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Local Fourier spaces and weighted Beurling density

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Abstract

We consider Banach spaces of functions or distributions on $\mathbb{R}^d$ for which the norm is defined in terms of a weighted L^p-norm of the Fourier transform of the elements and the weight w in question is assumed to be tempered and moderate. We study in particular subspaces of these spaces obtained by taking the closure in the corresponding norm of the test functions with compact support in a fixed open subset U of $\mathbb{R}^d$, usually assumed to be bounded. We consider weighted inequalities involving the L^p-norm of the Fourier transform of the elements of the subspace with respect to a positive Borel measure μ on $\mathbb{R}^d$ and the original norm defined on the subspace. We obtain, in particular, an exact characterization for these inequalities to hold in the case where U is a ball with a small enough radius using a suitable weighted version of the Beurling density. Exploiting duality, we then use these results to characterize the positive Borel measures μ having the property that the inverse Fourier transform of any measure $F\,d\mu$, where $F \in L^q(\mu)$, agrees on any open ball B of sufficiently small radius with the inverse Fourier transform of a tempered function g, where $g \in L^q(\tilde{w})$, for a weight $\tilde{w}$ related to w and, if it is the case, we also obtain a necessary and sufficient condition for the associated mapping $\mathcal{F}^{-1}L^q(\mu)|_B \to \mathcal{F}^{-1}L^q(\tilde{w})|_B$ to be surjective.

Keywords Sampling measures · Weighted Beurling densities · Moderate weights

Mathematics Subject Classification 42C15 · 44A35

Dedicated to Professor F. H. Szafraniec on the occasion of his 80th birthday.

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1 Introduction

In Lemma 4.2 of [20], R. Strichartz proved that if μ is a translation-bounded Borel measure on $\mathbb{R}^d$ (see (12) for the definition) and $F \in L^2(\mu)$, then the Fourier transform of the (complex) measure $F\,d\mu$ is locally square-integrable on $\mathbb{R}^d$. Although this is not mentioned in [20], the converse of this statement is also true: if, for any $F \in L^2(\mu)$, the Fourier transform of $F\,d\mu$ is in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$, then μ must be translation-bounded. We note that the condition that μ is translation-bounded is equivalent to the upper-Beurling density of μ, $\mathcal{D}^+(\mu)$, being finite (see Sect. 5, for more details and for the definition of Beurling densities). If, in addition, the lower-Beurling density of μ, $\mathcal{D}^-(\mu)$ is strictly positive, then any locally square-integrable function h on $\mathbb{R}^d$ can in fact be expressed, in any ball having sufficiently small radius, as the Fourier transform of a measure $F\,d\mu$, for some $F \in L^2(\mu)$ (which depends on the ball). As an example, consider in one dimension the measure $\mu = \sum_{n\in\mathbb{Z}} \delta_n$, where δ_a denotes the Dirac mass concentrated at the point a. It satisfies $\mathcal{D}^-(\mu) = \mathcal{D}^+(\mu) = 1$ and the set of inverse Fourier transforms of measures $F\,d\mu$, where $F \in L^2(\mu)$, consists exactly of the 1-periodic locally square-integrable functions on the line. It is then clear that, on any open ball B of radius 1/2, any square integrable function f can be expressed as the inverse Fourier transform of $F\,d\mu$, for some function $F \in L^2(\mu)$.

One of our main goals in this paper, is to generalize these results to spaces more general than $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. We will consider here Banach spaces or functions or distributions on $\mathbb{R}^d$ for which the corresponding norm is defined using the weighted L^p-norm of the Fourier transform of the elements, where the associated weight is assumed to be moderate and tempered (see Sect. 2, for the exact definitions). When $p = 2$, the corresponding spaces have been studied by the author in [8] and many of the results in [8] are generalized here in the case $1 \le p < \infty$. It turns out that multiplication by a function in the Schwartz class, $\mathcal{S}(\mathbb{R}^d)$, defines a continuous linear map on these spaces and this will allow us to define a "local" version of this spaces, in analogy with the relationship between $L^2(\mathbb{R}^d)$ and $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. We will be mostly interested in subspaces of these spaces obtained by taking the closure in the corresponding norm of the test functions with compact support in a fixed open set U. Given a locally finite positive measure μ on $\mathbb{R}^d$ as well as a moderate and tempered weight w defined on $\mathbb{R}^d$ and p with $1 \le p < \infty$, we will be interested in comparing the norms

$$\|\varphi\|_{p,w} := \left(\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p\, w(\xi)\, d\xi\right)^{1/p} \quad \text{and} \quad \|\varphi\|_{p,\mu} := \left(\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p\, d\mu(\xi)\right)^{1/p},$$

where φ ranges over all test functions with compact support in the open set U. As we will show in Theorem 9 (see also Theorem 8 for the unweighted case $w = 1$), the fact that $\|\varphi\|_{p,\mu} \le B\, \|\varphi\|_{p,w}$ for some positive constant B and for all test functions φ supported in a ball of sufficiently small radius is equivalent to having $\mathcal{D}^+(w^{-1}\,\mu) < \infty$. If this is the case, a duality argument shows that if $F \in L^q(\mu)$, where q is the dual exponent of p, then the inverse Fourier transform of the measure $F\,d\mu$ (in the sense of tempered distributions) coincides on any fixed ball with the inverse Fourier transform of some tempered function h (that depends on the ball) satisfying $\int_{\mathbb{R}^d} |h(\xi)|^q\,\tilde{w}(\xi)\,d\xi < \infty$, where $\tilde{w} = w^{1-q}$ if $1 < p < \infty$, or $\|h\,\tilde{w}\|_\infty < \infty$ if $p = 1$ where $\tilde{w} = w^{-1}$. This generalizes thus the result of Strichartz mentioned above which corresponds to the case $p = 2$ and $w = 1$, since the required condition $\mathcal{D}^+(\mu) < \infty$ is equivalent to μ being translation-bounded by Proposition 6. We will also prove in Theorem 7, that the two norms above are equivalent in the case where U is a ball in $\mathbb{R}^d$ with sufficiently small radius if and only if $\mathcal{D}^-(w^{-1}\,\mu) > 0$ and $\mathcal{D}^+(w^{-1}\,\mu) < \infty$ (see also Theorem 6 for the unweighted case). This implies, again by a duality argument, that, if both these conditions are met, the inverse Fourier transforms of the tempered measures $F\,d\mu$ with $F \in L^q(\mu)$ and those of the tempered functions h satisfying $\int_{\mathbb{R}^d} |h(\xi)|^q\,\tilde{w}(\xi)\,d\xi < \infty$ if $1 < p < \infty$ or $\|h\,\tilde{w}\|_\infty < \infty$ if $p = 1$, where $\tilde{w}$ is as above, generate the same space of distributions when restricted to any ball of sufficiently small radius. This generalizes the fact mentioned earlier that if a positive measure μ satisfies $\mathcal{D}^-(\mu) > 0$ and $\mathcal{D}^+(\mu) < \infty$, the restrictions to any ball with sufficiently small radius of the inverse Fourier transform of measures of the form $F\,d\mu$ where $F \in L^2(\mu)$, generate exactly the space of square-integrable functions on that ball.

The paper is organized as follows. We consider Banach spaces of functions or tempered distributions where the norm of an element is defined by a weighted L^p-norm of their Fourier transform in Sect. 2 and prove some of their basic properties and characterize their dual spaces. In Sect. 3, we prove that if a positive Borel measure μ on $\mathbb{R}^d$ has the property that its associated L^p-space contains the Fourier transform of all the test functions supported in a small ball, then μ is necessarily a tempered measure, i.e.

$$\int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^M}\,d\mu(\xi) < \infty$$

for some $M > 0$. We then define certain weighted inequalities associated with a positive Borel measure and show that they are equivalent to some properties of the adjoint of certain operators defined by the Fourier transform on those spaces. These type of inequalities have been considered by researchers in sampling theory in various frameworks such as Gabor frames or Fock spaces (e.g. [1, 17]) and the measures giving rise to these inequalities are often called "sampling measures".

In Sect. 4, we prove a useful result which allows us, for example, to deduce a weighted inequality from an unweighted one (i.e. for the weight $w = 1$) and vice-versa. Finally, the last section, Sect. 5, is the most technical. Here, we prove our main results which generalize Strichartz's result mentioned above.

Let us mention some notations and definitions used in this paper. If U is an open subset of $\mathbb{R}^d$, we denote by $C_0^\infty(U)$ the space of test-functions supported in U, i.e. the infinitely differentiable functions compactly supported in U and if $K \subset \mathbb{R}^d$ is compact, $C_0^\infty(K)$ denotes the space of functions in $C_0^\infty(\mathbb{R}^d)$ whose support is contained in K. The Schwartz class, denoted by $\mathcal{S}(\mathbb{R}^d)$, consists of all functions ψ on $\mathbb{R}^d$, such that

$$\sup_{x\in\mathbb{R}^d} |D^\alpha\psi(x)\,(1+|x|^2)^N| < \infty$$

for any multi-index α. $C_0(\mathbb{R}^d)$ is the space of continuous functions on $\mathbb{R}^d$ that vanish at infinity.

If $w > 0$ is a weight on $\mathbb{R}^d$ and $1 \le p < \infty$, the space $L^p_w(\mathbb{R}^d)$ is the Lebesgue space of measurable functions f on $\mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d} |f(\xi)|^p\, w(\xi)\, d\xi < \infty.$$

If $w = 1$, $L^p_w(\mathbb{R}^d)$ is denoted by $L^p(\mathbb{R}^d)$ and we let $\|f\|_p = \left(\int_{\mathbb{R}^d} |f(\xi)|^p\, d\xi\right)^{1/p}$ and $\|f\|_\infty = \text{ess}\,\sup_{\xi\in\mathbb{R}^d}|f(\xi)|$. If A, B are subsets of $\mathbb{R}^d$ and $t \in \mathbb{R}^d$, we will denote by $A + B$ the set $\{a + b:\ a \in A,\ b \in B\}$ and by $t + A$ the set $\{t + a:\ a \in A\}$. We also denote by $B(a, r)$ the open ball of center $a \in \mathbb{R}^d$ with radius $r > 0$, i.e. $\{x \in \mathbb{R}^d:\ |x - a| < r\}$. If $f \in L^1(\mathbb{R}^d)$, we denote its Fourier transform by $\hat{f}$ or $\mathcal{F}(f)$. It is defined by

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot x} f(x)\, dx, \quad \xi \in \mathbb{R}^d.$$

This definition extends in the usual way to the dual of $\mathcal{S}(\mathbb{R}^d)$, the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions on $\mathbb{R}^d$. If U is open, we also denote by $\mathcal{D}'(U)$, the space of distributions on U (which is the dual of the space $C_0^\infty(U)$ defined earlier). If X is a Banach space, its dual, the space of continuous linear functionals on X, is denoted by X' (see [19] for more details on these various spaces).

2 Weighted Fourier L^p-spaces

A *moderate weight* on $\mathbb{R}^d$ is a continuous function $w > 0$ defined on $\mathbb{R}^d$ and satisfying

$$w(\xi + \eta) \le w(\xi)\, v(\eta), \quad \xi, \eta \in \mathbb{R}^d, \tag{1}$$

for some function $v > 0$ on $\mathbb{R}^d$. In the following, we will always assume that v is tempered, i.e. that there exists a constants $C, M > 0$ such that

$$v(\xi) \le C\,(1 + |\xi|^2)^M, \quad \xi \in \mathbb{R}^d. \tag{2}$$

This implies, in particular, that w is tempered as well and, in fact, it is easy to see that, for some integer $M \ge 0$, the function $w(\xi)\,(1 + |\xi|^2)^{-M}$ is bounded, and so is the function $w^{-1}(\xi)\,(1 + |\xi|^2)^{-M}$ (since w^{-1} satisfies the inequality (1) with $v(\eta)$ replaced by $v(-\eta)$). We will assume that v is *submultiplicative*, i.e. that $v(\xi + \tau) \le v(\xi)\, v(\tau)$ for any $\xi, \tau \in \mathbb{R}^d$. This is not a restriction since v can be defined as

$v(\tau) = \sup_{\xi\in\mathbb{R}^d} w(\xi+\tau)/w(\xi)$, for $\tau \in \mathbb{R}^d$. It is easily checked that any power of w, w^α with $\alpha \in \mathbb{R}$, defines a moderate weight which is also tempered. An example of a weight w satisfying (1) and (2) is the weight

$$w(\xi) = (1+|\xi|^2)^s, \quad \xi \in \mathbb{R}^d,$$

with $s \in \mathbb{R}$, which is used in the definition of the standard Sobolev space $H^s(\mathbb{R}^d)$ corresponding to the case $p = 2$ below. Using Peetre's inequality, it is easily seen that the corresponding v satisfies

$$v(\xi) \sim (1+|\xi|^2)^{|s|},$$

where $w_1 \sim w_2$ means that $A\,w_1 \le w_2 \le B\,w_1$ pointwise for two positive constant A and B. We refer the reader to Gröchenig's paper [11] for more examples of weights satisfying (1) as well as an extensive overview of their properties and applications in harmonic analysis (see also [6, 8, 13]).

Definition 1 If $1 \le p < \infty$, let

$$\mathcal{F}^{-1}L^p_w(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d),\ \hat{u} \in L^1_{\text{loc}}(\mathbb{R}^d) \text{ and } \int_{\mathbb{R}^d} |\hat{u}(\xi)|^p\, w(\xi)\, d\xi < \infty \right\} \tag{3}$$

and let

$$\mathcal{F}^{-1}L^\infty_w(\mathbb{R}^d) = \{ u \in \mathcal{S}'(\mathbb{R}^d),\ \hat{u} \in L^1_{\text{loc}}(\mathbb{R}^d) \text{ and } \hat{u}\,w \in L^\infty(\mathbb{R}^d)\}.$$

The norm of an element $u \in \mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$ is defined by

$$\|u\|_{p,w} = \left(\int_{\mathbb{R}^d} |\hat{u}(\xi)|^p\, w(\xi)\, d\xi \right)^{1/p},\ 1 \le p < \infty, \quad \text{and} \quad \|u\|_{\infty,w} = \|\hat{u}\,w\|_\infty.$$

Proposition 1 *For any p with $1 \le p \le \infty$, $\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$ is a Banach space. When $p = 2$, $\mathcal{F}^{-1}L^2_w(\mathbb{R}^d)$ is Hilbert space with inner product*

$$\langle h, g\rangle_w = \int_{\mathbb{R}^d} \hat{h}(\xi)\,\overline{\hat{g}(\xi)}\, w(\xi)\, d\xi, \quad h, g \in \mathcal{F}^{-1}L^2_w(\mathbb{R}^d).$$

We have the continuous embeddings

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{F}^{-1}L^p_w(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$$

and $\mathcal{F} : \mathcal{F}^{-1}L^p_w(\mathbb{R}^d) \to L^p_w(\mathbb{R}^d)$ an isometric isomorphism, where $\mathcal{F}$ is the Fourier transform.

Proof As the statements to prove are easily checked, we just verify the continuity of the embeddings above as well as the completeness property. To show the continuous embedding $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$, choose $M > 0$ large enough so that both functions $w(\xi)^{-1}\,(1+|\xi|^2)^{-M}$ and $w(\xi)\,(1+|\xi|^2)^{-M}$ are bounded on $\mathbb{R}^d$. For $p=\infty$ and $u\in\mathcal{S}(\mathbb{R}^d)$, we have the estimate

$$\|\hat{u}\,w\|_\infty \le \|\hat{u}(\xi)\,(1+|\xi|^2)^M\|_\infty\,\|\,w(\xi)\,(1+|\xi|^2)^{-M}\|_\infty.$$

and for $1\le p<\infty$, by Hölder's inequality, we have

$$\int_{\mathbb{R}^d} |\hat{u}(\xi)|^p\,w(\xi)\,d\xi \le C\,\|\hat{u}(\xi)\,(1+|\xi|^2)^s\|_\infty^p$$

where $C:=\int_{\mathbb{R}^d}\frac{w(\xi)}{(1+|\xi|^2)^{sp}}\,d\xi<\infty$ if $s\,p>M+d/2$. If $u\in\mathcal{F}^{-1}L^\infty_w(\mathbb{R}^d)$, the inequality

$$\int_{\mathbb{R}^d}\frac{|\hat{u}(\xi)|}{(1+|\xi|^2)^s}\,d\xi\le C\,\|u\|_{\infty,w},$$

where $C=\int_{\mathbb{R}^d}\frac{w^{-1}(\xi)}{(1+|\xi|^2)^s}\,d\xi<\infty$ if $s>M+d/2$ shows the continuity of the embedding $\mathcal{F}^{-1}L^\infty_w(\mathbb{R}^d)\hookrightarrow\mathcal{S}'(\mathbb{R}^d)$. If $1<p<\infty$ and $s\,q>Mq/p+d/2$ where q is the dual exponent of p defined by $1/p+1/q=1$, Hölder's inequality shows that, for any $u\in\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{|\hat{u}(\xi)|}{(1+|\xi|^2)^s}\,d\xi &\le\left(\int_{\mathbb{R}^d}|\hat{u}(\xi)|^p\,w(\xi)\,d\xi\right)^{1/p}\left(\int_{\mathbb{R}^d}\frac{w^{-q/p}(\xi)}{(1+|\xi|^2)^{sq}}\,d\xi\right)^{1/q}\\ &\le C\,\|u\|_{p,w},\end{aligned}$$

where

$$C=\left(\sup_{\xi\in\mathbb{R}^d}\left(w(\xi)^{-1}\,(1+|\xi|^2)^{-M}\right)\right)^{1/p}\left(\int_{\mathbb{R}^d}\frac{1}{(1+|\xi|^2)^{sq-Mq/p}}\,d\xi\right)^{1/q}<\infty.$$

For $p=1$, we have the estimate

$$\int_{\mathbb{R}^d}\frac{|\hat{u}(\xi)|}{(1+|\xi|^2)^M}\,d\xi\le\sup_{\xi\in\mathbb{R}^d}\left(w(\xi)^{-1}\,(1+|\xi|^2)^{-M}\right)\|u\|_{1,w}.$$

This shows that the space $\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$ is continuously embedded in $\mathcal{S}'(\mathbb{R}^d)$ if $1\le p<\infty$.

In particular, if $\{u_n\}$ is a Cauchy sequence in $\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$, the sequence $\{\hat{u}_n\}$ is Cauchy in $L^p_w(\mathbb{R}^d)$. Since this last space is complete, there exists $h\in L^p_w(\mathbb{R}^d)$ such that $\hat{u}_n\to h$ in $L^p_w(\mathbb{R}^d)$ and the previous estimate yields

$$\int_{\mathbb{R}^d}\frac{|h(\xi)|}{(1+|\xi|^2)^s}\,d\xi<\infty$$

if s is large enough, showing that h defines a tempered distribution. If the element $u\in\mathcal{S}'(\mathbb{R}^d)$ is defined by the equation $\hat{u}=h$, it follows thus that u belongs to

$\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$ and $u_n \to u$ in $\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$, proving the required completeness property. □

We should point out that, for a general moderate weight w as above, it might not be true that $\overline{u}$, the complex conjugate of u, belongs to $\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$ whenever u does. This will be the case, however, if w is even ($w(-\xi) = w(\xi)$), or more generally, if $w(-\xi) \le C\, w(\xi)$ for some constant $C > 0$. The following two results show that the dual of $\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$ can be identified with $\mathcal{F}^{-1}L^q_{w^{1-q}}(\mathbb{R}^d)$ if $1 < p < \infty$ and with $\mathcal{F}^{-1}L^\infty_{w^{-1}}(\mathbb{R}^d)$ if $p = 1$.

Proposition 2 *Let p with $1 < p < \infty$ and let $h \in \mathcal{F}^{-1}L^q_{w^{1-q}}(\mathbb{R}^d)$, where q is the dual exponent of p. Then, the mapping*

$$\ell_h(u) = \int_{\mathbb{R}^d} \hat{u}(\xi)\, \overline{\hat{h}(\xi)}\, d\xi, \quad u \in \mathcal{F}^{-1}L^p_w(\mathbb{R}^d), \tag{4}$$

is well defined as an element of $\left(\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)\right)'$. Furthermore, we have $\|\ell_h\| = \|h\|_{q,w^{1-q}}$. Conversely any element ℓ of the dual space $\left(\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)\right)'$ is of the form $\ell = \ell_h$ as in (4) for some $h \in \mathcal{F}^{-1}L^q_{w^{1-q}}(\mathbb{R}^d)$.

Proof If $h \in \mathcal{F}^{-1}L^q_{w^{1-q}}(\mathbb{R}^d)$ and ℓ_h is defined by (4), we have, using Hölder's inequality, that

$$\begin{aligned}
|\ell_h(u)| &\le \int_{\mathbb{R}^d} |\hat{u}(\xi)|\, |\hat{h}(\xi)|\, d\xi = \int_{\mathbb{R}^d} |\hat{u}(\xi)|\, |\hat{h}(\xi)|\, w(\xi)^{1/p}\, w(\xi)^{-1/p}\, d\xi \\
&\le \left(\int_{\mathbb{R}^d} |\hat{u}(\xi)|^p\, w(\xi)\, d\xi \right)^{1/p} \left(\int_{\mathbb{R}^d} |\hat{h}(\xi)|^q\, w(\xi)^{-q/p}\, d\xi \right)^{1/q} \\
&= \|u\|_{p,w} \left(\int_{\mathbb{R}^d} |\hat{h}(\xi)|^q\, w(\xi)^{1-q}\, d\xi \right)^{1/q},
\end{aligned}$$

showing that $\ell_h \in \left(\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)\right)'$ and $\|\ell_h\| \le \|h\|_{q,w^{1-q}}$. Furthermore, defining $u \in \mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$ by the formula

$$\hat{u}(\xi) = \begin{cases} \hat{h}(\xi)\, |\hat{h}(\xi)|^{q-2}\, w(\xi)^{-q/p} & \text{if } h(\xi) \ne 0, \\ 0 & \text{if } h(\xi) = 0, \end{cases}$$

we have $\ell_h(u) = \|u\|_{p,w}\, \|h\|_{q,w^{1-q}}$, showing that $\|\ell_h\| = \|h\|_{q,w^{1-q}}$. Conversely, since the mapping $\mathcal{F} : \mathcal{F}^{-1}L^p_w(\mathbb{R}^d) \to L^p_w(\mathbb{R}^d)$ is an isometric isomorphism and $\left(L^p_w(\mathbb{R}^d)\right)' = L^q_w(\mathbb{R}^d)$ in the sense that any continuous linear functional $\tilde{\ell}$ on $L^p_w(\mathbb{R}^d)$ has the form

$$\tilde{\ell}(f) = \int_{\mathbb{R}^d} f(\xi)\, \overline{g(\xi)}\, w(\xi)\, d\xi, \quad f \in L^p_w(\mathbb{R}^d),$$

for some $g \in L^q_w(\mathbb{R}^d)$, it follows that any element ℓ of $\left(\mathcal{F}L^p_w(\mathbb{R}^d)\right)'$ has the form

$$\ell(u) = \int_{\mathbb{R}^d} \hat{u}(\xi)\,\overline{G(\xi)}\,w(\xi)\,d\xi, \quad f \in L^p_w(\mathbb{R}^d),$$

for some $G \in L^q_w(\mathbb{R}^d)$. Defining h by the formula $\hat{h} = \overline{G}\,w$, it is easily checked that $h \in \mathcal{F}^{-1}L^q_{w^{1-q}}(\mathbb{R}^d)$ and that $\ell = \ell_h$, as above. This proves our claim. □

We can deal with the case $p = 1$ in a similar way. The proof of the next proposition is left to the reader.

Proposition 3 *If $h \in \mathcal{F}^{-1}L^\infty_{w^{-1}}(\mathbb{R}^d)$, the mapping*

$$\ell_h(u) = \int_{\mathbb{R}^d} \hat{u}(\xi)\,\overline{\hat{h}(\xi)}\,d\xi, \quad u \in \mathcal{F}^{-1}L^1_w(\mathbb{R}^d), \tag{5}$$

is well defined as an element of $\left(\mathcal{F}^{-1}L^1_w(\mathbb{R}^d)\right)'$. Furthermore, we have $\|\ell_h\| = \|h\|_{\infty,w^{-1}}$. Conversely any element ℓ of $\left(\mathcal{F}^{-1}L^1_w(\mathbb{R}^d)\right)'$ is of the form $\ell = \ell_h$ as in (5) *for some $h \in \mathcal{F}^{-1}L^\infty_{w^{-1}}(\mathbb{R}^d)$.*

We can use the previous duality characterization to prove the density of the test functions $\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$ when $1 \le p < \infty$.

Proposition 4 *If $1 \le p < \infty$, the space $C_0^\infty(\mathbb{R}^d)$ is dense in $\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$.*

Proof We argue by contradiction. If $C_0^\infty(\mathbb{R}^d)$ were not dense in $\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$, the Hahn-Banach theorem would show the existence of a non-zero element ℓ of $\left(\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)\right)'$ satisfying $\ell(\varphi) = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^d)$. If $1 < p < \infty$, Proposition 2 would imply the existence of $h \in \mathcal{F}^{-1}L^q_{w^{1-q}}(\mathbb{R}^d)$ with $h \neq 0$ such that

$$\ell(\varphi) = \int_{\mathbb{R}^d} \hat{\varphi}(\xi)\,\overline{\hat{h}(\xi)}\,d\xi = 0, \quad \varphi \in C_0^\infty(\mathbb{R}^d).$$

Since the space $\mathcal{F}^{-1}L^q_{w^{1-q}}(\mathbb{R}^d)$ is continuously embedded in $\mathcal{S}'(\mathbb{R}^d)$ and the space $C_0^\infty(\mathbb{R}^d)$ is dense is $\mathcal{S}(\mathbb{R}^d)$, it would follow that $h = 0$, a contradiction. If $p = 1$, the proof is similar and uses Proposition 3. □

Definition 2 If U is an open set of $\mathbb{R}^d$ and $1 \le p < \infty$, we will denote by $\mathcal{F}^{-1}L^p_w(U)$ the closure of the space $C_0^\infty(U)$ in $\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$.

Note that Proposition 4 shows that there is no ambiguity between Definitions 1 and 2 in the case where $U = \mathbb{R}^d$.

Lemma 1 *If $1 \le p \le \infty$ and $\psi \in \mathcal{S}(\mathbb{R}^d)$, the mapping $u \mapsto \psi\, u$ is a continuous linear mapping from $\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$ to itself.*

Proof If $1 \le p < \infty$ and $u \in \mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$, the integral form of Minkowski's inequality yields

$$
\begin{aligned}
\|\psi\, u\|_{p,w} &= \left(\int_{\mathbb{R}^d} |(\hat{u} * \hat{\psi})(\xi)|^p\, w(\xi)\, d\xi\right)^{1/p}\\
&= \left(\int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} \hat{u}(\xi-\eta)\, \hat{\psi}(\eta)\, d\eta\right|^p w(\xi)\, d\xi\right)^{1/p}\\
&\leq \int_{\mathbb{R}^d} |\hat{\psi}(\eta)| \left(\int_{\mathbb{R}^d} |\hat{u}(\xi-\eta)|^p\, w(\xi)\, d\xi\right)^{1/p} d\eta\\
&\leq \int_{\mathbb{R}^d} |\hat{\psi}(\eta)| \left(\int_{\mathbb{R}^d} |\hat{u}(\xi-\eta)|^p\, w(\xi-\eta)\, v(\eta)\, d\xi\right)^{1/p} d\eta = C\, \|u\|_{p,w},
\end{aligned}
$$

where $C := \int_{\mathbb{R}^d} |\hat{\psi}(\eta)|\, v(\eta)^{1/p}\, d\eta < \infty$. The case $p = \infty$ follows from a similar argument. $\square$

The next lemma will help us define tempered distributions on $\mathbb{R}^d$ which are *locally* in $\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$.

Lemma 2 *Let $1 \leq p \leq \infty$ and let w be a weight on $\mathbb{R}^d$ satisfying* (1) *and* (2)*. Then, given $T \in \mathcal{S}'(\mathbb{R}^d)$, the following are equivalent.*

(a) *For any $\varphi \in C_0^\infty(\mathbb{R}^d)$, the distribution φT belongs to $\mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$.*
(b) *For any bounded open set $U \subset \mathbb{R}^d$, there exists $u \in \mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$* such that $u = T$ on U.

Proof If (a) holds and $U \subset \mathbb{R}^d$ is a bounded open set, we can find $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi \equiv 1$ on U. Then, $u = \varphi T \in \mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$ and $u = T$ on U. Conversely, if (b) holds and $\varphi \in C_0^\infty(\mathbb{R}^d)$, let U be a bounded open set containing the support of φ. If $T \in \mathcal{S}'(\mathbb{R}^d)$, let $u \in \mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$ with $u = T$ on U. We have then $\varphi T = \varphi u \in \mathcal{F}^{-1}L^p_w(\mathbb{R}^d)$ by Lemma 1, which proves our claim. $\square$

We will denote by $\mathcal{F}^{-1}_{\mathrm{loc}}L^p_w(\mathbb{R}^d)$ the set of tempered distributions on $\mathbb{R}^d$ which satisfy any of the equivalent statements of the previous lemma.

3 Weighted inequalities in measure spaces

We now introduce certain weighted inequalities which will play a central role in the following sections.

Definition 3 Let $w > 0$ be a weight on $\mathbb{R}^d$ satisfying (1) and (2) and let μ be a positive, locally finite Borel measure on $\mathbb{R}^d$. Let $U \subset \mathbb{R}^d$ be open and non-empty, let p with $1 \leq p < \infty$ and let $A, B > 0$.

(a) We say that the couple (μ, w) belongs to $\mathcal{B}^p(U, B)$ if we have the inequality

$$\int_{\mathbb{R}^d} |\hat{u}(\xi)|^p\, d\mu(\xi) \leq B \int_{\mathbb{R}^d} |\hat{u}(\xi)|^p\, w(\xi)\, d\xi, \quad u \in \mathcal{F}^{-1}L^p_w(U). \tag{6}$$

(b) We say that the couple (μ, w) belongs to $\mathcal{F}^p(U, A, B)$ if, for any u in the space $\mathcal{F}^{-1}L^p_w(U)$, we have the inequalities

$$A \int_{\mathbb{R}^d} |\hat{u}(\xi)|^p\, w(\xi)\, d\xi \le \int_{\mathbb{R}^d} |\hat{u}(\xi)|^p\, d\mu(\xi) \le B \int_{\mathbb{R}^d} |\hat{u}(\xi)|^p\, w(\xi)\, d\xi. \tag{7}$$

Since, by definition, the space $C_0^\infty(U)$ is dense in $\mathcal{F}^{-1}L^p_w(U)$, in order to establish that a couple (μ, w) belongs to $\mathcal{B}^p(U, B)$ or $\mathcal{F}^p(U, A, B)$, it is thus sufficient to verify the inequalities in (6) or (7), respectively, for test functions $u = \varphi \in C_0^\infty(U)$. Note that, for any $a \in \mathbb{R}^d$, we have

$$\mathcal{B}^p(U, B) = \mathcal{B}^p(U + a, B) \quad \text{and} \quad \mathcal{F}^p(U, A, B) = \mathcal{F}^p(U + a, A, B). \tag{8}$$

Clearly, if μ is any tempered positive Borel measure on $\mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} |\hat{\varphi}(\lambda)|^p\, d\mu(\lambda) < \infty$$

if $1 \le p < \infty$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$. However, it is not immediately obvious that a positive Borel measure μ must be tempered if the previous integral is finite for all the test functions in $C_0^\infty(U)$ where $U \subset \mathbb{R}^d$ is a non-empty open set. The next lemma will be needed to show that it is indeed the case.

Lemma 3 *Let q with $1 < q < \infty$, let μ be a positive, locally finite Borel measure on $\mathbb{R}^d$ and suppose that, for every $F \in L^q(\mu)$ with $F \ge 0$, the measure $F\, d\mu$ is tempered, i.e. there exists an integer $m = m(F)$ such that*

$$\int_{\mathbb{R}^d} \frac{F(\xi)}{(1 + |\xi|^2)^m}\, d\mu(\xi) < \infty.$$

Then, the measure μ must itself be tempered, i.e. there exists an integer M such that

$$\int_{\mathbb{R}^d} \frac{1}{(1 + |\xi|^2)^M}\, d\mu(\xi) < \infty.$$

Proof We first show that there exist an integer $m_0 \ge 0$ such that

$$\int_{\mathbb{R}^d} \frac{F(\xi)}{(1 + |\xi|^2)^{m_0}}\, d\mu(\xi) < \infty$$

for all $F \in L^q(\mu)$ with $F \ge 0$. Indeed, if it weren't the case, we could find a sequence $\{F_k\}_{k \ge 1} \in L^q(\mu)$ with $F_k \ge 0$ and

$$\int_{\mathbb{R}^d} \frac{F_k(\xi)}{(1 + |\xi|^2)^k}\, d\mu(\xi) = \infty.$$

Letting $F = \sum_{k=1}^{\infty} 2^{-k} \|F_k\|_{q,\mu}^{-1} F_k$, we have $F \in L^q(\mu)$, $F \geq 0$ and, for any $k \geq 1$,

$$\int_{\mathbb{R}^d} \frac{F(\xi)}{(1+|\xi|^2)^k} \, d\mu(\xi) \geq 2^{-k} \|F_k\|_{p,\mu}^{-1} \int_{\mathbb{R}^d} \frac{F_k(\xi)}{(1+|\xi|^2)^k} \, d\mu(\xi) = \infty,$$

in contradiction with our hypothesis. Letting thus m_0 be as above, we can define the linear mapping $\ell : L^q(\mu) \to \mathbb{C}$ by

$$\ell(G) = \int_{\mathbb{R}^d} \frac{G(\xi)}{(1+|\xi|^2)^{m_0}} \, d\mu(\xi), \quad G \in L^q(\mu).$$

Define also for $N \geq 1$, the linear mappings $\ell_N : L^q(\mu) \to \mathbb{C}$ by

$$\ell_N(G) = \int_{\{|\xi| \leq N\}} \frac{G(\xi)}{(1+|\xi|^2)^{m_0}} \, d\mu(\xi), \quad G \in L^q(\mu).$$

Each ℓ_N defines a bounded linear map since, for any $G \in L^q(\mu)$, we have

$$|\ell_N(G)| \leq \left(\int_{\mathbb{R}^d} |G(\xi)|^q \, d\mu(\xi) \right)^{1/q} \left(\int_{\{|\xi| \leq N\}} \frac{1}{(1+|\xi|^2)^{pm_0}} \, d\mu(\xi) \right)^{1/p}.$$

Furthermore, for any $G \in L^q(\mu)$, the sequence $\{\ell_N(G)\}$ is bounded since $\ell_N(G) \to \ell(G)$, $N \to \infty$, by the Lebesgue dominated convergence theorem. It follows thus from the uniform boundedness principle that the sequence of operators $\{\ell_N\}$ is bounded, i.e. there exists $B > 0$ such that $\|\ell_N\| \leq B$ for all N. This implies that $\|\ell\| \leq B$, i.e. the linear functional ℓ is continuous. The fact that the dual of $L^q(\mu)$ is $L^p(\mu)$ (with $1/p + 1/q = 1$) shows that

$$\int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^{pm_0}} \, d\mu(\xi) < \infty,$$

proving that our claim holds with $M = pm_0$. □

Note that the previous result does not hold for $q = 1$ since the measure $F \, d\mu$ is automatically bounded if $F \in L^1_\mu$.

Proposition 5 *Let p with* $1 \leq p < \infty$, *let* μ *be a positive Borel measure on* $\mathbb{R}^d$ *(not necessarily locally finite) and suppose that, for some* $\epsilon > 0$, *we have*

$$\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, d\mu(\xi) < \infty, \quad \varphi \in C_0^\infty(B(0,\epsilon)),$$

where $B(0,\epsilon) = \{x \in \mathbb{R}^d, \ |x| < \epsilon\}$. *Then,* μ *is a tempered measure, i.e. there exists* $M > 0$ *such that*

$$\int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^M} \, d\mu(\xi) < \infty.$$

Proof We first show that μ is locally finite. By a compactness argument, it is enough to show that each point in $\mathbb{R}^d$ is the center of a ball with finite μ-measure. Let $\varphi_0 \in C_0^\infty(B(0,\epsilon))$ with $\varphi_0 \neq 0$ and choose $\xi_0 \in \mathbb{R}^d$ such that $|\hat{\varphi}_0(\xi_0)| := 2\,r > 0$. By continuity, we have thus, for some $\epsilon > 0$, that $|\hat{\varphi}_0(\xi)| \geq r$ if $|\xi - \xi_0| < \epsilon$. If $\xi_1 \in \mathbb{R}^d$, the function φ_1, defined by $\varphi_1(x) = e^{2\pi i x\cdot(\xi_1 - \xi_0)}\,\varphi_0(x)$, belongs to $C_0^\infty(B(0,\epsilon))$ and we have the inequality $|\hat{\varphi}_1(\xi)| \geq r$ if $|\xi - \xi_1| < \epsilon$. Hence,

$$\mu(B(\xi_1,\epsilon)) \leq \frac{1}{r^2}\int_{\{|\xi-\xi_1|<\epsilon\}} |\hat{\varphi}_1(\xi)|^2\,d\mu(\xi) \leq \frac{1}{r^2}\int_{\mathbb{R}^d} |\hat{\varphi}_1(\xi)|^2\,d\mu(\xi) < \infty,$$

showing that μ is locally finite. To show that μ is actually tempered, we consider first the case where $1 < p < \infty$. Using Lemma 3, it suffices to show that for any $F \in L^q(\mu)$ with $F \geq 0$, the measure $F\,d\mu$ is tempered. For such an F, let $N \geq 1$ and define, for any $\varphi \in C_0^\infty(B(0,\epsilon))$,

$$\langle T,\varphi\rangle = \int_{\mathbb{R}^d} F(\xi)\,\hat{\varphi}(\xi)\,d\mu(\xi) \quad \text{and} \quad \langle T_N,\varphi\rangle = \int_{B(0,N)} F(\xi)\,\hat{\varphi}(\xi)\,d\mu(\xi).$$

It is easy to check that for each $N \geq 1$, $T_N \in \mathcal{D}'(B(0,\epsilon))$ since it is the restriction to the ball $B(0,\epsilon)$ of the inverse Fourier transform of the bounded measure $F\,\chi_{B(0,N)}\,d\mu$. Furthermore, the sequence $\{T_N\}_{N\geq 1}$ is bounded in $\mathcal{D}'(B(0,\epsilon))$ since this is equivalent to the boundedness of each sequence $\{\langle T_N,\varphi\rangle\}_{N\geq 1}$ with $\varphi \in C_0^\infty(B(0,\epsilon))$ and, for such φ, we have

$$|\langle T_N,\varphi\rangle| \leq \int_{B(0,N)} F(\xi)\,|\hat{\varphi}(\xi)|\,d\mu(\xi) \leq \left(\int_{\mathbb{R}^d} F^q(\xi)\,d\mu(\xi)\right)^{1/q} \|\varphi\|_{p,\mu},$$

where $\|\varphi\|_{p,\mu} = \left(\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p\,d\mu(\xi)\right)^{1/p}$. It follows then from elementary distribution theory, that there exist an integer $K \geq 0$ and a constant $C > 0$, such that

$$|\langle T_N,\varphi\rangle| \leq C \sum_{|\alpha|\leq K} \|\partial^\alpha\varphi\|_\infty, \quad \varphi \in C_0^\infty(B(0,\epsilon/2)).$$

Hence, $T \in \mathcal{D}'(B(0,\epsilon))$ and satisfies

$$|\langle T,\varphi\rangle| \leq C \sum_{|\alpha|\leq K} \|\partial^\alpha\varphi\|_\infty, \quad \varphi \in C_0^\infty(B(0,\epsilon/2)).$$

Let $\rho \in C_0^\infty(B(0,1))$ satisfy $\hat{\rho}(0) = 1$ and for $r > 0$, define $\rho_r(x) = \rho(x/r)$ and $\tilde{\rho}_r(x) = \overline{\rho(-x/r)}$. Clearly, $\rho_r * \tilde{\rho}_r \in C_0^\infty(B(0,\epsilon/2))$ if $0 < r \leq \epsilon/4$. Furthermore, for any $x \in \mathbb{R}^d$, we have

$$\partial^\alpha(\rho_r * \tilde{\rho}_r)(x) = ((\partial^\alpha\rho_r) * \tilde{\rho}_r)(x) = r^{-|\alpha|}\int_{\mathbb{R}^d} (\partial^\alpha\rho)((y-x)/r)\,\overline{\rho(y/r)}\,dy.$$

Hence, for any multi-index α, we have

$$\|\partial^\alpha(\rho_r * \tilde{\rho}_r)\|_\infty \leq r^{-|\alpha|+d}\,\|\partial^\alpha\rho\|_2\,\|\rho\|_2.$$

It follows that, there exists a constant $C_1 > 0$ such that

$$|\langle T, \rho_r * \tilde{\rho}_r \rangle| \leq C_1\, r^{d-K} \quad \text{if } 0 < r \leq \epsilon/4 < 1.$$

Since $\hat{\rho}_r(\xi) = r^d\, \hat{\rho}(r\xi)$, we have thus the inequality

$$\int_{\mathbb{R}^d} F(\xi)\, |\hat{\rho}(r\xi)|^2\, d\mu(\xi) \leq C_1\, r^{-d-K}, \quad 0 < r \leq \epsilon/4 < 1.$$

Let $\delta > 0$ be small enough so that $|\hat{\rho}(\xi)|^2 \geq 1/2$ if $|\xi| \leq \delta$. We have then

$$\int_{\{|\xi| \leq \delta/r\}} F(\xi)\, d\mu(\xi) \leq 2\, C_1\, r^{-d-K}, \quad 0 < r \leq \epsilon/4 < 1,$$

and

$$\int_{\mathbb{R}^d} \frac{F(\xi)}{(1+|\xi|^2)^M}\, d\mu(\xi) \leq \int_{\{|\xi| \leq 1\}} \frac{F(\xi)}{(1+|\xi|^2)^M}\, d\mu(\xi) + \sum_{k=1}^{\infty} k^{-2M} \int_{\{k < |\xi| \leq k+1\}} F(\xi)\, d\mu(\xi).$$

For k large enough, we have

$$\int_{\{k < |\xi| \leq k+1\}} F(\xi)\, d\mu(\xi) \leq 2\, C_1 \left(\frac{k+1}{\delta}\right)^{d+K}$$

and the series above converges to a finite value if $M > (d+K+1)/2$, proving our claim. If $p = 1$, it suffices to reproduce the above argument with $F \equiv 1$. □

Corollary 1 *Under the previous assumptions, if the couple* (μ, w) *belongs to* $\mathcal{B}^p(U, B)$ *for some p with* $1 \leq p < \infty$, *then* μ *must be a tempered measure and so is the (complex) measure* $F\, d\mu$ *if* $F \in L^q(\mu)$ *with* $1 \leq q \leq \infty$.

Proof The open set U contains a ball of radius $\epsilon > 0$, which we can assume to be centered at the origin. Since $C_0^\infty(B(0,\epsilon))$ is contained in $\mathcal{F}^{-1}L^p_w(U)$, Proposition 5 shows that μ is tempered if $1 \leq p < \infty$. If $M > 0$ is such that

$$\int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^M}\, d\mu(\xi) < \infty$$

and $F \in L^q(\mu)$, where $1 < q < \infty$, we have, letting p be the conjugate exponent of q and $C = \left(\int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^{sp}}\, d\mu(\xi)\right)^{1/p}$, that

$$\int_{\mathbb{R}^d} \frac{|F(\xi)|}{(1+|\xi|^2)^s}\, d\mu \leq C \left(\int_{\mathbb{R}^d} |F(\xi)|^q\, d\mu(\xi)\right)^{1/q} < \infty$$

if $s \geq M/p$. If $q = 1$, $F\, d\mu$ is a bounded measure and is thus also tempered. If

$q=\infty$, the fact that $F\,d\mu$ is tempered follows from the fact that μ is tempered together with the inequality $|F|\,d\mu \le \|F\|_\infty\,d\mu$. $\square$

The next result gives a different interpretation of the fact that (μ, w) belongs to $\mathcal{B}^p(U,B)$ or $\mathcal{F}^p(U,A,B)$. We will use the property that for any $F \in L^q(\mu)$ with $1 \le q \le \infty$, $\mathcal{F}^{-1}\{F\,d\mu\}$ is well-defined as a tempered distribution by the previous corollary.

Theorem 1 *Let μ be a tempered positive Borel measure on $\mathbb{R}^d$ and let w be a moderate $\mathbb{R}^d$ satisfying (1) and (2). Let p with $1 \le p < \infty$, let q be the conjugate exponent of p and let U be a non-empty open subset of $\mathbb{R}^d$. Then, the following are equivalent.*

(a) *$(\mu, w) \in \mathcal{B}^p(U,B)$ for some $B > 0$.*
(b) *For any $F \in L^q(\mu)$, there exists $h \in \mathcal{F}^{-1}L^q_{\tilde{w}}(\mathbb{R}^d)$ with $\mathcal{F}^{-1}\{F\,d\mu\} = h$ on the open set U, where $\tilde{w} = w^{1-q}$* if *$1 < p < \infty$ and $\tilde{w} = w^{-1}$ if $p = 1$.*

Proof Assume first that $(\mu, w) \in \mathcal{B}^p(U,B)$. This means that mapping $T : \mathcal{F}^{-1}L^p_w(U) \to L^p(\mu) : u \mapsto \hat{u}$ is bounded and, thus, so is the adjoint mapping $T^* : L^p(\mu)' \to \left(\mathcal{F}^{-1}L^p_w(U)\right)'$. Using the $(L^p(\mu), L^q(\mu))$ duality, given any $F \in L^q(\mu)$, there exists thus an element ℓ_F of $\left(\mathcal{F}^{-1}L^p_w(U)\right)'$ such that

$$\int_{\mathbb{R}^d} \hat{u}(\xi)\,\overline{F(\xi)}\,d\mu(\xi) = \ell_F(u), \quad u \in \mathcal{F}^{-1}L^p_w(U).$$

By the Hahn–Banach theorem, the continuous linear form ℓ_F can be extended to an element of $\left(\mathcal{F}L^p_w(\mathbb{R}^d)\right)'$ and using the duality results in Proposition 2 and Proposition 3, this means that, given any $F \in L^q(\mu)$, there exists a corresponding element $h \in \mathcal{F}^{-1}L^q_{\tilde{w}}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \hat{\varphi}(\xi)\,\overline{\hat{h}(\xi)}\,d\xi = \int_{\mathbb{R}^d} \hat{\varphi}(\xi)\,\overline{F(\xi)}\,d\mu(\xi), \quad \varphi \in C_0^\infty(U). \tag{9}$$

This last identity means exactly that $\mathcal{F}^{-1}\{F\,d\mu\} = h$ as distributions on the open set U. Conversely, if (b) holds, given any $F \in L^q(\mu)$, there exists $h \in \mathcal{F}^{-1}L^q_{\tilde{w}}(\mathbb{R}^d)$ such that $\mathcal{F}^{-1}\{F\,d\mu\} = h$ on U. In particular, there exists a constant $C(F) > 0$ such that

$$\left|\int_{\mathbb{R}^d} \hat{\varphi}(\xi)\,\overline{F(\xi)}\,d\mu(\xi)\right| \le C(F)\,\|\varphi\|_{p,w}, \quad \varphi \in C_0^\infty(U).$$

Thus, for any $F \in L^q(\mu)$, the linear functional ℓ_F defined by

$$\ell_F(\varphi) = \int_{\mathbb{R}^d} \hat{\varphi}(\xi)\,\overline{F(\xi)}\,d\mu(\xi), \quad \varphi \in C_0^\infty(U),$$

can be extended to a continuous linear functional on $\mathcal{F}^{-1}L^p_w(U)$, i.e. an element of

$\big(\mathcal{F}^{-1}L^p_w(U)\big)'$. Note that if $F_n \to F$ in $L^q(\mu)$ and $\ell_{F_n} \to \ell$ in $\big(\mathcal{F}^{-1}L^p_w(U)\big)'$, we have, for any $\varphi \in C_0^\infty(U)$,

$$\ell(\varphi) = \lim_{n\to\infty} \int_{\mathbb{R}^d} \hat{\varphi}(\xi)\, \overline{F_n(\xi)}\, d\mu(\xi) = \int_{\mathbb{R}^d} \hat{\varphi}(\xi)\, \overline{F(\xi)}\, d\mu(\xi) = \ell_F(\varphi),$$

and thus $\ell = \ell_F$, using the density of $C_0^\infty(U)$ in $\mathcal{F}^{-1}L^p_w(U)$. It follows that the linear mapping $L^q(\mu) \to \big(\mathcal{F}^{-1}L^p_w(U)\big)'$: $F \mapsto \ell_F$ is closed and thus continuous, using the closed graph theorem. There exists thus a constant $B > 0$ such that

$$\left|\int_{\mathbb{R}^d} \hat{\varphi}(\xi)\, \overline{F(\xi)}\, d\mu(\xi)\right| \le B \left(\int_{\mathbb{R}^d} |F(\xi)|^q\, d\mu(\xi)\right)^{1/q} \|\varphi\|_{p,w},$$

whenever $\varphi \in C_0^\infty(U)$ and $F \in L^q(\mu)$. Hence, we obtain the inequality

$$\left(\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p\, d\mu(\xi)\right)^{1/p} = \sup_{\substack{F \in L^q(\mu)\\ \left(\int |F|^q\, d\mu\right)^{1/q} = 1}} \left|\int_{\mathbb{R}^d} \hat{\varphi}(\xi)\, \overline{F(\xi)}\, d\mu(\xi)\right| \le B\, \|\varphi\|_{p,w},$$

for any $\varphi \in C_0^\infty(U)$, proving (a). □

Corollary 2 *Under the previous assumptions, the following are equivalent.*

(a) $(\mu, w) \in \mathcal{B}^p(U, B)$ *for some* $B > 0$ *and some non-empty bounded open set* $U \subset \mathbb{R}^d$.

(b) $(\mu, w) \in \mathcal{B}^p(U, B(U))$ *for all non-empty bounded open set* $U \subset \mathbb{R}^d$ *where* $B(U) > 0$ *depends on* U.

(c) *For any* $F \in L^q(\mu)$, $\mathcal{F}^{-1}\{F\, d\mu\} \in \mathcal{F}^{-1}_{\text{loc}}L^q_{\tilde{w}}(\mathbb{R}^d)$ *where* $\tilde{w} = w^{1-q}$ if $1 < p < \infty$ *and* $\tilde{w} = w^{-1}$ *if* $p = 1$.

Proof Clearly (b) implies (a). Conversely, if (a) holds, there exists $\epsilon > 0$ such that $(\mu, w) \in \mathcal{B}^p(B(0,\epsilon), B)$ for some $B > 0$, using (8). If U is a bounded open set in $\mathbb{R}^d$, we can use a partition of unity argument to construct N functions $\zeta_1, \ldots, \zeta_N \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(\zeta_i) \subset B(a_i, \epsilon)$, where $a_i \in \mathbb{R}^d$ such that $\sum_{i=1}^N \zeta_i = 1$ on a neighborhood of U. We have then, for $\varphi \in C_0^\infty(U)$, that

$$\begin{aligned}\left(\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p\, d\mu(\xi)\right)^{1/p} &= \left(\int_{\mathbb{R}^d} \Big|\sum_{i=1}^N (\hat{\varphi} * \hat{\zeta}_i)(\xi)\Big|^p\, d\mu(\xi)\right)^{1/p}\\ &\le \sum_{i=1}^N \left(\int_{\mathbb{R}^d} \big|(\hat{\varphi} * \hat{\zeta}_i)(\xi)\big|^p\, d\mu(\xi)\right)^{1/p} \le B \sum_{i=1}^N \|\varphi\, \zeta_i\|_{p,w} \le B(U)\, \|\varphi\|_{p,w},\end{aligned}$$

where Lemma 1 was used in the last step, showing that (a) holds. The equivalence of (b) and (c) then follows directly from Theorem 1. □

Theorem 2 *Let μ be a tempered positive Borel measure on $\mathbb{R}^d$ and let w be a moderate weight on $\mathbb{R}^d$ satisfying (1) and (2). Let p with $1 \le p < \infty$, let q be the conjugate exponent of p and let U be a non-empty subset of $\mathbb{R}^d$. Define the weight $\tilde{w}$ by $\tilde{w} = w^{1-q}$ if $1 < p < \infty$ and $\tilde{w} = w^{-1}$ if $p = 1$. Then, $(\mu, w) \in \mathcal{F}^p(U, A, B)$ for some $A, B > 0$ if and only if*

(a) *For any $F \in L^q(\mu)$, there exists $v \in \mathcal{F}^{-1}L^q_{\tilde{w}}(\mathbb{R}^d)$* with *$\mathcal{F}^{-1}\{F\,d\mu\} = v$ on the open set U.*
(b) *For any $h \in \mathcal{F}^{-1}L^q_{\tilde{w}}(\mathbb{R}^d)$, there exists $F \in L^q(\mu)$ such that $\mathcal{F}^{-1}\{F\,d\mu\} = h$ on the open set U.*

Proof If $(\mu, w) \in \mathcal{F}^p(U, A, B)$, then $(\mu, w) \in \mathcal{B}^p(U, B)$ and (a) follows from Theorem 1. Since the linear mapping $T : \mathcal{F}^{-1}L^p_w(U) \to L^p(\mu) : u \mapsto \hat{u}$ is bounded and also bounded below the adjoint mapping $T^* : (L^p(\mu))' \to \left(\mathcal{F}^{-1}L^p_w(U)\right)'$ is bounded and surjective. If $h \in \mathcal{F}^{-1}L^q_{\tilde{w}}(\mathbb{R}^d)$, the linear mapping

$$l(\varphi) = \int_{\mathbb{R}^d} \varphi(\xi)\,\overline{\hat{h}(\xi)}\,d\xi, \quad \varphi \in C_0^\infty(U),$$

can be extended uniquely to an element of $\left(\mathcal{F}^{-1}L^p_w(U)\right)'$. Hence, using the surjectivity of T^*, there exists $F \in L^q(\mu)$ such that

$$\int_{\mathbb{R}^d} \hat{\varphi}(\xi)\,\overline{\hat{h}(\xi)}\,d\xi = \int_{\mathbb{R}^d} \hat{\varphi}(\xi)\,\overline{F(\xi)}\,d\mu(\xi)$$

which is equivalent to the identity $\mathcal{F}^{-1}\{F\,d\mu\} = h$ on the open set U, so (b) holds. Conversely, the statement in (a) is equivalent to the boundedness of T^* and thus of T and the statement in (b) is equivalent to the surjectivity of T^* which is equivalent to the topological injectivity of T, i.e. to the lower-bound inequality in (7), showing thus that $(\mu, w) \in \mathcal{F}^p(U, A, B)$ for some $A, B > 0$. □

4 Perturbation by multiplication

In this section, we consider the following natural problem. Suppose we know, for example that a weight w_1 satisfying(1) and (2) and a tempered measure μ_1 are such that the couple (μ, w_1) belongs to $\mathcal{F}^p(U, A, B)$ for some open set $U \subset \mathbb{R}^d$. Does it follow that the couple $(w^{-1}\,\mu_1, w^{-1}\,w_1)$ also belongs to $\mathcal{F}^p(U, A, B)$ if w also satisfies (1) and (2)? Simple examples with $p = 2$ ([8]) show that this is not the case in general. However, we will show that $(w^{-1}\,\mu_1, w^{-1}\,w_1)$ belongs to a larger class $\mathcal{F}^p(V, A', B')$ if V is an open set slightly smaller than U in the sense that $V + B(0, \epsilon) \subset U$, for some $\epsilon > 0$. Note that, letting $\mu_2 = w^{-1}\,\mu_1$ and $w_2 = w^{-1}\,w_1$, we have then $w_1^{-1}\,d\mu_1 = w_2^{-1}\,d\mu_2$. Our goal in this section can then be rephrased more generally as follows. Given two moderate weights w_1, w_2 on $\mathbb{R}^d$ satisfying both (1) (with $v = v_i$ for w_i, $i = 1, 2$) and (2), we will show that if the couple (μ_1, w_1) belongs to $\mathcal{B}^p(U, B)$ (resp. $\mathcal{F}^p(U, A, B)$) for some open set U, then the couple (μ_2, w_2) belongs to $\mathcal{B}^p(V, B')$ (resp. $\mathcal{F}^p(V, A', B')$) if the open set V is as above.

We will need the following lemma which can be found in ([8]). We reproduce it here for the reader's convenience.

Lemma 4 *Let w and v satisfy (1) and (2) and let $F \geq 0$ be a measurable function on $\mathbb{R}^d$ satisfying*

$$\int_{\mathbb{R}^d} F(\xi)\, v(-\xi)\, d\xi < \infty.$$

Then, for any $\xi \in \mathbb{R}^d$, we have the inequalities

$$w(\xi)\left(\int_{\mathbb{R}^d} F(\gamma)\, v^{-1}(\gamma)\, d\gamma\right) \leq (w * F)(\xi) \leq w(\xi)\left(\int_{\mathbb{R}^d} F(\gamma)\, v(-\gamma)\, d\gamma\right). \tag{10}$$

Proof Note first that, since $1 = v(0) \leq v(\gamma)\, v(-\gamma)$, we have

$$\int_{\mathbb{R}^d} F(\gamma)\, v^{-1}(\gamma)\, d\gamma \leq \int_{\mathbb{R}^d} F(\gamma)\, v(-\gamma)\, d\gamma < \infty,$$

Hence,

$$(w * F)(\xi) = \int_{\mathbb{R}^d} F(\xi - \gamma)\, w(\gamma)\, d\gamma \leq w(\xi) \int_{\mathbb{R}^d} F(\gamma)\, v(-\gamma)\, d\gamma,$$

and

$$(w * F)(\xi) = \int_{\mathbb{R}^d} F(\xi - \gamma)\, w(\gamma)\, d\gamma \geq w(\xi) \int_{\mathbb{R}^d} F(\gamma)\, v^{-1}(\gamma)\, d\gamma,$$

which proves the inequalities in (10). □

The next result can be used to deduce weighted inequalities, such as (6) or (7), from unweighted ones (i.e. with $w = 1$) holding for a slightly larger space and vice-versa. The case $p = 2$ of this theorem was proved in ([8]). The case $1 \leq p < \infty$ is proved below by a similar method.

Theorem 3 *Let $\epsilon > 0$ and consider open sets V and U in $\mathbb{R}^d$ such that $V + B(0, \epsilon) \subset U$. Let p with $1 \leq p < \infty$ and let $w_1, w_2 > 0$ be two moderate weights on $\mathbb{R}^d$ satisfying*

$$w_i(\xi + \eta) \leq w_i(\xi)\, v_i(\eta), \quad \xi, \eta \in \mathbb{R}^d,$$

where v_i is tempered for $i = 1, 2$. Let $U, V \subset \mathbb{R}^d$ be open and suppose that, for some $\epsilon > 0$, $V + B(0, \epsilon) \subset U$. Let *$\mu_1, \mu_2$ be positive Borel measures on $\mathbb{R}^d$ satisfying*

$$w_1^{-1}\, d\mu_1 = w_2^{-1}\, d\mu_2$$

and, letting $v := v_1\, v_2$, define the quantity

$$M(\epsilon) = \inf\left\{\frac{\int_{\mathbb{R}^d} |\hat{\psi}(\xi)|^p\, v(-\xi)\, d\xi}{\int_{\mathbb{R}^d} |\hat{\psi}(\xi)|^p\, v^{-1}(\xi)\, d\xi},\ \psi \in C_0^\infty(B(0,\epsilon))\backslash\{0\}\right\} \geq 1. \tag{11}$$

(a) *If* $(\mu_1, w_1) \in \mathcal{B}^p(U,B)$, *then* $(\mu_2, w_2) \in \mathcal{B}^p(V, B(\epsilon))$ *where* $B(\epsilon) = B\, M(\epsilon)$.

(b) *If* $(\mu_1, w_1) \in \mathcal{F}^p(U,A,B)$, *then* $(\mu_2, w_2) \in \mathcal{F}^p(V, A(\epsilon), B(\epsilon))$ *where* $A(\epsilon) = B\, M(\epsilon)^{-1}$ and $B(\epsilon) = B\, M(\epsilon)$.

Proof As we mentioned before, it suffices to prove the required inequalities for test functions in $C_0^\infty(V)$ instead of general elements of $\mathcal{F}^{-1}L^p_w(V)$. Letting $w = w_1\, w_2^{-1}$ and $v = v_1\, v_2$, it is easily checked that both (1) and (2) hold. Suppose that $\psi \in C_0^\infty(B(0,\epsilon))\backslash\{0\}$. Since v is tempered, so is v^{-1} using the inequality $1 = v(0) \leq v(\xi)\, v(-\xi)$ for $\xi \in \mathbb{R}^d$. Furthermore, we have

$$0 < \int_{\mathbb{R}^d} |\hat{\psi}(\xi)|^p\, v^{-1}(\xi)\, d\xi \leq \int_{\mathbb{R}^d} |\hat{\psi}(\xi)|^p\, v(-\xi)\, d\xi < \infty.$$

Using Lemma 4 with w and v replaced with w^{-1} and $v(-\cdot)$, respectively, , we have the pointwise inequalities

$$w^{-1} \leq C_1 \left(|\hat{\psi}|^p * w^{-1}\right) \quad \text{and} \quad |\hat{\psi}|^p * w^{-1} \leq C_2\, w^{-1} \quad \text{on } \mathbb{R}^d,$$

where

$$C_1 = \left(\int_{\mathbb{R}^d} |\hat{\psi}(\xi)|^p\, v^{-1}(-\xi)\, d\xi\right)^{-1} \quad \text{and} \quad C_2 = \int_{\mathbb{R}^d} |\hat{\psi}(\xi)|^p\, v(\xi)\, d\xi.$$

Suppose that $(\mu_1, w_1) \in \mathcal{B}^p(U,B)$. If $\varphi \in C_0^\infty(V)$, we have thus

$$\begin{aligned}
\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p\, d\mu_2(\xi) &= \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p\, w_2(\xi)\, w_1^{-1}(\xi)\, d\mu_1(\xi) \\
&= \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p\, w^{-1}(\xi)\, d\mu_1(\xi) \\
&\leq C_1 \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \left(|\hat{\psi}|^p * w^{-1}\right)(\xi)\, d\mu_1(\xi) \\
&= C_1 \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \left(\int_{\mathbb{R}^d} |\hat{\psi}(\xi-\tau)|^p\, w^{-1}(\tau)\, d\tau\right) d\mu_1(\xi) \\
&= C_1 \int_{\mathbb{R}^d} w^{-1}(\tau) \left(\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p\, |\hat{\psi}(\xi-\tau)|^p\, d\mu_1(\xi)\right) d\tau.
\end{aligned}$$

Since for fixed τ, the function $\xi \mapsto \hat{\varphi}(\xi)\, \hat{\psi}(\xi - \tau)$ is the Fourier transform of the convolution of φ with $e^{2\pi i \tau x}\, \psi$, a function which belongs to $C_0^\infty(V + B(0,\epsilon)) \subset C_0^\infty(U)$ and $(\mu_1, w_1) \in \mathcal{B}(U,B)$, we obtain that

$$
\begin{aligned}
\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, d\mu_2(\xi) &\leq C_1 \, B \int_{\mathbb{R}^d} w^{-1}(\tau) \left(\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, |\hat{\psi}(\xi - \tau)|^p \, w_1(\xi) \, d\xi \right) d\tau \\
&= C_1 \, B \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \left(\int_{\mathbb{R}^d} |\hat{\psi}(\xi - \tau)|^p \, w^{-1}(\tau) \, d\tau \right) w_1(\xi) \, d\xi \\
&= C_1 \, B \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \left(|\hat{\psi}|^p * w^{-1} \right)(\xi) \, w_1(\xi) \, d\xi \\
&\leq C_1 \, B \, C_2 \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, w^{-1}(\xi) \, w_1(\xi) \, d\xi \\
&= C_1 \, B \, C_2 \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, w_2(\xi) \, d\xi
\end{aligned}
$$

from which the conclusion of statement (a) immediately follows.

Suppose now that $(\mu_1, w_1) \in \mathcal{F}^p(U, A, B)$. By part (a), it suffices to prove the first inequality in (7). If $\varphi \in C_0^\infty(V)$, we have thus

$$
\begin{aligned}
\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, d\mu_2(\xi) &= \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, w^{-1}(\xi) \, d\mu_1(\xi) \\
&\geq C_2^{-1} \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \left(|\hat{\psi}|^2 * w^{-1} \right)(\xi) \, d\mu_1(\xi) \\
&= C_2^{-1} \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \left(\int_{\mathbb{R}^d} |\hat{\psi}(\xi - \tau)|^p \, w^{-1}(\tau) \, d\tau \right) d\mu_1(\xi) \\
&= C_2^{-1} \int_{\mathbb{R}^d} w^{-1}(\tau) \left(\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, |\hat{\psi}(\xi - \tau)|^p \, d\mu_1(\xi) \right) d\tau.
\end{aligned}
$$

Since for fixed τ, we have $\varphi(\xi) \, \hat{\psi}(\xi - \tau) = \hat{\phi}(\xi)$ with $\phi \in C_0^\infty(U)$, we obtain, using our assumption, that

$$
\begin{aligned}
\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, d\mu_2(\xi) &\geq C_2^{-1} A \int_{\mathbb{R}^d} w^{-1}(\tau) \left(\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, |\hat{\psi}(\xi - \tau)|^p \, w_1(\xi) \, d\xi \right) d\tau \\
&= C_2^{-1} A \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \left(\int_{\mathbb{R}^d} |\hat{\psi}(\xi - \tau)|^p \, w^{-1}(\tau) \, d\tau \right) w_1(\xi) \, d\xi \\
&= C_2^{-1} A \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \left(|\hat{\psi}|^p * w^{-1} \right)(\xi) \, w_1(\xi) \, d\xi \\
&\geq C_2^{-1} A \, C_1^{-1} \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, w^{-1}(\xi) \, w_1(\xi) \, d\xi \\
&= C_2^{-1} A \, C_1^{-1} \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, w_2(\xi) \, d\xi.
\end{aligned}
$$

Our proof is completed by noticing that the constants $A(\epsilon)$ and $B(\epsilon)$ can be obtained by taking the infimum of the quantity $C_1 \, C_2$ as ψ vary over all non-zero functions in $C_0^\infty(B(0, \epsilon))$ and by using the inequalities obtained above. □

5 Weighted inequalities for functions with a spectrum in a small ball

In this section, our main goal will be to characterize when a pair (μ, w) belongs to $\mathcal{B}^p(U,B)$ (for some $B>0$) or to $\mathcal{F}^p(U,A,B)$ (for some $A,B>0$) in the case where U is a ball with a sufficiently small radius. Using Theorem 3, we can reduce the problem to the unweighted case, i.e. the case where $w=1$. We first need to define the upper and lower-Beurling density of a positive Borel measure on $\mathbb{R}^d$. If $r>0$, we let $I_r=\{x\in\mathbb{R}^d,\ |x_i|\le r/2,\ i=1,\ldots,d\}$, the closed hypercube of side length r centered at the origin in $\mathbb{R}^d$. We will write I for I_1 for simplicity. If μ is a positive Borel measure on $\mathbb{R}^d$, the quantities

$$\mathcal{D}^+(\mu)=\limsup_{R\to\infty}\ \sup_{z\in\mathbb{R}^d}\frac{\mu(z+I_R)}{R^d}\quad\text{and}\quad \mathcal{D}^-(\mu)=\liminf_{R\to\infty}\ \inf_{z\in\mathbb{R}^d}\frac{\mu(z+I_R)}{R^d}$$

are called the *upper and lower-Beurling density* of the measure μ, respectively. If both these densities are equal and finite, we say that the Beurling density of the measure μ exists and we define it to be the quantity $\mathcal{D}(\mu):=\mathcal{D}^+(\mu)=\mathcal{D}^-(\mu)$. Note that the notion of Beurling density, particularly that of a discrete set of points in $\mathbb{R}^d$ (which corresponds to a mesure that assigns a mass of 1 unit at each of these points), is a very useful tool in sampling theory where the type of inequalities we are considering in the case $p=2$ play an essential role (see [4–10, 12, 14, 16, 18]). A positive Borel measure μ is called *translation-bounded* if there exists a constant $C>0$ such that

$$\mu(x+[0,1]^d)\le C\quad \forall x\in\mathbb{R}^d. \tag{12}$$

Note that the space of (complex) measures σ whose total variation $|\sigma|$ is translation-bounded is a special case of amalgam space and is denoted by $W(M,l^\infty)$ (see [2, 3] for more details).

Proposition 6 *([5]) Let μ be a positive Borel measure on $\mathbb{R}^d$. Then, the following are equivalent:*

(a) *μ is translation bounded.*
(b) $\mathcal{D}^+(\mu)<\infty$.
(c) *There exists $f\in L^1(\mathbb{R}^d)$ with $f\ge 0$, $\int f\,dx=1$ and a constant $C>0$ such that $\mu * f\le C$ a.e. on $\mathbb{R}^d$.*

As the last condition in the previous proposition shows, the notion of upper-Beurling density is related to certain convolution inequalities satisfied by the measure μ. The following result will also be used in the proof of our main result in this section.

Theorem 4 ([5]) *Let μ be a positive Borel measure on $\mathbb{R}^d$ and let $h\in L^1(\mathbb{R}^d)$ with $h\ge 0$. Let $A,B>0$ be constants. Then*

(a) *If $\mu * h\le B$ a.e. on $\mathbb{R}^d$, then $\mathcal{D}^+(\mu)\int h\,dx\le B$.*

(b) *If μ is translation-bounded and the inequality $A \le \mu * h$ holds a.e. on $\mathbb{R}^d$, then $A \le \mathcal{D}^-(\mu) \int h\,dx$.*

Lemma 5 *Let μ be a positive Borel measure on $\mathbb{R}^d$. For any $r > 0$, let E_r be a Borel measurable subset of $\mathbb{R}^d$, such that*

$$\{x \in \mathbb{R}^d,\ |x_i| < r/2,\ i = 1, \ldots, d\} \subset E_r \subset I_r.$$

Then

$$\mathcal{D}^+(\mu) = \limsup_{R\to\infty} \sup_{\xi\in\mathbb{R}^d} \frac{\mu(\xi + E_R)}{R^d} \quad \text{and} \quad \mathcal{D}^-(\mu) = \liminf_{R\to\infty} \inf_{\xi\in\mathbb{R}^d} \frac{\mu(\xi + E_R)}{R^d}.$$

Proof Let $0 < \delta < 1$, using the inclusion $I_{\delta R} \subset E_R \subset I_R$, we have the inequalities

$$\delta^d \frac{\mu(\xi + I_{\delta R})}{(\delta R)^d} \le \frac{\mu(\xi + E_R)}{R^d} \le \frac{\mu(\xi + I_R)}{R^d}$$

which imply that

$$\delta^d \mathcal{D}^+(\mu) \le \limsup_{R\to\infty} \sup_{\xi\in\mathbb{R}^d} \frac{\mu(\xi + E_R)}{R^d} \le \mathcal{D}^+(\mu)$$

and

$$\delta^d \mathcal{D}^-(\mu) \le \liminf_{R\to\infty} \inf_{\xi\in\mathbb{R}^d} \frac{\mu(\xi + E_R)}{R^d} \le \mathcal{D}^-(\mu).$$

The result follows by letting $\delta \to 1^-$ in the previous inequalities. □

The following lemma will also be needed. It shows, in particular, the continuous embedding of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ in the amalgam space $W(C, \ell^1)$. (see [2, 3] for the precise definition of this last space and for an overview of applications of general amalgam spaces in Fourier analysis).

Lemma 6 *Let $\psi \in \mathcal{S}(\mathbb{R}^d)$. Then, there exists $C > 0$ such that*

$$\delta^d \sum_{k\in\mathbb{Z}^d} \sup_{\gamma\in I_\delta} |\psi(\xi - k\delta - \gamma)| \le C, \quad \xi \in \mathbb{R}^d,\ 0 < \delta \le 1.$$

Proof Let us define

$$g(\gamma) = \frac{1}{1 + \gamma^2}, \quad \gamma \in \mathbb{R}.$$

and suppose that $0 < \delta \le 1$. If $\xi \in [-\delta/2, \delta/2]$ and $k \in \mathbb{Z}\backslash\{0\}$,

$$\inf_{|\gamma|\le\delta/2}|\xi-\delta k-\gamma| = \min\{|\xi-k\delta-\delta/2|,|\xi-k\delta+\delta/2|\}\ge\delta\,(|k|-1).$$

Hence,

$$\delta\sum_{k\in\mathbb{Z}}\sup_{|\gamma|\le\delta/2} g(\xi-k\delta-\gamma)\le\left(\delta+\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{\delta}{1+\delta^2\,(|k|-1)^2}\right)$$
$$=3\,\delta+2\sum_{n=1}^{\infty}\frac{\delta}{1+\delta^2\,n^2}\le 3+2\int_0^\infty\frac{1}{1+x^2}\,dx=c<\infty.$$

Since the left-hand side of the previous expression is δ-periodic as a function of ξ, if follows that the inequality holds for all $\xi\in\mathbb{R}$. If $\psi\in\mathcal{S}(\mathbb{R}^d)$, we have the estimate

$$|\psi(\gamma)|\le C_1\prod_{i=1}^d g(\gamma_i),\quad \gamma\in\mathbb{R}^d.$$

Therefore, for any $\xi\in\mathbb{R}^d$, we obtain

$$\delta^d\sum_{k\in\mathbb{Z}^d}\sup_{\gamma\in I_\delta}|\psi(\xi-\delta k-\gamma)|\le\sum_{k\in\mathbb{Z}^d}C_1\prod_{i=1}^d\delta\sup_{|\gamma_i|\le\delta/2}g(\xi_i-\delta k_i-\gamma_i)$$
$$=C_1\prod_{i=1}^d\delta\sum_{k_i\in\mathbb{Z}}\sup_{|\gamma_i|\le\delta/2}g(\xi_i-\delta k_i-\gamma_i)\le C_1\,c^d=C<\infty.$$

□

The inequalities (13) in the following theorem are known as the Plancherel–Polya inequalities (see [21]) and one can show that they hold for $\delta<1$ (i.e. one can take $\delta_0=1$ in Theorem 5). For the convenience of the reader, we provide a quick proof for the weaker result stated below as this is all we will need. Furthermore,we do not know of a reference for (14) which gives the limiting values for the best constants in the inequalities as $\delta\to 0^+$. These will be used in the proof of Theorem 6.

Theorem 5 (*Plancherel–Polya*) *Let p with $1\le p<\infty$. Then, there exists δ_0 with $0<\delta_0\le 1$ such that, if with $0<\delta<\delta_0$, there exists constants $C_1(\delta), C_2(\delta)>0$ such that*

$$C_1(\delta)\,\|\hat\varphi\|_p\le\left(\sum_{k\in\mathbb{Z}^d}\delta^d\,|\hat\varphi(\delta k))|^p\right)^{1/p}\le C_2(\delta)\,\|\hat\varphi\|_p,\quad \varphi\in C_0^\infty(I).\tag{13}$$

Furthermore, if $C_1(\delta)$ and $C_2(\delta)$ are the best constants in the inequality (13), we have

$$\lim_{\delta\to 0^+}C_1(\delta)=\lim_{\delta\to 0^+}C_2(\delta)=1.\tag{14}$$

Proof As usual we let q be the dual exponent of p. We give the proof for the case $1<p<\infty$, as the case $p=1$ (where $q=\infty$) can be dealt with in a similar way by replacing by 1 any term raised to the power $\frac{p}{q}$ or $\frac{1}{q}$ in the proof below.

We have, using Minkowski's inequality,

$$\begin{aligned}\left(\sum_{k\in\mathbb{Z}^d}\delta^d\,|\hat{\varphi}(\delta k)|^p)\right)^{1/p} &= \left(\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}|\hat{\varphi}(\delta k)|^p\,d\gamma\right)^{1/p}\\ &= \left(\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}|\hat{\varphi}(\delta k)-\hat{\varphi}(\gamma)+\hat{\varphi}(\gamma)|^p\,d\gamma\right)^{1/p}\\ &\leq \left(\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}|\hat{\varphi}(\delta k)-\hat{\varphi}(\gamma)|^p\,d\gamma\right)^{1/p}+\left(\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}|\hat{\varphi}(\gamma)|^p\,d\gamma\right)^{1/p}\\ &\leq \left(\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}|\hat{\varphi}(\delta k)-\hat{\varphi}(\gamma)|^p\,d\gamma\right)^{1/p}+\left(\int_{\mathbb{R}^d}|\hat{\varphi}(\gamma)|^p\,d\gamma\right)^{1/p}.\end{aligned}$$

Similarly, we have also

$$\begin{aligned}\left(\int_{\mathbb{R}^d}|\hat{\varphi}(\gamma)|^p\,d\gamma\right)^{1/p} &= \left(\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}|\hat{\varphi}(\gamma)|^p\,d\gamma\right)^{1/p}\\ &= \left(\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}|\hat{\varphi}(\gamma)-\hat{\varphi}(\delta k)+\hat{\varphi}(\delta k)|^p\,d\gamma\right)^{1/p}\\ &\leq \left(\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}|\hat{\varphi}(\delta k)-\hat{\varphi}(\gamma)|^p\,d\gamma\right)^{1/p}+\left(\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}|\hat{\varphi}(\delta k)|^p\,d\gamma\right)^{1/p}\\ &= \left(\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}|\hat{\varphi}(\delta k)-\hat{\varphi}(\gamma)|^p\,d\gamma\right)^{1/p}+\left(\sum_{k\in\mathbb{Z}^d}\delta^d\,|\hat{\varphi}(\delta k))|^p)\right)^{1/p}.\end{aligned}$$

Hence, to prove (13) and (14), it suffices to show that

$$\left(\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}|\hat{\varphi}(\delta k)-\hat{\varphi}(\gamma)|^p\,d\gamma\right)^{1/p}\leq C(\delta)\,\|\hat{\varphi}\|_p,\quad \varphi\in C_0^\infty(I). \tag{15}$$

where $C(\delta)\to 0$ as $\delta\to 0$. Choosing $\beta\in C_0^\infty(\mathbb{R}^d)$ so that $\beta=1$ on a neighborhood of I and letting $\psi=\hat{\beta}$, we have $\hat{\varphi}=\hat{\varphi}*\psi$ if $\varphi\in C_0^\infty(\mathbb{R}^d)$ is supported in I.

We have thus, using Hölder's inequality,

$$\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}|\hat{\varphi}(\delta k)-\hat{\varphi}(\gamma)|^p\,d\gamma$$
$$=\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}\left|\int_{\mathbb{R}^d}[\psi(\delta k-\tau)-\psi(\gamma-\tau)]\,\hat{\varphi}(\tau)\,d\tau\right|^p d\gamma$$
$$\leq\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}\left(\int_{\mathbb{R}^d}|\psi(\delta k-\tau)-\psi(\gamma-\tau)|\,|\hat{\varphi}(\tau)|^p\,d\tau\right)S(\gamma)\,d\gamma,$$

where $S(\gamma)=\left(\int_{\mathbb{R}^d}|\psi(\delta k-\tau)-\psi(\gamma-\tau)|\,d\tau\right)^{p/q}\leq(2\,\|\psi\|_1)^{p/q}$. It follows thus that

$$\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}|\hat{\varphi}(\delta k)-\hat{\varphi}(\gamma)|^p\,d\gamma$$
$$\leq(2\,\|\psi\|_1)^{p/q}\int_{\mathbb{R}^d}\left\{\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}|\psi(\delta k-\tau)-\psi(\gamma-\tau)|\,d\gamma\right\}|\hat{\varphi}(\tau)|^p\,d\tau.$$

Let

$$H_\delta(\tau)=\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}|\psi(\delta k-\tau)-\psi(\gamma-\tau)|\,d\gamma,\quad \tau\in\mathbb{R}^d.$$

By the mean-value theorem, if $\gamma\in\delta k+I_\delta$, we have

$$|\psi(\delta k-\tau)-\psi(\gamma-\tau)|\leq\delta\sqrt{d}\sum_{1\leq i\leq d}\sup_{\xi\in I_\delta}|\psi_{\xi_i}(\xi+\delta k-\tau)|.$$

Hence,

$$H_\delta(\tau)\leq\sum_{k\in\mathbb{Z}^d}\delta^{d+1}\sqrt{d}\sum_{1\leq i\leq d}\sup_{\xi\in\delta k+I_\delta}|\psi_{\xi_i}(\xi-\tau)|\,d\gamma$$
$$=\delta\sqrt{d}\sum_{1\leq i\leq d}\delta^d\sum_{k\in\mathbb{Z}^d}\sup_{\xi\in\delta k+I_\delta}|\psi_{\xi_i}(\xi-\tau)|,\quad \tau\in\mathbb{R}^d.$$

Applying Lemma 6 to each of the functions $\psi_{\xi_i}\in\mathcal{S}(\mathbb{R}^d)$, $i=1,\ldots,d$, we deduce the existence of a constant $A>0$ such that $H_\delta(\tau)\leq A\,\delta$. It follows that the inequality (15) holds with $C(\delta)=(2\,\|\psi\|_1)^{1/q}A^{1/p}\,\delta^{1/p}\to 0$ as $\delta\to 0^+$, proving our claim. □

The following theorem is related to the Logvinenko–Sereda theorem ([15]; see also Proposition 3.34 in [18]) in which the measure μ in the next theorem is of the form $d\mu=\chi_E(\xi)\,d\xi$ where E is a measurable subset of $\mathbb{R}^d$.

Theorem 6 *Let μ be a locally finite, positive Borel measure on $\mathbb{R}^d$ and let p with $1\leq p<\infty$. Then, the following are equivalent.*

(a) *There exist constants $A,B>0$ and $\epsilon>0$ such that*

$$A\,\|\hat{\rho}\|_p^p\leq\int_{\mathbb{R}^d}|\hat{\rho}(\xi)|^p\,d\mu(\xi)\leq B\,\|\hat{\rho}\|_p^p,\quad \rho\in C_0^\infty(I_\epsilon).\tag{16}$$

(b) *We have $0<\mathcal{D}^-(\mu)\leq\mathcal{D}^+(\mu)<\infty$.*

Moreover, if (a) *holds for* $\epsilon > 0$ *and we denote by* $A(\eta)$ *and* $B(\eta)$ *respectively the best constants* A *and* B *such that the inequalities in* (16) *holds for all functions* $\rho \in C_0^\infty(I_\eta)$, *where* $0 < \eta \leq \epsilon$, *then these constants satisfy the inequalities* $A(\eta) \leq \mathcal{D}^-(\mu) \leq \mathcal{D}^+(\mu) \leq B(\eta)$ *and*

$$\lim_{\eta \to 0^+} A(\eta) = \mathcal{D}^-(\mu) \quad \text{while} \quad \lim_{\eta \to 0^+} B(\eta) = \mathcal{D}^+(\mu).$$

Proof The proof below deals only with the case $1 < p < \infty$, as the case $p = 1$ (where $q = \infty$) can be dealt with in a similar way by replacing by 1 any term raised to the power $\frac{p}{q}$ or $\frac{1}{q}$. Suppose first that (a) holds for some $\epsilon > 0$. Then, letting $\rho(x) = \overline{\rho_0(x)}\, e^{2\pi i \eta x}$, where $\rho_0 \in C_0^\infty(I_\epsilon)$ and $\rho_0 \neq 0$, we have $|\hat{\rho}(\xi)| = |\hat{\rho}_0(\eta - \xi)|$ and using the inequalities in (16), we obtain that

$$A\, \|\hat{\rho}_0\|_p^p \leq \int_{\mathbb{R}^d} |\hat{\rho}_0(\eta - \xi)|^p \, d\mu(\xi) \leq B\, \|\hat{\rho}_0\|_p^p, \quad \eta \in \mathbb{R}^d,$$

or, equivalently, that

$$A\, \|\hat{\rho}_0\|_p^p \leq (\mu * |\hat{\rho}_0|^p)(\eta) \leq B\, \|\hat{\rho}_0\|_p^p, \quad \eta \in \mathbb{R}^d.$$

This implies, using Proposition 6 and Theorem 4, that

$$A \leq \mathcal{D}^-(\mu) \leq \mathcal{D}^+(\mu) \leq B$$

and thus that (b) holds. Conversely, if (b) holds, and $\epsilon > 0$ is given, note that any function $\rho \in C_0^\infty(I_\epsilon)$ can be written in the form $\rho(x) = \epsilon^{-d(1-1/p)}\, \varphi(x/\epsilon)$, where $\varphi \in C_0^\infty(I)$ and $\|\hat{\rho}\|_p = \|\hat{\varphi}\|_p$.

It follows that the inequalities in (16) are equivalent to

$$A\, \|\hat{\varphi}\|_p^p \leq \int_{\mathbb{R}^d} \epsilon^d\, |\hat{\varphi}(\epsilon\, \xi)|^p \, d\mu(\xi) \leq B\, \|\hat{\varphi}\|_p^p, \quad \varphi \in C_0^\infty(I). \tag{17}$$

For any $\epsilon > 0$, let μ_ϵ be the measure defined by

$$\int_{\mathbb{R}^d} \phi(\xi)\, d\mu_\epsilon(\xi) = \int_{\mathbb{R}^d} \epsilon^d\, \phi(\epsilon\, \xi)\, d\mu(\xi), \quad \phi \in C_0(\mathbb{R}^d).$$

The inequalities in (16) can thus also be written using (17) as

$$A\, \|\hat{\varphi}\|_p^p \leq \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, d\mu_\epsilon(\xi) \leq B\, \|\hat{\varphi}\|_p^p, \quad \varphi \in C_0^\infty(I). \tag{18}$$

Note that if $\delta > 0$ and $\xi \in \mathbb{R}^d$, we have

$$\mu_\epsilon(\xi + I_\delta) = \epsilon^d\, \mu(\xi/\epsilon + I_{\delta/\epsilon}) = \delta^d\, \frac{\mu(\xi/\epsilon + I_{\delta/\epsilon})}{(\delta/\epsilon)^d}$$

and, in particular,

$$\delta^d \inf_{\xi' \in \mathbb{R}^d} \frac{\mu(\xi' + I_{\delta/\epsilon})}{(\delta/\epsilon)^d} \leq \mu_\epsilon(\xi + I_\delta) \leq \delta^d \sup_{\xi' \in \mathbb{R}^d} \frac{\mu(\xi' + I_{\delta/\epsilon})}{(\delta/\epsilon)^d}. \tag{19}$$

Let $\beta \in C_0^\infty(\mathbb{R}^d)$ with $\beta = 1$ on a neighborhood of I and let $\psi = \hat{\beta}$. As before, we have then $\hat{\varphi} = \hat{\varphi} * \psi$, for $\varphi \in C_0^\infty(I)$ and, in particular, for any $\xi \in \mathbb{R}^d$, using Hölder's inequality, we have

$$\begin{aligned} |\hat{\varphi}(\xi)|^p &= \left| \int_{\mathbb{R}^d} \psi(\xi - \gamma)\, \hat{\varphi}(\gamma)\, d\gamma \right|^p \\ &\leq \int_{\mathbb{R}^d} |\psi(\xi - \gamma)|\, |\hat{\varphi}(\gamma)|^p\, d\gamma \left(\int_{\mathbb{R}^d} |\psi(\xi - \gamma)|\, d\gamma \right)^{p/q} \\ &= \|\psi\|_1^{p/q} \int_{\mathbb{R}^d} |\psi(\xi - \gamma)|\, |\hat{\varphi}(\gamma)|^p\, d\gamma. \end{aligned}$$

Letting $\delta = 1$, we can use (19) and the fact that $\mathcal{D}^+(\mu) < \infty$, to find a number $M_0 > 0$ and $\epsilon_0 > 0$ such that

$$\mu_\epsilon(\xi + I) \leq M_0 \quad \xi \in \mathbb{R}^d,\ 0 < \epsilon \leq \epsilon_0.$$

We have, in particular,

$$\mu_{\epsilon_0}(\xi + I) \leq M_0, \quad \xi \in \mathbb{R}^d.$$

Hence, letting $C = \|\psi\|_1^{p/q}$ and using Fubini's theorem, we have, for any $\varphi \in C_0^\infty(I)$, that

$$\begin{aligned} \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p\, d\mu_{\epsilon_0}(\xi) &\leq C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\psi(\xi - \gamma)|\, |\hat{\varphi}(\gamma)|^p\, d\gamma\, d\mu_{\epsilon_0}(\xi) \\ &= C \int_{\mathbb{R}^d} \left\{ \int_{\mathbb{R}^d} |\psi(\xi - \gamma)|\, d\mu_{\epsilon_0}(\xi) \right\} |\hat{\varphi}(\gamma)|^p\, d\gamma. \end{aligned}$$

Using Lemma 6, there exists thus a number $M_1 > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^d} |\psi(\xi - \gamma)|\, d\mu_{\epsilon_0}(\xi) &\leq \sum_{k \in \mathbb{Z}^d} \int_{k+I} |\psi(\xi - \gamma)|\, d\mu_{\epsilon_0}(\xi) \\ &\leq M_0 \sum_{k \in \mathbb{Z}^d} \sup_{\gamma \in I} |\psi(\xi - k - \gamma)| \leq M_1. \end{aligned}$$

Hence, it follows that there exists thus a number $M > 0$ such that

$$\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p\, d\mu_{\epsilon_0}(\xi) \leq M\, \|\varphi\|_p^p, \quad \varphi \in C_0^\infty(I).$$

If $\delta > 0$, define the set Q_δ as $\{\xi \in \mathbb{R}^d,\ -\delta/2 \leq \xi_i < \delta/2,\ i = 1, \ldots, d\}$. If $\varphi \in C_0^\infty(I)$, let $Y_{k,\delta,\epsilon}(\varphi) = \int_{\delta k + Q_\delta} |\hat{\varphi}(\delta k)|^p\, d\mu_\epsilon(\xi)$. If $0 < \epsilon < \epsilon_0$, we can write, using Minkoswki's inequality twice, that

$$
\begin{aligned}
\left(\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, d\mu_\epsilon(\xi)\right)^{1/p} &= \left(\sum_{k\in\mathbb{Z}^d} \int_{\delta k+Q_\delta} |\hat{\varphi}(\xi)|^p \, d\mu_\epsilon(\xi)\right)^{1/p} \\
&= \left(\sum_{k\in\mathbb{Z}^d} \int_{\delta k+Q_\delta} |\hat{\varphi}(\delta k) + (\hat{\varphi}(\xi) - \hat{\varphi}(\delta k))|^p \, d\mu_\epsilon(\xi)\right)^{1/p} \\
&\le \left(\sum_{k\in\mathbb{Z}^d} \left[\left(Y_{k,\delta,\epsilon}(\varphi)\right)^{1/p} + \left(\int_{\delta k+Q_\delta} |\hat{\varphi}(\xi) - \hat{\varphi}(\delta k))|^p \, d\mu_\epsilon(\xi)\right)^{1/p}\right]^p\right)^{1/p} \\
&\le \left(\sum_{k\in\mathbb{Z}^d} Y_{k,\delta,\epsilon}(\varphi)\right)^{1/p} + \left(\sum_{k\in\mathbb{Z}^d} \int_{\delta k+Q_\delta} |\hat{\varphi}(\xi) - \hat{\varphi}(\delta k))|^p \, d\mu_\epsilon(\xi)\right)^{1/p}.
\end{aligned}
$$

Similarly, letting $Z_{k,\delta,\epsilon}(\varphi) = \int_{\delta k+Q_\delta} |\hat{\varphi}(\xi)|^p \, d\mu_\epsilon(\xi)$, we have

$$
\begin{aligned}
&\left(\sum_{k\in\mathbb{Z}^d} \int_{\delta k+Q_\delta} |\hat{\varphi}(\delta k))|^p \, d\mu_\epsilon(\xi)\right)^{1/p} \\
&\quad \le \left(\sum_{k\in\mathbb{Z}^d} Z_{k,\delta,\epsilon}(\varphi)\right)^{1/p} + \left(\sum_{k\in\mathbb{Z}^d} \int_{\delta k+Q_\delta} |\hat{\varphi}(\xi) - \hat{\varphi}(\delta k))|^p \, d\mu_\epsilon(\xi)\right)^{1/p} \\
&\quad = \left(\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, d\mu_\epsilon(\xi)\right)^{1/p} + \left(\sum_{k\in\mathbb{Z}^d} \int_{\delta k+Q_\delta} |\hat{\varphi}(\xi) - \hat{\varphi}(\delta k))|^p \, d\mu_\epsilon(\xi)\right)^{1/p},
\end{aligned}
$$

showing that

$$
G(\delta,\epsilon,\varphi) - I(\delta,\epsilon,\varphi) \le \left(\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, d\mu_\epsilon(\xi)\right)^{1/p} \le G(\delta,\epsilon,\varphi) + I(\delta,\epsilon,\varphi) \tag{20}
$$

where

$$
G(\delta,\epsilon,\varphi) = \left(\sum_{k\in\mathbb{Z}^d} \int_{\delta k+Q_\delta} |\hat{\varphi}(\delta k))|^p \, d\mu_\epsilon(\xi)\right)^{1/p}
$$

and

$$
I(\delta,\epsilon,\varphi) = \left(\sum_{k\in\mathbb{Z}^d} \int_{\delta k+Q_\delta} |\hat{\varphi}(\xi) - \hat{\varphi}(\delta k))|^p \, d\mu_\epsilon(\xi)\right)^{1/p}.
$$

We first estimate $I(\delta,\epsilon,\varphi)$. We have, using the inclusion $Q_\delta \subset I_\delta$,

$$
\begin{aligned}
(I(\delta,\epsilon,\varphi))^p &\le \sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}|\hat\varphi(\xi)-\hat\varphi(\delta k)|^p\,d\mu_\epsilon(\xi)\\
&=\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}\left|\int_{\mathbb{R}^d}[\psi(\xi-\gamma)-\psi(\delta k-\gamma)]\,\hat\varphi(\gamma)\,d\gamma\right|^p d\mu_\epsilon(\xi).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^d}(\psi(\xi-\gamma)-\psi(\delta k-\gamma))\,\hat\varphi(\gamma)\,d\gamma\right|^p\\
&\quad\le C_1\int_{\mathbb{R}^d}|\psi(\xi-\gamma)-\psi(\delta k-\gamma)|\,|\hat\varphi(\gamma)|^p\,d\gamma,
\end{aligned}
$$

where $C_1=(2\,\|\psi\|_1)^{p/q}$, Fubini's theorem yields

$$
\begin{aligned}
(I(\delta,\epsilon,\varphi))^p &\le C_1\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}\int_{\mathbb{R}^d}|\psi(\xi-\gamma)-\psi(\delta k-\gamma)|\,|\hat\varphi(\gamma)|^p\,d\gamma\;d\mu_\epsilon(\xi)\\
&=C_1\int_{\mathbb{R}^d}\left\{\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}|\psi(\xi-\gamma)-\psi(\delta k-\gamma)|\,d\mu_\epsilon(\xi)\right\}|\hat\varphi(\gamma)|^p\,d\gamma
\end{aligned}
$$

Let

$$
H_\delta(\gamma)=C_1\sum_{k\in\mathbb{Z}^d}\int_{\delta k+I_\delta}|\psi(\xi-\gamma)-\psi(\delta k-\gamma)|\,d\mu_\epsilon(\xi),\quad \gamma\in\mathbb{R}^d.
$$

We have

$$
H_\delta(\gamma)\le C_1\sum_{k\in\mathbb{Z}^d}\sup_{\xi\in\delta k+I_\delta}|\psi(\xi-\gamma)-\psi(\delta k-\gamma)|\,\mu_\epsilon(\delta k+I_\delta).
$$

By the mean-value theorem, if $\xi\in\delta k+I_\delta$, we have

$$
|\psi(\xi-\gamma)-\psi(\delta k-\gamma)|\le\delta\sqrt{d}\sum_{1\le i\le d}\sup_{\xi'\in\delta k+I_\delta}|\psi_{\xi_i}(\xi'-\gamma)|.
$$

Using (19), it follows that

$$
\begin{aligned}
H_\delta(\gamma)&\le C_1\sum_{k\in\mathbb{Z}^d}\delta\sqrt{d}\sum_{1\le i\le d}\sup_{\xi'\in\delta k+I_\delta}|\psi_{\xi_i}(\xi'-\gamma)|\,\delta^d\sup_{\zeta\in\mathbb{R}^d}\frac{\mu(\zeta+I_{\delta/\epsilon})}{(\delta/\epsilon)^d}\\
&=C_1\,\delta\sqrt{d}\sup_{\zeta\in\mathbb{R}^d}\frac{\mu(\zeta+I_{\delta/\epsilon})}{(\delta/\epsilon)^d}\sum_{1\le i\le d}\sum_{k\in\mathbb{Z}^d}\delta^d\sup_{\xi'\in\delta k+I_\delta}|\psi_{\xi_i}(\xi'-\gamma)|.
\end{aligned}
$$

Applying Lemma 6 to each of the functions $\psi_{\xi_i}\in\mathcal{S}(\mathbb{R}^d)$, $i=1,\ldots,d$, we deduce the existence of a constant $C>0$ such that

$$H_\delta(\gamma) \leq C\,\delta \sup_{\zeta\in\mathbb{R}^d} \frac{\mu(\zeta + I_{\delta/\epsilon})}{(\delta/\epsilon)^d}, \quad \gamma \in \mathbb{R}^d.$$

It follows that, for any $\delta > 0$, we have the inequality

$$(I(\delta,\epsilon,\varphi))^p \leq C\,\delta \sup_{\zeta\in\mathbb{R}^d} \frac{\mu(\zeta + I_{\delta/\epsilon})}{(\delta/\epsilon)^d} \|\hat{\varphi}\|_p^p, \quad \varphi \in C_0^\infty(I).$$

We now consider $G(\delta,\epsilon,\varphi)$ and assume that $\delta < \delta_0$ where δ_0 is as in Theorem 5 . Let $C_1(\delta)$ and $C_2(\delta)$ be the best constants in the inequalities (13). Since $Q_\delta \subset I_\delta$, we have, for any $\varphi \in C_0^\infty(I)$, using (19),

$$\begin{aligned}(G(\delta,\epsilon,\varphi))^p &= \sum_{k\in\mathbb{Z}^d} \int_{\delta k + Q_\delta} |\hat{\varphi}(\delta k)|^p \, d\mu_\epsilon(\xi) \leq \sum_{k\in\mathbb{Z}^d} |\hat{\varphi}(\delta k)|^p \, \mu_\epsilon(\delta k + I_\delta) \\ &\leq \left(\sup_{\zeta\in\mathbb{R}^d} \frac{\mu(\zeta + I_{\delta/\epsilon})}{(\delta/\epsilon)^d}\right) \sum_{k\in\mathbb{Z}^d} \delta^d \, |\hat{\varphi}(\delta k)|^p \leq (C_2(\delta))^p \left(\sup_{\zeta\in\mathbb{R}^d} \frac{\mu(\zeta + I_{\delta/\epsilon})}{(\delta/\epsilon)^d}\right) \|\hat{\varphi}\|_p^p.\end{aligned}$$

Similarly, letting $E_\delta = \{x \in \mathbb{R}^d,\ |x_i| < \delta/2,\ i = 1,\ldots,d\}$ for $\delta > 0$, we have

$$\begin{aligned}(G(\delta,\epsilon,\varphi))^p &= \sum_{k\in\mathbb{Z}^d} \int_{\delta k + Q_\delta} |\hat{\varphi}(\delta k)|^p \, d\mu_\epsilon(\xi) \geq \sum_{k\in\mathbb{Z}^d} |\hat{\varphi}(\delta k)|^p \, \mu_\epsilon(\delta k + E_\delta) \\ &\geq \sum_{k\in\mathbb{Z}^d} \delta^d \, |\hat{\varphi}(\delta k)|^p \inf_{\zeta\in\mathbb{R}^d} \frac{\mu(\zeta + E_{\delta/\epsilon})}{(\delta/\epsilon)^d} \geq (C_1(\delta))^p \inf_{\zeta\in\mathbb{R}^d} \frac{\mu(\zeta + E_{\delta/\epsilon})}{(\delta/\epsilon)^d} \|\hat{\varphi}\|_p^p.\end{aligned}$$

Using (20), we obtain thus, for $\varphi \in C_0^\infty(I)$, the inequalities

$$\left(\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, d\mu_\epsilon(\xi)\right)^{1/p} \leq \left[C_2(\delta) + C^{1/p}\,\delta^{1/p}\right] \left(\sup_{\zeta\in\mathbb{R}^d} \frac{\mu(\zeta + I_{\delta/\epsilon})}{(\delta/\epsilon)^d}\right)^{1/p} \|\hat{\varphi}\|_p, \quad (21)$$

and

$$\begin{aligned}&\left(\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, d\mu_\epsilon(\xi)\right)^{1/p} \\ &\geq \left[C_1(\delta)\left(\inf_{\zeta\in\mathbb{R}^d} \frac{\mu(\zeta + E_{\delta/\epsilon})}{(\delta/\epsilon)^d}\right)^{1/p} - C^{1/p}\delta^{1/p}\left(\sup_{\zeta\in\mathbb{R}^d} \frac{\mu(\zeta + I_{\delta/\epsilon})}{(\delta/\epsilon)^d}\right)^{1/p}\right] \|\hat{\varphi}\|_p. \end{aligned} \quad (22)$$

Fix ρ with $0 < \rho < \mathcal{D}^-(\mu)$. Since $C_i(\delta) \to 1$ as $\delta \to 0^+$, for $i = 1,2$, by Theorem 5, we obtain, letting $\delta = \sqrt{\epsilon}$ in (21) and (22), the existence of $\epsilon_1 > 0$ such that

$$\left(\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, d\mu_\epsilon(\xi)\right)^{1/p} \leq [\mathcal{D}^+(\mu) + \rho]^{1/p} \|\hat{\varphi}\|_p, \quad 0 < \epsilon \leq \epsilon_1, \quad \varphi \in C_0^\infty(I), \quad (23)$$

and, using Lemma 5,

$$\left(\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, d\mu_\epsilon(\xi) \right)^{1/p} \geq [\mathcal{D}^-(\mu) - \rho]^{1/p} \, \|\hat{\varphi}\|_p, \quad 0 < \epsilon \leq \epsilon_1, \quad \varphi \in C_0^\infty(I). \tag{24}$$

Using the first part of the proof, we deduce the inequalities

$$\mathcal{D}^-(\mu) - \rho \leq A(\epsilon) \leq \mathcal{D}^-(\mu) \leq \mathcal{D}^+(\mu) \leq B(\epsilon) \leq \mathcal{D}^+(\mu) + \rho, \quad 0 < \epsilon \leq \epsilon_1.$$

This proves our claim. □

Since every set I_ϵ contains the translate of a small ball centered at the origin, we can replace the set I_ϵ by the ball $B(0,\epsilon)$ in the statement of the previous theorem. A consequence of the previous result, of the statement (b) in Theorem 3 and of Theorem 2, is the following characterization.

Theorem 7 *Let μ be a tempered positive Borel measure on $\mathbb{R}^d$ and let w be a moderate $\mathbb{R}^d$ satisfying (1) and (2). Define the weight $\tilde{w} = w^{1-q}$ if $1 < p < \infty$ and $\tilde{w} = w^{-1}$ if $p = 1$. Then, the following are equivalent.*

(a) *There exists $\epsilon > 0$ such that $(\mu, w) \in \mathcal{F}^p(B(0,\epsilon), A, B)$ for some $A, B > 0$.*
(b) *For any $F \in L^q(\mu)$, $\mathcal{F}^{-1}\{F \, d\mu\} \in \mathcal{F}^{-1}_{\text{loc}} L^q_{\tilde{w}}(\mathbb{R}^d)$, and if $\epsilon > 0$ is small enough, for any $h \in \mathcal{F}^{-1} L^q_{\tilde{w}}(\mathbb{R}^d)$ and any $a \in \mathbb{R}^d$, there exists $F \in L^q(\mu)$ such that $\mathcal{F}^{-1}\{F \, d\mu\} = h$ on the open set $B(a,\epsilon)$.*
(c) *We have the inequalities $0 < \mathcal{D}^-(w^{-1}\mu) \leq \mathcal{D}^+(w^{-1}\mu) < \infty$.*

Proof The equivalence of (a) and (b) following directly from Theorem 2 and Corollary 2, it suffices to prove the equivalence of (a) and (c). Assume first that (a) holds. Using (b) of Theorem 3 with $\mu_1 = \mu$, $w_1 = w$, $d\mu_2 = w^{-1} \, d\mu$ and $w_2 = 1$ and using the inclusion $B(0,\epsilon/2) + B(0,\epsilon/2) \subset B(0,\epsilon)$, we deduce that $(w^{-1}\mu, 1) \in \mathcal{F}^p(B(0,\epsilon/2), A', B')$ for some $A', B' > 0$. This implies (c) using Theorem 6. Conversely, if (c) holds, then Theorem 6 shows the existence of $\epsilon > 0$ such that $(w^{-1}\mu, 1) \in \mathcal{F}^p(B(0,\epsilon), A, B)$ for some $A, B > 0$. Using (b) of Theorem 3 with $d\mu_1 = w^{-1} \, d\mu$ $w_1 = 1$, $\mu_2 = \mu$ and $w_2 = w$ and using again the inclusion $B(0,\epsilon/2) + B(0,\epsilon/2) \subset B(0,\epsilon)$, we deduce that $(\mu, w) \in \mathcal{F}^p(B(0,\epsilon/2), A', B')$ for some $A', B' > 0$, yielding (a). □

There is also a version of the Theorem 6 above where we only assume the inequality on the right-hand side. The proof is similar to that of the previous theorem. Alternatively, one can also prove it by applying the previous theorem to the measure $d\mu + s \, d\xi$ where $s > 0$ is a small constant and letting s approach zero.

Theorem 8 *Let μ be a positive Borel measure on $\mathbb{R}^d$ which is locally finite and let p with $1 \leq p < \infty$. Then, the following are equivalent.*

(a) *There exist constants $B > 0$ and $\epsilon > 0$ such that*

$$\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, d\mu(\xi) \leq B \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^p \, d\xi, \quad \varphi \in C_0^\infty(I_\epsilon). \tag{25}$$

(b) *We have* $\mathcal{D}^+(\mu) < \infty$.

Moreover, if (a) *holds for* $\epsilon > 0$ *and we denote by* $B(\eta)$ *the best constant* B *such that the inequalities in* (25) *holds for all functions* $\varphi \in C_0^\infty(I_\eta)$, *where* $0 < \eta \leq \epsilon$, *we have the inequality* $\mathcal{D}^+(\mu) \leq B(\eta)$ and

$$\lim_{\eta \to 0^+} B(\eta) = \mathcal{D}^+(\mu).$$

Combining the previous theorem, the statement (a) in Theorem 3 as well as the equivalence of (a) and (b) in Corollary 2, we can prove following result, following arguments similar to those used in the proof of Theorem 7. The details are left to the reader.

Theorem 9 *Let* μ *be a tempered positive Borel measure on* $\mathbb{R}^d$ *and let* w *be a weight on* $\mathbb{R}^d$ *satisfying* (1) *and* (2). *Let* $U \subset \mathbb{R}^d$ *be a bounded open set. Then, the following are equivalent.*

(a) $(\mu, w) \in \mathcal{B}^p(U, B)$ *for some* $B > 0$.

(b) *For any* $F \in L^q(\mu)$, $\mathcal{F}^{-1}\{F\,d\mu\} \in \mathcal{F}^{-1}_{\text{loc}} L^q_{\tilde{w}}(\mathbb{R}^d)$, *where* $\tilde{w} = w^{1-q}$ *in the case where* $1 < p < \infty$ *and* $\tilde{w} = w^{-1}$ *if* $p = 1$.

(c) $\mathcal{D}^+(w^{-1}\mu) < \infty$.

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References

1. Ascensi, G.: Sampling measures for the Gabor transform. J. Approx. Theory **200**, 40–67 (2015)
2. Feichtinger, H.G.: New results on regular and irregular sampling based on Wiener amalgams in Function spaces (Edwardsville, IL, 1990), 107–121, Lecture Notes in Pure and Appl. Math., 136, Dekker, New York (1992)
3. Feichtinger, H.G.: Wiener amalgams over Euclidean spaces and some of their applications in Function spaces (Edwardsville, IL, 1990), 123–137, Lecture Notes in Pure and Appl. Math., 136, Dekker, New York (1992)
4. Führ, H., Gröchenig, K., Haimi, A., Klotz, A., Romero, J.L.: Density of sampling and interpolation in reproducing kernel Hilbert spaces. J. Lond. Math. Soc. (2) **96**(3), 663–686 (2017)
5. Gabardo, J.-P.: Convolution inequalities for positive Borel measures on $\mathbb{R}^d$ and Beurling density. In: Excursions in harmonic analysis. Volume 2, 23–47, Appl. Numer. Harmon. Anal., Birkhäuser/ Springer, New York (2013)
6. Gabardo, J.-P.: Weighted convolution inequalities and Beurling density. In: Frames and harmonic analysis, 175–200, Contemp. Math., 706, Amer. Math. Soc., Providence, RI (2018)
7. Gabardo, J.-P.: Convolution inequalities in locally compact groups and unitary systems. Numer. Funct. Anal. Optim. **33**(7–9), 100–1030 (2012)
8. Gabardo, J.-P.: Sampling and interpolation in weighted L^2-spaces of band-limited functions. Sampl. Theory Signal Image Process. **17**(2), 197–224 (2018)
9. Gabardo, J.-P., Lai, C.-K.: Frames of multi-windowed exponentials on subsets of $\mathbb{R}^d$. Appl. Comput. Harmon. Anal. **36**(3), 461–472 (2014)

10. Gabardo, J.-P., Lai, C.-K.: Beurling densities and frames of exponentials on the union of small hypercubes **(preprint)**
11. Gröchenig, K.: Weight functions in time-frequency analysis. In: Pseudo-differential operators: partial differential equations and time–frequency analysis. Fields Inst. Commun., vol. 52, pp. 343–366. Amer. Math. Soc., Providence, RI (2007)
12. Gröchenig, K., Razafinjatovo, H.: On Landau's necessary density conditions for sampling and interpolation of band-limited functions. J. Lond. Math. Soc. (2) **54**(3), 557–565 (1996)
13. Heil, C.: An introduction to weighted Wiener amalgams. In: Krishna, M., Radha, R., Thangavelu, S. (eds.) Wavelets and Their Applications (Chennai, January 2002), pp. 183–216. Allied Publishers, New Delhi (2003)
14. Landau, H.J.: Necessary density conditions for sampling and interpolation of certain entire functions. Acta Math. **117**, 37–52 (1967)
15. Logvinenko, V.N., Sereda, J.F.: Equivalent norms in spaces of entire functions of exponential type. Teor. Funkciĭ Funkcion il. Anal. i Priloȓen. Vyp. **20**, 102–111, 175 (1974). (Russian)
16. Nitzan, S., Olevskii, A.: Revisiting Landau's density theorems for Paley-Wiener spaces. C. R. Math. Acad. Sci. Paris **350**(9–10), 509–512 (2012)
17. Ortega-Cerdá, J.: Sampling measures. Publ. Mat. **42**(2), 559–566 (1998)
18. Olevskii, A.M., Ulanovskii, A.: Functions with Disconnected Spectrum, University Lecture Series, 65. American Mathematical Society, Providence, RI (2016)
19. Rudin, W.: Functional analysis, second edition, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York (1991)
20. Strichartz, R.S.: Fourier asymptotics of fractal measures. J. Funct. Anal. **89**(1), 154–187 (1990)
21. Triebel, H.: Theory of function spaces. Reprint of 1983 edition, Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel (2010)