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ORIGINAL PAPER

The local spectrum, property (β) and decomposability of operator matrices

Weihong Zhang[1] · Guojun Hai[1] · Alatancang Chen[2]

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Abstract
We mainly deal with the relationship between the local spectrum as well as propery (β) and decomposability of certain 2×2 operator matrices with that of its individual entries. As application of these results, we obtain several relevant conclusions of Hamilton type operator.

Keywords Local spectrum · Local spectral property · Operator matrices

Mathematics Subject Classification 47A11 · 47B99

1 Introduction

The study of operator matrices arises naturally from the following fact: if T is a bounded linear operator on a Hilbert space and M is an invariant subspace for T, then T has a 2×2 oparetor matrix representation of the form

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} : M \oplus M^{\perp} \longrightarrow M \oplus M^{\perp},$$

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✉ Weihong Zhang
zhangwh0202@163.com

Guojun Hai
3695946@163.com

Alatancang Chen
alatanca@imu.edu.cn

1 School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China

2 Inner Mongolia Normal University, Hohhot 010022, China

and one way to study operators is to see them as entries of simpler operators (see [10]). This is a working theory which is based on the problem that studied operator matrices.

Throughout this paper, let X be a Hilbert space and let $\mathcal{B}(X)$ denote the set of all bounded linear operators from X to X. For $T \in \mathcal{B}(X)$, the local resolvent set $\rho_T(x)$ of T at the point $x \in X$ is defined as the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f : U \to X$ which satisfies $(T - \lambda I)f(\lambda) = x$ for all $\lambda \in U$. The local spectrum $\sigma_T(x)$ of T at x is then defined as $\sigma_T(x) := \mathbb{C} \backslash \rho_T(x)$. Evidently, $\rho_T(x)$ is open, and $\sigma_T(x)$ is closed. As already mentioned, the resolvent set $\rho(T)$ is always a subset of $\rho_T(x)$, so $\sigma_T(x) \subseteq \sigma(T)$. For each subset F of $\mathbb{C}$, we define the local spectral subspace of T, $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$. Evidently, $X_T(F)$ is a hyperinvariant subspace of T, but not always closed. An operator $T \in \mathcal{B}(X)$ has Dunford's property (C), if the local spectral subspace $X_T(F)$ is closed for every closed set $F \subseteq C$.

An operator $T \in \mathcal{B}(X)$ is decomposable if every open cover $\mathbb{C} = \{U, V\}$ of the complex plane $\mathbb{C}$ by two open sets U and V effects a splitting of the spectrum $\sigma(T)$ and of the space X, in the sense that there exist T-invariant closed linear subspaces M and N of X for which $X = M + N, \sigma(T|_M) \subseteq U$ and $\sigma(T|_N) \subseteq V$. An operator $T \in \mathcal{B}(X)$ has property (δ) if $X = \mathcal{X}_T(\bar{U}) + \mathcal{X}_T(\bar{V})$ for every open cover $\{U, V\}$ of $\mathbb{C}$, where $\mathcal{X}_T(\bar{U})$ consist of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \backslash \bar{U} \to X$ that satisfies $(T - \lambda I)f(\lambda) = x$ for all $\lambda \in \mathbb{C} \backslash \bar{U}$. An operator $T \in \mathcal{B}(X)$ has Bishop's property (β), if, for every open subset $U \subseteq \mathbb{C}$ and every sequence of analytic functions $f_n : U \to X$ with the property that $(T - \lambda I)f_n(\lambda) \to 0$ as $n \to \infty$, uniformly on all compact subsets of U, it follows that $f_n(\lambda) \to 0$ as $n \to \infty$, again locally uniformly on U.

Remark 1 The following table summarizes the implications between the above various conditions from local spectral theory. For related results, we refer to [1, 2, 9, 13, 14]

$$\begin{array}{ccccc} property(\beta) & \Leftarrow & decomposable & \Rightarrow & property(\delta) \\ \Downarrow & & & & \Downarrow \\ property(C) & & & & T^* : property(\beta). \end{array}$$

Recently, many authors have studied the local spectral properties of different operators, see [4, 8, 13]. In [3], Bračič and Müller investigated the local spectrum of an operator at a fixed vector. In [11], the author have studied local spectrum of a family of operators. In this paper, we find several kind of operator matrices and discuss the local spectrum and local spectral properties of these operator matrices. In addition, some local spectral properties are studied with certain entry operators are Hamilton type operator, including property (C), property (δ), property (β) and decomposability.

2 The local spectrum, property (β) and decomposability of operator matrices

First, we study local spectrum, property (β) and decomposability of skew diagonal operator matrices.

Theorem 1 *Let* $T = \begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix} \in \mathcal{B}(X \oplus X)$.

(i) $\sigma_T(x \oplus y) \subseteq \{\pm\lambda \in \mathbb{C} : \lambda^2 \in \sigma_{T_2T_3}(x) \cup \sigma_{T_3T_2}(y)\}$ *for any* $x, y \in X$.
(ii) *If* $T_2T_3 = T_3T_2$, T_2*has property* (β)*and* T_3 *is algebraic, then* T *has property* (β).

Proof (i) Let $\lambda_0^2 \in \rho_{T_2T_3}(x) \cap \rho_{T_3T_2}(y)$. Then there exist open subsets U and V of $\mathbb{C}$ and analytic functions $f_1 : U \to X$ and $f_2 : V \to X$ such that $\lambda_0^2 \in U \cap V$ and

$$(T_2T_3 - sI)f_1(s) = x, \forall s \in U; \quad (T_3T_2 - tI)f_2(t) = y, \forall t \in V.$$

Put

$$W = \{\lambda \in \mathbb{C} : \lambda^2 \in U \cap V\}.$$

It is clear that W is an open subset containing λ_0. Since

$$(T - \lambda I)(T + \lambda I) = (T + \lambda I)(T - \lambda I) = \begin{bmatrix} T_2T_3 - \lambda^2 I & 0 \\ 0 & T_3T_2 - \lambda^2 I \end{bmatrix},$$

so if we choose

$$f(\lambda) = (T + \lambda I)(f_1(\lambda^2) \oplus f_2(\lambda^2), \lambda \in W,$$

then $f : W \to X \oplus X$ is analytic and

$$(T - \lambda I)f(\lambda) = ((T_2T_3 - \lambda^2 I)f_1(\lambda^2)) \oplus ((T_3T_2 - \lambda^2 I)f_2(\lambda^2)) = x \oplus y$$

for every $\lambda \in W$, which means that $\lambda_0 \in \rho_T(x \oplus y)$. Similarly, $-\lambda_0 \in \rho_T(x \oplus y)$. This completes the proof that $\pm\lambda_0 \in \rho_T(x \oplus y)$.

(ii) It suffices to show that T_2T_3 has property (β), since $T^2 = T_2T_3 \oplus T_2T_3$. Because T_3 is algebraic, then there exist $\mu_1, \mu_2, \mu_3, \cdots, \mu_k$ on $\mathbb{C}$ such that

$$(T_3 - \mu_1 I)(T_3 - \mu_2 I)(T_3 - \mu_3 I) \cdots (T_3 - \mu_k I) = 0,$$

where k is a fixed positive integer. Let $p_0(\lambda) = \lambda$, $p_i(\lambda) = (\lambda - \mu_1)(\lambda - u_2)(\lambda - \mu_3) \cdots (\lambda - \mu_i)$ for $i = 1, 2, 3, \cdots k$. Assume that, for every open set $U \subseteq \mathbb{C}$ and every sequence of analytic functions $f_n : U \to X$ we have

$$\lim_{n \to \infty} \|(T_2T_3 - \lambda I)f_n(\lambda)\| = 0$$

uniformly on every compact sets in U. Then $\lim_{n\to\infty} \|p_i(T_3)f_n(\lambda)\| = 0$ uniformly on every compact sets in U for $i = 0, 1, 2, \ldots, k$. Indeed, we can use mathematical induction on i. Since $p_k(T_3) = 0$, then it is true for $i = k$. Assume that there exist i with $0 < i \leq k$ such that $\lim_{n\to\infty} \|(p_i(T_3)f_n(\lambda)\| = 0$ uniformly on every compact sets in U. Since $T_2T_3 = T_3T_2$, we have

$$\lim_{n\to\infty} \|(T_2T_3 - \lambda I)f_n(\lambda)\| = \lim_{n\to\infty} \|(T_3 - \mu_i I)T_2 f_n(\lambda) + \mu_i T_2 f_n(\lambda) - \lambda f_n(\lambda)\| = 0$$

uniformly on every compact sets in U. Therefore, it follows that

$$\lim_{n\to\infty} \|(\mu_i T_2 - \lambda I)p_{i-1}(T_3)f_n(\lambda)\| = 0$$

uniformly on every compact sets in U by induction hypothesis. Hence $\lim_{n\to\infty} \|p_{i-1}(T_3)f_n(\lambda)\| = 0$ uniformly on every compact sets in U since T_2 has property (β), that is to say, $\lim_{n\to\infty} \|p_i(T_3)f_n(\lambda)\| = 0$ uniformly on every compact sets in U for $i = 0, 1, 2, \ldots, k$. So T_2T_3 has property (β). □

Now, we consider upper triangular operator matrices. In particular, we first consider the case of main diagonal operator matrices.

Theorem 2 *Let* $T = \begin{bmatrix} T_1 & 0 \\ 0 & T_4 \end{bmatrix} \in \mathcal{B}(X \oplus X)$.

(i) $\sigma_T(x \oplus y) = \sigma_{T_1}(x) \cup \sigma_{T_4}(y)$ *for any* $x, y \in X$.

(ii) $(X \oplus X)_T(F) = \{x \oplus y : x \in X_{T_1}(F), y \in X_{T_4}(F)\}$ *for any subset* F *of* $\mathbb{C}$.

(iii) *If* T_1 *has the property that, property* (β) *implies decomposability, then*

  (a) T *is decomposable implies* T_1 *is decomposable.*

  (b) *If* T_4 *is decomposable, then* T *is decomposable if and only if* T_1 *is decomposable.*

Proof (i) It is easy to see that $\sigma_T(x \oplus y) \subseteq \sigma_{T_1}(x) \cup \sigma_{T_4}(y)$ and so we just have to prove $\sigma_{T_1}(x) \cup \sigma_{T_4}(y) \subseteq \sigma_T(x \oplus y)$. Let $\lambda_0 \in \rho_T(x \oplus y)$. Then there exist open subset U and analytic function $f = f_1 \oplus f_2 : U \to X \oplus X$ such that $(T - \lambda I)f(\lambda) = x \oplus y$ for all $\lambda \in U$, it follows that

$$\begin{cases} (T_1 - \lambda I)f_1(\lambda) = x, \\ (T_4 - \lambda I)f_2(\lambda) = y \end{cases}$$

for all $\lambda \in U$. Therefore, $\lambda_0 \in \rho_{T_1}(x) \cap \rho_{T_4}(y)$. Hence, $\sigma_{T_1}(x) \cup \sigma_{T_4}(y) = \sigma_T(x \oplus y)$.

(ii) For any $x \oplus y \in (X \oplus X)_T(F)$, we have $\sigma_T(x \oplus y) \subseteq F$, it follows that $\sigma_{T_1}(x) \cup \sigma_{T_4}(y) \subseteq F$ by statement (i). Then we get that $x \in X_{T_1}(F)$ and $y \in X_{T_4}(F)$ for any $x \oplus y \in (X \oplus X)_T(F)$. Conversely, we can obtain $\{x \oplus y : x \in X_{T_1}(F), y \in X_{T_4}(F)\} \subseteq (X \oplus X)_T(F)$ by similar method.

(iii) (a) Assume that, for every open set U in $\mathbb{C}$ and every sequence of analytic functions $f_n : U \to X$ such that

$$\lim_{n\to\infty} \|(T_1 - \lambda I)f_n(\lambda)\| = 0$$

uniformly on every compact sets in U. Then

$$\lim_{n\to\infty}\|(T-\lambda I)(f_n(\lambda)\oplus 0)\| = \lim_{n\to\infty}\|(T_1-\lambda I)f_n(\lambda)\| = 0$$

uniformly on every compact sets in U. Since T is decomposable, then

$$\lim_{n\to\infty}\|f_n(\lambda)\| = \lim_{n\to\infty}\|(f_n(\lambda)\oplus 0)\| = 0$$

uniformly on every compact sets in U, it follows that T_1 has property (β). Therefore, T_1 is decomposable by the assumption of T_1.

(b) Assume that T_1 and T_4 are decomposable, then they both have property (β). Let g_n be analytic functions defined on an open set D in $\mathbb{C}$ such that

$$\lim_{n\to\infty}\|(T-\lambda I)g_n(\lambda)\| = 0$$

uniformly on every compact sets in D. Let $g_n = g_{n,1}\oplus g_{n,2}$, then we get that

$$\begin{cases} \lim_{n\to\infty}\|(T_1-\lambda I)g_{n,1}(\lambda)\| = 0, \\ \lim_{n\to\infty}\|(T_4-\lambda I)g_{n,2}(\lambda)\| = 0 \end{cases}$$

uniformly on every compact sets in D. On the other hand, since T_1 and T_4 have property (β), then

$$\lim_{n\to\infty}\|g_{n,1}(\lambda)\| = \lim_{n\to\infty}\|g_{n,2}(\lambda)\| = 0$$

uniformly on every compact sets in D. Therefore, T has property (β). Hence, it follows from Remark 1 that T has property (C).

To show that T is decomposable, it remains to prove that the identity $X\oplus X = (X\oplus X)_T(\bar{U}_1) + (X\oplus X)_T(\bar{U}_2)$ holds for every open cover $\{U_1, U_2\}$ of $\mathbb{C}$ by [9, Theorem 1.2.23]. Since T_1 and T_4 are decomposable, then we have, again by [9, Theorem 1.2.23], the identities

$$X = X_{T_1}(\bar{U}_1) + X_{T_1}(\bar{U}_2)$$

and

$$X = X_{T_4}(\bar{U}_1) + X_{T_4}(\bar{U}_2)$$

hold for every open cover $\{U_1, U_2\}$ of $\mathbb{C}$. Let $x\oplus y\in X\oplus X$, then there are $x_i \in X_{T_1}(\bar{U}_i)$ and $y_i \in X_{T_4}(\bar{U}_i)$ for which $x = x_1 + x_2$ and $y = y_1 + y_2$ for $i = 1, 2$. Therefore, $x\oplus y = (x_1\oplus y_1) + (x_2\oplus y_2) \in X_T(\bar{U}_1) + X_T(\bar{U}_2)$ by (ii). Thus T is decomposable. □

Corollary 1 *Let* $T = \begin{bmatrix} T_1 & 0 \\ 0 & T_4 \end{bmatrix} \in \mathcal{B}(X\oplus X)$, T_1 *has the property that, property* (β) *implies decomposability, and* T_4 *is decomposable. Then the following assertions hold*:

(i) T_1 *has property* (β) *if and only if* T *has property* (β).

(ii) *T_1 has property (δ) if and only if T has property (δ).*
(iii) *T_1 has property (C) if and only if T has property (C).*

Proof We can derive these results from Theorem 1.2.29 of [9] and Remark 1 and Theorem 2. □

Theorem 3 *Let* $T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_4 \end{bmatrix} \in \mathcal{B}(X \oplus X)$.

(i) *If $T_2^2 = 0$ and $T_1T_2 = T_2T_1$, then*

  (a) $\sigma_{T_1}(T_2x) \cup \sigma_{T_4}(y) \subseteq \sigma_T(x \oplus y)$ *for any* $x, y \in X$.
  (b) $(T_2 \oplus I)((X \oplus X)_T(F)) \subseteq ((X \oplus X)_{T_1 \oplus T_4}(F))$ *for any subset F of* $\mathbb{C}$.

(ii) *If T_1 and T_4 have property (β), then T has property (β).*
(iii) *T is decomposable and T_1 has the property that, property (β) implies decomposability, then T_1 is decomposable.*

Proof (i) (a) Assume that $\lambda_0 \in \rho_T(x \oplus y)$, then there exist a neighborhood U of λ_0 and an analytic function $f = f_1 \oplus f_2 : U \to X \oplus X$ such that

$$(T - \lambda I)f(\lambda) = x \oplus y$$

on U. Thus we have

$$\begin{cases} (T_1 - \lambda I)f_1(\lambda) + T_2f_2(\lambda) = x, \\ (T_4 - \lambda I)f_2(\lambda) = y. \end{cases}$$

Then it follows that $\lambda \in \rho_{T_4}(y)$. On the other hand, since $T_2^2 = 0$ and $T_1T_2 = T_2T_1$, it follows that $\lambda \in \rho_{T_1}(T_2x)$. So we complete the proof.

(b) Let $x \oplus y \in (X \oplus X)_T(F)$, then we have $\sigma_T(x \oplus y) \subseteq F$. Since

$$\sigma_{T_1}(T_2x) \cup \sigma_{T_4}(y) = \sigma_{T_1 \oplus T_4}(T_2x \oplus y)$$

by Theorem 2, we get that $\sigma_{T_1 \oplus T_4}(T_2x \oplus y) \subseteq F$ from (a). Hence, $(T_2 \oplus I)(x \oplus y) \in X_{T_1 \oplus T_4}(F)$. Therefore, it is easy to see $(T_2 \oplus I)((X \oplus X)_T(F)) \subseteq (X \oplus X)_{T_1 \oplus T_4}(F)$ for any subset F of $\mathbb{C}$.

(ii) Let $f_n = f_{n,1} \oplus f_{n,2}$ be a sequence of analytic functions defined on open set U in $\mathbb{C}$ such that

$$\lim_{n \to \infty} \|(T - \lambda I)f_n(\lambda)\| = 0$$

uniformly on every compact sets in U. Then we get that

$$\begin{cases} \lim_{n \to \infty} \|(T_1 - \lambda I)f_{n,1}(\lambda) + T_2f_{n,2}(\lambda)\| = 0, \\ \lim_{n \to \infty} \|(T_4 - \lambda I)f_{n,2}(\lambda)\| = 0 \end{cases}$$

uniformly on every compact sets in U. Since T_1 and T_4 have property (β), it follows that $\lim_{n\to\infty}\|f_n(\lambda)\| = 0$ uniformly on every compact sets in U. So T has property (β).

(iii) The proof is similar to the statement (iii) of Theorem 2. □

Finally, we focus on the more general case, the local spectrum, property (β) and decomposability of 2×2 operator matrices.

Theorem 4 *Let* $T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \in \mathcal{B}(X\oplus X)$. *Suppose that* $T_2^2 = 0$ *and* $T_3^2 = 0$, *then we have*

(i) *If* $T_1T_2 = T_2T_1$ *and* $T_3T_4 = T_4T_3$, *then*

(a) $\sigma_{T_1}(T_2x)\cup\sigma_{T_4}(T_3y) \subseteq \sigma_T(x\oplus y)$ *for every* $x, y\in X$.
(b) $(T_2\oplus T_3)((X\oplus X)_T(F)) \subseteq (X\oplus X)_{T_1\oplus T_4}(F)$ *for any subset* F *of* $\mathbb{C}$.

(ii) *If* $T_1T_3 = T_3T_1$ *and* $T_2T_4 = T_4T_2$, *then* T_1 *and* T_4 *have property* (β) *if and only if* T *has property* (β).

Proof (i)(a) Let $\lambda_0 \in \rho_T(x\oplus y)$, then there exist a neighborhood U of λ_0 and an analytic function $f = f_1\oplus f_2 : U\to X\oplus X$ such that $(T-\lambda I)f(\lambda) = x\oplus y$ for any $\lambda\in U$. Thus we have

$$\begin{cases} (T_1-\lambda I)f_1(\lambda) + T_2f_2(\lambda) = x, \\ T_3f_1(\lambda) + (T_4-\lambda I)f_2(\lambda) = y. \end{cases}$$

Since $T_2^2 = 0, T_3^2 = 0$ and $T_1T_2 = T_2T_1, T_3T_4 = T_4T_3$, it follows that

$$(T_1-\lambda I)T_2f_1(\lambda) = T_2x$$

on U and

$$(T_4-\lambda I)T_3f_2(\lambda) = T_3y$$

on U. Therefore $\lambda\in\rho_{T_1}(T_2x)\cap\rho_{T_4}(T_3y)$, thus $\sigma_{T_1}(T_2x)\cup\sigma_{T_4}(T_3y)\subseteq\sigma_T(x\oplus y)$ for every $x, y\in X$.

(b) Let $x\oplus y\in(X\oplus X)_T(F)$, then we get that $\sigma_T(x\oplus y)\subseteq F$. Since

$$\sigma_{T_1}(T_2x)\cup\sigma_{T_4}(T_3y) = \sigma_{T_1\oplus T_4}(T_2x\oplus T_3y)$$

by Theorem 2, we have, from statement (i),

$$\sigma_{T_1\oplus T_4}(T_2x\oplus T_3y)\subseteq\sigma_T(x\oplus y)\subseteq F.$$

Hence, $T_2x\oplus T_3y\in(X\oplus X)_{T_1\oplus T_4}(F)$, thus we have $(T_2\oplus T_3)((X\oplus X)_T(F))\subseteq(X\oplus X)_{T_1\oplus T_4}(F)$ for any subset F of $\mathbb{C}$.

(ii) Suppose that $T_2, T_3\neq 0$ and T has property (β). Let U be an open subset of $\mathbb{C}$, and $f_n : U\to X$ be a sequence of analytic functions such that

$$\lim_{n\to\infty} \|(T_1 - \lambda I)f_n(\lambda)\| = 0$$

uniformly on every compact sets of U. Then

$$\lim_{n\to\infty} \|(T - \lambda I)(T_3 f_n(\lambda) \oplus 0)\| = \lim_{n\to\infty} \|T_3(T_1 - \lambda I)f_n(\lambda) \oplus 0\| = 0$$

uniformly on every compact sets of U. Therefore, $\|T_3 f_n(\lambda)\| \to 0$, Since T is assumed to have the property (β). It follows that

$$\lim_{n\to\infty} \|(T - \lambda I)(f_n(\lambda) \oplus 0)\| = \lim_{n\to\infty} \|((T_1 - \lambda I)f_n(\lambda)) \oplus (T_3 f_n(\lambda))\| = 0$$

uniformly on compact sets in U. As a result, $\|f_(\lambda)\| = \|f_n(\lambda) \oplus 0\| \to 0$ uniformly on compact sets in U. This shows that T_1 has property (β). Similarly, T_4 also has property (β).

Conversely, Assume that T_1 and T_4 have property (β). we obtain T has property (β) follows easily from assumption. □

Remark 2 The different results in statement (i) of Theorems 3 and 4 show that Theorem 3 is not the special case of Theorem 4 under the same condition, which should be noticed in the comparison. In fact, it can be attributed to the difference of the proofs. We can draw a conclusion as follows when $T_3 = 0$ in statement (i) of Theorem 4:

$$\sigma_{T_1}(T_2 x) \cup \sigma_{T_4}(0) \subseteq \sigma_T(x \oplus y)$$

for every $x, y \in X$ and

$$(T_2 \oplus 0)((X \oplus X)_T(F)) \subseteq (X \oplus X)_{T_1 \oplus T_4}(F)$$

for any subset F of $\mathbb{C}$. Obviously, it is stronger than the statement (i) of Theorem 3.

3 Applications to Hamilton type operator

In this section, we consider Hamilton type operator and give applications of the previous results. For this, let us introduce research background of Hamilton operator and some basic notions which will be used later.

The Hamilton system is an important branch in dynamical systems, all real physical processes with negligible dissipations, no matter whether they are classical, quantum, or relativistic, and of finite or infinite degress of freedom, can always be cast in the suitable Hamiltonian form. While infinite dimensional Hamilton operators come from the infinite dimensional Hamilton systems, and have deep mechanical background [5–7]. Let $T : D(T) \subseteq X \times X \to X \times X$ be a densely defined closed operator. If T satisfies $(JT)^* = JT$, where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, then T is called an infinite dimensional Hamilton operator. Now we introduce an operator related to Hamilton operator. An operator $T \in \mathcal{B}(X)$ is called a Hamilton type operator if there exists an unitary operator J on X such that $J^2 = -I$ and $(JT)^* = JT$.

In this case, we say that T is a Hamilton type operator with J. Clearly, infinite dimensional Hamilton operator is Hamilton type operator. In what follows we always assume that J is a linear unitary operator which satisfies $J^2 = -I$.

The following two Lemma are taken from [12].

Lemma 1 *Let $T \in \mathcal{B}(X)$ be a Hamilton type operator with J. Then*

(i) $\sigma_{T^*}(Jx) = -\sigma_T(x)$ *and* $\sigma_T(J^*x) = -\sigma_{T^*}(x)$ *for all* $x \in X$.
(ii) $JX_T(-F) = X_{T^*}(F)$ *for any subset* F *of* $\mathbb{C}$.

Lemma 2 *Let $T \in \mathcal{B}(X)$ be a Hamilton type operator. Then T has property (β) or property (δ) if and only if T is decomposable.*

Proposition 1 *Let* $T = \begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix} \in \mathcal{B}(X \oplus X)$ *where T_2 and T_3 are Hamilton type operator with same J.*

(i) $\sigma_T(x \oplus y) = -\sigma_{T^*}(Jy \oplus Jx)$ *for any* $x, y \in X$.
(ii) $(X \oplus X)_T(F) = \mathcal{J}(X \oplus X)_{T^*}(-F)$ *for any subset F of $\mathbb{C}$, where* $\mathcal{J} = \begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix}$.
(iii) If $T_2T_3 = T_3T_2$, *T_2 has property (β) and T_3 is algebraic, then T is decomposable.*

Proof It is easy to prove that T is Hamilton type operator with $\mathcal{J} = \begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix}$. Therefore, (i) and (ii) hold from Lemma 1.

(iii) To show T is decomposable, it remains to prove that T has property (β) from Lemma 2. In fact, it is immediate consequence of Theorem 1. □

Proposition 2 *Let* $T = \begin{bmatrix} T_1 & T_2 \\ 0 & JT_1^*J \end{bmatrix} \in \mathcal{B}(X \oplus X)$.

(i) *If T_1 is a Hamilton type operator and T is decomposable, then T_1 is decomposable.*
(ii) *If T_2 is a Hamilton type operator with J, then we have*
   (a) $\sigma_T(x \oplus y) = -\sigma_{T^*}(Jy \oplus Jx)$ *for any* $x, y \in X$.
   (b) $(X \oplus X)_T(F) = \mathcal{J}(X \oplus X)_{T^*}(-F)$ *for any subset F of $\mathbb{C}$, where* $\mathcal{J} = \begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix}$.
   (c) *If T_1 is decomposable, then T is decomposable.*

Proof (i) The proof is immediate conclusion of the statement (iii) of Theorem 3.

(ii) In fact, by simple calculation, we can obtain that T is Hamilton type operator with unitary operator $\mathcal{J} = \begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix}$. Therefore, We can derive the results (a) and

(b) from Lemma 1.

(c) Let T_1 be decomposable, then both T_1 and T_1^* have property (β). Moreover, we can deduce that JT_1^*J has property (β). Therefore, T has property (β). Hence, T is decomposable by Lemma 2. □

Remark 3 In the statement (i) of Proposition 2, T_1 is Hamilton type operator with some unitary operators not necessarily J.

Proposition 3 *Let* $T = \begin{bmatrix} T_1 & T_2 \\ -T_2 & T_1 \end{bmatrix} \in \mathcal{B}(X \oplus X)$ *where* T_1 *and* T_2 *are Hamilton type operator with same* J.

(i) $\sigma_T(x \oplus y) = -\sigma_{T^*}(\frac{1}{\sqrt{2}}(Jx + Jy) \oplus \frac{1}{\sqrt{2}}(Jx - Jy))$ *for any* $x, y \in X$.

(ii) $(X \oplus X)_{T_1 \oplus T_1}(F) = (J^* \oplus J^*)((X \oplus X)_{T_1^* \oplus T_1^*}(-F))$ *for any subset* F *of* $\mathbb{C}$.

(iii) *If* $T_2^2 = 0$ *and* $T_1T_2 = T_2T_1$. *Then* T_1 *has property* (β) *if and only if* T *is decomposable.*

Proof It is easy to prove that T is Hamilton type operator with $\mathcal{J} = \frac{1}{\sqrt{2}}\begin{bmatrix} J & J \\ J & -J \end{bmatrix}$. Therefore, we have $\sigma_T(x \oplus y) = -\sigma_{T^*}(\frac{1}{\sqrt{2}}(Jx + Jy) \oplus \frac{1}{\sqrt{2}}(Jx - Jy))$ for any $x, y \in X$ and $(X \oplus X)_{T_1 \oplus T_1}(F) = (J^* \oplus J^*)((X \oplus X)_{T_1^* \oplus T_1^*}(-F))$ for any subset F of $\mathbb{C}$ by Lemma 1. So we complete the proof of (i) and (ii).

(iii) As an immediate corollary of Theorem 4 we have T_1 has property (β) if and only if T has property (β). Since T is Hamilton type operator with $\mathcal{J}$, T is decomposable by Lemma 1. □

Corollary 2 *Let* $T = \begin{bmatrix} T_1 & T_2 \\ -T_2 & T_1 \end{bmatrix} \in \mathcal{B}(X \oplus X)$ *where* T_1 *and* T_2 *are Hamilton type operator with same* J, *and* $T_2^2 = 0$ *and* $T_1T_2 = T_2T_1$. *If* T_1 *has property* (β), *then the following assertions hold*:

(i) T *has property* (β), *property* (δ), *property* (C), *and SVEP.*

(ii) T^* *has property* (β), *property* (δ), *property* (C), *and SVEP.*

(iii) *If* T *is a supercyclic operator, then* $|\lambda| = r(T)$ *for every* $\lambda \in \sigma(T)$, *where* $r(T)$ *is spectral radius of* T.

(iv) *Suppose that* $f : U \to \mathbb{C}$ *is an analytic function on an open neighbourhood* U *of* $\sigma(T)$. *Then* $f(T)$ *is decomposable. Moreover, if* T *has real spectrum on* $X \oplus X$, *then* $exp(iT)$ *is decomposable.*

(v) $(X \oplus X)^*_{T^*}(F) = (X \oplus X)_T(\mathbb{C} \backslash F)^{\perp}$ *for all closed sets* $F \subseteq \mathbb{C}$.

(vi) $(X \oplus X)_T(0) = \{x \oplus y \in X \oplus X : \lim_{n\to\infty} \|T^n(x \oplus y)\|^{\frac{1}{n}} = 0\}$.

Proof (i) It is immediately from Proposition 3 and Remark 1.

(ii) Since T is Hamilton type operator with $\mathcal{J} = \frac{1}{\sqrt{2}}\begin{bmatrix} J & J \\ J & -J \end{bmatrix}$, from (i) and [12] we conclude that T^* has property (β), property (δ), property (C), and SVEP.

The proof of (iii), (iv), (v) are immediately follow from Proposition 3 and page 239,225,151 of [9], respectively.

(vi) Since T has the SVEP, then, by [9, Proposition 3.3.13], we conclude that

$$X_T(0) = \mathcal{X}_T(0) = \{x \oplus y \in X \oplus X : r_T(x \oplus y) = 0\},$$

where $r_T(x)$ is the local spectral radius of T at x. Moreover, we have $r_T(x \oplus y) = \lim_{n\to\infty} \|T^n(x \oplus y)\|^{\frac{1}{n}} = 0$ by [9, Proposition 3.3.17] since T has property (β). Hence, $(X \oplus X)_T(0) = \{x \oplus y \in X \oplus X : \lim_{n\to\infty} \|T^n(x \oplus y)\|^{\frac{1}{n}} = 0\}$, which completes the proof. □

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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