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ORIGINAL PAPER

Logarithmic refinements of a power weighted Hardy–Rellich-type inequality

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Abstract

The principal purpose of this note is to prove a logarithmic refinement of the power weighted Hardy–Rellich inequality on n-dimensional balls, valid for the largest variety of underlying parameters and for all dimensions $n \in \mathbb{N}$, $n \geq 2$.

Keywords Weighted Hardy–Rellich-type inequality · Logarithmic refinements · Differential operators

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1 Introduction

In the recent paper [9], we reconsidered the following sharp inequality, first derived by Caldiroli and Musina [4, Theorem 3.1],

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Dedicated, with admiration, to Fedor Sukochev, mathematician extraordinaire

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$$\int_{\mathbb{R}^n} |x|^{\gamma} |(\Delta f)(x)|^2 \, d^n x \geq C_{n,\gamma} \int_{\mathbb{R}^n} |x|^{\gamma-4} |f(x)|^2 \, d^n x,$$
$$\gamma \in \mathbb{R}, \ f \in C_0^{\infty}(\mathbb{R}^n \backslash \{0\}), \ n \in \mathbb{N}, \ n \geq 2, \tag{1.1}$$

where

$$C_{n,\gamma} = \min_{j \in \mathbb{N}_0} \left\{ \left(\frac{(n-2)^2}{4} - \frac{(\gamma-2)^2}{4} + j(j+n-2) \right)^2 \right\}. \tag{1.2}$$

In addition, we also derived the sharp inequality (sometimes called the Hardy–Rellich inequality),

$$\int_{\mathbb{R}^n} |x|^{\gamma} |(\Delta f)(x)|^2 \, d^n x \geq A_{n,\gamma} \int_{\mathbb{R}^n} |x|^{\gamma-2} |(\nabla f)(x)|^2 \, d^n x,$$
$$\gamma \in \mathbb{R}, \ f \in C_0^{\infty}(\mathbb{R}^n \backslash \{0\}), \ n \in \mathbb{N}, \ n \geq 2, \tag{1.3}$$

where

$$A_{n,\gamma} = \min_{j \in \mathbb{N}_0} \{\alpha_{n,\gamma,\lambda_j}\}, \tag{1.4}$$

with

$$\alpha_{n,\gamma,\lambda_0} = \alpha_{n,\gamma,0} = 4^{-1}(n-\gamma)^2,$$
$$\alpha_{n,\gamma,\lambda_j} = \big[4^{-1}(n+\gamma-4)(n-\gamma) + \lambda_j\big]^2 \Big/ \big[4^{-1}(n+\gamma-4)^2 + \lambda_j\big], \quad j \in \mathbb{N}. \tag{1.5}$$

In the unweighted case $\gamma = 0$, this simplifies to the known fact,

$$A_{n,0} = \begin{cases} n^2/4, & n \geq 5, \\ 3, & n = 4, \\ 25/36, & n = 3, \\ 0, & n = 2. \end{cases} \tag{1.6}$$

In the special case where $C_{n,\gamma}$ in (1.1), or $A_{n,\gamma}$ in (1.3), vanishes, the resulting inequality is rendered trivial (e.g., there is no nontrivial inequality of the type (1.3) in the case $n = 2, \gamma = 0$) and hence one wonders about the possibility of logarithmically refining these inequalities to prevent them from becoming insignificant.

In this connection, we recall that logarithmic refinements of (1.1) were already known. Indeed, as discussed in [4], whenever, $4^{-1}\big[(\gamma-2)^2 - (n-2)^2\big]$ equals one of the eigenvalues of $-\Delta_{\mathbb{S}^{n-1}}$ (i.e., one of the numbers $j(j+n-2)$, $j \in \mathbb{N}_0$), then $C_{n,\gamma}$ vanishes, rendering inequality (1.1) trivial. In this context we recall the following result from [8, Theorem 1.3]:

$$\int_{B_n(0;R)} |x|^{\gamma} |(-\Delta f)(x)|^2 \, d^n x \geq C_{n,\gamma} \int_{B_n(0;R)} |x|^{\gamma-4} |f(x)|^2 \, d^n x$$
$$+ \big\{\big[(n-\gamma)^2 + (n+\gamma-4)^2\big]/16\big\}$$

$$\times \int_{B_n(0;R)} |x|^{\gamma-4} \Bigg(\sum_{k=1}^{N} \prod_{p=1}^{k} [\ln_p(\eta/|x|)]^{-2} \Bigg) |f(x)|^2 \, d^n x,$$

$$R \in (0,\infty),\ \gamma \in \mathbb{R},\ N \in \mathbb{N},\ \eta \in [e_N R, \infty),\ f \in C_0^\infty(B_n(0;R)\backslash\{0\}), \tag{1.7}$$

which yields an appropriate logarithmic refinement even if $C_{n,\gamma}$ vanishes. Here, $B_n(0;R)$ denotes the open ball in $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, centered at the origin 0 of radius $R \in (0,\infty)$, the iterated logarithms $\ln_k(\,\cdot\,)$, $k \in \mathbb{N}$, are given by

$$\ln_1(\,\cdot\,) = \ln(\,\cdot\,), \quad \ln_{k+1}(\,\cdot\,) = \ln\big(\ln_k(\,\cdot\,)\big), \quad k \in \mathbb{N}, \tag{1.8}$$

and the iterated exponentials e_j, $j \in \mathbb{N}_0$, are introduced via

$$e_0 = 0, \quad e_{j+1} = e^{e_j}, \quad j \in \mathbb{N}_0. \tag{1.9}$$

Given the logarithmic refinement (1.7) of (1.1), it is natural to ask if a corresponding analogous logarithmic refinement of (1.3) exists that prevents it from becoming insignificant if $A_{n,\gamma}$ vanishes. Answering this question in the affirmative is the principal purpose of this note. In particular, we will prove the following inequality in Theorem 2.3:

$$\int_{B_n(0;R)} |x|^{\gamma} |(-\Delta f)(x)|^2 \, d^n x \geq A_{n,\gamma} \int_{B_n(0;R)} |x|^{\gamma-2} |(\nabla f)(x)|^2 \, d^n x$$
$$+ 4^{-1} \int_{B_n(0;R)} |x|^{\gamma-2} \Bigg(\sum_{k=1}^{N} \prod_{p=1}^{k} [\ln_p(\eta/|x|)]^{-2} \Bigg) |(\nabla f)(x)|^2 \, d^n x$$
$$+ 4^{-1} \int_{B_n(0;R)} |x|^{\gamma-4} \Bigg(\sum_{k=1}^{N} \prod_{p=1}^{k} [\ln_p(\eta/|x|)]^{-2} \Bigg) |(\nabla_{\mathbb{S}^{n-1}} f)(x)|^2 \, d^n x,$$
$$R \in (0,\infty),\ \gamma \in \mathbb{R},\ N, n \in \mathbb{N},\ n \geq 2,\ \eta \in [e_N R, \infty),\ f \in C_0^\infty(B_n(0;R)\backslash\{0\}). \tag{1.10}$$

Once again, this inequality remains meaningful even if $A_{n,\gamma}$ vanishes.

Given the enormity of the literature on (power weighted) Rellich and Hardy–Rellich-type inequalities, we will not repeat the extensive list (still necessarily incomplete) of references cited in [9], and so refer the reader to the latter. However, more specifically, we mention that Caldiroli and Musina [4] proved in 2012 that the constant $C_{n,\gamma}$ in (1.1) is optimal. (For various restricted ranges of γ see also Adimurthi, Grossi, and Santra [1], Ghoussoub and Moradifam [10], [11, Sects. 6.3, 6.5, Ch. 7], and Tertikas and Zographopoulos [13].) The special unweighted case $\gamma = 0$ was settled for $n \geq 5$ by Herbst [12] in 1977 and subsequently by Yafaev [14] in 1999 for $n \geq 3$, $n \neq 4$ (both authors consider much more general fractional inequalities).

Under various restrictions on γ, Tertikas and Zographopoulos [13] obtained in 2007 optimality of $A_{n,\gamma}$ for $n \geq 5$ and $\mathbb{R}^n$ replaced by appropriate open bounded domains Ω with $0 \in \Omega$. This is revisited in Ghoussoub and Moradifam [10], [11, Part 2]. Similarly, Tertikas and Zographopoulos [13] obtained optimality of $A_{n,0}$ for $n \geq 5$; Beckner [3] (see also [2]), and subsequently, Ghoussoub and Moradifam [10], [11, Sects. 6.3, 6.5, Ch. 7] and Cazacu [5], obtained optimality of $A_{n,0}$ for $n \geq 3$.

As a notational comment, we remark that we abbreviate $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$.

2 A logarithmically modified Hardy–Rellich-type inequality

We begin by recalling the following simplified version of [7, Theorem 3.1 (iii)].

Lemma 2.1 *Let $R \in (0, \infty)$, $\alpha \in \mathbb{R}$, $N \in \mathbb{N}$, $\eta \in [e_N R, \infty)$, and $f \in C_0^\infty((0, R))$. Then*

$$\int_0^R r^\alpha |f'(r)|^2 \, dr \geq 4^{-1}(1-\alpha)^2 \int_0^R r^{\alpha-2} |f(r)|^2 \, dr + 4^{-1} \int_0^R r^{\alpha-2} \left(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\eta/r)]^{-2} \right) |f(r)|^2 \, dr, \tag{2.1}$$

where the iterated logarithms $\ln_k(\,\cdot\,)$, $k \in \mathbb{N}$, are given by

$$\ln_1(\,\cdot\,) = \ln(\,\cdot\,), \quad \ln_{k+1}(\,\cdot\,) = \ln\big(\ln_k(\,\cdot\,)\big), \quad k \in \mathbb{N}, \tag{2.2}$$

and the iterated exponentials e_j, $j \in \mathbb{N}_0$, are introduced via

$$e_0 = 0, \quad e_{j+1} = e^{e_j}, \quad j \in \mathbb{N}_0. \tag{2.3}$$

Proof As the current investigation came about while studying factorizations in [9], we provide a factorization proof of this lemma in the spirit of [9] (see also [6] for related higher dimensional unweighted factorizations with log refinements).

Given $R \in (0, \infty)$, $\alpha \in \mathbb{R}$, $N \in \mathbb{N}$, $\eta \in [e_N R, \infty)$, one defines the differential expression

$$T_{N,\alpha} = r^{\alpha/2} \frac{d}{dr} + \frac{\alpha - 1}{2} r^{(\alpha-2)/2} + \frac{1}{2} r^{(\alpha-2)/2} \sum_{k=1}^N \prod_{p=1}^k [\ln_p(\eta/r)]^{-1}, \quad r \in (0, R). \tag{2.4}$$

Then, after applying appropriate integration by parts and combining similar terms, one confirms that for $f \in C_0^\infty((0, R))$,

$$0 \leq \int_0^R |(T_{N,\alpha} f)(r)|^2 \, dr = \int_0^R r^\alpha |f'(r)|^2 \, dr - 4^{-1}(1-\alpha)^2 \int_0^R r^{\alpha-2} |f(r)|^2 \, dr$$

$$- 4^{-1} \int_0^R r^{\alpha-2} \bigg(\sum_{k=1}^{N} \prod_{p=1}^{k} [\ln_p(\eta/r)]^{-2} \bigg) |f(r)|^2 \, dr, \tag{2.5}$$

proving (2.1). □

Before deriving our next result, we recall some standard notation and facts. Let $\mathbb{S}^{n-1}$ denote the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, with $d^{n-1}\omega := d^{n-1}\omega(\theta)$ the usual volume measure on $\mathbb{S}^{n-1}$. We denote by $-\Delta_{\mathbb{S}^{n-1}}$ the nonnegative, self-adjoint Laplace–Beltrami operator in $L^2(\mathbb{S}^{n-1}; d^{n-1}\omega)$. Let

$$\lambda_j = j(j + n - 2), \quad j \in \mathbb{N}_0, \tag{2.6}$$

be the eigenvalues of $-\Delta_{\mathbb{S}^{n-1}}$, that is, $\sigma(-\Delta_{\mathbb{S}^{n-1}}) = \{j(j+n-2)\}_{j\in\mathbb{N}_0}$, of multiplicity

$$m(\lambda_j) = (2j + n - 2)(j + n - 2)^{-1} \binom{j + n - 2}{n - 2}, \quad j \in \mathbb{N}_0, \tag{2.7}$$

with corresponding eigenfunctions $\varphi_{j,\ell}$, $j \in \mathbb{N}_0$, $\ell \in \{1, \dots, m(\lambda_j)\}$. We may (and will) assume that $\{\varphi_{j,\ell}\}_{j\in\mathbb{N}_0,\ \ell\in\{1,\dots,m(\lambda_j)\}}$ is an orthonormal basis of $L^2(\mathbb{S}^{n-1}; d^{n-1}\omega)$, and let

$$\begin{aligned} &F_{f,j,\ell}(r) = (\varphi_{j,\ell}, f(r, \cdot))_{L^2(\mathbb{S}^{n-1};d^{n-1}\omega)} = \int_{\mathbb{S}^{n-1}} \overline{\varphi_{j,\ell}(\theta)} f(r, \theta) \, d^{n-1}\omega(\theta), \\ &f \in C_0^\infty(\mathbb{R}^n \backslash \{0\}), \quad r > 0, \quad j \in \mathbb{N}_0, \ \ell \in \{1, \dots, m(\lambda_j)\}. \end{aligned} \tag{2.8}$$

Finally, let $B_n(0; R)$ denote the open ball in $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, centered at the origin 0 of radius $R \in (0, \infty)$.

We are now in the position to prove the following lemma which will be combined with Lemma 2.1 to prove our main result.

Lemma 2.2 *Let $R \in (0, \infty)$, $f \in C_0^\infty(B_n(0; R)\backslash\{0\})$, and $g \in C((0, R))$ satisfy $g(r) > 0$ for all $r \in (0, R)$. Then*

$$\begin{aligned} &\int_{B_n(0;R)} g(|x|)|(\nabla f)(x)|^2 \, d^n x \\ &\quad = \sum_{j=0}^{\infty} \sum_{\ell=1}^{m(\lambda_j)} \int_0^R g(r)\big[|F'_{f,j,\ell}(r)|^2 r^{n-1} + \lambda_j |F_{f,j,\ell}(r)|^2 r^{n-3}\big] \, dr. \end{aligned} \tag{2.9}$$

Proof We begin by recalling that

$$|(\nabla f)(x)|^2 = |(\partial f/\partial r)(r, \theta)|^2 + r^{-2} |(\nabla_{\mathbb{S}^{n-1}} f(r, \cdot))(\theta)|^2, \tag{2.10}$$

where $\nabla_{\mathbb{S}^{n-1}}$ denotes the gradient operator on $\mathbb{S}^{n-1}$. Thus applying (2.10) and [8, Lemma 2.1] yields

$$\begin{aligned}
&\int_{B_n(0;R)} g(|x|)|(\nabla f)(x)|^2\, d^n x = \int_0^R g(r) \int_{\mathbb{S}^{n-1}} |(\nabla f)(r,\theta)|^2\, d^{n-1}\omega(\theta)\, r^{n-1} dr \\
&= \int_0^R g(r) \int_{\mathbb{S}^{n-1}} \big[|(\partial f/\partial r)(r,\theta)|^2 \\
&\quad + r^{-2}|(\nabla_{\mathbb{S}^{n-1}} f(r,\,\cdot\,))(\theta)|^2\big]\, d^{n-1}\omega(\theta)\, r^{n-1} dr \\
&= \int_0^R g(r)\bigg\{ \int_{\mathbb{S}^{n-1}} \bigg| \sum_{j=0}^{\infty} \sum_{\ell=1}^{m(\lambda_j)} F'_{f,j,\ell}(r)\varphi_{j,\ell}(\theta)\bigg|^2 d^{n-1}\omega(\theta) \\
&\quad + r^{-2} \int_{\mathbb{S}^{n-1}} \overline{(-\Delta_{\mathbb{S}^{n-1}} f)(r,\theta)}\, f(r,\theta)\, d^{n-1}\omega(\theta)\bigg\}\, r^{n-1} dr \\
&= \int_0^R g(r)\bigg\{ \sum_{j=0}^{\infty} \sum_{\ell=1}^{m(\lambda_j)} |F'_{f,j,\ell}(r)|^2 \\
&\quad + r^{-2} \int_{\mathbb{S}^{n-1}} \bigg[\sum_{j=0}^{\infty} \sum_{\ell=1}^{m(\lambda_j)} \lambda_j \overline{F_{f,j,\ell}(r)\varphi_{j,\ell}(\theta)}\bigg] \\
&\quad \times \bigg[\sum_{j=0}^{\infty} \sum_{\ell=1}^{m(\lambda_j)} F_{f,j,\ell}(r)\varphi_{j,\ell}(\theta)\bigg]\, d^{n-1}\omega(\theta)\bigg\}\, r^{n-1} dr \\
&= \sum_{j=0}^{\infty} \sum_{\ell=1}^{m(\lambda_j)} \int_0^R g(r)\big[|F'_{f,j,\ell}(r)|^2 + \lambda_j r^{-2} |F_{f,j,\ell}(r)|^2\big]\, r^{n-1} dr, \qquad (2.11)
\end{aligned}$$

proving (2.9) □

Explicitly, (2.11) yields

$$\begin{aligned}
&\int_0^R g(r) \int_{\mathbb{S}^{n-1}} |(\partial f/\partial r)(r,\theta)|^2\, d^{n-1}\omega(\theta)\, r^{n-1} dr \\
&\quad = \sum_{j=0}^{\infty} \sum_{\ell=1}^{m(\lambda_j)} \int_0^R g(r)|F'_{f,j,\ell}(r)|^2\, r^{n-1} dr, \qquad (2.12) \\
&\int_0^R g(r) \int_{\mathbb{S}^{n-1}} r^{-2}|(\nabla_{\mathbb{S}^{n-1}} f(r,\,\cdot\,))(\theta)|^2\, d^{n-1}\omega(\theta)\, r^{n-1} dr \\
&\quad = \sum_{j=0}^{\infty} \sum_{\ell=1}^{m(\lambda_j)} \lambda_j \int_0^R g(r) r^{-2} |F_{f,j,\ell}(r)|^2\, r^{n-1} dr. \qquad (2.13)
\end{aligned}$$

The previous results now allow us to prove the main result in this note in the form of the following Hardy–Rellich-type inequality with logarithmic refinements.

Theorem 2.3 *Let $R \in (0, \infty)$, $\gamma \in \mathbb{R}$, $N, n \in \mathbb{N}$, with $n \geq 2$, $\eta \in [e_N R, \infty)$, and $f \in C_0^\infty(B_n(0; R)\backslash\{0\})$. Then*

$$\begin{aligned}
&\int_{B_n(0;R)} |x|^\gamma |(-\Delta f)(x)|^2 \, d^n x \geq A_{n,\gamma} \int_{B_n(0;R)} |x|^{\gamma-2} |(\nabla f)(x)|^2 \, d^n x \\
&\quad + 4^{-1} \int_{B_n(0;R)} |x|^{\gamma-2} \Bigg(\sum_{k=1}^{N} \prod_{p=1}^{k} [\ln_p(\eta/|x|)]^{-2} \Bigg) |(\nabla f)(x)|^2 \, d^n x \\
&\quad + 4^{-1} \int_{B_n(0;R)} |x|^{\gamma-4} \Bigg(\sum_{k=1}^{N} \prod_{p=1}^{k} [\ln_p(\eta/|x|)]^{-2} \Bigg) |(\nabla_{\mathbb{S}^{n-1}} f)(x)|^2 \, d^n x, \qquad (2.14)
\end{aligned}$$

where

$$A_{n,\gamma} = \min_{j \in \mathbb{N}_0} \{\alpha_{n,\gamma,\lambda_j}\}, \qquad (2.15)$$

with

$$\begin{aligned}
&\alpha_{n,\gamma,\lambda_0} = \alpha_{n,\gamma,0} = 4^{-1}(n-\gamma)^2, \\
&\alpha_{n,\gamma,\lambda_j} = \big[4^{-1}(n+\gamma-4)(n-\gamma) + \lambda_j\big]^2 \Big/ \big[4^{-1}(n+\gamma-4)^2 + \lambda_j\big], \quad j \in \mathbb{N}. \qquad (2.16)
\end{aligned}$$

Excluding the cases (α) $n = 2$, $\gamma = 2$ and (β) $n = 3$, $\gamma = 1$, the constant $A_{n,\gamma}$ on the right-hand side of inequality (2.14) is optimal.

Proof By [9, Eq. (A.25)] and [8, Lemmas 2.3 and B.3 (i)], one has

$$\begin{aligned}
&\int_{B_n(0;R)} |x|^\gamma |(-\Delta f)(x)|^2 \, d^n x \\
&= \sum_{j=0}^{\infty} \sum_{\ell=1}^{m(\lambda_j)} \int_0^R r^{\gamma+n-1} \Big| - r^{1-n}\big[d/dr\big(r^{n-1} F'_{f,j,\ell}(r)\big)\big] + \lambda_j r^{-2} F_{f,j,\ell}(r)\Big|^2 \, dr \\
&= \sum_{j=0}^{\infty} \sum_{\ell=1}^{m(\lambda_j)} \Bigg\{ \int_0^R r^{\gamma+n-1} \big|F''_{f,j,\ell}(r)\big|^2 \, dr \\
&\quad + [2\lambda_j + (n-1)(1-\gamma)] \int_0^R r^{\gamma+n-3} \big|F'_{f,j,\ell}(r)\big|^2 \, dr \\
&\quad + \lambda_j[\lambda_j + (\gamma+n-4)(2-\gamma)] \int_0^R r^{\gamma+n-5} |F_{f,j,\ell}(r)|^2 \, dr \Bigg\}. \qquad (2.17)
\end{aligned}$$

Furthermore, note that (2.15) implies

$$\lambda_j A_{n,\gamma} \leq \big[4^{-1}(n+\gamma-4)(n-\gamma) + \lambda_j\big]^2 - 4^{-1}(n+\gamma-4)^2 A_{n,\gamma}, \quad j \in \mathbb{N}_0, \qquad (2.18)$$

or equivalently,

$$\begin{aligned} \lambda_j A_{n,\gamma} &\leq 4^{-1}(n+\gamma-4)^2\big[4^{-1}(n-\gamma)^2 + 2\lambda_j - A_{n,\gamma}\big] \\ &\quad + \lambda_j[\lambda_j + (n+\gamma-4)(2-\gamma)], \quad j \in \mathbb{N}_0. \end{aligned} \tag{2.19}$$

Applying Lemma 2.1 and (2.19) to (2.17) yields

$$\begin{aligned} &\int_{B_n(0;R)} |x|^\gamma |(-\Delta f)(x)|^2 \, d^n x \\ &\geq \sum_{j=0}^{\infty} \sum_{\ell=1}^{m(\lambda_j)} \bigg\{ 4^{-1}(2-n-\gamma)^2 \int_0^R r^{\gamma+n-3} \big|F'_{f,j,\ell}(r)\big|^2 \, dr \\ &\quad + 4^{-1} \int_0^R r^{\gamma+n-3} \big|F'_{f,j,\ell}(r)\big|^2 \bigg(\sum_{k=1}^{N} \prod_{p=1}^{k} [\ln_p(\eta/r)]^{-2} \bigg) \, dr \\ &\quad + [2\lambda_j + (n-1)(1-\gamma)] \int_0^R r^{\gamma+n-3} \big|F'_{f,j,\ell}(r)\big|^2 \, dr \\ &\quad + \lambda_j[\lambda_j + (\gamma+n-4)(2-\gamma)] \int_0^R r^{\gamma+n-5} |F_{f,j,\ell}(r)|^2 \, dr \bigg\} \\ &= \sum_{j=0}^{\infty} \sum_{\ell=1}^{m(\lambda_j)} \bigg\{ \big[4^{-1}(n-\gamma)^2 + 2\lambda_j\big] \int_0^R r^{\gamma+n-3} \big|F'_{f,j,\ell}(r)\big|^2 \, dr \\ &\quad + 4^{-1} \int_0^R r^{\gamma+n-3} \big|F'_{f,j,\ell}(r)\big|^2 \bigg(\sum_{k=1}^{N} \prod_{p=1}^{k} [\ln_p(\eta/r)]^{-2} \bigg) \, dr \\ &\quad + \lambda_j[\lambda_j + (\gamma+n-4)(2-\gamma)] \int_0^R r^{\gamma+n-5} |F_{f,j,\ell}(r)|^2 \, dr \bigg\} \\ &= \sum_{j=0}^{\infty} \sum_{\ell=1}^{m(\lambda_j)} \bigg\{ A_{n,\gamma} \int_0^R r^{\gamma+n-3} \big|F'_{f,j,\ell}(r)\big|^2 \, dr \\ &\quad + 4^{-1} \int_0^R r^{\gamma+n-3} \big|F'_{f,j,\ell}(r)\big|^2 \bigg(\sum_{k=1}^{N} \prod_{p=1}^{k} [\ln_p(\eta/r)]^{-2} \bigg) \, dr \\ &\quad + \big[4^{-1}(n-\gamma)^2 + 2\lambda_j - A_{n,\gamma}\big] \int_0^R r^{\gamma+n-3} \big|F'_{f,j,\ell}(r)\big|^2 \, dr \\ &\quad + \lambda_j[\lambda_j + (\gamma+n-4)(2-\gamma)] \int_0^R r^{\gamma+n-5} |F_{f,j,\ell}(r)|^2 \, dr \bigg\} \\ &\geq \sum_{j=0}^{\infty} \sum_{\ell=1}^{m(\lambda_j)} \bigg\{ A_{n,\gamma} \int_0^R r^{\gamma+n-3} \big|F'_{f,j,\ell}(r)\big|^2 \, dr \end{aligned}$$

$$
\begin{aligned}
&+ 4^{-1} \int_0^R r^{\gamma+n-3} \big|F'_{f,j,\ell}(r)\big|^2 \Bigg(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\eta/r)]^{-2} \Bigg) \, \mathrm{d}r \\
&+ \big\{ 4^{-1}(\gamma+n-4)^2 \big[4^{-1}(n-\gamma)^2 + 2\lambda_j - A_{n,\gamma} \big] \\
&+ \lambda_j [\lambda_j + (\gamma+n-4)(2-\gamma)] \big\} \int_0^R r^{\gamma+n-5} |F_{f,j,\ell}(r)|^2 \, \mathrm{d}r \\
&+ 4^{-1} \big[4^{-1}(n-\gamma)^2 + 2\lambda_j - A_{n,\gamma} \big] \\
&\times \int_0^R r^{\gamma+n-5} \big|F_{f,j,\ell}(r)\big|^2 \Bigg(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\eta/r)]^{-2} \Bigg) \, \mathrm{d}r \Bigg\} \\
\geq & \sum_{j=0}^{\infty} \sum_{\ell=1}^{m(\lambda_j)} \Bigg\{ A_{n,\gamma} \int_0^R \Big[r^{\gamma+n-3} \big|F'_{f,j,\ell}(r)\big|^2 + \lambda_j r^{\gamma+n-5} |F_{f,j,\ell}(r)|^2 \Big] \, \mathrm{d}r \\
&+ 4^{-1} \int_0^R r^{\gamma-2} \Bigg(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\eta/r)]^{-2} \Bigg) \\
&\times \Big[\big|F'_{f,j,\ell}(r)\big|^2 r^{n-1} + \lambda_j |F_{f,j,\ell}(r)|^2 r^{n-3} \Big] \, \mathrm{d}r \Bigg\} \\
&+ \sum_{j=0}^{\infty} \sum_{\ell=1}^{m(\lambda_j)} 4^{-1} \lambda_j \int_0^R r^{\gamma+n-5} \big|F_{f,j,\ell}(r)\big|^2 \Bigg(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\eta/r)]^{-2} \Bigg) \, \mathrm{d}r, \qquad (2.20)
\end{aligned}
$$

where we used the fact that $4^{-1}(n-\gamma)^2 \geq A_{n,\gamma}$ (following from letting $j=0$ in (2.15)) in the last inequality. Finally, applying Lemma 2.2 to the last inequality in (2.20) with

$$
g(r) = r^{\gamma-2} \text{ and } g(r) = r^{\gamma-2} \Bigg(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\eta/r)]^{-2} \Bigg), \quad r \in (0, R), \qquad (2.21)
$$

one obtains, employing (2.12) and (2.13),

$$
\begin{aligned}
&\int_{B_n(0;R)} |x|^{\gamma} |(-\Delta f)(x)|^2 \, \mathrm{d}^n x \geq A_{n,\gamma} \int_{B_n(0;R)} |x|^{\gamma-2} |(\nabla f)(x)|^2 \, \mathrm{d}^n x \\
&+ 4^{-1} \int_{B_n(0;R)} |x|^{\gamma-2} \Bigg(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\eta/|x|)]^{-2} \Bigg) |(\nabla f)(x)|^2 \, \mathrm{d}^n x \\
&+ 4^{-1} \int_{B_n(0;R)} |x|^{\gamma-4} \Bigg(\sum_{k=1}^N \prod_{p=1}^k [\ln_p(\eta/|x|)]^{-2} \Bigg) |(\nabla_{\mathbb{S}^{n-1}} f)(x)|^2 \, \mathrm{d}^n x. \qquad (2.22)
\end{aligned}
$$

To prove optimality of $A_{n,\gamma}$ (excluding the cases (α) $n=2$, $\gamma=2$ and (β) $n=3$, $\gamma=1$), one can modify the proof of optimality found in [9, Theorem A.7], and we now recall the major steps of the latter. That proof begins by choosing a sequence $\{f_m\}_{m\in\mathbb{N}} \subset C_0^\infty((0,\infty))$ such that f_m is real-valued and $f_m \not\equiv 0$ for all $m \in \mathbb{N}$, and

$$\lim_{m\to\infty} \left(\int_0^\infty r^{\gamma+n-1}|f_m''(r)|^2\,dr\right)\left(\int_0^\infty r^{\gamma+n-5}|f_m(r)|^2\,dr\right)^{-1} = \frac{(2-\gamma-n)^2(4-\gamma-n)^2}{16}. \tag{2.23}$$

Next, depending on the values of γ and n, one chooses an eigenfunction, φ, of $-\Delta_{\mathbb{S}^{n-1}}$ and defines $g_m \in C_0^\infty(\mathbb{R}^n\backslash\{0\}),\ m \in \mathbb{N}$, by

$$g_m(x) = g_m(r,\theta) = f_m(r)\varphi(\theta), \quad x \in \mathbb{R}^n\backslash\{0\}. \tag{2.24}$$

One can then show that

$$\frac{\int_{\mathbb{R}^n} |x|^\gamma|(-\Delta g_m)(x)|^2\,d^nx}{\int_{\mathbb{R}^n} |x|^{\gamma-2}|(\nabla g_m)(x)|^2\,d^nx} \underset{m\to\infty}{\longrightarrow} A_{n,\gamma}, \tag{2.25}$$

completing the proof of optimality in [9, Theorem A.7]. To modify this proof for the current purpose of proving optimality of $A_{n,\gamma}$ in (2.14), one needs to choose a new sequence in $C_0^\infty((0,R))$ rather than $C_0^\infty((0,\infty))$ to begin with. To this end, we choose $\{f_m\}_{m\in\mathbb{N}} \subset C_0^\infty((0,\infty))$ as above and let $f_m \in C_0^\infty((0,\rho_m))$ (e.g., $\rho_m \geq [\sup(\mathrm{supp}(f_m))] + 1$) for all $m \in \mathbb{N}$. We then define, for all $m \in \mathbb{N}$, $\widehat{f}_m \in C_0^\infty((0,R))$ by

$$\widehat{f}_m(y) = f_m(\rho_m y/R), \quad 0 < y < R. \tag{2.26}$$

One then readily verifies that

$$\lim_{m\to\infty} \left(\int_0^\infty r^{\gamma+n-1}|\widehat{f}_m''(r)|^2\,dr\right)\left(\int_0^\infty r^{\gamma+n-5}|\widehat{f}_m(r)|^2\,dr\right)^{-1} = \frac{(2-\gamma-n)^2(4-\gamma-n)^2}{16}. \tag{2.27}$$

Thus, replacing $\{f_m\}_{m\in\mathbb{N}} \subset C_0^\infty((0,\infty))$ by $\{\widehat{f}_m\}_{m\in\mathbb{N}} \in C_0^\infty((0,R))$ in the proof of [9, Theorem A.7] shows optimality of $A_{n,\gamma}$ in (2.14), once again, excluding the cases (α) $n = 2,\ \gamma = 2$ and (β) $n = 3,\ \gamma = 1$. □

Remark 2.4 (i) The proof of Theorem 2.3 is similar to proofs found in [11, Chs. 6, 7], but due to our application of [7, Theorem 3.1 (iii)] in Lemma 2.1, the range of parameters has now been greatly extended in Theorem 2.3, in particular, the two-dimensional case $n = 2$ in inequality (2.14) appears to have no precedent.

(ii) In [9, Theorems A.5 and A.7], the authors proved Theorem 2.3 without the log refinement terms (i.e., without the last two terms on the right side of (2.14)) and for a larger function space $C_0^\infty(\mathbb{R}^n\backslash\{0\})$. But even with this larger function space and without the log refinement terms, due to the method of proof, the authors were unable to show optimality of $A_{n,\gamma}$ in the two excluded cases in Theorem 2.3, that is, for (α) $n = 2$ and $\gamma = 2$, and (β) $n = 3$ and $\gamma = 1$. Therefore, the optimality of $A_{n,\gamma}$ for those two cases remains open.

(iii) We note that the inequality (2.14) was formulated for the smallest natural function space $f \in C_0^\infty(B_n(0; R)\backslash\{0\})$. Thus, at least in principle, the optimal constants could have increased in the process when compared to the function spaces $f \in C_0^\infty(B_n(0; R))$ typically employed in [1, 3, 5, 10], [11, Part 2], [13], etc. Interestingly enough, Theorem 2.3 demonstrates this possible increase in optimal constants is not happening with $A_{n,\gamma}$. In this context we note that [11, Ch. 6] derive optimality of $A_{n,\gamma}$ for $f \in C_0^\infty(B_n(0; R))$.

(iv) Of course, by restriction, the principal inequalities in this paper (such as (2.14)–(2.16)) extend to the case where $f \in C_0^\infty(B_n(0; R)\backslash\{0\})$, $n \in \mathbb{N}$, $n \geq 2$, is replaced by $f \in C_0^\infty(\Omega\backslash\{0\})$, where $\Omega \subseteq B_n(0; R)$ is open and bounded with $0 \in \Omega$, without changing the constants in these inequalities. $\diamond$

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