Ann. Funct. Anal. (2024) 15:63
https://doi.org/10.1007/s43034-024-00368-3

Tusi Mathematical Research Group

ORIGINAL PAPER

A new uniform structure for Hilbert C^*-modules

Denis Fufaev[1] · **Evgenij Troitsky**[1]

Received: 28 February 2024 / Accepted: 14 May 2024 / Published online: 3 June 2024

Abstract

We introduce and study some new uniform structures for Hilbert C^*-modules over a C^*-algebra $\mathcal{A}$. In particular, we prove that in some cases they have the same totally bounded sets. To define one of them, we introduce a new class of $\mathcal{A}$-functionals: locally adjointable functionals, which have interesting properties in this context and seem to be of independent interest. A relation between these uniform structures and the theory of $\mathcal{A}$-compact operators is established.

Keywords Hilbert C^*-module · Uniform structure · Totally bounded set · Multiplier · Compact operator · $\mathcal{A}$-compact operator

Mathematics Subject Classification 46L08 · 47B10 · 47L80 · 54E15

Introduction

In the theory of Hilbert C^*-modules, there are problems in which the necessity to construct uniform structures arises naturally. More precisely, this is the case for the theory of $\mathcal{A}$-compact operators, where $\mathcal{A}$ is a C^*-algebra as everywhere in the article. In the case of Hilbert spaces, i.e. in the case $\mathcal{A} = \mathbb{C}$, the geometric description of such operators is well known: the operator is compact if and only if the image of the unit ball is totally bounded in norm. In general, this is not true for Hilbert C^*-modules: even if we take any infinite-dimensional unital C^*-algebra as a module over itself and the identity operator, it is $\mathcal{A}$-compact (it has $\mathcal{A}$-rank one), but the unit ball is not totally bounded due to infinite dimension. Therefore, to describe the $\mathcal{A}$-compactness

To Professor Fedor Sukochev.

Communicated by Vladimir Manuilov.

✉ Evgenij Troitsky
evgenij.troitsky@math.msu.ru

Denis Fufaev
fufaevdv@rambler.ru; denis.fufaev@math.msu.ru

1 Department of Mechanics and Mathematics, Moscow Center for Fundamental and Applied Mathematics, Lomonosov Moscow State University, 119991 Moscow, Russia

property in geometric terms, it is necessary to construct a new geometric structure on the Hilbert C^*-module, for example, a uniform structure, i.e. a system of pseudometrics or seminorms. This problem was considered to be unsolvable in reasonable generality for a long time. Only partial advances were obtained in [12, 14]. Nevertheless in [24] a uniform structure was discovered that gave a solution in the case of any algebra and any countably generated module as the range module of the operator under consideration. Namely, if $F : \mathcal{M} \to \mathcal{N}$ is an adjointable operator and $\mathcal{N}$ is countably generated then F is $\mathcal{A}$-compact if and only if the image of the unit ball is totally bounded with respect to each defining seminorm for the uniform structure. In [25] the result was strengthened: the necessity of the condition was established for arbitrary modules, the sufficiency was established for modules with some analogue of the projectivity property (it turns out that this property is equivalent to the existence of a standard frame). However, by using this uniform structure the problem cannot be completely solved. In particular, in [7], a counterexample was constructed: a specific C^*-algebra, considered as a module over itself, for which the identity operator is not $\mathcal{A}$-compact, but the unit ball is totally bounded with respect to the introduced uniform structure (in [8], this work was continued with a close relation to the theory of frames). This close connection with the theory of frames has its origin in the fact that Bessel sequences in the module context are involved in the construction of the above seminorms.

The attempts to solve the above problem in full generality lead to the problem of search for more general uniform structures analogous to that considered in above papers. The idea is to take in the definition of a Bessel sequence elements not from the module itself, but from some larger module. In particular, it is possible to replace elements of the module by $\mathcal{A}$-linear functionals.

In the present paper, we introduce some new uniform structures constructed in this way and establish that in some cases they have the same totally bounded sets as the old uniform structure [24].

We also define a new class of $\mathcal{A}$-functionals, slightly more general than the class of adjointable functionals—locally adjointable functionals. By their properties, they are similar to left multipliers, but in some cases, they can be described simply in terms of multipliers.

In Sect. 1, we first recall some facts about C^*-algebras and Hilbert C^*-modules which we need. Then introduce new uniform structures which generalize the old one in a natural way. Also we obtain some useful properties (Lemmas 1.12, 1.23, 1.24). In particular, we prove, that boundedness with respect to any of these uniform structures implies boundedness in norm (that is not true typically for uniform structures, for example, for weak topology).

In Sect. 2, we deal with the uniform structure which is constructed via multipliers, and prove that any set is totally bounded with respect to it if and only if it is totally bounded with respect to the old one. This result holds for arbitrary module $\mathcal{N}$.

In Sect. 3, we work with the uniform structure which is constructed using a more general class of functionals, the locally adjointable functionals, and prove a similar result but only for standard and countably generated modules. It turns out that the results on the structure of functionals on the standard module, obtained in [2], as well as the Kasparov stabilization theorem, which allows us to reduce the problem to the case of the standard module, play a significant role here.

1 Preliminaries and formulation of results

We start with several statements about states on C^*-algebras.

Lemma 1.1 [19, Theorem 3.3.2] *For any state φ on $\mathcal{A}$ and any $a \in \mathcal{A}$ one has $|\varphi(a)|^2 \leq \varphi(a^*a)$.*

Lemma 1.2 [24, Lemma 1.2] *For any $a \in \mathcal{A}$ there is a state φ such that $\|a\| \leq 2|\varphi(a)|$.*

Lemma 1.3 [9, Lemma 2.1] *Let φ be an arbitrary state on C^*-algebra $\mathcal{A}$, $\{e_\lambda\}$—an approximate identity in $\mathcal{A}$. Then $\varphi(x - e_\lambda x) \to 0$ uniformly on bounded sets. Moreover, for any $n \in \mathbb{N}$ there exists a positive $g_n \in \mathcal{A}$, $\|g_n\| \leq 1$, such that $|\varphi(x) - \varphi(g_n x)| \leq \frac{\|x\|}{n}$ for any $x \in \mathcal{A}$.*

One can find basics of the Hilbert C^*-modules theory and their morphisms in books [13, 17] and the survey paper [16] (for other directions of the theory see [4, 6, 22, 23]).

Definition 1.4 A (right) pre-Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ is an $\mathcal{A}$-module equipped with an $\mathcal{A}$-*inner product* $\langle .,. \rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ being a sesquilinear form on the underlying linear space and restricted to satisfy:

1. $\langle x, x \rangle \geq 0$ for any $x \in \mathcal{M}$;
2. $\langle x, x \rangle = 0$ if and only if $x = 0$;
3. $\langle y, x \rangle = \langle x, y \rangle^*$ for any $x, y \in \mathcal{M}$;
4. $\langle x, y \cdot a \rangle = \langle x, y \rangle a$ for any $x, y \in \mathcal{M}$, $a \in \mathcal{A}$.

We will say that a module is countably generated if there exists a countable subset with dense linear span with coefficients from $\mathcal{A}$.

A pre-Hilbert C^*-module over $\mathcal{A}$ is a *Hilbert C^*-module* if it is complete w.r.t. its norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$.

The Hilbert sum of Hilbert C^*-modules (defined in evident sense) is denoted by $\oplus$.

The following analogue of the Cauchy-Schwartz inequality (see [21] or [17, Proposition 1.2.4]) for any $x, y \in \mathcal{M}$ holds:

$$\langle x, y \rangle \langle y, x \rangle \leq \|y\|^2 \langle x, x \rangle. \tag{1.1}$$

Definition 1.5 The *standard* Hilbert C^*-module $\ell^2(\mathcal{A})$ is a Hilbert sum of countably many copies of $\mathcal{A}$ considered as a module over itself, with the inner product $\langle a, b \rangle = \sum_i (a_i)^* b_i$, where $b = (b_1, b_2, \ldots)$.

If $\mathcal{A}$ is unital, then $\ell^2(\mathcal{A})$ is countably generated.

Using of the following famous property of countably generated modules is significant for our purposes (see [11] or [17, Theorem 1.4.2])

Theorem 1.6 (Kasparov stabilization theorem) *Any countably generated Hilbert C^*-module $\mathcal{M}$ over C^*-algebra $\mathcal{A}$ can be represented as an orthogonal direct summand of the standard module, i.e. there exists an isomorphism $\mathcal{M} \oplus \ell^2(\mathcal{A}) \cong \ell^2(\mathcal{A})$.*

Now recall the main concepts and results of [24].

Definition 1.7 Let $\mathcal{N}$ be a Hilbert C^*-module over $\mathcal{A}$. A countable system $X = \{x_i\}$ of its elements is called *admissible* for a submodule $\mathcal{N}_0 \subseteq \mathcal{N}$ (or $\mathcal{N}^0$-admissible) if for each $x \in \mathcal{N}^0$ partial sums of the series $\sum_i \langle x, x_i\rangle\langle x_i, x\rangle$ are bounded by $\langle x, x\rangle$ and the series is convergent. Also, we require $\|x_i\| \le 1$ for any i.

Example 1.8 For the standard module $\ell^2(\mathcal{A})$ over a unital algebra $\mathcal{A}$ one can take for X the natural base $\{e_i\}$. In the case of $\ell^2(\mathcal{A})$ over a general algebra $\mathcal{A}$, one can take x_i having only the i-th component nontrivial and of norm ≤ 1. The other important example is X with only finitely many non-zero elements and an appropriate normalization (this works for any module).

Denote by Φ a countable collection $\{\varphi_1, \varphi_2, \ldots\}$ of states on $\mathcal{A}$. For each pair (X, Φ) with an $\mathcal{N}^0$-admissible X, consider the following seminorms

$$\nu_{X,\Phi}(x)^2 := \sup_k \sum_{i=k}^{\infty} |\varphi_k(\langle x, x_i\rangle)|^2, \quad x \in \mathcal{N}^0, \tag{1.2}$$

and corresponding pseudo-metrics

$$d_{X,\Phi}(x, y)^2 := \sup_k \sum_{i=k}^{\infty} |\varphi_k(\langle x - y, x_i\rangle)|^2, \quad x, y \in \mathcal{N}^0.$$

In [24], it is proved that these pseudo-metrics define a uniform structure and the following definition is introduced.

Definition 1.9 A set $Y \subseteq \mathcal{N}^0 \subseteq \mathcal{N}$ is *totally bounded* with respect to this uniform structure, if for any (X, Φ), where $X \subseteq \mathcal{N}$ is $\mathcal{N}^0$-admissible, and any $\varepsilon > 0$ there exists a finite collection $y_1, \ldots, y_n$ of elements of Y such that the sets

$$\left\{y \in Y \mid d_{X,\Phi}(y_i, y) < \varepsilon\right\}$$

form a cover of Y. This finite collection is an *ε-net in Y for $d_{X,\Phi}$*.
If so, we will say briefly that Y is $(\mathcal{N}, \mathcal{N}^0)$*-totally bounded.*

The main result of [24] is the following theorem.

Theorem 1.10 *Suppose that $F : \mathcal{M} \to \mathcal{N}$ is an adjointable operator and $\mathcal{N}$ is countably generated. Then, F is $\mathcal{A}$-compact if and only if $F(B)$ is $(\mathcal{N}, \mathcal{N})$-totally bounded, where B is the unit ball of $\mathcal{M}$.*

In the present paper, we consider a natural generalization of that uniform structure.

Definition 1.11 Let $\mathcal{N}$ be a Hilbert C^*-module over $\mathcal{A}$. A countable system $F = \{f_i\}$ of elements of the dual module $\mathcal{N}'$ (i.e. $\mathcal{A}$-linear maps $\mathcal{N} \to \mathcal{A}$) is called $*$-*admissible* for a submodule $\mathcal{N}_0 \subseteq \mathcal{N}$ (or $*$-$\mathcal{N}^0$-admissible) if

(1) for each $x \in \mathcal{N}^0$ partial sums of the series $\sum_i (f_i(x))^* f_i(x)$ are bounded by $\langle x, x\rangle$;

(2) this series is norm convergent;
(3) $\|f_i\| \le 1$ for any i.

Lemma 1.12 *If* $\mathcal{N}^0 = \mathcal{N}$, *then condition* (3) *follows from conditions* (1) *and* (2).

Proof For any $i \in \mathbb{N}$, we have $(f_i(x))^* f_i(x) \le \sum_i (f_i(x))^* f_i(x) \le \langle x, x\rangle$ for any $x \in \mathcal{N}$. Hence, $\|(f_i(x))^* f_i(x)\| \le \|\langle x, x\rangle\|$, i.e. $\|f_i(x)\|^2 \le \|x\|^2$ and $\|f_i(x)\| \le \|x\|$, so $\|f_i\| = \sup_{\|x\|\le 1} \|f_i(x)\| \le \|x\| = 1$.

Remark 1.13 If $\mathcal{N}^0 \ne \mathcal{N}$, this is not true in general even for $\mathcal{N}^0$-admissible case. Indeed, for any (non-trivial) modules $\mathcal{Z}_1$ and $\mathcal{Z}_2$ take $\mathcal{N} = \mathcal{Z}_1 \oplus \mathcal{Z}_2$, $\mathcal{N}^0 = \mathcal{Z}_1 \oplus \{0\} \subset \mathcal{Z}_1 \oplus \mathcal{Z}_2$ and $X = \{x_1, 0, \dots\}$, where $x_1 = (0, w)$, $\|w\| > 1$. Then for any $x = (z, 0) \in \mathcal{N}^0$, we have that $\langle x, x_1\rangle = 0$ so X is $\mathcal{N}^0$-admissible, but $\|x_1\| > 1$.

It turns out that this question is closely related to the following problem attracted attention recently. Suppose that $\mathcal{N}_0 \subset \mathcal{N}$ is a Hilbert C^*-submodule such that its orthogonal complement is trivial: $(\mathcal{N}_0)^{\perp}_{\mathcal{N}} = \{0\}$. Is it true that the norm of $x \in \mathcal{N}$ is equal to its norm as an $\mathcal{A}$-functional on $\mathcal{N}_0$, i.e. $\sup\{|\langle x, y\rangle| : y \in \mathcal{N}_0,\ \|y\| \le 1\}$? The answer is generally "no" (see [10, 15]), but in some cases, for example for $\mathcal{A}$ being a commutative von Neumann algebra, the answer is "yes" [15].

Denote by Φ a countable collection $\{\varphi_1, \varphi_2, \dots\}$ of states on $\mathcal{A}$. For each pair (F, Φ) with a $*$-$\mathcal{N}^0$-admissible F, consider the following seminorms

$$\nu_{F,\Phi}(x)^2 := \sup_k \sum_{i=k}^{\infty} |\varphi_k\left(f_i(x)\right)|^2, \quad x \in \mathcal{N}^0,$$

and corresponding pseudo-metrics

$$d_{F,\Phi}(x, y)^2 := \sup_k \sum_{i=k}^{\infty} |\varphi_k\left(f_i(x-y)\right)|^2, \quad x, y \in \mathcal{N}^0. \tag{1.3}$$

Let us observe that this is indeed a generalization of the seminorms $\nu_{X,\Phi}$ defined by (1.2) since $\mathcal{N}$ is included in $\mathcal{N}'$ by formula $\widehat{x}(z) = \langle x, z\rangle$ and since $\langle x, z\rangle = \langle z, x\rangle^*$ and $\varphi(a^*) = \overline{\varphi(a)}$ for any $a \in \mathcal{A}$ and any state φ on $\mathcal{A}$, so $|\varphi(\langle x, x_i\rangle)| = |\varphi(\langle x_i, x\rangle^*)| = |\varphi(\langle x_i, x\rangle)| = |\varphi(\widehat{x_i}(x))|$, where $\widehat{x_i} \in \mathcal{N}$ (note that in [18] the inclusion $\mathcal{N} \to \mathcal{N}'$ is described by another order in the $\mathcal{A}$-inner product but this difference in not significant).

Note that $\widehat{xa}(z) = \langle xa, z\rangle = a^*\langle x, z\rangle$. This corresponds to the structure of a right $\mathcal{A}$-module on $\mathcal{N}'$, namely, $fa(z) = a^* f(z)$.

To prove that this is a seminorm, let us start with a remark that this is a finite non-negative number. Indeed, by Lemma 1.1

$$\begin{aligned}\sum_{i=k}^{s} |\varphi_k\left(f_i(x-y)\right)|^2 &\le \varphi_k\left(\sum_{i=k}^{s} (f_i(x-y))^* f_i(x-y)\right)\\ &\le \left\|\sum_{i=k}^{s} (f_i(x-y))^* f_i(x-y)\right\| \le \|\langle x-y, x-y\rangle\| = \|x-y\|^2.\end{aligned}$$

Since in (1.3) we have a series of non-negative numbers, this estimation implies its convergence and the estimation

$$d_{F,\Phi}(x, y) \le \|x - y\|.$$

As it was noted in [24, Proposition 2.8], for $x \ne y$ there exists (X, Φ) such that $d_{X,\Phi}(x, y) > \frac{1}{2}\|x - y\|$ so this uniform structure is Hausdorff.

Let us verify the triangle inequality:

$$\nu_{F,\Phi}(z + x) \le \nu_{F,\Phi}(z) + \nu_{F,\Phi}(x). \tag{1.4}$$

Take an arbitrary $\varepsilon > 0$ and choose k and m such that

$$\nu_{F,\Phi}(z + x) < \sqrt{\sum_{i=k}^{m} |\varphi_k(f_i(z + x))|^2} + \varepsilon. \tag{1.5}$$

We have

$$\sqrt{\sum_{i=k}^{m} |\varphi_k(f_i(z + x))|^2} \le \sqrt{\sum_{i=k}^{m} (|\varphi_k(f_i(z))| + |\varphi_k(f_i(x)|)^2}. \tag{1.6}$$

By the triangle inequality for the standard norm in $\mathbb{C}^{m-k+1}$, we have

$$\sqrt{\sum_{i=k}^{m} (|\varphi_k(f_i(z))| + |\varphi_k(f_i(x)|)^2} \le \sqrt{\sum_{i=k}^{m} |\varphi_k(f_i(z))|^2} + \sqrt{\sum_{i=k}^{m} |\varphi_k(f_i(x))|^2}$$
$$\le \nu_{F,\Phi}(z) + \nu_{F,\Phi}(x).$$

Since ε in (1.5) is arbitrary, together with (1.6) the last estimation gives (1.4). Therefore, we have verified that $d_{F,\Phi}$ satisfy the conditions for seminorms and now can define a uniform structure on the unit ball of $\mathcal{N}^0$.

We will introduce several variants of uniform structure. For this purpose, we will consider several classes of functionals.

One of them is the case of only adjointable functionals, which have special description given by multipliers. For any C^*-algebra $\mathcal{A}$ one can define the C^*-algebra of *multipliers* $M(\mathcal{A})$ (see [20, §3.12] for details). Then, for any Hilbert $\mathcal{A}$-module $\mathcal{N}$, a Hilbert $M(\mathcal{A})$-module $M(\mathcal{N})$ (which is called the *multiplier module* of $\mathcal{N}$) can be defined in a natural way. This is the set of all adjointable morphisms m from $\mathcal{A}$ to $\mathcal{N}$ with the inner product $\langle m_1, m_2\rangle = m_1^* m_2 \in M(\mathcal{A})$ containing $\mathcal{N}$ as an ideal submodule associated with $\mathcal{A}$, i.e. $\mathcal{N} = M(\mathcal{N})\mathcal{A}$ (see [3, Sect. 1] and [1, Sect. 3] for more details; in [3] it is denoted by $\mathcal{N}_d$ and it is proved that, for a full module $\mathcal{N}$, it has some natural universal property [3, Theorem 1.2]). The adjoint of m is an adjointable $\mathcal{A}$-functional $m^* : \mathcal{N} \to \mathcal{A}$, i.e. the correspondence $m \mapsto m^*$, together with the property $m^{**} = m$, gives us a proof of the following statement.

Theorem 1.14 *There is a natural identification of the multiplier module $M(\mathcal{N})$ with the module $\mathcal{N}^*$ of all adjointable $\mathcal{A}$-linear functionals from $\mathcal{N}$ to $\mathcal{A}$.*

If $f \in \mathcal{N}'$ is represented by some element of multiplier module we will denote it also as $f \in M(\mathcal{N})$ if it does not cause a confusion.

We will call a functional $f : \mathcal{N} \to \mathcal{A}$ *locally adjointable,* if for any adjointable $\mathcal{A}$-linear operator $g : \mathcal{A} \to \mathcal{N}$, the composition $f \circ g : \mathcal{A} \to \mathcal{A}$ is an adjointable functional. Of course, any adjointable functional is locally adjointable.

Definition 1.15 A set $Y \subseteq \mathcal{N}^0 \subseteq \mathcal{N}$ is $(\mathcal{N}, \mathcal{N}^0)^*$-*totally bounded* if for any (F, Φ), where $F \subseteq \mathcal{N}'$ is $*$-$\mathcal{N}^0$-admissible, and any $\varepsilon > 0$ there exists a finite collection $y_1, \dots, y_n$ of elements of Y such that the sets

$$\left\{ y \in Y \mid d_{F,\Phi}(y_i, y) < \varepsilon \right\}$$

form a cover of Y. This finite collection is an ε-*net in* Y for $d_{F,\Phi}$. Uniform structure form by pseudometrics $d_{F,\Phi}$ we will call $(\mathcal{N}, \mathcal{N}^0)^*$-uniform structure.

If in this definition we take for F only adjointable functionals or locally adjointable, then we will say that Y is $(\mathcal{N}, \mathcal{N}^0)^*_{\mathrm{ad}}$-*totally bounded* or $(\mathcal{N}, \mathcal{N}^0)^*_{\mathrm{lad}}$-*totally bounded* respectively, and corresponding uniform structures we will call $(\mathcal{N}, \mathcal{N}^0)^*_{\mathrm{ad}}$-uniform structure or $(\mathcal{N}, \mathcal{N}^0)^*_{\mathrm{lad}}$-uniform structure respectively.

In adjointable case, it is equivalent (due to Theorem 1.14) to take for F only multipliers, so we can call Y in this case $(\mathcal{N}, \mathcal{N}^0)^*_{\mathrm{out}}$-*totally bounded*, and the uniform structure coincides with $(\mathcal{N}, \mathcal{N}^0)^*_{\mathrm{out}}$-uniform structure (in these terms it was previously introduced in [7]).

Lemma 1.16 *Suppose, a set $Y \subseteq \mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ is $(\mathcal{N}, \mathcal{N}^0)^*$-totally bounded. Then $p_1 Y$ and $p_2 Y$ are $(\mathcal{N}_1, \mathcal{N}_1^0)^*$- and $(\mathcal{N}_2, \mathcal{N}_2^0)^*$-totally bounded, respectively, where $p_1 : \mathcal{N}_1 \oplus \mathcal{N}_2 \to \mathcal{N}_1$, $p_2 : \mathcal{N}_1 \oplus \mathcal{N}_2 \to \mathcal{N}_2$ are the orthogonal projections, $\mathcal{N}_1^0 = p_1 \mathcal{N}^0$ and $\mathcal{N}_2^0 = p_2 \mathcal{N}^0$.*

Conversely, if $p_1 Y$ and $p_2 Y$ are respectively $(\mathcal{N}_1, \mathcal{N}_1^0)^*$- and $(\mathcal{N}_2, \mathcal{N}_2^0)^*$-totally bounded for some submodules $\mathcal{N}_1^0$ and $\mathcal{N}_2^0$, respectively, then Y is $(\mathcal{N}, \mathcal{N}_1^0 \oplus \mathcal{N}_2^0)^*$-totally bounded.

This is similarly true for $(\mathcal{N}, \mathcal{N}^0)^*_{\mathrm{ad}}$ and $(\mathcal{N}, \mathcal{N}^0)^*_{\mathrm{lad}}$-totally boundedness.

Proof Denote by $J_j = p_j^*$ the corresponding inclusions $J_j : \mathcal{N}_j \hookrightarrow \mathcal{N}_1 \oplus \mathcal{N}_2$, $j = 1, 2$.

Also we introduce the map $p'_j : \mathcal{N}'_j \to (\mathcal{N}_1 \oplus \mathcal{N}_2)'$ defined by the formula $p'_j(f)(x) = f(p_j(x))$, and the map $J'_j : (\mathcal{N}_j \oplus \mathcal{N}_2)' \to \mathcal{N}'_j$, $J'_j(f)(x) = f(J_j(x))$.

Suppose, Y is $(\mathcal{N}, \mathcal{N}^0)^*$-totally bounded and $F = \{f_i\}$ is a $*$-admissible system for a submodule $\mathcal{N}_1^0 \subseteq \mathcal{N}_1$. Then $p'_1 F = \{p'_1(f_i)\}$ is admissible for $\mathcal{N}^0$ because

$$(p'_1(f_i)(x))^* p'_1(f_i)(x) = (f_i(p_1(x)))^* f_i(p_1(x)).$$

Let $y_1, \dots, y_s$ be an ε-net in Y for $d_{p'_1 F,\Phi}$. Then $p_1 y_1, \dots, p_1 y_s$ is an ε-net in $p_1 Y$ for $d_{F,\Phi}$. Indeed, consider an arbitrary $z \in p_1 Y$. Then $z = p_1 y$ for some $y \in Y$. Find

y_k such that $d_{p_1' F,\Phi}(y, y_k) < \varepsilon$. Then

$$\begin{aligned} d^2_{F,\Phi}(z, p_1 y_k) &= \sup_k \sum_{i=k}^{\infty} |\varphi_k(f_i(z - p_1 y_k))|^2 \\ &= \sup_k \sum_{i=k}^{\infty} |\varphi_k(f_i(p_1(y - y_k)))|^2 \\ &= \sup_k \sum_{i=k}^{\infty} |\varphi_k\left((J_1' f_i)(y - y_k)\right)|^2 = d^2_{p_1' F,\Phi}(y, y_k) < \varepsilon^2. \end{aligned}$$

Similarly for $j = 2$.

Conversely, suppose that $p_j Y$ are $(\mathcal{N}_j, \mathcal{N}_j^0)^*$-totally bounded, $j = 1, 2$. Let $F = \{f_i\}$ be an admissible system in $\mathcal{N}'$ for $\mathcal{N}_1^0 \oplus \mathcal{N}_2^0$ and $\varepsilon > 0$ is arbitrary. Then $F_j := \{J_j'(f_i)\}$ is an admissible system in $\mathcal{N}_j'$ for $\mathcal{N}_j^0$. Indeed, this follows from

$$(J_j'(f_i)(x))^* J_j'(f_i)(x) = (f_i(J_j(x)))^* f_i(J_j(x)).$$

For $u, v \in p_j Y$, we have

$$d_{F,\Phi}(J_j u, J_j v) = d_{F_j,\Phi}(u, v). \tag{1.7}$$

Indeed, we obtain the convergence and can estimate the sum using (as above) the equality

$$\sum_{i=1}^{s} (f_i(J_j(u - v)))^* f_i(J_j(u - v)) = \sum_{i=1}^{s} ((J_j' f_i)(u - v))^* (J_j' f_i)(u - v)$$

and, quite similarly, (1.7) follows from the equality

$$f_i(J_j u - J_j v) = (J_j' f_i)(u - v), \quad j = 1, 2.$$

Suppose that $z_1, \ldots, z_m$ is an $\varepsilon/4$-net in $p_1 Y$ for $d_{F_1,\Phi}$ and $w_1, \ldots, w_r$ is an $\varepsilon/4$-net in $p_2 Y$ for $d_{F_2,\Phi}$. Consider $\{z_k + w_s\}$, $k = 1, \ldots, m$, $s = 1, \ldots, r$. Then $\{J_1 z_k + J_2 w_s\}$ is an $\varepsilon/2$-net in $p_1 Y \oplus p_2 Y$ for $d_{F,\Phi}$. Indeed, for any $J_1 p_1 y_1 + J_2 p_2 y_2$, $y_1, y_2 \in Y$, one can find z_k and w_s such that

$$d_{F_1,\Phi}(p_1 y_1, z_k) < \varepsilon/4, \quad d_{F_2,\Phi}(p_2 y_2, w_s) < \varepsilon/4.$$

Then by (1.4) and (1.7)

$$\begin{aligned} &d_{F,\Phi}(J_1 p_1 y_1 + J_2 p_2 y_2, J_1 z_k + J_2 w_s) \\ &\quad \le d_{F,\Phi}(J_1 p_1 y_1, J_1 z_k) + d_{F,\Phi}(J_2 p_2 y_2, J_2 w_s) \\ &\quad = d_{F_1,\Phi}(p_1 y_1, z_k) + d_{F_2,\Phi}(p_2 y_2, w_s) < \varepsilon/2. \end{aligned}$$

Now find a subset $\{u_l\} \subset \{z_k + w_s\}$ formed by all elements of $\{z_k + w_s\}$ such that there exists an element $u^* \in Y \subseteq p_1 Y \oplus p_2 Y$ with $d_{F,\Phi}(u^*, z_k + w_s) < \varepsilon/2$. Denote these u^* by u_l^*, $l = 1, \dots, L$. Therefore,

(1) for any $y \in Y$, there exists $l \in 1, \dots, L$ such that $d_{F,\Phi}(y, u_l) < \varepsilon/2$;
(2) for each $l \in 1, \dots, L$, we have $d_{F,\Phi}(u_l^*, u_l) < \varepsilon/2$.

By the triangle inequality, $\{u_l^*\}$ is a finite ε-net in Y for $d_{F,\Phi}$ and we are done.

Remark 1.17 From [24, Lemma 2.15], it follows that $\mathcal{N}_1^0 \oplus \mathcal{N}_2^0$ is countably generated if and only if $\mathcal{N}_1^0$ and $\mathcal{N}_2^0$ are countably generated.

Evidently we have the following statements.

Proposition 1.18 *A functional $f : \mathcal{A} \to \mathcal{A}$ is locally adjointable if and only if it is adjointable.*

Proposition 1.19 *If $LM(\mathcal{A}) = M(\mathcal{A})$, hence $RM(\mathcal{A}) = M(\mathcal{A})$, then any functional is locally adjointable (the details on LM and RM, left and right multipliers respectively, one can find, for example, in* [20, §3.12]).

This is fulfilled, in particular, for commutative and unital algebras.
Now, we are able to formulate our main result.

Theorem 1.20 (Main Theorem) *Let Y be a subset of $\mathcal{N}^0 \subset \mathcal{N}$. Then Y is $(\mathcal{N}, \mathcal{N}^0)^*_{\text{lad}}$-totally bounded if and only if it is $(\mathcal{N}, \mathcal{N}^0)$-totally bounded in the following cases:*

(1) $\mathcal{N} = \mathcal{A}$;
(2) $\mathcal{N} = \ell^2(\mathcal{A})$;
(3) *$\mathcal{N}$ is countably generated.*

Corollary 1.21 *Suppose that $LM(\mathcal{A}) = M(\mathcal{A})$ and Y is a subset of $\mathcal{N}^0 \subset \mathcal{N}$. Then Y is $(\mathcal{N}, \mathcal{N}^0)^*$-totally bounded if and only if it is $(\mathcal{N}, \mathcal{N}^0)$-totally bounded and if and only if it is $(\mathcal{N}, \mathcal{N}^0)^*_{\text{ad}}$-totally bounded in the following cases:*

(1) $\mathcal{N} = \mathcal{A}$;
(2) $\mathcal{N} = \ell^2(\mathcal{A})$;
(3) *$\mathcal{N}$ is countably generated.*

From Theorems 1.10 and 1.20, we immediately obtain

Theorem 1.22 *Suppose, $F : \mathcal{M} \to \mathcal{N}$ is an adjointable operator and $\mathcal{N}$ is countably generated. Then F is $\mathcal{A}$-compact if and only if $F(B)$ is $(\mathcal{N}, \mathcal{N})^*_{\text{lad}}$-totally bounded, where B is the unit ball of $\mathcal{M}$.*

A similar statement based on Corollary 1.21 can be obtained as well.
For outer systems, we will be able to prove an analogue of Theorem 1.20 (see Theorem 2.2 below) without the countability restriction. This is not surprising because $\mathcal{N}^*$ is rather close to $\mathcal{N}$ (and coincides with it, e.g. for a unital $\mathcal{A}$).
The following fact is very useful.

Lemma 1.23 *Suppose that $F = \{f_i\}$ is a $*$-$\mathcal{N}^0$-admissible system and $\{g_i\}_{i\in\mathbb{N}}$ is an arbitrary sequence of elements of $\mathcal{A}$ such that $\|g_i\| \leq 1$ for any i. Then, $\{(f_i g_i)\}$ is also a $*$-$\mathcal{N}^0$-admissible system. Also, if F is outer $\mathcal{N}^0$-admissible system (i.e. all $f_i \in M(\mathcal{N})$), then $\{(f_i g_i)\}$ is $\mathcal{N}^0$-admissible (i.e. $f_i g_i \in \mathcal{N}$).*

Proof Indeed,

$$((f_i g_i)(x))^*(f_i g_i)(x) = ((g_i)^* f_i(x))^*((g_i)^* f_i(x)) = (f_i(x))^* g_i g_i^* f_i(x)$$
$$\leq \|g_i\|^2 (f_i(x))^* f_i(x) \leq (f_i(x))^* f_i(x).$$

This implies properties (1) and (2) of Definition 1.11. The property (3) is evident.

Lemma 1.24 *Suppose $Y \subset \mathcal{N}^0 \subset \mathcal{N}$ is bounded w.r.t. any seminorm of $(\mathcal{N}, \mathcal{N}^0)$-uniform structure. Then, Y is bounded in norm.*

Proof Suppose that Y is not norm-bounded. Then, there exists a sequence $\{z_k\}_{k\in\mathbb{N}} \subset \mathcal{N}^0$ such that $\|z_k\| \geq 3^k$. Define $x_k = \frac{z_k}{\|z_k\| 2^k}$. The collection $X = \{x_k\}$ is $\mathcal{N}^0$-admissible, even $\mathcal{N}$-admissible, by the following inequality: $\sum_k \langle x, x_k\rangle\langle x_k, x\rangle \leq \sum_k \frac{1}{4^k}\langle x, x\rangle \leq \langle x, x\rangle$ for any $x \in \mathcal{N}$. For any $k \in \mathbb{N}$ take a state φ_k such that $\varphi_k(\langle z_k, z_k\rangle) \geq \frac{1}{2}\|\langle z_k, z_k\rangle\| = \frac{1}{2}\|z_k\|^2$. Then

$$\nu_{X,\Phi}(z_k) \geq |\varphi_k(\langle z_k, x_k\rangle)| = \frac{1}{\|z_k\| 2^k}|\varphi_k(\langle z_k, z_k\rangle)| \geq \frac{1}{\|z_k\| 2^{k+1}}\|z_k\|^2 \geq \frac{3^k}{2^{k+1}},$$

i.e. Y is not bounded w.r.t. the seminorm $\nu_{X,\Phi}$. A contradiction.

Corollary 1.25 *Suppose $Y \subset \mathcal{N}^0 \subset \mathcal{N}$ is bounded w.r.t. any seminorm of $(\mathcal{N}, \mathcal{N}^0)^*_{\mathrm{lad}}$, $(\mathcal{N}, \mathcal{N}^0)^*_{\mathrm{ad}}$ or $(\mathcal{N}, \mathcal{N}^0)^*$-uniform structure. Then Y is bounded in norm.*

Corollary 1.26 *Suppose $Y \subset \mathcal{N}^0 \subset \mathcal{N}$ is totally bounded w.r.t. any seminorm of $(\mathcal{N}, \mathcal{N}^0)$, $(\mathcal{N}, \mathcal{N}^0)^*_{\mathrm{lad}}$, $(\mathcal{N}, \mathcal{N}^0)^*_{\mathrm{ad}}$ or $(\mathcal{N}, \mathcal{N}^0)^*$-uniform structure. Then Y is bounded in norm.*

2 The case of multipliers

Lemma 2.1 *For any $\mathcal{A}$-module $\mathcal{N}$, any $f \in \mathcal{N}'$, any state φ on $\mathcal{A}$ and any $n \in \mathbb{N}$ there exists a positive $g_n \in \mathcal{A}$, $\|g_n\| \leq 1$, such that*

$$|\varphi(f(x)) - \varphi((fg_n)(x))| = |\varphi(f(x)) - \varphi(g_n f(x))| \leq \frac{\|f(x)\|}{n} \leq \frac{\|f\| \cdot \|x\|}{n}$$

for any $x \in \mathcal{N}$.

Proof From Lemma 1.3, we have that for any $n \in \mathbb{N}$ there exists a positive $g_n \in \mathcal{A}$, $\|g_n\| \leq 1$, such that $|\varphi(y) - \varphi(g_n y)| \leq \frac{\|y\|}{n}$ for any $y \in \mathcal{A}$. By taking $y = f(x)$ we have $|\varphi(f(x)) - \varphi(g_n f(x))| \leq \frac{\|f(x)\|}{n} \leq \frac{\|f\|\cdot\|x\|}{n}$ for any $x \in \mathcal{N}$, then note that $(fb)(\cdot) = b^* f(\cdot)$ for arbitrary $b \in \mathcal{A}$.

Now, for any $\varepsilon > 0$ and any x such that $\|x\| \le d$, there exists $n(i) \in \mathbb{N}$ such that

$$|\varphi(f(x)) - \varphi((fg_{n(i)})(x))| \le \frac{\varepsilon}{2^i}.$$

If $f \in M(\mathcal{N})$ (for example it is always so if f is locally adjointable and $\mathcal{N} = \mathcal{A}$ as a module over itself), then $fg_{n(i)} \in \mathcal{N}$ and $fg_{n(i)}(x) = \langle fg_{n(i)}, x\rangle = g_{n(i)}\langle f, x\rangle$.

Moreover, for every finite family of states $\{\varphi_j\}_{j=1}^L$, we can find $n(i)$ such that

$$|\varphi_j(f(x)) - \varphi_j((fg_{n(i)})(x))| \le \frac{\varepsilon}{2^i}$$

for all $j = 1, \dots, L$.

Theorem 2.2 *Let Y be a subset of $\mathcal{N}^0 \subset \mathcal{N}$. Then, Y is $(\mathcal{N}, \mathcal{N}^0)_{\text{out}}$-totally bounded if and only if it is $(\mathcal{N}, \mathcal{N}^0)$-totally bounded.*

Proof For any outer $\mathcal{N}^0$-admissible system $F = \{f_i\}$ of multipliers of $\mathcal{N}$ and any countable collection $\Phi = \{\varphi_j\}$ of states on $\mathcal{A}$ it is sufficient to find, for arbitrary $\varepsilon > 0$, a $\mathcal{N}^0$-admissible system $X = \{x_i\}$ in $\mathcal{N}$ such that for any $x \in \mathcal{N}^0$ with $\|x\| \le \operatorname{diam}(Y) =: d < \infty$ we have

$$d_{F,\Phi}(x, 0) \le d_{X,\Phi}(x, 0) + \varepsilon.$$

Indeed, this means that an ε-net on Y for $d_{X,\Phi}$ is a 2ε-net for $d_{F,\Phi}$. We may consider

$$\varepsilon < 1, \text{ hence } \varepsilon^4 < \varepsilon^2. \tag{2.1}$$

By Lemma 1.3, for each $i = 1, 2, \dots$ we can find $g_{n(i)} \ge 0$ such that, for all $x \in Y$, we have

$$|\varphi_k(\langle f_i, x\rangle) - \varphi_k(g_{n(i)}\langle f_i, x\rangle)| \le \frac{\varepsilon^2}{2^i \cdot 4\max\{1, d\}}, \quad k = 1, \dots, i, \tag{2.2}$$

i.e. $x_i = f_i g_{n(i)} \in \mathcal{N}$. The system $\{x_i\}$ is $\mathcal{N}^0$-admissible due to Lemma 1.23. For arbitrary $x \in Y$ and arbitrary $k, i \in \mathbb{N}$, $k \le i$, we have

$$|\varphi_k(\langle f_i, x\rangle)| \le |\varphi_k(\langle f_i, x\rangle) - \varphi_k(\langle x_i, x\rangle)| + |\varphi_k(\langle x_i, x\rangle)|.$$

Hence, by (2.2) and (2.1)

$$\begin{aligned}|\varphi_k(\langle f_i, x\rangle)|^2 &\le |\varphi_k(\langle f_i, x\rangle) - \varphi_k(\langle x_i, x\rangle)|^2 \\ &\quad + 2|\varphi_k(\langle f_i, x\rangle) - \varphi_k(\langle x_i, x\rangle)||\varphi_k(\langle x_i, x\rangle)| + |\varphi_k(\langle x_i, x\rangle)|^2 \\ &\le \frac{\varepsilon^2}{2^i \cdot 4} + 2\frac{\varepsilon^2}{2^i \cdot 4d}d + |\varphi_k(\langle x_i, x\rangle)|^2 \le \frac{\varepsilon^2}{2^i} + |\varphi_k(\langle x_i, x\rangle)|^2.\end{aligned}$$

Then,

$$\sum_{i=k}^{\infty} |\varphi_k(\langle f_i, x\rangle)|^2 \le \varepsilon^2 + \sum_{i=k}^{\infty} |\varphi_k(\langle x_i, x\rangle)|^2.$$

Thus, using $\sqrt{s+t} \le \sqrt{s + 2\sqrt{st} + t} = \sqrt{s} + \sqrt{t}$, for $s, t \ge 0$, we obtain

$$\sqrt{\sum_{i=k}^{\infty} |\varphi_k(\langle f_i, x\rangle)|^2} \le \varepsilon + \sqrt{\sum_{i=k}^{\infty} |\varphi_k(\langle x_i, x\rangle)|^2}.$$

Taking at first the supremum on the right hand side and then on the left hand side, we obtain

$$\sup_k \sqrt{\sum_{i=k}^{\infty} |\varphi_k(\langle f_i, x\rangle)|^2} \le \varepsilon + \sup_k \sqrt{\sum_{i=k}^{\infty} |\varphi_k(\langle x_i, x\rangle)|^2},$$

i.e. $d_{F,\Phi}(x, 0) \le d_{X,\Phi}(x, 0) + \varepsilon$ as desired.

3 Proof of the main theorem

3.1 Proof of the main theorem for $\mathcal{N} = \mathcal{A}$

By Proposition 1.18, in this case locally adjointable $\mathcal{A}$-functionals are exactly adjointable ones, that are multipliers by Theorem 1.14 (in fact in this case the identification is trivial). Hence, the statement follows Theorem 2.2.

3.2 Proof of the main theorem for $\mathcal{N} = \ell^2(\mathcal{A})$

To reduce this case to the case of $\mathcal{N} = \mathcal{A}$, recall the description of arbitrary functional $f : \ell^2(\mathcal{A}) \to \mathcal{A}$ from [2, Theorem 2.3].

In that paper, the inner product is defined to be anti-linear on second variable, and embedding $\ell^2(\mathcal{A}) \to \ell^2(\mathcal{A})'$ is defined by the formula $x \mapsto \widehat{x}(\cdot) = \langle \cdot, x\rangle$, so it is extended for left multipliers x by the formula $\widehat{x}(y) = \sum_{s=1}^{\infty} y_s x_s^*$ as an isometric isomorphism $\ell^2_{\text{strong}}(LM(\mathcal{A})) \to \ell^2(\mathcal{A})'$. Hence x needs to satisfy the condition $\sup_N \left\| \sum_{s=1}^{N} x_s x_s^* \right\| < \infty$.

In our case we define the inner product to be anti-linear on first variable and define embedding $\ell^2(\mathcal{A}) \to \ell^2(\mathcal{A})'$ by the formula $x \mapsto \widehat{x}(\cdot) = \langle x, \cdot\rangle$, hence after extension the embedding in our case x needs to be a sequence of right multipliers acting by the formula $\widehat{x}(y) = \sum_{s=1}^{\infty} x_s^* y_s$ and it must satisfy the condition $\sup_N \left\| \sum_{s=1}^{N} x_s^* x_s \right\| < \infty$.

Therefore, any functional $f : \ell^2(\mathcal{A}) \to \mathcal{A}$ can be described as a sequence $f_s \in RM(\mathcal{A})$, $s = 1, 2, \dots$, such that the partial sums of the series $\sum_s f_s^* f_s$ are uniformly bounded. If f is locally adjointable, then $f_s \in M(\mathcal{A})$, $s = 1, 2, \dots$, because the inclusion of $\mathcal{A}$ into $\ell^2(\mathcal{A})$ (as the sth summand) is adjointable.

Since the partial sums of the series $\sum_s f_s^* f_s$ are uniformly bounded, the series $\sum_s \varphi(f_s^* f_s)$ is convergent, where φ is an arbitrary state on $\mathcal{A}$. Here we consider any state as a state on $M(\mathcal{A})$ due to [5, 2.3.24].

By Lemma 1.24, there exists $d < \infty$ such that $\|x\| \le d$ for any $x \in Y$.

Therefore, for a $*$-$\mathcal{N}^0$-admissible system $F = \{f_i\}$, $f_i = (f_{i,1}, f_{i,2}, \dots)$ we first choose a finite part of each functional $(f_{i,1}, f_{i,2}, \dots, f_{i,r(i)}, 0, \dots)$ such that

$$\sum_{s=r(i)+1}^{\infty} \varphi_k(f_{i,s}^* f_{i,s}) < \frac{\varepsilon^2}{2^i \cdot 4d^2} \quad \text{for all } k = 1, \dots, i.$$

Note that for any $k, i, p, q \in \mathbb{N}$, $k \le i$, $r(i) \le p \le q$, the function $(a, b) \mapsto \varphi_k(\sum_{s=p}^{q} a_s^* b_s)$ is a complex inner product on $\ell^2_{\text{strong}}(M(\mathcal{A}))$, so by the Cauchy-Schwartz inequality, we have

$$\begin{aligned}\left|\varphi_k\left(\sum_{s=p}^{q} f_{i,s}^* x_s\right)\right|^2 &\le \varphi_k\left(\sum_{s=p}^{q} f_{i,s}^* f_{i,s}\right)\varphi_k\left(\sum_{s=p}^{q} x_s^* x_s\right)\\ &\le \varphi_k\left(\sum_{s=r(i)+1}^{\infty} f_{i,s}^* f_{i,s}\right)\|x\|^2 \le \frac{\varepsilon^2 d^2}{2^i \cdot 4d^2} = \frac{\varepsilon^2}{2^i \cdot 4}.\end{aligned}$$

Second, we approximate each multiplier $f_{i,s}$, $s = 1, \dots, r(i)$, by choosing $g_{n(i)}$ for $f_{i,s} g_{n(i)}$ as in the previous section (by Lemma 1.3):

$$|\varphi_k(f_{i,s}^* x) - \varphi_k(g_{n(i)} f_{i,s}^* x)| \le \frac{\varepsilon}{2^i \cdot 4r(i)}, \quad \text{for all } k = 1, \dots, i.$$

Hence, by using the inequality for scalars $|a+b|^2 \le 2|a|^2 + 2|b|^2$ we have for all $k \le i$

$$\begin{aligned}|\varphi_k(f_i(x))|^2 &= \left|\varphi_k\left(\sum_{s=1}^{\infty} f_{i,s}^* x_s\right)\right|^2\\ &\le 2\left|\varphi_k\left(\sum_{s=1}^{r(i)} f_{i,s}^* x_s\right)\right|^2 + 2\left|\varphi_k\left(\sum_{s=r(i)+1}^{\infty} f_{i,s}^* x_s\right)\right|^2\\ &\le 4\left|\varphi_k\left(\sum_{s=1}^{r(i)} g_{n(i)} f_{i,s}^* x_s\right)\right|^2 + 4\left|\varphi_k\left(\sum_{s=1}^{r(i)} f_{i,s}^* x_s\right) - \varphi_k\left(\sum_{s=1}^{r(i)} g_{n(i)} f_{i,s}^* x_s\right)\right|^2\\ &\quad + \frac{\varepsilon^2}{2^i \cdot 2} \le\end{aligned}$$

$$\leq 4 \left| \varphi_k \left(\sum_{s=1}^{r(i)} g_{n(i)} f_{i,s}^* x_s \right) \right|^2 + 4 \frac{\varepsilon^2}{4^i \cdot 16} + \frac{\varepsilon^2}{2^i \cdot 2}.$$

Thus,

$$\sum_{i=k}^{\infty} |\varphi_k(f_i(x))|^2 \leq \sum_{i=k}^{\infty} 4 \left| \varphi_k \left(\sum_{s=1}^{r(i)} g_{n(i)} f_{i,s}^* x_s \right) \right|^2 + \varepsilon^2.$$

Hence, using again $\sqrt{s+t} \leq \sqrt{s + 2\sqrt{st} + t} = \sqrt{s} + \sqrt{t}$, for $s, t \geq 0$, we obtain

$$\sqrt{\sum_{i=k}^{\infty} |\varphi_k(f_i(x))|^2} \leq 2 \sqrt{\sum_{i=k}^{\infty} \left| \varphi_k \left(\sum_{s=1}^{r(i)} g_{n(i)} f_{i,s}^* x_s \right) \right|^2} + \varepsilon.$$

Taking at first the supremum on the right hand side and then on the left hand side, we obtain

$$\sup_k \sqrt{\sum_{i=k}^{\infty} |\varphi_k(f_i(x))|^2} \leq 2 \sup_k \sqrt{\sum_{i=k}^{\infty} \left| \varphi_k \left(\sum_{s=1}^{r(i)} g_{n(i)} f_{i,s}^* x_s \right) \right|^2} + \varepsilon,$$

i.e. if we denote $X = \{x_i\}$, $x_i = (f_{i,1} g_{n(i)}, f_{i,2} g_{n(i)}, \dots, f_{i,r(i)} g_{n(i)}, 0, \dots) \in \ell^2(\mathcal{A})$,

$$\sup_k \sqrt{\sum_{i=k}^{\infty} |\varphi_k(f_i(x))|^2} \leq 2 \sup_k \sqrt{\sum_{i=k}^{\infty} |\varphi_k(\langle x_i, x \rangle)|^2} + \varepsilon.$$

Thus,

$$d_{F,\Phi}(x, 0) \leq 2 d_{X,\Phi}(x, 0) + \varepsilon,$$

for any $x \in Y$, i.e. an ε-net on Y for $d_{X,\Phi}$ is a 3ε-net for $d_{F,\Phi}$. It is easy to see that X is $(\mathcal{N}, \mathcal{N}^0)$-admissible.

3.3 Proof of the main theorem in the general case

The general case of a countably generated module $\mathcal{N}$ can be reduced to the above considered case of $\mathcal{N} = \ell^2(\mathcal{A})$ using the Kasparov stabilization theorem by considering the module as a direct summand of the standard one and using the fact that uniform structures respect direct summand decomposition. More specifically, denote by S the map $S : \mathcal{N} \to \mathcal{N} \oplus \ell^2(\mathcal{A}) \cong \ell^2(\mathcal{A})$ given by the Kasparov theorem. Suppose that Y is $(\mathcal{N}, \mathcal{N}^0)$-totally bounded. Then, by Lemma 1.16, it follows that $S(Y)$ is $(\ell^2(\mathcal{A}), S(\mathcal{N}^0))$-totally bounded. Therefore, by previous section $S(Y)$ is $(\ell^2(\mathcal{A}), S(\mathcal{N}^0))^*_{\mathrm{lad}}$-totally bounded, hence, again by Lemma 1.16 we have that Y is $(\mathcal{N}, \mathcal{N}^0)^*_{\mathrm{lad}}$-totally bounded. The converse statement is obvious.

Funding This work is supported by the Russian Science Foundation under grant 23-21-00097.

Data availability Not applicable.

Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

Financial interest The authors have no relevant financial or non-financial interests to disclose.

References

1. Arambašić, L., Bakić, D.: Frames and outer frames for Hilbert C^*-modules. Linear Multilinear Algebra **65**(2), 381–431 (2017)
2. Bakić, D.: Weak frames in Hilbert C^*-modules with application in Gabor analysis. Banach J. Math. Anal. **13**(4), 1017–1075 (2019)
3. Bakić, D., Guljaš, B.: Extensions of Hilbert C^*-modules. Houst. J. Math. **30**, 537–558 (2004)
4. Blanchard, E., Gogić, I.: On unital $C(X)$-algebras and $C(X)$-valued conditional expectations of finite index. Linear Multilinear Algebra **64**(12), 2406–2418 (2016)
5. Bratteli, O., Robinson, D.W.: Operator Algebras and Quantum Statistical Mechanics. I. Springer, New York (1981)
6. Frank, M., Manuilov, V., Troitsky, E.: Hilbert C^*-modules from group actions: beyond the finite orbits case. Stud. Math. **200**, 131–148 (2010)
7. Fufaev, D.V.: A Hilbert C^*-module with extremal properties. Funct. Anal. Appl. **56**(1), 72–80 (2022)
8. Fufaev, D.V.: Topological and frame properties of certain pathological C*-algebras. Russ. J. Math. Phys. **29**(2), 170–182 (2022)
9. Fufaev, D.V.: Thick elements and states in C^*-algebras in view of frame theory. Russ. J. Math. Phys. **30**(2), 184–191 (2023)
10. Kaad, J., Skeide, M.: Kernels of Hilbert module maps: a counterexample. J. Oper. Theory **89**(2), 343–348 (2023)
11. Kasparov, G.G.: Hilbert C*-modules: theorems of Stinespring and Voiculescu. J. Oper. Theory **4**, 133–150 (1980)
12. Kečkić, D.J., Lazović, Z.: Compact and "compact" operators on standard Hilbert modules over W^*-algebras. Ann. Funct. Anal. **9**(2), 258–270 (2018)
13. Lance, E.C.: Hilbert C*-Modules—A Toolkit for Operator Algebraists. London Mathematical Society Lecture Note Series, vol. 210. Cambridge University Press, Cambridge (1995)
14. Lazović, Z.: Compact and "compact" operators on standard Hilbert modules over C^*-algebras. Adv. Oper. Theory **3**(4), 829–836 (2018)
15. Manuilov, V.: On extendability of functionals on Hilbert C^*-modules. Math. Nachr. **297**, 998–1005 (2024)
16. Manuilov, V.M., Troitsky, E.V.: Hilbert C^*- and W^*-modules and their morphisms. J. Math. Sci. N. Y. **98**(2), 137–201 (2000)
17. Manuilov, V.M., Troitsky, E.V.: Hilbert C^*-Modules. American Mathematical Society, Providence (2005)
18. Manuilov, V., Troitsky, E.: Hilbert C^*-modules with Hilbert dual and C^*-Fredholm operators. Integral Equ. Oper. Theory **95**(3), Paper No. 17, (2023)
19. Murphy, G.J.: C^*-Algebras and Operator Theory. Academic Press, San Diego (1990)
20. Pedersen, G.K.: C^*-Algebras and Their Automorphism Groups. London Mathematical Society Monographs, vol. 14. Academic Press, San Diego (1979)

21. Paschke, W.L.: Inner product modules over B*-algebras. Trans. Am. Math. Soc. **182**, 443–468 (1973)
22. Pavlov, A.A., Troitskii, E.V.: Quantization of branched coverings. Russ. J. Math. Phys. **18**(3), 338–352 (2011)
23. Troitsky, E.: Manuilov algebra, C^*-Hilbert modules, and Kuiper type theorems. Russ. J. Math. Phys. **25**(4), 534–544 (2018)
24. Troitsky, E.: Geometric essence of "compact" operators on Hilbert C*-modules. J. Math. Anal. Appl. **485**(2), 123842 (2020)
25. Troitsky, E.V., Fufaev, D.V.: Compact operators and uniform structures in Hilbert C*-modules. Funct. Anal. Appl. **54**, 287–294 (2020)