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ORIGINAL PAPER

Criterion for ellipticity on Heisenberg group

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Abstract
We provide a semi-constructive criterion for ellipticity of the differential operator on the Heisenberg group $\mathbb{H}^1$.

Keywords Elliptic operators · Heisenberg group

Mathematics Subject Classification 35J48 · 43A80

1 Introduction

The criterion for ellipticity of a differential operator in Euclidean space is well known. This criterion (the invertibility of the principal symbol of the operators) is perfectly constructive.

Criteria for ellipticity of differential operators on other Lie groups are much more involved. In this paper, we provide a criterion for ellipticity on the Heisenberg group $\mathbb{H}^1$ which is almost as constructive as the one for Euclidean space. It is possible that a (heavily modified) version of this proof works for an arbitrary stratified Lie group G. For background material on this topic, see [1–3].

The Heisenberg group $\mathbb{H}^1$ is the subgroup in $\mathrm{GL}(3, \mathbb{R})$ defined by

$$\mathbb{H}^1 = \left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, t \in \mathbb{R}. \right\}.$$

In other words, $\mathbb{H}^1$ is $\mathbb{R}^3$ equipped with the product

$$(x_1, y_1, t_1) \cdot (x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + x_1 y_2).$$

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The differential calculus on $\mathbb{H}^1$ consists of two left-invariant vector fields

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} + M_x \frac{\partial}{\partial t}.$$

In what follows, differential operators are defined on the Schwartz space $\mathcal{S}(\mathbb{H}^1) = \mathcal{S}(\mathbb{R}^3)$. Note that $X^w : \mathcal{S}(\mathbb{H}^1) \to \mathcal{S}(\mathbb{H}^1)$ for every word w in the alphabet with 2 letters. Here, X^w is the word w expressed in the alphabet $\{X_1, X_2\}$ and viewed as a differential operator. Hence, $\mathcal{S}(\mathbb{H}^1)$ serves as a natural domain for differential operators. Let $C_b^\infty(\mathbb{H}^1)$ denote the algebra of smooth functions f on $\mathbb{H}^1$, such that $X^w f$ is bounded for every word w.

Definition 1.1 A differential operator on $\mathbb{H}^1$ of order m is the mapping $P : \mathcal{S}(\mathbb{H}^1) \to \mathcal{S}(\mathbb{H}^1)$ of the shape

$$P = \sum_{\mathrm{len}(w) \le m} M_{a_w} X^w,$$

where the sum is taken over all words of length at most m and where each M_{a_w} is a multiplication operator with $a_w \in C_b^\infty(\mathbb{H}^1)$.

Clearly, a differential operator P extends to a mapping $P : \mathcal{S}'(\mathbb{H}^1) \to \mathcal{S}'(\mathbb{H}^1)$. The Lebesgue measure on $\mathbb{R}^3$ is a bivariant Haar measure for $\mathbb{H}^1$. We write $L_2(\mathbb{H}^1)$ for the L_2-space with this measure.

Definition 1.2 A differential operator P of order m on $\mathbb{H}^1$ is elliptic if, for every $f \in L_2(\mathbb{H}^1)$ with $Pf \in L_2(\mathbb{H}^1)$, we have $(1-\Delta)^{\frac{m}{2}} f \in L_2(\mathbb{H}^1)$ and

$$\|Pf\|_{L_2(\mathbb{H}^1)} + \|f\|_{L_2(\mathbb{H}^1)} \ge c_P \|(1-\Delta)^{\frac{m}{2}} f\|_{L_2(\mathbb{H}^1)}.$$

Here, $\Delta = X_1^2 + X_2^2$ and c_P is a strictly positive constant which only depends on P.

The paper [5] provides the following criterion (strictly speaking, only the sufficiency is established in [5]; however, the necessity is easy) for ellipticity of the differential operator P on a stratified Lie group. We state it here only for the Heisenberg group $\mathbb{H}^1$. This condition is related to the "maximal sub-ellipticity" of Helffer–Nourrigat [4, Chapter I, Definition 1.1]. Necessary and sufficient conditions for the hypoellipticity of left-invariant differential operators on the Heisenberg group were first discovered by Rockland [6].

Theorem 1.3 *Let P be a formally self-adjoint differential operator of order m on $\mathbb{H}^1$. The operator P is elliptic if and only if*

$$\|P_g f\|_{L_2(\mathbb{H}^1)} \ge c_P \|(-\Delta)^{\frac{m}{2}} f\|_{L_2(\mathbb{H}^1)}, \quad f \in \mathcal{S}(\mathbb{H}^1), \quad g \in \mathbb{H}^1.$$

Here, we are using the notation

$$P_g = \sum_{\mathrm{len}(w) = m} a_w(g) X^w, \quad g \in \mathbb{H}^1.$$

The latter condition is rather hard to deal with.

Definition 1.4 A differential operator P on $\mathbb{H}^1$ of order m is called formally elliptic if there exists a constant $c_P > 0$, such that

$$\left|\sum_{\mathrm{len}(w)=m} a_w(g)s^w\right| \geq c_P, \quad s \in \mathbb{S}^1, \quad g \in \mathbb{H}^1.$$

Here, $s \in \mathbb{S}^1$ means that $s = (s_0, s_1) \in \mathbb{R}^2$ and $s_0^2 + s_1^2 = 1$. The sum is taken over words in the alphabet $\{0, 1\}$ and s^w, $w = w_1 \cdots w_m$, is shorthand for $s_{w_1}s_{w_2}\cdots s_{w_m}$.

Equivalently, P is formally elliptic if $|\pi(P_g)| \geq c_P\pi((-\Delta)^{\frac{m}{2}})$ for every 1-dimensional representation π of $\mathbb{H}^1$.

A naive guess is that formal ellipticity is equivalent to ellipticity. This is not really the case: the operator $-X_1^2 - X_2^2 + i[X_1, X_2]$ is formally elliptic, but not elliptic (as it has a non-trivial kernel). However, the following weaker assertion is still true.

For each w, the bounded function $g \to a_w(g)$ extends to a continuous function on the Stone–Čech compactification $\beta\mathbb{H}^1$ of the topological space $(\mathbb{H}^1, \tau_{\mathrm{disc}})$ (here, τ_{disc} is the discrete topology on $\mathbb{H}^1$). Thus, we can also define P_g for every $g \in \beta\mathbb{H}^1$.

Theorem 1.5 *Let P be a formally self-adjoint differential operator of order m on $\mathbb{H}^1$. The operator P is elliptic if and only if the following conditions hold:*

1. *P is formally elliptic;*
2. *if $\xi \in \mathrm{dom}((-\Delta)^{\frac{m}{2}})$ and $g \in \beta\mathbb{H}^1$ are such that $P_g\xi = 0$, then $\xi = 0$;*

The second condition in Theorem 1.5 is stated in terms of the Stone–Čech compactification, because we do not want to introduce a topology on the set of left-invariant differential operators. If, instead, such a topology is introduced, then the second condition is the triviality of the kernel of Q for every Q in the closure of the set $\{P_g\}_{g\in\mathbb{H}^1}$.

2 Proof of the main theorem

In this section, we work in the Hilbert space $l_2(\mathbb{Z}_+)$, with standard orthonormal basis denoted $\{e_k\}_{k\geq 0}$. Denote $E_{j,k}$ for the matrix basis operator defined as $E_{j,k}e_n = e_j\delta_{k,n}$.

We work with the following operators:

$$ip = \frac{1}{\sqrt{2}}\sum_{k\geq 0}(k+1)^{\frac{1}{2}}(E_{k,k+1} - E_{k+1,k}), \quad q = \frac{1}{\sqrt{2}}\sum_{k\geq 0}(k+1)^{\frac{1}{2}}(E_{k+1,k} + E_{k,k+1}),$$
$$U = \sum_{k\geq 0} E_{k+1,k}, \quad H = \sum_{k\geq 0}(2k+1)E_{k,k}.$$

To be clear, p and q are self-adjoint unbounded operators, and $i = \sqrt{-1}$. We also identify $l_2(\mathbb{Z}_+)$ with a subspace in $l_2(\mathbb{Z})$. Let V denotes the right shift operator on $l_2(\mathbb{Z})$. For $n \in \mathbb{Z}_+$, let $E_n = \sum_{k=0}^{n-1} E_{k,k}$.

Theorem 2.1 *Let $\mathbb{J}$ be a set (discrete topological space) and let $(P_j)_{j\in\mathbb{J}}$ be a bounded family of homogeneous (of order m) polynomials in 2 non-commuting variables. The following conditions are equivalent:*

1. *there exists $c>0$, such that*

$$\|P_j(p,q)\xi\|_{l_2(\mathbb{Z}_+)} \geq c\|H^{\frac{m}{2}}\xi\|_{l_2(\mathbb{Z}_+)}, \quad \xi\in \mathrm{dom}(H^{\frac{m}{2}}), \quad j\in\mathbb{J};$$

2. *there exists $c>0$, such that*

$$|P_j(\Re(z),\Im(z))|\geq c, \quad z\in\mathbb{C}, \quad |z|=1, \quad j\in\mathbb{J},$$

and, for every $j\in\beta\mathbb{J}$, $P_j(p,q)\xi=0$, $\xi\in\mathrm{dom}(H^{\frac{m}{2}})$ implies $\xi=0$.

Lemma 2.2 *Let P be a homogeneous (of order m) polynomial in 2 non-commuting variables. The operator*

$$P(p,q)H^{-\frac{m}{2}} - P(-\Im(U),\Re(U))$$

is compact.

Proof We prove the assertion by induction on m. The base of the induction (i.e., the cases $m=1$ and $m=2$) is an easy computation.

Suppose the assertion is true for m. Let us prove it for $m+2$. Let P be a homogeneous (of order $m+2$) polynomial in 2 non-commuting variables. We write

$$P(p,q) = p^2P_1(p,q) + pqP_2(p,q) + qpP_3(p,q) + q^2P_4(p,q),$$

where $(P_k)_{k=1}^4$ are homogeneous (of order m) polynomial in 2 non-commuting variables. We have

$$\begin{aligned} P(p,q)H^{-\frac{m+2}{2}} &= H^{-1}P(p,q)H^{-\frac{m}{2}} + [P(p,q),H^{-1}]H^{-\frac{m}{2}} \\ &= H^{-1}P(p,q)H^{-\frac{m}{2}} - H^{-1}[P(p,q),H]H^{-\frac{m+2}{2}} \\ &= H^{-1}p^2\cdot P_1(p,q)H^{-\frac{m}{2}} + H^{-1}pq\cdot P_2(p,q)H^{-\frac{m}{2}} + H^{-1}qp\cdot P_3(p,q)H^{-\frac{m}{2}} \\ &\quad + H^{-1}q^2\cdot P_4(p,q)H^{-\frac{m}{2}} - H^{-1}\cdot[P(p,q),H]H^{-\frac{m+2}{2}}. \end{aligned}$$

The operators $[P(p,q),H]H^{-\frac{m+2}{2}}$ are bounded. Hence, the last summand is compact. By the inductive assumption (i.e., for m and for 2), the operator

$$P(p,q)H^{-\frac{m+2}{2}} - \Big((\Im(U))^2\cdot P_1(-\Im(U),\Re(U)) - \Im(U)\Re(U)\cdot P_2(-\Im(U),\Re(U)) \\ -\Re(U)\Im(U)\cdot P_3(-\Im(U),\Re(U)) + (\Re(U))^2\cdot P_4(-\Im(U),\Re(U))\Big)$$

is compact. Clearly

$$P(-\Im(U),\Re(U)) = (\Im(U))^2\cdot P_1(-\Im(U),\Re(U)) - \Im(U)\Re(U)\cdot P_2(-\Im(U),\Re(U)) \\ -\Re(U)\Im(U)\cdot P_3(-\Im(U),\Re(U)) + (\Re(U))^2\cdot P_4(-\Im(U),\Re(U)).$$

This yields the step of the induction and, hence, completes the proof. □

Lemma 2.3 *Let P be a homogeneous (of order m) polynomial in 2 non-commuting variables. We have*

$$P(-\Im(U),\Re(U))U^n\xi = P(-\Im(V),\Re(V))V^n\xi,\quad \xi\in l_2(\mathbb{Z}_+),\quad n\ge m.$$

Lemma 2.4 *Let P be a homogeneous (of order m) polynomial in 2 non-commuting variables. If*

$$\|P(p,q)\xi\|_{l_2(\mathbb{Z}_+)} \ge \|H^{\frac{m}{2}}\xi\|_{l_2(\mathbb{Z}_+)},\quad \xi\in \mathrm{dom}(H^{\frac{m}{2}}),$$

then

$$\|P(-\Im(V),\Re(V))\xi\|_{l_2(\mathbb{Z})} \ge \|\xi\|_{l_2(\mathbb{Z})},\quad \xi\in l_2(\mathbb{Z}_+).$$

Proof The assumption means

$$\|P(p,q)H^{-\frac{m}{2}}\xi\|_{l_2(\mathbb{Z}_+)} \ge \|\xi\|_{l_2(\mathbb{Z}_+)},\quad \xi\in l_2(\mathbb{Z}_+).$$

Substituting $U^n\xi$ instead of ξ, we obtain

$$\|P(p,q)H^{-\frac{m}{2}}U^n\xi\|_{l_2(\mathbb{Z}_+)} \ge \|\xi\|_{l_2(\mathbb{Z}_+)},\quad \xi\in l_2(\mathbb{Z}_+),\quad n\in\mathbb{Z}_+.$$

By the triangle inequality

$$\|\xi\|_{l_2(\mathbb{Z}_+)} \le \|P(-\Im(U),\Re(U))(U^n\xi)\|_{l_2(\mathbb{Z}_+)} + \\ +\Big\|\Big(P(p,q)H^{-\frac{m}{2}} - P(-\Im(U),\Re(U))\Big)(U^n\xi)\Big\|_{l_2(\mathbb{Z}_+)},\quad n\in\mathbb{Z}_+.$$

By Lemma 2.3

$$\|P(-\Im(U),\Re(U))(U^n\xi)\|_{l_2(\mathbb{Z}_+)} \\ = \|P(-\Im(V),\Re(V))V^n(\xi)\|_{l_2(\mathbb{Z})} = \|P(-\Im(V),\Re(V))(\xi)\|_{l_2(\mathbb{Z})},\quad n\ge m.$$

Thus

$$\|\xi\|_{l_2(\mathbb{Z}_+)} \le \|P(-\Im(V),\Re(V))\eta\|_{l_2(\mathbb{Z})} + \\ +\Big\|\Big(P(p,q)H^{-\frac{m}{2}} - P(-\Im(U),\Re(U))\Big)(U^n\xi)\Big\|_{l_2(\mathbb{Z}_+)},\quad n\in\mathbb{Z}_+.$$

By Lemma 2.2

$$\Big\|\Big(P(p,q)H^{-\frac{m}{2}} - P(-\Im(U),\Re(U))\Big)(U^n\eta)\Big\|_{l_2(\mathbb{Z}_+)} \to 0,\quad n\to\infty.$$

This suffices to complete the proof. □

Lemma 2.5 *Let P be a homogeneous (of order m) polynomial in 2 non-commuting variables. If*

$$\|P(-\Im(V), \Re(V))\xi\|_{l_2(\mathbb{Z})} \geq \|\xi\|_{l_2(\mathbb{Z})}, \quad \xi \in l_2(\mathbb{Z}_+),$$

then

$$|P(\Im(z), \Re(z))| \geq 1, \quad z \in \mathbb{C}, \quad |z| = 1.$$

Proof Let $\eta \in l_2(\mathbb{Z})$. Fix $\epsilon > 0$ and choose $\xi \in l_2(\mathbb{Z}_+)$ and $n \in \mathbb{Z}_+$, such that

$$\|\eta - V^{-n}\xi\|_{l_2(\mathbb{Z})} < \epsilon.$$

Set $\theta = \eta - V^{-n}\xi$. We have

$$\begin{aligned}\|P(-\Im(V), \Re(V))\eta\|_{l_2(\mathbb{Z})} &\geq \|P(-\Im(V), \Re(V))V^{-n}\xi\|_{l_2(\mathbb{Z})} - \|P(-\Im(V), \Re(V))\theta\|_{l_2(\mathbb{Z})} \\ &\geq \|P(-\Im(V), \Re(V))V^{-n}\xi\|_{l_2(\mathbb{Z})} - \epsilon\|P(-\Im(V), \Re(V))\|_\infty \\ &= \|P(-\Im(V), \Re(V))\xi\|_{l_2(\mathbb{Z})} - \epsilon\|P(\Im(V), \Re(V))\|_\infty.\end{aligned}$$

By the assumption, we have

$$\|P(-\Im(V), \Re(V))\xi\|_{l_2(\mathbb{Z})} \geq \|\xi\|_{l_2(\mathbb{Z})}.$$

Thus

$$\begin{aligned}\|P(-\Im(V), \Re(V))\eta\|_{l_2(\mathbb{Z})} &\geq \|\xi\|_{l_2(\mathbb{Z})} - \epsilon\|P(-\Im(V), \Re(V))\|_\infty \\ &= \|V^{-n}\xi\|_{l_2(\mathbb{Z})} - \epsilon\|P(-\Im(V), \Re(V))\|_\infty \\ &\geq \|\eta\|_{l_2(\mathbb{Z})} - \epsilon - \epsilon\|P(-\Im(V), \Re(V))\|_\infty.\end{aligned}$$

Since $\epsilon > 0$ is arbitrarily small, it follows that:

$$\|P(-\Im(V), \Re(V))\eta\|_{l_2(\mathbb{Z})} \geq \|\eta\|_{l_2(\mathbb{Z})}, \quad \eta \in l_2(\mathbb{Z}).$$

Since V is normal and since the spectrum of V is $\{z \in \mathbb{C} : |z| = 1\}$, the assertion immediately follows. □

Lemma 2.6 *Let $(P_j)_{j\in\mathbb{J}}$ be a family of homogeneous (of order m) polynomials in 2 non-commuting variables. Suppose the condition (1.) in* Theorem 2.1 *fails. There exists a sequence $(\eta_k)_{k\geq 0} \subset l_2(\mathbb{Z}_+)$ and a sequence $(j_k)_{k\geq 0} \subset \mathbb{J}$, such that*

1. $\|\eta_k\|_{l_2(\mathbb{Z}_+)} = 1$ *for every* $k \geq 0$.
2. $\eta_k \to \eta$ *weakly in* $l_2(\mathbb{Z}_+)$.
3. $P_{j_k}(p, q)H^{-\frac{m}{2}}\eta_k \to 0$ *in* $l_2(\mathbb{Z}_+)$ *as* $k \to \infty$.

Proof Indeed, assume the contrary and choose a sequence $(j_k)_{k\geq 0}\subset\mathbb{J}$ and $\xi_k\in\mathrm{dom}(H^{\frac{m}{2}})$, such that $\|H^{\frac{m}{2}}\xi_k\|_{l_2(\mathbb{Z}_+)}=1$ and such that $\|P_{j_k}(p,q)\xi_k\|_{l_2(\mathbb{Z}_+)}\to 0$.

Set $\eta_k=H^{\frac{m}{2}}\xi_k$, $k\geq 0$. The sequence $(\eta_k)_{k\geq 0}$ satisfies the first and third conditions. Since the unit ball in $l_2(\mathbb{Z}_+)$ is weakly compact, it follows that, passing to a subsequence if needed, we may also satisfy the second condition. □

Lemma 2.7 *Let $(P_j)_{j\in\mathbb{J}}$ be a bounded family of homogeneous (of order m) polynomials in 2 non-commuting variables. We have*

$$\sup_{j\in\mathbb{J}}\Big\|(1-E_n)\Big(P_j(p,q)H^{-\frac{m}{2}}-P_j(-\Im(U),\Re(U))\Big)\Big\|_\infty\to 0,\quad n\to\infty.$$

Proof By definition, we have

$$P_j=\sum_{\mathrm{len}(w)=m}a_{j,w}w(p,q).$$

Boundedness of the family means that

$$\sup_{j\in\mathbb{J}}|a_{j,w}|<\infty,\quad \mathrm{len}(w)=m.$$

By Lemma 2.2, the operator

$$w(p,q)H^{-\frac{m}{2}}-w(-\Im(U),\Re(U)),\quad \mathrm{len}(w)=m,$$

is compact. Hence

$$\Big\|(1-E_n)\Big(w(p,q)H^{-\frac{m}{2}}-w(-\Im(U),\Re(U))\Big)\Big\|_\infty\to 0,\quad n\to\infty. \tag{1.2}$$

By triangle inequality

$$\begin{aligned}&\sup_{j\in\mathbb{J}}\Big\|(1-E_n)\Big(P_i(p,q)H^{-\frac{m}{2}}-P_i(-\Im(U),\Re(U))\Big)\Big\|_\infty\\ &\leq\sup_{j\in\mathbb{J}}\sum_{\mathrm{len}(w)=m}|a_{j,w}|\cdot\Big\|(1-E_n)\Big(w(p,q)H^{-\frac{m}{2}}-w(-\Im(U),\Re(U))\Big)\Big\|_\infty\\ &\leq\sum_{\mathrm{len}(w)=m}\sup_{j\in\mathbb{J}}|a_{j,w}|\cdot\max_{\mathrm{len}(w)=m}\Big\|(1-E_n)\Big(w(p,q)H^{-\frac{m}{2}}-w(-\Im(U),\Re(U))\Big)\Big\|_\infty.\end{aligned}$$

Thus, the assertion follows from (1.2). □

Lemma 2.8 *Let P be a homogeneous (of order m) polynomial in 2 non-commuting variables. For every $n\geq m$, we have*

$$E_n\cdot P(-\Im(U),\Re(U))=E_n\cdot P(-\Im(U),\Re(U))\cdot E_{n+m}.$$
$$[P(p,q)H^{-\frac{m}{2}},E_n]=[P(p,q)H^{-\frac{m}{2}},E_n]\cdot(1-E_{n-m}).$$

Lemma 2.9 *Let $(P_j)_{j\in\mathbb{J}}$ be a bounded family of homogeneous (of order m) polynomials in 2 non-commuting variables. For every $\eta\in l_2(\mathbb{Z}_+)$, we have*

$$\sup_{j\in\mathbb{J}}\|(1-E_n)P_j(p,q)H^{-\frac{m}{2}}\eta\|_{l_2(\mathbb{Z}_+)}\to 0,\quad n\to\infty.$$

Proof By definition, we have

$$P_j=\sum_{\mathrm{len}(w)=m}a_{j,w}w(p,q).$$

Boundedness of the family means that

$$\sup_{j\in\mathbb{J}}|a_{j,w}|<\infty,\quad \mathrm{len}(w)=m.$$

By the triangle inequality

$$\begin{aligned}&\sup_{j\in\mathbb{J}}\|(1-E_n)P_j(p,q)H^{-\frac{m}{2}}\eta\|_{l_2(\mathbb{Z}_+)}\\ &\quad\leq\sup_{j\in\mathbb{J}}\sum_{\mathrm{len}(w)=m}|a_{j,w}|\|(1-E_n)w(p,q)H^{-\frac{m}{2}}\eta\|_{l_2(\mathbb{Z}_+)}\\ &\quad\leq\sum_{\mathrm{len}(w)=m}\sup_{j\in\mathbb{J}}|a_{j,w}|\cdot\max_{\mathrm{len}(w)=m}\|(1-E_n)w(p,q)H^{-\frac{m}{2}}\eta\|_{l_2(\mathbb{Z}_+)}.\end{aligned}$$

Since

$$\|(1-E_n)w(p,q)H^{-\frac{m}{2}}\eta\|_{l_2(\mathbb{Z}_+)}\to 0,\quad n\to\infty,$$

for every word w with $\mathrm{len}(w)=m$ and since there are finitely many (2^m, to be precise) words of length m, it follows that:

$$\max_{\mathrm{len}(w)=m}\|(1-E_n)w(p,q)H^{-\frac{m}{2}}\eta\|_{l_2(\mathbb{Z}_+)}\to 0,\quad n\to\infty.$$

This completes the proof. □

Lemma 2.10 *Let $(P_j)_{j\in\mathbb{J}}$ be a bounded family of homogeneous (of order m) polynomials in 2 non-commuting variables. Suppose that*

$$|P_j(\Im(z),\Re(z))|\geq c,\quad z\in\mathbb{C},\quad |z|=1,\quad j\in\mathbb{J}.$$

Suppose also that condition (1.) in Theorem 2.1 *fails. Let $(\eta_k)_{k\geq 0}$ and η be as in* Lemma 2.6*. We have $\eta_k\to\eta$ in $l_2(\mathbb{Z}_+)$.*

Proof By definition, we have

$$P_j = \sum_{\mathrm{len}(w)=m} a_{j,w} w(p, q).$$

Boundedness of the family means that

$$\sup_{j\in\mathbb{J}} |a_{j,w}| < \infty, \quad \mathrm{len}(w) = m.$$

We may assume without loss of generality that

$$\sum_{\mathrm{len}(w)=m} \sup_{j\in\mathbb{J}} |a_{j,w}| \le 1.$$

It follows from the preceding paragraph that:

$$\sup_{j\in\mathbb{J}} \|P_j(-\Im(U), \Re(U))\|_\infty \le 1. \tag{2.2}$$

Fix $\epsilon > 0$ and choose, using Lemmas 2.7 and 2.9, $n(\epsilon) \ge 2m$, such that

$$\sup_{j\in\mathbb{J}} \left\|(1 - E_{n(\epsilon)})\Big(P_j(p, q)H^{-\frac{m}{2}} - P_j(-\Im(U), \Re(U))\Big)\right\|_\infty < \epsilon, \tag{3.2}$$

$$\|(1 - E_{n(\epsilon)-2m})\eta\|_{l_2(\mathbb{Z}_+)} < \epsilon, \tag{4.2}$$

$$\sup_{j\in\mathbb{J}} \|(1 - E_{n(\epsilon)})P_j(p, q)H^{-\frac{m}{2}}\eta\|_{l_2(\mathbb{Z}_+)} < \epsilon. \tag{5.2}$$

Using the third and second conditions in Lemma 2.6, we can choose $k(\epsilon)$, such that

$$\|P_{j_k}(p, q)H^{-\frac{m}{2}}\eta_k\|_{l_2(\mathbb{Z}_+)} < \epsilon, \quad k \ge k(\epsilon), \tag{6.2}$$

$$\|E_{n(\epsilon)+m}(\eta_k - \eta)\|_{l_2(\mathbb{Z}_+)} < \epsilon, \quad k \ge k(\epsilon). \tag{7.2}$$

It follows from (6.2) that:

$$\|(1 - E_{n(\epsilon)})P_{j_k}(p, q)H^{-\frac{m}{2}}\eta_k\|_{l_2(\mathbb{Z}_+)} < \epsilon, \quad k \ge k(\epsilon).$$

Using (5.2) and the triangle inequality, we write

$$\|(1 - E_{n(\epsilon)})P_{j_k}(p, q)H^{-\frac{m}{2}}(\eta_k - \eta)\|_{l_2(\mathbb{Z}_+)} < 2\epsilon, \quad k \ge k(\epsilon).$$

Using (3.2), we write

$$\|(1-E_{n(\epsilon)})P_{j_k}(-\Im(U),\Re(U))(\eta_k-\eta)\|_{l_2(\mathbb{Z}_+)}<3\epsilon,\quad k\geq k(\epsilon).$$

It follows from Lemma 2.8 that:

$$E_{n(\epsilon)}\cdot P_{j_k}(-\Im(U),\Re(U))=E_{n(\epsilon)}\cdot P_{j_k}(-\Im(U),\Re(U))\cdot E_{n(\epsilon)+m}.$$

Thus

$$\begin{aligned}&\|E_{n(\epsilon)}P_{j_k}(-\Im(U),\Re(U))(\eta_k-\eta)\|_{l_2(\mathbb{Z}_+)}\\&\quad\leq\|P_{j_k}(-\Im(U),\Re(U))\|_\infty\|E_{n(\epsilon)+m}(\eta_k-\eta)\|_{l_2(\mathbb{R})}\\&\quad\overset{(7.2)}{<}\epsilon\|P_{j_k}(-\Im(U),\Re(U))\|_\infty\overset{(2.2)}{\leq}\epsilon,\quad k\geq k(\epsilon).\end{aligned}$$

Hence

$$\|P_{j_k}(-\Im(U),\Re(U))(\eta_k-\eta)\|_{l_2(\mathbb{Z}_+)}<4\epsilon,\quad k\geq k(\epsilon).$$

Again, using (7.2), we obtain

$$\|P_{j_k}(-\Im(U),\Re(U))(1-E_{n(\epsilon)})(\eta_k-\eta)\|_{l_2(\mathbb{Z}_+)}<5\epsilon,\quad k\geq k(\epsilon).$$

Taking into account that $n(\epsilon)\geq m$ and using Lemma 2.3, we write

$$\|P_{j_k}(-\Im(V),\Re(V))(1-E_{n(\epsilon)})(\eta_k-\eta)\|_{l_2(\mathbb{Z})}<5\epsilon,\quad k\geq k(\epsilon).$$

By the assumption, we have

$$\|P_j(-\Im(V),\Re(V))\xi\|_{l_2(\mathbb{Z})}\geq c\|\xi\|_{l_2(\mathbb{Z})},\quad \xi\in l_2(\mathbb{Z}).$$

Thus

$$c\|(1-E_{n(\epsilon)})(\eta_k-\eta)\|_{l_2(\mathbb{Z})}<5\epsilon,\quad k\geq k(\epsilon).$$

It follows now from (7.2):

$$\|\eta_k-\eta\|_{l_2(\mathbb{Z})}\leq(5c^{-1}+1)\epsilon,\quad k\geq k(\epsilon).$$

Since $\epsilon>0$ can be chosen arbitrarily small, the assertion follows. □

Proof of Theorem 2.1 If the condition (1.) holds, then

$$\|P_j(p,q)\xi\|_{l_2(\mathbb{Z}_+)}\geq c\|H^{\frac{m}{2}}\xi\|_{l_2(\mathbb{Z}_+)},\quad \xi\in\mathrm{dom}(H^{\frac{m}{2}}),\quad j\in\mathbb{J}.$$

Hence, exactly the same estimate holds for $j \in \beta\mathbb{J}$. In particular, if $P_j(p,q)\xi = 0$, $\xi \in \mathrm{dom}(H^{\frac{m}{2}})$, $j \in \beta\mathbb{J}$, then $H^{\frac{m}{2}}\xi = 0$ and, therefore, $\xi = 0$. Necessity of the condition (2.) follows now from Lemma 2.5.

Suppose now that the condition (1.) fails and that

$$|P_j(\Im(z), \Re(z))| \geq c, \quad z \in \mathbb{C}, \quad |z| = 1, \quad j \in \mathbb{J}.$$

Let $(\eta_k)_{k\geq 0}$ and η be as in Lemma 2.6. By Lemma 2.10, we have $\eta_k \to \eta$ in $l_2(\mathbb{Z}_+)$. Since $\|\eta_k\|_{l_2(\mathbb{Z}_+)} = 1$ for every $k \geq 0$, it follows that $\|\eta\|_{l_2(\mathbb{Z}_+)} = 1$. The third condition in Lemma 2.6 asserts that $P_{j_k}(p,q)H^{-\frac{m}{2}}\eta_k \to 0$ in $l_2(\mathbb{Z}_+)$ as $k \to \infty$. Since the family $\{P_j\}_{j\in\mathbb{J}}$ is bounded, it follows that $P_{j_k}(p,q)H^{-\frac{m}{2}}\eta \to 0$ in $l_2(\mathbb{Z}_+)$ as $k \to \infty$.

Set $\xi = H^{-\frac{m}{2}}\eta \in \mathrm{dom}(H^{\frac{m}{2}})$. We have $\xi \neq 0$ and $P_{j_k}(p,q)\xi \to 0$ in $l_2(\mathbb{Z}_+)$ as $k \to \infty$. Passing to a subsequence if needed, we may assume without loss of generality that $j_k \to j \in \beta\mathbb{J}$. Thus, $P_j\xi = 0$ and, hence, the condition (2.) fails. □

3 Proof of Theorem 1.5

Consider the position and momentum operators q and p on $L_2(\mathbb{R})$. Let $\{\psi_k\}_{k\geq 0}$ be the Hermite basis in $L_2(\mathbb{R})$. Recall that

$$ip\psi_k = \frac{1}{\sqrt{2}}(k^{\frac{1}{2}}\psi_{k-1} - (k+1)^{\frac{1}{2}}\psi_{k+1}), \quad q\psi_k = \frac{1}{\sqrt{2}}(k^{\frac{1}{2}}\psi_{k-1} + (k+1)^{\frac{1}{2}}\psi_{k+1})$$

for every $k \in \mathbb{Z}_+$. In what follows, we identify $\xi \in L_2(\mathbb{R})$ with the sequence $\{\langle \xi, \psi_k\rangle\}_{k\in\mathbb{Z}_+} \in l_2(\mathbb{Z}_+)$. In this way, we identify the Hilbert spaces $L_2(\mathbb{R})$ and $l_2(\mathbb{Z}_+)$ and, hence, we fall exactly into the setting of the preceding section. Consequently, Theorem 2.1 applies for *these* q and p.

Proof of Theorem 1.5 By continuity, the ellipticity condition holds for the closure of $\mathcal{S}(\mathbb{H}^1)$ in the graph norm of $(-\Delta)^{\frac{m}{2}}$. Hence, the condition

$$\|P_g f\|_{L_2(\mathbb{H}^1)} \geq c_P \|(-\Delta)^{\frac{m}{2}} f\|_{L_2(\mathbb{H}^1)}, \quad f \in \mathcal{S}(\mathbb{H}^1), \quad g \in \mathbb{H}^1$$

is equivalent to the condition

$$\|P_g f\|_{L_2(\mathbb{H}^1)} \geq c_P \|(-\Delta)^{\frac{m}{2}} f\|_{L_2(\mathbb{H}^1)}, \quad f \in \mathrm{dom}((-\Delta)^{\frac{m}{2}}), \quad g \in \mathbb{H}^1.$$

We want to apply Theorem 2.1 with $\mathbb{J} = \mathbb{H}^1 \times \{-1, 1\}$ and with

$$P_{(g,\pm 1)} = \sum_{\mathrm{len}(w)=m} a_w(g) w(\pm p, q), \quad g \in \mathbb{H}^1.$$

Recall the Plancherel decomposition

$$L_2(\mathbb{H}^1) = \int_{\mathbb{R}\setminus\{0\}}^{\oplus} L_2(\mathbb{R})\mathrm{d}\nu(s),$$

where ν is a Plancherel measure (in fact, $\mathrm{d}\nu(s) = sds$, up to a constant but the precise formula is not very important)

$$X_1 = i\int_{\mathbb{R}\setminus\{0\}}^{\oplus} \mathrm{sgn}(s)|s|^{\frac{1}{2}} p\mathrm{d}\nu(s), \quad X_2 = i\int_{\mathbb{R}\setminus\{0\}}^{\oplus} |s|^{\frac{1}{2}} q\mathrm{d}\nu(s).$$

Thus

$$(-\Delta)^{\frac{m}{2}} = \int_{\mathbb{R}\setminus\{0\}}^{\oplus} |s|^m H^{\frac{m}{2}}\mathrm{d}\nu(s),$$

$$P_g = \int_{\mathbb{R}\setminus\{0\}}^{\oplus} |s|^m (P_{(g,1)}\chi_{(0,\infty)}(s) + P_{(g,-1)}\chi_{(-\infty,0)}(s))\mathrm{d}\nu(s).$$

We have

$$\mathrm{dom}((-\Delta)^{\frac{m}{2}}) = \Big\{ \int_{\mathbb{R}\setminus\{0\}}^{\oplus} f_s\mathrm{d}\nu(s) : \int_{\mathbb{R}\setminus\{0\}} \|H^{\frac{m}{2}} f_s\|_{L_2(\mathbb{R})}^2 |s|^{2m}\mathrm{d}\nu(s) < \infty \Big\}.$$

Hence, the ellipticity condition

$$\|P_g f\|_{L_2(\mathbb{H}^1)} \geq c_P \|(-\Delta)^{\frac{m}{2}} f\|_{L_2(\mathbb{H}^1)}, \quad f \in \mathrm{dom}((-\Delta)^{\frac{m}{2}}), \quad g \in \mathbb{H}^1$$

can be equivalently rewritten as

$$\int_{\mathbb{R}\setminus\{0\}} |s|^{2m} \|P_{(g,\mathrm{sgn}(s))} f_s\|_{L_2(\mathbb{R})}^2 \mathrm{d}\nu(s) \geq c_P^2 \int_{\mathbb{R}\setminus\{0\}} \|H^{\frac{m}{2}} f_s\|_{L_2(\mathbb{R})}^2 |s|^{2m}\mathrm{d}\nu(s)$$

whenever the right-hand side is finite.

Fix $\xi \in \mathrm{dom}(H^{\frac{m}{2}})$ and set

$$f_s = \begin{cases} \xi, & s \in (0,1) \\ 0, & s \notin (0,1) \end{cases} \text{ or alternatively } f_s = \begin{cases} \xi, & s \in (-1,0) \\ 0, & s \notin (-1,0). \end{cases}$$

The ellipticity condition yields

$$\|P_{(g,\pm 1)}\xi\|_{L_2(\mathbb{R})} \geq c_P \|H^{\frac{m}{2}}\xi\|_{L_2(\mathbb{R})}, \quad \xi \in \mathrm{dom}(H^{\frac{m}{2}}). \tag{8.2}$$

Conversely, the condition (8.2) clearly yields the ellipticity condition.

By Theorem 2.1, the condition (8.2) is equivalent to formal ellipticity of P (this is exactly the condition (1.5) in Theorem (1.5) *and* the condition that $P_{(g,\pm 1)}(p,q)\xi = 0$,

$\xi \in \mathrm{dom}((-\Delta)^{\frac{m}{2}})$, $g \in \beta\mathbb{H}^1$, implies $\xi = 0$. Again, using the Plancherel decomposition, we see that the latter condition is equivalent to the condition (1.5) in Theorem 1.5.

Hence, the ellipticity condition is equivalent to the condition (8.2) which is, in turn, equivalent to the conditions (1.5) and (1.5) in Theorem 1.5. □

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