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ORIGINAL PAPER

Rajendra Bhatia and his mathematical achievements

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Abstract

In this paper, we provide a biography of Professor Rajendra Bhatia and discuss some of his influential mathematical works as one of the leading researchers in matrix analysis and linear algebra.

Keywords Rajendra Bhatia · Biography · Mathematical works

Mathematics Subject Classification 01A60 · 01A61 · 15-03 · 46-03 · 47-03

1 Biography

Rajendra Bhatia was born in 1952 in Rohtak in Northwest India. Both his parents were school teachers. He studied at Birla Public School Pilani, at St Stephen's College, University of Delhi, and at the Delhi Centre of the Indian Statistical Institute from where he obtained a Ph. D. with a thesis titled "Estimation of Spectral

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Variation" under the supervision of Professors Kalyan Mukherjea and Kalyanapuram Rangachari Parthasarathy. In 1979, he was awarded a Fulbright Fellowship, with which he came to the University of California, Berkeley. He spent the years 1980–1984 at the Tata Institute and the University of Bombay. After this, he came to the Indian Statistical Institute in Delhi as an Associate Professor and retired from there as a Distinguished Scientist in 2017. He is currently a Professor of Mathematics at Ashoka University, a Professor Emeritus at the Indian Statistical Institute and a Bhatnagar Fellow of CSIR, India.

(Rajendra Bhatia)

He has held visiting positions at various departments across the world, among them the Universities of Toronto and Guelph and the Fields Institute in Canada; Yale University and the University of Texas at Arlington in the US; Hokkaido University in Japan; Universities of Bielefeld, Ljubljana, Pisa, Como, Manchester, Lisbon, ICTP, and the Ecole Polytechnique in Europe; Sungkyunkwan University and Kyungpook National University in Korea; Shanghai University in China.

Rajendra has made essential contributions to matrix analysis: perturbation of eigenvalues and eigenvectors, matrix inequalities, operator functions, norm ideals of operators, connections with Fourier analysis, differential geometry, approximation problems, applications to numerical analysis, computations and mathematical physics; see also [4].

He has served on several editorial boards as

- Associate Editor, Linear Algebra and its Applications, 1988–2005; Senior Editor 2006–2017, Distinguished Editor 2018.
- Member of the Editorial Board, Linear and Multilinear Algebra, 1988–1992.
- Member of the Editorial Board, SIAM Journal on Matrix Analysis and its Applications, 1995–2006.
- Member of the Advisory Board, Indian Journal of Pure and Applied Mathematics, 1992–2011.
- Member of the Editorial Board, Operators and Matrices, 2006– 2017.
- Correspondent, Mathematical Intelligencer, 2007.
- Member of the Editorial Board, Kyungpook Mathematical Journal, 2008–2018.
- Associate Editor, Journal of the Ramanujan Mathematical Society, 2011–2017.

He is the Founder Editor of the book series "Texts and Readings in Mathematics" and "Culture and History of Mathematics". He was the Chief Editor of "Proceedings of the International Congress of Mathematicians", India, 2010.

He is the author of three well-known books on Matrix Analysis: "Perturbation Bounds for Matrix Eigenvalues", "Matrix Analysis", and "Positive Definite Matrices", and of two other books: "Fourier Series" and "Notes on Functional Analysis".

His awards and honours include

- CSIR Bhatnagar Fellow.
- Prasanta Chandra Mahalanobis Medal (INSA), 2017.
- Hans Schneider Prize in Linear Algebra, 2016.
- J. C. Bose National Fellow, 2007–2018.
- Fellow, Third World Academy of Sciences (TWAS).
- Fellow, Indian National Science Academy.
- Fellow, Indian Academy of Sciences.
- Fellow, National Academy of Sciences, India.
- Winegard Visiting Professor, University of Guelph, Canada, 1999.
- Shanti Swarup Bhatnagar Prize for Science and Technology, 1995.
- National Lecturer, University Grants Commission, 1990.
- Indian National Science Academy Medal for Young Scientists, 1982.
- Gold Medal of the Birla Education Trust, Pilani, 1968.

Some major international activities of Rajendra are

- Board of Directors, International Linear Algebra Society, 1995–1998.
- Academic Secretary, Ramanujan Mathematical Society, 2006–2009.
- National Committee for International Mathematical Union, Member 2008–2011, Chair 2012–2015.

His wife, Irpinder, is a writer and producer of TV films and their son, Gautam, is a lawyer and writer.

(Rajendra and Irpinder in South Korea during a conference celebrating his 60th birthday)

2 Influential publications

Several problems, conjectures, and inequalities carry his name. We discuss them in detail. Some of his mathematical works were surveyed in [4]. In this paper, we focus on other aspects of them and discuss some of his contributions attracting several mathematicians. In particular, we have a look at some recent problems having their origin in quantum information theory. See [19, 20, 30].

2.1 Matrix perturbation theory

After completing his thesis in 1979, Rajendra decided to move into a different branch of linear algebra, matrix perturbation theory. The central question is to bound the change that a small perturbation of a matrix can cause to its spectral resolution. Beside its theoretical interest, this has clear importance for numerical analysis. Classical results of Hermann Weyl, Alan Hoffman, Helmut Wielandt, V. B. Lidskii, Alexander Ostrowski, and others mostly dealt with the changes in the eigenvalues, but progress was underway in bounding the change in the eigenvectors as well. The then-recent "$\sin\theta$ theorem" of Davis and Kahan [33] simplified the picture by showing a perturbation bound resulted immediately from a bound on the solution X of a related operator equation $AX - XB = C$. Rajendra used his Fulbright Fellowship to make contact with Davis, proposing they do joint investigation in this area, and this was the start of their productive collaboration for the next three decades. The first fruits were their improvements on the $\sin\theta$ theorem (with essential contributions by Alan McIntosh, Paul Koosis, and K. R. Parthasarathy); see [13, 14]. The old problem of bounding change in the spectrum got attention too; for one, Rajendra and Davis proved that a classical perturbation result for hermitian matrices holds unchanged for unitaries (but not, it turned out later, for all normal matrices) [10].

These results were followed by many extensions, sometimes with other collaborators, treating different classes of operators and different unitarily invariant norms [12]. One of them, the perturbation of matrix functions, which is of great interest in diverse areas, has attracted the attention of Rajendra and his coworkers. These functions include operator monotone functions, power functions, the matrix absolute value, various matrix decompositions such as polar, QR, LR, SR, and tensor powers. Various kinds of perturbation problems related to Rajendra's research works include:

- Given a bounded linear operator A acting on a Hilbert space, seeking estimates for $|||f(A) - f(B)|||$ in terms of $|||A - B|||$ for some types of functions f is a perturbation problem. More generally, there are matrix perturbation problems comparing various norms of the generalized commutators $AX - XB$ and $f(A)X - Xf(B)$.
- A perturbation problem is to find bounds upon the solution X of an operator equation, say $AX - XB = C$, when A and B are normal operators acting on a Hilbert space, and a bound upon C and some geometric information about spectra of A and B are given.
- Let a matrix A have a specific factorization $A = ST$ (say, UP, QR, LR, SR). Suppose that A is subject to a perturbation ΔA, and that $A + \Delta A = (S + \Delta S)(T + \Delta T)$ is the same type of factorization of the perturbed matrix. A matrix factorization perturbation problem is to search upper bounds for $|||\Delta S|||$ and $|||\Delta T|||$, where $|||\cdot|||$ is a given unitarily invariant norm and $|||\Delta A||| \leq \varepsilon|||A|||$ for sufficiently small ε.

For a comprehensive account on Rajendra's works on matrix perturbation theory, we refer to [6, 9], and references therein.

2.2 Bhatia–Šemrl property

Let X be a normed space. A vector $x \in X$ is said to be *Birkhoff–James orthogonal* to $y \in X$ if $\|x + \lambda y\| \geq \|x\|$ for all scalars λ, and we write this as $x \perp_B y$. An operator T in the space $B(X, Y)$ of all bounded linear operators between normed spaces X and Y attains its norm at x_0 if there exists a unit vector $x_0 \in X$ such that $\|Tx_0\| = \|T\|$. Let $M_T := \{x \in X : \|T\| = \|Tx\| \text{and} \|x\| = 1\}$.

Suppose that there exists a vector $x \in M_T$ such that $Tx \perp_B Sx$ in Y. Then $T \perp_B S$ in $B(X, Y)$ since

$$\|T + \lambda S\| \geq \|Tx + \lambda Sx\| \geq \|Tx\| = \|T\|.$$

The converse is of special interest. An operator $T \in B(X, Y)$ is said to have the so-called *Bhatia–Šemrl property* (B–Š property) if for each S with $T \perp_B S$ there exists a vector $x \in M_T$ such that $Tx \perp_B Sx$. This definition goes back to the work of Bhatia and Šemrl [28] in which they proved that every operator in a finite-dimensional Hilbert space has the B–Š property. Li and Schneider [45] showed that there is a finite-dimensional normed space X and a bounded linear operator T on X without the B–Š property. It is proved by Sain et al. [50] that there are sufficiently many operators with the B–Š property in $B(X) := B(X, X)$ when X is a finite-dimensional strictly convex real Banach space. Kim [39] showed that for a Banach space X with the Radon–Nikodým property, the set of norm attaining operators having the B–Š property is dense in $B(X, Y)$. Wójcik [51] extended the above result of Bhatia and Šemrl when X is reflexive and Y is a smooth and strictly convex reflexive space subject to the condition that the set of points where T attains its norm is either connected or a doubleton. Zamani [52] obtained a B–Š type characterization of Birkhoff–James orthogonality for operators in a semi-Hilbertian space, that is a space equipped with a semi inner product induced by a positive linear operator A between Hilbert spaces via $[x, y] := \langle Ax, y\rangle$, see also [49].

2.3 A conjecture of Bhatia, Lim, and Yamazaki

Let $\mathbb{A} = (A_1, \ldots, A_m)$ be an m-tuple of positive definite matrices, let $\omega = (\omega_1, \ldots, \omega_m)$ be an m-tuple of positive weights such that $\sum_{i=1}^m \omega_i = 1$, and let t be a real number in [0, 1]. The power mean of $\mathbb{A}$ with weights ω and parameter t is defined by $Q_t(\omega, \mathbb{A}) := \left(\sum_{i=1}^m \omega_i A_i^t\right)^{1/t}$. Another power mean is defined as the unique solution $P_t(\omega, \mathbb{A})$ of the matrix equation $X = \sum_{i=1}^m \omega_i X \#_t A_i$, where $A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ stands for the operator weighted geometric mean of two positive definite matrices A and B; see [46]. A version of a conjecture of Bhatia et al. [25] states that for any unitarily invariant norm $|||\cdot|||$ and $0 \le t \le 1$, it holds that

$$|||P_t(\omega, \mathbb{A})||| \le |||Q_t(\omega, \mathbb{A})|||. \tag{2.1}$$

For two positive definite matrices A and B, this reads as follows:

$$|||A + B + 2(A\#B)||| \le |||A + B + A^{1/2}B^{1/2} + B^{1/2}A^{1/2}|||.$$

Inequality (2.1) is known to be valid for the p-Schatten norms $\|\cdot\|_p$ when $p = 1, 2$, and $p = \infty$, see [25]. Dinh et al. [35] proved that inequality (2.1) holds for all $p > 1$. The conjecture, in its general form, remains open.

2.4 Matrix arithmetic–geometric mean inequality

For a self-adjoint compact operator T on a separable Hilbert space, by $\lambda_j(T)$ we mean the j-th largest eigenvalue of T for $j = 1, 2, \ldots$. The j-th largest eigenvalue s_j of $|T| = (T^*T)^{1/2}$ is called the *j-th singular value of T*.

Bhatia and Kittaneh [24] established the first arithmetic–geometric type inequality involving singular values of a compact operator on a separable Hilbert space. They proved that

$$2s_j(A^*B) \le s_j(AA^* + BB^*), \quad j \ge 1$$

for compact operators A and B (see [53] for a new proof and an equivalent inequality).

Later, Bhatia and Davis [11] provided another version for unitarily invariant norms as

$$2|||A^*XB||| \le |||AA^*X + XBB^*|||,$$

where A, B, and X are arbitrary square matrices; see the comprehensive survey [23] for other possible matrix extensions of the arithmetic–geometric mean inequality.

A problem raised from [24] is whether for all positive semidefinite $n \times n$ matrices A and B, it is true that

$$\sqrt{s_j(AB)} \le \frac{1}{2}\lambda_j(A + B) \quad (1 \le j \le n)$$

The question was affirmatively solved by Drury [36]; see also [47] for a simplified proof.

2.5 An interpolation inequality involving the Heinz means

The Heinz mean $H_\nu(a,b)$ of two real numbers $a, b \ge 0$ is defined for $\nu \in [0,1]$ by $H_\nu(a,b) = \frac{a^{1-\nu}b^\nu + a^\nu b^{1-\nu}}{2}$. These means interpolate between the arithmetic and geometric means, that is, $\sqrt{ab} \le H_\nu(a,b) \le \frac{a+b}{2}$; see [7]. Bhatia and Davis [11] proved that if A, B, X are $n \times n$ matrices such that A and B are positive semidefinite, and $|||\cdot|||$ is a unitarily invariant norm, then the function $f(p) = |||A^{1+p}XB^{1-p} + A^{1-p}XB^{1+p}|||$ is convex on $[-1,1]$ and takes its minimum at $p = 0$. As a consequence,

$$2|||A^{1/2}XB^{1/2}||| \le |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu||| \le |||AX + XB||| \tag{2.2}$$

which gives a matrix version of the above interpolation. The right side of (2.2) is known as Heinz inequality; see [6, p. 265]. The double inequality drew attention of many mathematicians; cf. [3, 34, 37, 38, 40, 41].

2.6 Monotonicity of Karcher means

Kubo and Ando [43] essentially developed the theory of means of two positive definite matrices. Several mathematicians have tried to extend this theory, in particular the notion of the matrix geometric mean, to three or more matrices. This was first done by Ando et al. [2] via a recursive method. A different approach, based on Riemannian geometry, was developed in two papers by Moakher [48] and by Bhatia and Holbrook [16]. This has stimulated a lot of work by others.

The space $\mathbb{P}$ consisting of the $m \times m$ positive definite matrices is a Riemannian manifold equipped with the metric $\delta_2(A,B) = \left(\sum_{i=1}^m \log^2 \lambda_i(A^{-1}B)\right)^{1/2}$. The geometric mean $G(A_1,A_2\ldots,A_n)$ of n matrices $A_1,\ldots,A_n\in\mathbb{P}$ is defined as the unique matrix $X\in\mathbb{P}$ which minimizes the sum

$$\arg\min_X \sum_{i=1}^{n}\frac{1}{n}\delta_2^2(X,A_i).$$

and it is called the *Karcher mean* (or *least squares mean*) of $A_1,\ldots,A_n\in\mathbb{P}$. This matrix is indeed the unique positive definite solution of the so-called *Karcher equation* $\sum_{i=1}^{n}\left(\log(X^{1/2}A_i^{-1}X^{1/2}\right)=0$.

Bhatia and Holbrook [16] conjectured the monotonicity of the Karcher mean with respect to Löwner order in the sense that

$$G(A_1,A_2,\ldots,A_m)\le G(B_1,B_2,\ldots,B_m) \text{ whenever} A_j\le Bj \text{ for all } 1\le j\le m.$$

This was affirmatively answered by Lawson and Lim [44] via utilizing a result of Sturm on probability measures on metric spaces of nonpositive curvature, and by Bhatia and Karandikar [22] employing some counting arguments and basic inequalities for the metric δ_2.

2.7 Inertia of Löwner matrices

Let f be a continuously differentiable real-valued function on $(0,\infty)$, and let $p_1,\ldots,p_n$ be distinct positive real numbers. The $n\times n$ matrix

$$L_f=\left[\frac{f(p_i)-f(p_j)}{p_i-p_j}\right]$$

is called a *Löwner matrix* associated with f. Here the ith diagonal entry is given by $f'(p_i)$. The power functions t^r, $r\in\mathbb{R}$ are of special interest, and we denote the associated Löwner matrices by L_r, i.e.,

$$L_r=\left[\frac{p_i^r-p_j^r}{p_i-p_j}\right].$$

Löwner in his classical work on operator monotone functions showed that f is operator monotone if and only if every $n\times n$ Löwner matrix L_f is positive semidefinite for every $n\in\mathbb{N}$. He also showed that t^r is operator monotone if and only if $0\le r\le 1$. In [5] Bhatia showed that

$$\|DA^r\|=r\|A^{r-1}\|, \tag{2.3}$$

where DA^r denotes the derivative of the map A^r for a positive definite matrix A. He used this to derive a perturbation bound for the operator absolute value map. The relation (2.3) was extended to all r in $(-\infty,0)\cup[2,\infty)$ in [29], where the authors also showed that mysteriously, the relation does not hold for any $r\in(1,\sqrt{2})$. In 2000, Bhatia and Holbrook resolved the remaining case in [17], and showed that (2.3) holds for $r\in[\sqrt{2},2]$. In the process they studied the Löwner matrices L_r for $1<r<2$, and showed that every L_r has exactly one positive eigenvalue in this case. Hence, by

Löwner's theory, we see that L_r has all nonnegative eigenvalues for $0<r<1$ and by [17] we get that L_r has exactly one positive eigenvalue for $1<r<2$. This contrast as r moves from (0, 1) to (1, 2) is intriguing and raises a natural question on the behaviour of the sign distribution of eigenvalues of the Löwner matrices L_r as r varies over the real line. Bhatia and Holbrook conjectured the behaviour of the inertia of L_r, $r \in \mathbb{R}$ in [17]. They, in addition to solving the problem for $1<r<2$, also computed the inertia for $r = 0, 1, \ldots, n-1$. Bhatia and Sano revisited this problem in [26, 27] and computed the inertia for $2<r<3$. This problem was fully solved in 2015 in the paper [15]. In this process, Bhatia and Jain in [18] also computed the inertia of the matrices $P_r = \left[(p_i + p_j)^r \right]$ for $r \in \mathbb{R}$ and distinct positive numbers $p_1, \ldots, p_n$. An interesting observation that comes from this analysis is that the inertia of P_r is equal to that of L_{r+1} for all $r > 0$. A natural question arises whether there is a deeper connection between the two matrix families. The question is open and may lead to some interesting development in the area.

2.8 Symplectic eigenvalues of positive definite matrices

Let J denote the $2n \times 2n$ matrix

$$J = \begin{bmatrix} O & I_n \\ -I_n & O \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix. A $2n \times 2n$ real matrix M is called a *symplectic matrix* if $M^T J M = J$. By a theorem of Williamson, see [30], we know that for every $2n \times 2n$ real positive definite matrix A there exists a symplectic matrix M such that

$$M^T A M = \begin{bmatrix} D & O \\ O & D \end{bmatrix},$$

where D is an $n \times n$ positive diagonal matrix with the diagonal entries $d_1 \leq \cdots \leq d_n$. These n positive numbers are complete invariants for A under the action of the symplectic group and are called the *symplectic eigenvalues* of A. These are important quantities in different areas such as classical and quantum mechanics, symplectic geometry, symplectic topology and in the newer area of quantum information. See [30] and references therein. Bhatia and Jain initiated the study of symplectic eigenvalues from the perspective of matrix analysis in [30] and proved some fundamental results including some variational principles, interlacing theorem, majorisation inequalities and perturbation bounds. In spite of some major differences between the classical eigenvalue and symplectic eigenvalue theories, we see that many fundamental results on eigenvalues can be extended to analogous results on symplectic eigenvalues with suitable modifications and interpretations. For instance, in [21] Bhatia and Jain proved an analogue of the classical Schur–Horn theorem for symplectic eigenvalues.

2.9 Bures–Wasserstein distance and barycenter

Let $\mathbb{P}(n)$ denote the set of all $n \times n$ positive definite matrices. Define

$$d(A,B)=\left[\operatorname{tr}A+\operatorname{tr}B-2\operatorname{tr}\left(A^{1/2}BA^{1/2}\right)^{1/2}\right]^{1/2}.$$

This is a metric on $\mathbb{P}(n)$ that occurs in different areas of mathematics and physics. It is known as the *Bures distance* in quantum information and is called the *Wasserstein distance* in the theory of optimal transport and statistics. We call this the *Bures–Wasserstein distance*. This distance is of interest in differential geometry as it is the distance corresponding to a Riemannian metric. It is also closely related to a well-known problem in factor analysis and multidimensional scaling called the *orthogonal Procrustes problem*. The quantity $\operatorname{tr}\left(A^{1/2}BA^{1/2}\right)^{1/2}$ is the fidelity between two states A and B. This is an important quantity in quantum information processes. The Bures–Wasserstein distance is also related to a measure of separation between two subspaces of $\mathbb{C}^n$. If A and B are diagonal matrices, d reduces to the Hellinger distance between probability distributions. Bhatia et al. [19] studied the Bures–Wasserstein distance from a matrix analysis perspective. They unified many known facts and simplified many of the existing proofs, leading to a better understanding of the subject. It is remarkable that an exact expression for the geodesic between any two elements A and B in $\mathbb{P}(n)$ with respect to the Bures–Wasserstein distance can be computed. This in turn gives us the mean of two positive definite matrices with respect to this metric, called the *Wasserstein mean*. We can also define the barycentre of m positive definite matrices $A_1,\ldots,A_m$: consider the minimisation problem

$$\underset{X>0}{\operatorname{argmin}}\ \sum_{j=1}^{m} w_j d^2(X,A_j). \tag{2.4}$$

This problem was first treated in [42]. The general problem for several probability measures on $\mathbb{R}^n$ was studied in [1] as a part of the multimarginal transport problem or the m-coupling problem. The special case of Gaussian measures is the problem (2.4). It was shown in [1] that the problem has a unique solution. In [19], the authors gave a simpler proof using the techniques of matrix analysis. We call this unique solution of (2.4) the *Wasserstein barycentre* of $A_1,\ldots,A_m$. This is the unique positive definite matrix $\Omega(w;A_1,\ldots,A_m)$ that satisfies the matrix equation

$$X=\sum_{j=1}^{m} w_j (X^{1/2}A_jX^{1/2})^{1/2},$$

where $w=(w_1,\ldots,w_m)$ are nonnegative numbers with $\sum w_j=1$. When $m=2$ and $(w=((1-t),t)$, this is the Wasserstein mean of A_1 and A_2, and has the closed form

$$A_1\diamond_t A_2=(1-t)^2A_1+t^2A_2+t(1-t)\left((A_1A_2)^{1/2}+(A_2A_1)^{1/2}\right).$$

Many basic properties of $\Omega(w;A_1,\ldots,A_m)$ were investigated in [19]. In [20] many inequalities involving the Wasserstein barycentre, Cartan mean, log Euclidean mean and the power mean were established. In particular, inequalities parallel to those studied in [25] were addressed in this paper.

2.10 Applications in brain–computer interfaces

The work of Rajendra has been instrumental for the introduction of Riemannian geometry methods in the field of brain–computer interfaces (BCIs) based on electroencephalography (EEG). Such BCIs are systems for translating EEG signals directly into commands for a computerized system, that is, to allow the user to send commands without using the muscles or the eyes. Seen this way, a BCI is a system designed to 'interpret' the intention of the subject.

Such interpretation is a machine learning problem. Encoding EEG signals by means of some form of covariance matrices has allowed the introduction of very efficient classifiers acting in the manifold of positive-definite matrices [31], which are now state-of-the-art in the BCI domain [32]. These advances have been made possible thanks to the 'matrix' interpretation of Riemannian geometry illustrated with exceptional clarity by Rajendra in a number of papers and collected in the monograph [8], which have provided neuroscientists and data scientists with a workable framework for applying differential geometry concepts such as geodesics, barycenters and parallel transport in the realm of brain neuroimaging data.

2.11 Scientometrics

We present a scientometrics analysis of research activity and collaboration of Rajendra, based on MathSciNet (MR) and Zentralblatt MATH (zbMATH).

His first publication is

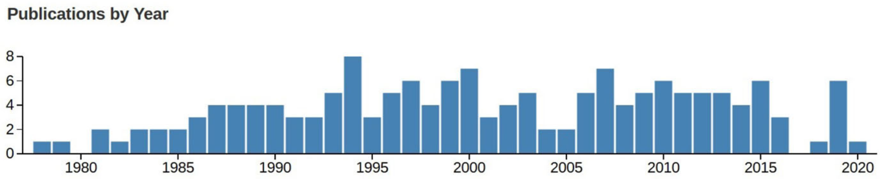
- Bhatia, Rajendra; Parthasarathy, K. R. Lectures on functional analysis. Part I. Perturbation by bounded operators. ISI Lecture Notes, 3. Macmillan Co. of India, Ltd., New Delhi, 1978. 146 pp.

The five most cited articles of Rajendra in MR are:

- Bhatia, Rajendra; Holbrook, John. Riemannian geometry and matrix geometric means. *Linear Algebra Appl.* 413 (2006), no. 2-3, 594618. (101 citations)
- Bhatia, Rajendra; Rosenthal, Peter. How and why to solve the operator equation $AX - XB = Y$. *Bull. London Math. Soc.* 29 (1997), no. 1, 121. (96 citations)
- Bhatia, Rajendra; Davis, Chandler. More matrix forms of the arithmetic–geometric mean inequality. *SIAM J. Matrix Anal. Appl.* 14 (1993), no. 1, 132136. (87 citations)
- Bhatia, Rajendra; Kittaneh, Fuad. On the singular values of a product of operators. *SIAM J. Matrix Anal. Appl.* 11 (1990), no. 2, 272277. (76 citations)
- Bhatia, Rajendra; Kittaneh, Fuad. Notes on matrix arithmetic–geometric mean inequalities. *Linear Algebra Appl.* 308 (2000), no. 1-3, 203211. (66 citations)

The number of Rajendra's publications recorded in MR is 148, which are cited 3928 times by 2568 authors. The mathematics subject classification in which Rajendra has published the most number of his papers is "linear and multilinear algebra and matrix theory", which is also subject in which his works have most citations.

On zbMATH, one can see that the journal that published the most number of Rajendra's papers is "Linear Algebra and its Applications" (43 papers). He collaborated with 57 mathematicians. Fuad Kittaneh (with 16 papers), Tanvi Jain (with 15 papers), and Chandler Davis (with 13 papers) have the largest number of joint papers with him. The paper [16] by Bhatia and Holbrook is included in the list of "10 notable papers from *Linear Algebra and Its Applications* over the last 50 years" offered by the editors of the journal to celebrate its golden anniversary in 2018.



(Bhatia's publications by year in zbMATH
Ref. https://zbmath.org/authors/?q=Rajendra+Bhatia)

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