Therefore,

$$\left(\sum_{s=0}^{\infty}(s+1)^{\alpha\theta}\left\|\sigma_s(f_2)\right\|_{\psi,\tau_1}^{\theta}\right)^{\frac{1}{\theta}} = (l+2)^{\alpha}\left\|\sigma_{l+1}(f_2)\right\|_{\psi_1,\tau_1} \leq C_2.$$

Therefore, the function $F_2 = C_2^{-1}f_2 \in \mathbf{B}^{0,\alpha}_{\psi_1,\tau_1,\theta}$.

Further, taking into account the definition of the best approximation and using inequality (3.12), we have

$$E_n(F_2)_{\psi_2,\tau_2} \geq E_{2^l}(F_2)_{\psi_2,\tau_2} = \left\|C_2^{-1}f_2\right\|_{\psi_2,\tau_2} > > \frac{\psi_2(2^{-l})}{\psi_1(2^{-l})}(l+1)^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})}$$
$$> > \frac{\psi_2(n^{-1})}{\psi_1(n^{-1})}(\log(n+1))^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})}$$

in the case $\theta \leq \tau_2$. Hence,

$$E_n(\mathbf{B}^{0,\alpha}_{\psi_2,\tau_2,\theta})_{\psi_1,\tau_1} \geq C\frac{\psi_2(n^{-1})}{\psi_1(n^{-1})}(\log(n+1))^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})}$$

in the case $\theta \leq \tau_2$.

Let $\tau_2 < \theta < \infty$. We will consider the function

$$f_3(2\pi\bar{x}) = (n+1)^{-\frac{1}{\theta}}\sum_{s=n+1}^{2n}(s+1)^{-(\alpha+\frac{1}{\tau_1})}\prod_{j=2}^{m}e^{i2\pi x_j 2^{s-1}}$$
$$\times\sum_{k_1=2^{s-1}}^{2^s-1}\frac{\cos k_1 2\pi x_1}{(k_1-2^{s-1}+1)\psi_1(\frac{1}{k_1-2^{s-1}+1})}$$

where $\bar{x} \in \mathbf{I}^m$, $n \in \mathbf{N}_0$. Then

$$\begin{aligned}\left\|\sigma_s(f_3)\right\|_{\psi_1,\tau_1} &= (n+1)^{-\frac{1}{\theta}}(s+1)^{-(\alpha+\frac{1}{\tau_1})}\\ &\times\left\|\prod_{j=2}^{m}e^{i2\pi x_j 2^{s-1}}\sum_{k_1=2^{s-1}}^{2^s-1}\frac{\cos k_1 2\pi x_1}{(k_1-2^{s-1}+1)\psi_1(\frac{1}{k_1-2^{s-1}+1})}\right\|_{\psi_1,\tau_1}\\ &= (n+1)^{-\frac{1}{\theta}}(s+1)^{-(\alpha+\frac{1}{\tau_1})}\left\|\sum_{k_1=2^{s-1}}^{2^s-1}\frac{\cos k_1 2\pi x_1}{(k_1-2^{s-1}+1)\psi_1(\frac{1}{k_1-2^{s-1}+1})}\right\|_{\psi_1,\tau_1}\\ &\asymp (n+1)^{-\frac{1}{\theta}}(s+1)^{-\alpha}\end{aligned} \tag{4.9}$$

for $1 < p, \tau_1 < \infty$, $s \in \mathbf{N}_0$. By continuity, the function $f_3 \in L_{\psi_1,\tau_1}(\mathbf{T}^m)$ and using relations (3.12) and (4.9) we obtain

$$\Big(\sum_{s=0}^{\infty} s^{\alpha\theta}\Big\|\sigma_s(f_3)\Big\|_{\psi_1,\tau_1}^{\theta}\Big)^{\frac{1}{\theta}} = \Big(\sum_{s=n+1}^{2n} s^{\alpha\theta}\Big\|\sigma_s(f_3)\Big\|_{\psi_1,\tau_1}^{\theta}\Big)^{\frac{1}{\theta}} \leq C_3.$$

Hence, the function $F_3 = C_3^{-1} f_3 \in \mathbf{B}^{0,\alpha}_{\psi_1,\tau_1,\theta}$. Further, by definition of the best approximation of the function, we have

$$\begin{aligned} E_{2^n}(F_3)_{\psi_2,\tau_2} =& C_3^{-1}\|f_3\|_{\psi_2,\tau_2} = (n+1)^{-\frac{1}{\theta}} \\ & \times \Big\| \sum_{s=n+1}^{2n} (s+1)^{-(\alpha+\frac{1}{\tau_1})} \sum_{k_1=2^{s-1}}^{2^s-1} \frac{\cos k_1 2\pi x_1}{(k_1-2^{s-1}+1)\psi_1(\frac{1}{k_1-2^{s-1}+1})}\Big\|_{\psi_2,\tau_2}. \end{aligned} \tag{4.10}$$

If $2^{1/2} < \alpha_\psi, \beta_\psi < 2$ and $1 < \tau_2 \leq 2$, then using Lemma 2.6 we get

$$\begin{aligned} \|f_3\|_{\psi_2,\tau_2} \geq & C\Big(\sum_{s=n+1}^{2n} \|\sigma_s(f_3)\|^2_{\psi_2,\tau_2}\Big)^{\frac{1}{2}} = C(n+1)^{-\frac{1}{\theta}} \\ & \times \Big(\sum_{s=n+1}^{2n} (s+1)^{-(\alpha+\frac{1}{\tau_1})2} \Big\| \sum_{k_1=2^{s-1}}^{2^s-1} \frac{\cos k_1 2\pi x_1}{(k_1-2^{s-1}+1)\psi_1(\frac{1}{k_1-2^{s-1}+1})}\Big\|^2_{\psi_2,\tau_2}\Big)^{\frac{1}{2}} \\ & \geq C(n+1)^{-\frac{1}{\theta}} \Bigg[\sum_{s=n+1}^{2n} (s+1)^{-(\alpha+\frac{1}{\tau_1})2} \Big(\frac{\psi_2(1/2^s)}{\psi_1(1/2^s)} (\log 2^s)^{1/\tau_2}\Big)^2\Bigg]^{\frac{1}{2}}. \end{aligned}$$

Further, taking into account that $\frac{\psi_2}{\psi_1} \in SVL$ we get

$$\|f_3\|_{\psi_2,\tau_2} \geq C\frac{\psi_2(1/2^n)}{\psi_1(1/2^n)}(n+1)^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})+\frac{1}{2}-\frac{1}{\theta}}. \tag{4.11}$$

Now, from equality (4.10) and inequality (4.11) it follows that

$$E_{2^n}(F_3)_{\psi_2,\tau_2} \geq C\frac{\psi_2(1/2^n)}{\psi_1(1/2^n)}(n+1)^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})+\frac{1}{2}-\frac{1}{\theta}}.$$

in the case $2^{1/2} < \alpha_{\psi_2}, \beta_{\psi_2} < 2$ and $1 < \tau_2 \leq 2$.

If $1 < \alpha_{\psi_2}, \beta_{\psi_2} < 2^{1/2}$ and $2 \leq \tau_2 < \infty$, then using Lemma 2.4 and after similar reasoning we get

$$E_{2^n}(F_3)_{\psi_2,\tau_2} \geq C\frac{\psi_2(1/2^n)}{\psi_1(1/2^n)}(n+1)^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})+\frac{1}{\tau_2}-\frac{1}{\theta}}.$$

Hence,

$$E_l(\mathbf{B}^{0,\alpha}_{\psi_1,\tau_1,\theta})_{\psi_2,\tau_2} \geq C\frac{\psi_2(1/l)}{\psi_1(1/l)}(\log(l+1))^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})+\frac{1}{2}-\frac{1}{\theta}}.$$

in the case $2^{1/2} < \alpha_{\psi_2}, \beta_{\psi_2} < 2$ and $1 < \tau_2 \leq 2$ and

$$E_l(\mathbf{B}^{0,\alpha}_{\psi_1,\tau_1,\theta})_{\psi_2,\tau_2} \geq C\frac{\psi_2(1/l)}{\psi_1(1/l)}(\log(l+1))^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})+\frac{1}{\tau_2}-\frac{1}{\theta}}$$

in the case $1<\alpha_{\psi_2}, \beta_{\psi_2}<2^{1/2}$ and $2<\tau_2<\infty$, $\theta<\infty$.

If $\theta=\infty$, then we will consider the function

$$f_4(2\pi\bar{x}) = \sum_{s=1}^{\infty}(s+1)^{-(\alpha+\frac{1}{\tau_1})}\prod_{j=2}^{m}e^{i2\pi x_j 2^{s-1}}\sum_{k_1=2^{s-1}}^{2^s-1}\frac{\cos k_1 2\pi x_1}{(k_1-2^{s-1}+1)\psi_1(\frac{1}{k_1-2^{s-1}+1})},$$

where $\bar{x}\in\mathbf{I}^m$. Then taking into account (3.12) we get

$$\sup_{s\in\mathbf{N}_0}(s+1)^{\alpha}\|\sigma_s(f_4)\|_{\psi_1,\tau_1}\leq C_4.$$

Hence, the function $F_4=C_4^{-1}f_4\in\mathbf{B}^{0,\alpha}_{\psi_1,\tau_1,\infty}$.

If $1<\alpha_{\psi_2}, \beta_{\psi_2}<2^{1/2}$ and $2\leq\tau_2<\infty$, then using Lemma 2.4 and in the case $2^{1/2}<\alpha_{\psi}, \beta_{\psi}<2$ and $1<\tau_2\leq 2$, using Lemma 2.6 we get

$$E_{2^n}(F_4)_{\psi_2,\tau_2} >> \left(\sum_{s=n+1}^{2n}\|\sigma_s(f_4)\|^{\gamma}_{\psi_2,\tau_2}\right)^{\frac{1}{\gamma}} >> \frac{\psi_2(1/2^n)}{\psi_1(1/2^n)}(n+1)^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})+\frac{1}{\gamma}},$$

where $\gamma=\max\{\tau_2,2\}$. Hence,

$$E_l(\mathbf{B}^{0,\alpha}_{\psi_1,\tau_1,\theta})_{\psi_2,\tau_2} >> \frac{\psi_2(1/l)}{\psi_1(1/l)}(\log(l+1))^{-(\alpha+\frac{1}{\tau_1}-\frac{1}{\tau_2})+\frac{1}{\gamma}}$$

in the case $\theta=\infty$. □

Theorem 4.2 *Let* $1<\alpha_{\psi}\leq\beta_{\psi}<2$ *and* $1<\tau\leq 2$ *or* $1<\alpha_{\psi}\leq\beta_{\psi}<2^{1/2}$, $2\leq\tau<\infty$, $1\leq\theta\leq\infty$, $\tau_0=\min\{\tau,2\}$. *If* $\alpha>(\frac{1}{\tau_0}-\frac{1}{\theta})_+$, *then*

$$E_M(\mathbf{B}^{0,\alpha}_{\psi,\tau,\theta})_{\psi,\tau}\asymp(\log(M+1))^{-\alpha+(\frac{1}{\tau_0}-\frac{1}{\theta})_+},$$

where $a_+=\max\{a,0\}$.

Proof Let $f\in\mathbf{B}^{0,\alpha}_{\psi,\tau,\theta}$ and a positive integer n such that $2^{n-1}\leq M<2^n$. It follows from Lemma 2.1 and Lemma 2.4 that

$$\|f\|_{\psi,\tau}\leq C\left(\sum_{s=0}^{\infty}\|\sigma_s(f)\|^{\tau_0}_{\psi,\tau}\right)^{1/\tau_0}. \tag{4.12}$$

Now applying this inequality to the function $f-\sum_{s=0}^{n}\sigma_s(f)\in L_{\psi,\tau}(\mathbf{T}^m)$ we will have

$$E_M(f)_{\psi,\tau} \le E_{2^n}(f)_{\psi,\tau} \le \|f - \sum_{s=0}^{n} \sigma_s(f)\|_{\psi,\tau} \le C\Big(\sum_{s=n}^{\infty} \|\sigma_s(f)\|_{\psi,\tau}^{\tau_0}\Big)^{\frac{1}{\tau_0}}. \tag{4.13}$$

If $\theta \le \tau_0$, then applying Jensen's inequality (see [24, Lemma 3.3.3]) from (4.13) we obtain

$$E_M(f)_{\psi,\tau} << \Big(\sum_{s=n}^{\infty} \|\sigma_s(f)\|_{\psi,\tau}^{\theta}\Big)^{\frac{1}{\theta}} << (n+1)^{-\alpha} \asymp (\log(M+1))^{-\alpha}$$

for any function $f \in \mathbf{B}_{\psi,\tau,\theta}^{0,\alpha}$ in the case $\theta \le \tau_0$. Hence,

$$E_M(\mathbf{B}_{\psi,\tau,\theta}^{0,\alpha})_{\psi,\tau} << (\log(M+1))^{-\alpha},$$

in the case $\theta \le \tau_0$.

Let $\tau_0 < \theta$. Then applying the Hölder's inequality $(\beta = \frac{\theta}{\tau_0} > 1, \frac{1}{\beta} + \frac{1}{\beta'} = 1)$ and taking into account the inequality $\alpha > \frac{1}{\tau_0} - \frac{1}{\theta}$ from (4.13) we have

$$E_M(f)_{\psi,\tau} << \Big(\sum_{s=n}^{\infty} (s+1)^{\alpha\theta} \|\sigma_s(f)\|_{\psi,\tau}^{\theta}\Big)^{\frac{1}{\theta}} \Big(\sum_{s=n}^{\infty} (s+1)^{-\alpha\tau_0\beta'}\Big)^{\frac{1}{\tau_0\beta'}}$$
$$<< (n+1)^{-\alpha+\frac{1}{\tau_0}-\frac{1}{\theta}} \asymp (\log(M+1))^{-\alpha+\frac{1}{\tau_0}-\frac{1}{\theta}}.$$

Therefore,

$$E_M(\mathbf{B}_{\psi,\tau,\theta}^{0,\alpha})_{\psi,\tau} << (\log(M+1))^{-\alpha+\frac{1}{\tau_0}-\frac{1}{\theta}},$$

in the case $\tau_0 < \theta$.

If $\theta = \infty$, then taking into account the inequality $\alpha > \frac{1}{\tau_0}$ from (4.13) we have

$$E_M(f)_{\psi,\tau} << \Big(\sum_{s=n}^{\infty} (s+1)^{-}\alpha\tau_0\Big)^{\frac{1}{\tau_0}} \sup_{s\in\mathbf{N}_0} (s+1)^{\alpha} \|\sigma_s(f)\|_{\psi,\tau}$$
$$<< (n+1)^{-\alpha+\frac{1}{\tau_0}} \sup_{s\in\mathbf{N}_0} (s+1)^{\alpha} \|\sigma_s(f)\|_{\psi,\tau} << (\log(M+1))^{-\alpha+\frac{1}{\tau_0}}.$$

This proves the upper bound.

Let us prove the lower bounds. Let $\tau_0 < \theta$. We will consider the function

$$f_5(2\pi\bar{x}) = (n+1)^{-\frac{1}{\theta}} \sum_{s=n+1}^{2n} (s+1)^{-\alpha} \frac{2^{-sm}}{\psi(2^{-sm})} \sum_{\bar{k}\in\Delta_{2^s}\setminus\Delta_{2^{s-1}}} e^{i\langle\bar{k},2\pi\bar{x}\rangle},$$

for $\bar{x} \in \mathbf{I}^m$, $n \in \mathbf{N}_0$.

By the estimate of the norm of the Dirichlet kernel in the generalized Lorentz space (3.2), we have

$$\left\{\sum_{s=0}^{\infty}(s+1)^{\alpha\theta}||\sigma_s(f_5)||_{\psi,\tau}^{\theta}\right\}^{\frac{1}{\theta}}=\left\{\sum_{s=n+1}^{2n}(s+1)^{\alpha\theta}||\sigma_s(f_5)||_{\psi,\tau}^{\theta}\right\}^{\frac{1}{\theta}}$$

$$<<(n+1)^{-\frac{1}{\theta}}\left\{\sum_{s=n+1}^{2n}1\right\}^{\frac{1}{\theta}}\leq C_5.$$

Thus, the function $F_5=C_5^{-1}f_5\in\mathbf{B}_{p,\tau,\theta}^{0,\alpha}$ for $1<p,\tau<\infty$, $1\leq\theta<\infty$.

Let $1<\alpha_\psi\leq\beta_\psi<2$, $1<\tau\leq 2$, i.e. $\tau_0=\tau$. We select the number $q>(\log_2\alpha_\psi)^{-1}$ i.e. $2^{1/q}<\alpha_\psi$. Then, using Theorem 2.3 and the method of proving Lemma 2.6, we can prove that

$$\left\{\sum_{s=0}^{\infty}(2^{\frac{sm}{q}}\psi(2^{-nm}))^{\tau}\|\sigma_s(f)\|_q^{\tau}\right\}^{\frac{1}{\tau}}\leq C\|f\|_{\psi,\tau},\tag{4.14}$$

for $f\in L_{\psi,\tau}(\mathbf{T}^m)$, $1<\tau<\infty$.

Now we apply this inequality to the function $F_5=C_5^{-1}f_5\in\mathbf{B}_{\psi,\tau,\theta}^{0,\alpha}$. Then, given the estimate of the norm of the Dirichlet kernel (see relation (3.2)), we obtain

$$E_{2^n}(F_5)_{\psi,\tau}=C_5^{-1}\|f_5\|_{\psi,\tau}>>\left\{\sum_{s=n+1}^{2n}(2^{\frac{sm}{q}}\psi(2^{-sm}))^{\tau}\|\sigma_s(f_5)\|_q^{\tau}\right\}^{\frac{1}{\tau}}$$

$$>>(n+1)^{-\frac{1}{\theta}}\left\{\sum_{s=n+1}^{2n}(s+1)^{-\alpha\tau}\right\}^{\frac{1}{\tau}}\geq C(n+1)^{-\alpha+\frac{1}{\tau}-\frac{1}{\theta}}.$$

Thus, $E_{2^n}(F_5)_{\psi,\tau}>>(n+1)^{-\alpha+\frac{1}{\tau}-\frac{1}{\theta}}$ for $1<\alpha_\psi\leq\beta_\psi<2$, $1<\tau<\infty$. Hence,

$$E_M(\mathbf{B}_{\psi,\tau,\theta}^{0,\alpha})_{\psi,\tau}\geq E_M(F_5)_{\psi,\tau}\geq E_{2^n}(F_5)_{\psi,\tau}>>(n+1)^{-\alpha+\frac{1}{\tau}-\frac{1}{\theta}}$$

for $1<\alpha_\psi\leq\beta_\psi<2$, $1<\tau<\infty$. This inequality shows the exactness of the estimate in Theorem 4.2 for $1<\tau\leq 2$, $\tau_0=\min\{\tau,2\}<\theta$, $1<\alpha_\psi\leq\beta_\psi<2$.

If $\theta=\infty$, then we will consider the function

$$f_6(2\pi\bar{x})=\sum_{s=1}^{\infty}(s+1)^{-\alpha}\frac{2^{-sm}}{\psi(2^{-sm})}\sum_{\bar{k}\in\Delta_{2^s}\setminus\Delta_{2^{s-1}}}e^{i\langle\bar{k},2\pi\bar{x}\rangle},\ \bar{x}\in\mathbf{I}^m$$

for $1<\alpha_\psi\leq\beta_\psi<2$, $1<\tau\leq 2$.

Using relation (3.2) and inequality (4.12), it is easy to verify that $f_6\in L_{\psi,\tau}(\mathbf{T}^m)$ and

$$\sup_{s\in\mathbf{N}_0}(s+1)^{\alpha}||\sigma_s(f_6)||_{\psi,\tau}\leq C_6.$$

Hence, the function $F_6=C_6^{-1}f_6\in\mathbf{B}_{p,\tau,\infty}^{0,\alpha}$.

Further, using inequality (4.14), we can verify that

$$E_{2^n}(F_6)_{\psi,\tau} = C_6^{-1}\|f_6\|_{\psi,\tau} >\, > \left\{\sum_{s=n+1}^{\infty}(2^{\frac{sm}{q}}\psi(2^{-sm}))^{\tau}\|\sigma_s(f_6)\|_q^{\tau}\right\}^{\frac{1}{\tau}}$$
$$>\, > (n+1)^{-\frac{1}{\theta}}\left\{\sum_{s=n+1}^{2n}(s+1)^{-\alpha\tau}\right\}^{\frac{1}{\tau}} >\, > (n+1)^{-\alpha+\frac{1}{\tau}-\frac{1}{\theta}}.$$

Hence,

$$E_M(\mathbf{B}^{0,\alpha}_{\psi,\tau,\theta})_{\psi,\tau} \geq E_M(F_6)_{\psi,\tau} \geq E_{2^n}(F_6)_{\psi,\tau} >\, > (\log M)^{-\alpha+\frac{1}{\tau}}$$

for $1 < \alpha_\psi \leq \beta_\psi < 2$, $1 < \tau \leq 2$.

Now we prove lower bounds for $\beta_\psi < 2^{1/2}$, $2 \leq \tau < \infty$ and $2 = \tau_0 < \theta$. We will consider the function

$$f_7(2\pi\bar{x}) = (n+1)^{-\frac{1}{\theta}}\sum_{s=n+1}^{2n}(s+1)^{-\alpha}2^{-\frac{sm}{2}}\prod_{j=1}^{m}R_s(x_j),$$

where $R_s(x_j) = \sum_{k=2^{s-1}}^{2^s-1}\varepsilon_k e^{ik2\pi x_j}$ is the Rudin–Shapiro polynomial and $\varepsilon_k = \pm 1$. It is known that $\|R_s\|_\infty < < 2^{s/2}$ (see [19, p. 146]). Therefore,

$$||\sigma_s(f_7)||_{\psi,\tau} = (n+1)^{-\frac{1}{\theta}}(s+1)^{-\alpha}2^{-\frac{sm}{2}}||\prod_{j=1}^{m}R_s(x_j)||_{\psi,\tau}$$
$$\leq (n+1)^{-\frac{1}{\theta}}(s+1)^{-\alpha}2^{-\frac{sm}{2}}\prod_{j=1}^{m}\|R_s(x_j)\|_\infty \leq (n+1)^{-\frac{1}{\theta}}(s+1)^{-\alpha}.$$

Hence,

$$\left\{\sum_{s=0}^{\infty}(s+1)^{\alpha\theta}||\sigma_s(f_7)||_{\psi,\tau}^{\theta}\right\}^{\frac{1}{\theta}} = \left\{\sum_{s=n+1}^{2n}(s+1)^{\alpha\theta}||\sigma_s(f_7)||_{\psi,\tau}^{\theta}\right\}^{\frac{1}{\theta}} \leq C_7$$

i.e. the function $F_7 = C_7^{-1}f_7 \in \mathbf{B}^{0,\alpha}_{\psi,\tau,\theta}$. Since $\beta_\psi < 2^{1/2}$, $2 \leq \tau < \infty$, then $L_{\psi,\tau}(\mathbf{T}^m) \subset L_2(\mathbf{T}^m)$ and $\|f\|_2 \leq C\|f\|_{\psi,\tau}$, for $f \in L_{\psi,\tau}(\mathbf{T}^m)$. Therefore, according to Parseval's equality, we obtain

$$E_{2^n}(F_7)_{\psi,\tau} = C_7^{-1}\|f_7\|_{\psi,\tau} >\, > \|f_7\|_2$$
$$>\, > (n+1)^{-\frac{1}{\theta}}\left\{\sum_{s=n+1}^{2n}(s+1)^{-2\alpha}\right\}^{\frac{1}{2}} \geq C(n+1)^{-\alpha+\frac{1}{2}-\frac{1}{\theta}} \tag{4.15}$$

for $\beta_\psi < 2^{1/2}$. Hence,

$$E_M(\mathbf{B}^{0,\alpha}_{\psi,\tau,\theta})_{\psi,\tau} \ge E_M(F_7)_{\psi,\tau} \ge E_{2^n}(F_7)_{\psi,\tau} \ge C(n+1)^{-\alpha+\frac{1}{2}-\frac{1}{\theta}}$$

in the case $\beta_\psi < 2^{1/2}$, $2 \le \tau < \infty$ and $2 = \tau_0 < \theta$.

Now we prove the lower bound for $\theta \le \tau_0$. We will consider the function

$$f_8(2\pi\bar{x}) = (n+1)^{-\alpha} \frac{2^{-nm}}{\psi(2^{-nm})} \sum_{\bar{k}\in\Delta_{2^{n+1}}\setminus\Delta_{2^n}} e^{i\langle\bar{k},2\pi\bar{x}\rangle}, \ \bar{x} \in \mathbf{I}^m, \ n \in \mathbf{N}_0.$$

Then, taking into account relation (3.2), we have

$$\left\{\sum_{s=0}^{\infty}(s+1)^{\alpha\theta}\|\sigma_s(f_8)\|^{\theta}_{\psi,\tau}\right\}^{\frac{1}{\theta}} = \frac{2^{-nm}}{\psi(2^{-nm})}\left\|\sum_{\bar{k}\in\Delta_{2^{n+1}}\setminus\Delta_{2^n}} e^{i\langle\bar{k},2\pi\bar{x}\rangle}\right\|_{\psi,\tau} \le C_8.$$

Therefore, the function $F_8 = C_8^{-1}f_8 \in \mathbf{B}^{0,\alpha}_{p,\tau,\theta}$. Now using relation (3.2) will have

$$E_{2^n}(F_8)_{\psi,\tau} = \|C_8^{-1}f_8\|_{\psi,\tau} >> (n+1)^{-\alpha}.$$

Hence,

$$E_M(\mathbf{B}^{0,\alpha}_{\psi,\tau,\theta})_{\psi,\tau} \ge E_M(F_8)_{\psi,\tau} \ge E_{2^n}(F_8)_{\psi,\tau} >> (n+1)^{-\alpha}$$

in the case $\theta \le \tau_0$, for $1 < \alpha_\psi \le \beta_\psi < 2$ $1 < \tau < \infty$.

If $\theta = \infty$, then we will consider the function

$$f_9(2\pi\bar{x}) = \sum_{s=1}^{\infty}(s+1)^{-\alpha}\frac{2^{-sm}}{\psi(2^{-sm})}\sum_{\bar{k}\in\Delta_{2^s}\setminus\Delta_{2^{s-1}}} e^{i\langle\bar{k},2\pi\bar{x}\rangle},$$

for $\bar{x} \in \mathbf{I}^m$.

By the property of the Rudin-Shapiro polynomial, we obtain $\sup_{s\in\mathbf{N}_0}(s+1)^{\alpha}\|\sigma_s(f_9)\|_{\psi,\tau} \le C_9$. Hence, the function $F_9 = C_9^{-1}f_9 \in \mathbf{B}^{0,\alpha}_{\psi,\tau,\infty}$. Further, as in the proof of (4.15), one can verify that

$$\begin{aligned} E_{2^n}(F_9)_{\psi,\tau} &= C_9^{-1}\|f_9\|_{\psi,\tau} >> \|f_9\|_2 \\ &>> \left\{\sum_{s=n+1}^{2n}(s+1)^{-2\alpha}\right\}^{\frac{1}{2}} >> (n+1)^{-\alpha+\frac{1}{2}}. \end{aligned}$$

Hence,

$$E_M(\mathbf{B}^{0,\alpha}_{\psi,\tau,\infty})_{\psi,\tau} >> (n+1)^{-\alpha+\frac{1}{2}} >> (\log M)^{-\alpha+\frac{1}{2}}$$

for $\beta_\psi < 2^{1/2}$, $2 \le \tau < \infty$. □

Remark 4.1 In the case $\psi(t) = t^{1/p}$ and $1 < \tau = p < \infty$, $1 \le \theta \le \min\{2, p\}$ from Theorem 4.2 we obtain [30, Theorem, item (i)]. For the function $\psi(t) = t^{1/p}$ and $1 < \tau, p < \infty$, $1 \le \theta \le \infty$ Theorem 4.2 is proved in [4, Theorem 2.1].

In the case $\psi_1(t) = \psi_2(t) = t^{1/p}$, $t \in [0,1]$, $1 < p < \infty$ Theorem 4.1 was proved in [4, Theorem 3.4].

Remark 4.2 In the case, $1 < \alpha_{\psi_2} \leq \beta_{\psi_2} < \alpha_{\psi_1} \leq \beta_{\psi_1} < 2$ and $1 < \tau_1, \tau_2 < \infty$ using Theorem 2.4, we can study the estimate of the quantity $E_M(\mathbf{B}^r_{\psi_1,\tau_1,\theta})_{\psi_2,\tau_2}$. In particular, for $\psi_1(t) = t^{1/p}$, $\psi_2(t) = t^{1/q}$ and $\tau_1 = p$, $\tau_2 = q$, $1 < p, q < \infty$ this problem was investigated by A. S. Romanyuk [25, Theorem 1].

Remark 4.3 We note that the results of this paper can be applied to estimate the best $M-$ term approximations, trigonometric widths, linear widths of classes $\mathbf{B}^r_{\psi_1,\tau_1,\theta}$ and $\mathbf{B}^{0,\alpha}_{\psi_1,\tau_1,\theta}$ in the Lorentz space $L_{\psi_2,\tau_2}(\mathbf{T}^m)$ (see special cases for example in [3]).

Remark 4.4 Let $1 < \tau_2 < \tau_1 < \infty$, $1 < \alpha_{\psi_1} = \beta_{\psi_2} < 2$ and

$$\int_0^1 \left(\frac{\psi_2(t)}{\psi_1(t)}\right)^{\frac{\tau_1\tau_2}{\tau_1-\tau_2}} \frac{dt}{t} < \infty,$$

then $L_{\psi_1,\tau_1}(\mathbf{T}^m) \subset L_{\psi_2,\tau_2}(\mathbf{T}^m)$ (see [29, p. 916]). Therefore, we can consider an analogue of Theorem 2.2, Theorem 3.2, Theorem 4.1, and Theorem 4.2 in this case.

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