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Characterization of *a*-Birkhoff–James orthogonality in C^*-algebras and its applications

Hooriye Sadat Jalali Ghamsari[1] · **Mahdi Dehghani**[1]

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Abstract

Let $\mathcal{A}$ be a unital C^*-algebra with unit $1_{\mathcal{A}}$ and let $a \in \mathcal{A}$ be a positive and invertible element. Suppose that $\mathcal{S}(\mathcal{A})$ is the set of all states on $\mathcal{A}$ and let

$$\mathcal{S}_a(\mathcal{A}) = \left\{ \frac{f}{f(a)} \ : \ f \in \mathcal{S}(\mathcal{A}), \ f(a) \neq 0 \right\}.$$

The norm $\|x\|_a$ for every $x \in \mathcal{A}$ is defined by

$$\|x\|_a = \sup_{\varphi \in \mathcal{S}_a(\mathcal{A})} \sqrt{\varphi(x^*ax)}.$$

In this paper, we aim to investigate the notion of Birkhoff–James orthogonality with respect to the norm $\|\cdot\|_a$ in $\mathcal{A}$, namely a-Birkhoff–James orthogonality. The characterization of a-Birkhoff–James orthogonality in $\mathcal{A}$ by means of the elements of generalized state space $\mathcal{S}_a(\mathcal{A})$ is provided. As an application, a characterization for the best approximation to elements of $\mathcal{A}$ in a subspace $\mathcal{B}$ with respect to $\|\cdot\|_a$ is obtained. Moreover, a formula for the distance of an element of $\mathcal{A}$ to the subspace $\mathcal{B} = \mathbb{C}1_{\mathcal{A}}$ is given. We also study the strong version of a-Birkhoff–James orthogonality in $\mathcal{A}$. The classes of C^*-algebras in which these two types orthogonality relationships coincide are described. In particular, we prove that the condition of the equivalence between the strong a-Birkhoff–James orthogonality and $\mathcal{A}$-valued inner product orthogonality in $\mathcal{A}$ implies that the center of $\mathcal{A}$ is trivial. Finally, we show that if the (strong)

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✉ Mahdi Dehghani
m.dehghani@kashanu.ac.ir; e.g.mahdi@gmail.com

Hooriye Sadat Jalali Ghamsari
jalali.hooriyesadat@gmail.com

1 Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan, Iran

a-Birkhoff–James orthogonality is right-additive (left-additive) in $\mathcal{A}$, then the center of $\mathcal{A}$ is trivial.

Keywords C^*-algebras · State space of C^*-algebras · Birkhoff–James orthogonality · a-Birkhoff–James orthogonality · Best approximation · Strong Birkhoff–James orthogonality

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1 Introduction and preliminaries

Let $\mathcal{A}$ be a unital C^*-algebra with unit $1_{\mathcal{A}}$. We denote by $\mathcal{A}'$ and $\mathcal{Z}(\mathcal{A})$ the topological dual space and the center of $\mathcal{A}$, respectively. The adjoint of any element $x \in \mathcal{A}$ is denoted, as usual, by x^*. Also, $\mathrm{Re}(x) = \frac{1}{2}(x + x^*)$ is reserved to indicate the real part of x. An element a of $\mathcal{A}$ is called positive (written by $a \geq 0$), if a is selfadjoint whose spectrum $\sigma(a)$ is contained in $[0, \infty)$. It is known that if $a \in \mathcal{A}$ is positive, then there exists a unique positive element $b \in \mathcal{A}$ such that $a = b^2$. Such an element b is called the positive square root of a and is denoted by $a^{\frac{1}{2}}$. The symbol $\mathcal{A}^+$ stands for the cone of positive elements in $\mathcal{A}$. If in addition a is invertible, then $a^{\frac{1}{2}}$ is invertible too and its inverse is denoted by $a^{-\frac{1}{2}}$. A linear functional f on $\mathcal{A}$ is called positive if $f(a) \geq 0$ for every positive element $a \in \mathcal{A}$. Given a positive functional f on $\mathcal{A}$, the following well-known version of the Cauchy–Schwarz inequality holds for every $x, y \in \mathcal{A}$: $|f(x^*y)|^2 \leq f(x^*x)f(y^*y)$.

A state on $\mathcal{A}$ is a positive linear functional whose norm is equal to one. It is well-known that a linear functional on $\mathcal{A}$ is positive if and only if $f(1_{\mathcal{A}}) = \|f\|$; see [17, Corollary 3.3.4]. Let $\mathcal{S}(\mathcal{A})$ be the set of all states on $\mathcal{A}$. Then

$$\mathcal{S}(\mathcal{A}) = \{f \in \mathcal{A}' : f(1_{\mathcal{A}}) = \|f\| = 1\}.$$

Birkhoff–James orthogonality of elements in a normed linear space was introduced by Birkhoff in [11] and developed by James [14] to generalize the concept of orthogonality in inner product spaces. If x and y are vectors of a normed linear space $(X, \|\cdot\|)$, then x is said to be Birkhoff–James orthogonal to y, in short $x \perp_{BJ} y$, if

$$\|x + \lambda y\| \geq \|x\| \quad (\forall \lambda \in \mathbb{C}).$$

The concept of the strong Birkhoff–James orthogonality in C^*-algebras as a natural generalization of Birkhoff–James orthogonality was introduced and studied in [4, 6]. Let $x, y \in \mathcal{A}$. Then x is said to be strong Birkhoff–James orthogonal to y, denoted by $x \perp_{S-BJ} y$, if

$$\|x + yb\| \geq \|x\| \quad (\forall b \in \mathcal{A}).$$

We also recall that two elements x and y of $\mathcal{A}$ are orthogonal with respect to the $\mathcal{A}$-valued inner product $\langle x, y\rangle := x^*y$ if $\langle x, y\rangle = 0$. It was shown in [4] the following relation between the strong and the classical Birkhoff–James orthogonality:

$$\langle x, y\rangle = 0 \Rightarrow x \perp_{S-BJ} y \Rightarrow x \perp_{BJ} y \quad (\forall x, y \in \mathcal{A}).$$

It is well-known that the Birkhoff–James orthogonality of vectors in normed linear spaces can be characterized in terms of linear functionals [14]. Over the years, the problem of finding characterizations of Birkhoff–James orthogonality of matrices and generally of the elements of C^*-algebras has been considered by many mathematicians. A complete characterization of Birkhoff–James orthogonality of bounded linear operators defined on Hilbert spaces obtained by Bhatia and Šemrl [10] (see also, [3, 9, 18]). Some famous and useful Characterizations of the (strong) Birkhoff–James orthogonality in C^*-algebra $\mathcal{A}$ and in a more general setting Hilbert C^*-modules over $\mathcal{A}$ in terms of the elements of state space $\mathcal{S}(\mathcal{A})$ have been obtained in [5, 9, 16]. The characterization of the (strong) Birkhoff–James orthogonality for elements of a C^*-algebra by means of its state space were obtained as follows:

Theorem 1.1 [5, Theorem 2.7] *An element $x \in \mathcal{A}$ is Birkhoff–James orthogonal to another element $y \in \mathcal{A}$, if and only if there is $f \in \mathcal{S}(\mathcal{A})$ such that $f(x^*x) = \|x\|^2$ and $f(x^*y) = 0$.*

Theorem 1.2 [4, Theorem 2.5] *An element $x \in \mathcal{A}$ is strong Birkhoff–James orthogonal to another element $y \in \mathcal{A}$ if and only if there is $f \in \mathcal{S}(\mathcal{A})$ such that $f(x^*x) = \|x\|^2$ and $f(\langle x, y\rangle \langle y, x\rangle) = 0$ if and only if there is $f \in \mathcal{S}(\mathcal{A})$ such that $f(x^*x) = \|x\|^2$ and $f(\langle x, y\rangle b) = 0$ for all $b \in \mathcal{A}$.*

The classes of C^*-algebras in which any two of these orthogonality relationships coincide have been described in [4, 6]. More precisely,

Theorem 1.3 [6, Corollary 4.10] *Let $\mathcal{A}$ be a nonzero C^*-algebra. Then the following statements are equivalent*:

(1) *For all $x, y \in \mathcal{A}$, $x \perp_{S-BJ} y$ if and only if $\langle x, y\rangle = 0$.*
(2) *For all $x, y \in \mathcal{A}$, $x \perp_{BJ} y$ if and only if $x \perp_{S-BJ} y$.*
(3) *$\mathcal{A}$ is isomorphic to $\mathbb{C}$.*

Let a be a nonzero positive element of $\mathcal{A}$. A generalization of state space of $\mathcal{A}$ was introduced in [1] as follows:

$$\mathcal{S}_a(\mathcal{A}) := \{\varphi \in \mathcal{A}^{'} : \varphi \geq 0,\ \varphi(a) = 1\} = \left\{\frac{f}{f(a)} : f \in \mathcal{S}(\mathcal{A}),\ f(a) \neq 0\right\}.$$

Observe that if $a = 1_{\mathcal{A}}$, then $\mathcal{S}_a(\mathcal{A}) = \mathcal{S}(\mathcal{A})$. It has been proved in [1] that $\mathcal{S}_a(\mathcal{A})$ is a nonempty convex and w^*-closed subset of $\mathcal{A}'$. But, unlike $\mathcal{S}(\mathcal{A})$, the set $\mathcal{S}_a(\mathcal{A})$ may not be w^*-compact. In fact, according to the following result, $\mathcal{S}_a(\mathcal{A})$ is w^*-compact if and only if a is invertible.

Proposition 1.4 [1, Proposition 2.3] *Let $\mathcal{A}$ be a unital C^*-algebra and let $a \in \mathcal{A}$ be a positive element. Then the following statement are equivalent*:

(1) *$\mathcal{S}_a(\mathcal{A})$ is w^*-compact.*
(2) *a is invertible.*

For any element $x \in \mathcal{A}$, the a-operator semi-norm $\|\cdot\|_a : \mathcal{A} \to [0, \infty)$ is defined by

$$\|x\|_a := \sup\left\{\sqrt{\varphi(x^*ax)} : \varphi \in \mathcal{S}_a(\mathcal{A})\right\}.$$

Due to the Proposition 1.4, if a is not invertible, then $\mathcal{S}_a(\mathcal{A})$ is not w^*-compact, and so it may happen that $\|x\|_a = \infty$ for some $x \in \mathcal{A}$; see [1, Example 3.2]. Denote by $\mathcal{A}^a := \{x \in \mathcal{A} : \|x\|_a < \infty\}$. It was shown in [1] that $\|\cdot\|_a$ is a submultiplicative semi-norm on $\mathcal{A}^a$; i.e., $\|xy\|_a \leq \|x\|_a\|y\|_a$ for all $x, y \in \mathcal{A}^a$. Also, $\|x\|_a = 0$ if and only if $ax = 0$. In addition, if a is invertible, then $\|\cdot\|_a$ is a norm on $\mathcal{A}$. Consequently, $\|\cdot\|_{1_\mathcal{A}}$ is equal to the C^*-norm $\|\cdot\|$ of $\mathcal{A}$.

An element $x^\sharp \in \mathcal{A}$ is called an a-adjoint of $x \in \mathcal{A}$ if $ax^\sharp = x^*a$. The set of all a-adjointable elements of $\mathcal{A}$ is denoted by $\mathcal{A}_a$. Note that $\mathcal{A}_a = \mathcal{A}$ if $\mathcal{A}$ is commutative.

In [1, Corollary 4.9] it was proved that if $x \in \mathcal{A}_a$ and $x^\sharp$ is an a-adjoint of it, then

$$\|x\|_a^2 = \|xx^\sharp\|_a = \|x^\sharp x\|_a = \|x^\sharp\|_a^2. \tag{1.1}$$

An element $x \in \mathcal{A}$ is said to be a-selfadjoint if ax is selfadjoint; i.e., $ax = x^*a$. Moreover, any element $x \in \mathcal{A}_a$ can be written as $x = x_1 + ix_2$, where x_1 and x_2 are a-selfadjoint. In fact, if $x^\sharp$ is an a-adjoint of x, then

$$x = \frac{x + x^\sharp}{2} + i\frac{x - x^\sharp}{2i}. \tag{1.2}$$

This decomposition is not unique, since there might be many (or none) a-adjoints $x^\sharp$ of x; see e.g., [1, 8]. Note that if we assume that a is invertible, then x has the unique a-adjoint $x^\sharp = a^{-1}x^*a$, and, therefore, the decomposition (1.2) is unique.

The notions, $\mathcal{S}_a(\mathcal{A})$ and $\|\cdot\|_a$ were introduced in [1] to generalize algebraic numerical range and algebraic numerical radius of elements of C^*-algebra $\mathcal{A}$. To study abundant results related to these concepts the reader is referred to [1, 2, 5].

In this paper, we investigate the notions of Birkhoff–James orthogonality and its strong version in an unital C^*-algebra $\mathcal{A}$ with respect to the norm $\|\cdot\|_a$, whenever $a \in \mathcal{A}$ is a positive and invertible element.

In Sect. 2 first, the main properties of a-Birkhoff–James orthogonality are studied and a variety of examples in simple C^*-algebra $\mathbb{M}_n(\mathbb{C})$ are presented to illustrate the relationship between a-Birkhoff–James orthogonality and Birkhoff–James orthogonality. Next, a complete characterization of a-Birkhoff–James orthogonality in terms of elements of the generalized state space $\mathcal{S}_a(\mathcal{A})$ is presented. As an application, a characterization for the best approximation to elements of $\mathcal{A}$ in a subspace $\mathcal{B}$ is obtained with respect to $\|\cdot\|_a$. Moreover, a generalization of the well-known distance formula which obtained by Williams in [19] is given.

Section 3 is devoted to the study of strong a-Birkhoff–James orthogonality in unital C^*-algebras. The classes of unital C^*-algebras in which the a-Birkhoff–James orthogonality coincides with the strong a-Birkhoff–James orthogonality are described.

In particular, we prove that if $x \perp^a_{S-BJ} y$ implies $\langle x, y\rangle_a := x^*ay = 0$, for all $x, y \in \mathcal{A}$, then the center of $\mathcal{A}$ is trivial, i.e., the only central elements of $\mathcal{A}$ are multiplies of the identity. Moreover, we prove that the right additivity (left additivity) of (strong) a-Birkhoff–James orthogonality in $\mathcal{A}$ concludes that $\mathcal{Z}(\mathcal{A}) \cong \mathbb{C}1_{\mathcal{A}}$.

2 *a*-Birkhoff–James orthogonality in *C**-algebras

Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$. If $\dim\mathcal{H} = n$, then we identify $\mathcal{B}(\mathcal{H})$ with the simple C^*-algebra $\mathbb{M}_n(\mathbb{C})$ of all $n \times n$ complex matrices and denote the identity matrix by I_n. Assume that $A \in \mathcal{B}(\mathcal{H})$ is a positive operator, which induces a positive semi-definite sesquilinear form $\langle\cdot,\cdot\rangle_A : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ defined by $\langle x, y\rangle_A = \langle Ax, y\rangle$. The semi-norm $\|\cdot\|_A$ induced by $\langle\cdot,\cdot\rangle_A$ is defined by $\|x\|_A = \sqrt{\langle Ax, x\rangle}$ for every $x \in \mathcal{H}$. Furthermore, the set of all A-bounded operators on $\mathcal{H}$ is defined by

$$\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}) := \{T \in \mathcal{B}(\mathcal{H}) \; : \; \exists\, c > 0,\ \|Tx\|_A \le c\|x\|_A, \quad \forall x \in \mathcal{H}\}.$$

In fact, $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ is a unital subalgebra of $\mathcal{B}(\mathcal{H})$ which is equipped with the semi-norm

$$\gamma_A(T) := \sup_{\|x\|_A=1} \sqrt{\langle ATx, Tx\rangle} \quad (T \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})).$$

The Birkhoff–James orthogonality with respect to the semi-norm $\gamma_A(\cdot)$ (called A-Birkhoff–James orthogonality) was studied by Zamani in [20]. An operator $T \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ is called A-Birkhoff–James orthogonal to the operator $S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, denoted by $T \perp^A_{BJ} S$, if $\gamma_A(T + \lambda S) \ge \gamma_A(T)$ for all $\lambda \in \mathbb{C}$. The following characterization of A-Birkhoff–James orthogonality which extends the Bhatia and Šemrl Theorem for A-bounded operators has been obtained as follow:

Theorem 2.1 [20, Theorem 2.2] *Let $T, S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$. The following conditions are equivalent*:

(1) $T \perp^A_{BJ} S$.
(2) *There exists a sequence $\{h_n\}$ of A-unit vectors ($\|h_n\|_A = 1$) in $\mathcal{H}$ such that*

$$\lim_{n\to\infty} \|Th_n\|_A = \gamma_A(T) \quad \textit{and} \quad \lim_{n\to\infty} \langle Th_n, Sh_n\rangle_A = 0. \tag{2.1}$$

We recall that by the Gelfand–Naimark Theorem, any unital C^*-algebra $\mathcal{A}$ can be considered as a norm closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. In fact, there exists an unital faithful $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $\|x\| = \|\pi(x)\|$ for all $x \in \mathcal{A}$; see e.g., [12, 17]. It was proved that in [1], if $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a unital faithful $*$-representation of $\mathcal{A}$, then

$$\|x\|_a = \gamma_{\pi(a)}(\pi(x)) \tag{2.2}$$

for any $x \in \mathcal{A}$. As a direct consequence of this fact, we have $\mathcal{A}^A = \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ for $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and all positive operator $A \in \mathcal{B}(\mathcal{H})$.

Let $\mathcal{A}$ be a unital C^*-algebra and let $a \in \mathcal{A}$ be a nonzero positive element. It was proved in [1, Theorem 3.9] that $\mathcal{A}_a \subset \mathcal{A}^a$. Now, if we assume that a is a positive and invertible element of $\mathcal{A}$, then for every $x \in \mathcal{A}$ the equation $ax^\sharp = x^*a$ has the unique solution $x^\sharp = a^{-1}x^*a$, and so every $x \in \mathcal{A}$ is a-adjointable. Therefore $\mathcal{A}^a = \mathcal{A}$.

From now on we assume that $\mathcal{A}$ is a unital C^*-algebra and $a \in \mathcal{A}$ is a positive and invertible element. Let us introduce the concept of a-Birkhoff–James orthogonality with respect to the norm $\|\cdot\|_a$ in C^*-algebras.

Definition 2.2 Let $\mathcal{A}$ be a unital C^*-algebra and $a \in \mathcal{A}$ be a positive and invertible element. We say that an element $x \in \mathcal{A}$ is Birkhoff–James orthogonal with respect to the norm $\|\cdot\|_a$ (a-Birkhoff–James orthogonal) to an element $y \in \mathcal{A}$, in short $x \perp_{BJ}^a y$, if

$$\|x + \lambda y\|_a \geq \|x\|_a \quad (\forall \lambda \in \mathbb{C}).$$

First, note that a-Birkhoff–James orthogonality reduces to the Birkhoff–James orthogonality when $a = 1_{\mathcal{A}}$. Also, it is easy to see that a-Birkhoff–James orthogonality is homogenous; i.e., if $x \perp_{BJ}^a y$, then $\alpha x \perp_{BJ}^a \beta y$ for all $\alpha, \beta \in \mathbb{C}$. It is trivial for $\alpha = 0$ or $\beta = 0$. So, suppose that α and β are nonzero complex numbers. For each $\lambda \in \mathbb{C}$, we have

$$\|\alpha x + \lambda\beta y\|_a = \left\|\alpha\left(x + \frac{\beta}{\alpha}\lambda y\right)\right\|_a = |\alpha| \left\|\left(x + \frac{\beta}{\alpha}\lambda y\right)\right\|_a \geq |\alpha|\, \|x\|_a = \|\alpha x\|_a.$$

It follows that $\alpha x \perp_{BJ}^a \beta y$.

Also, a-Birkhoff–James orthogonality is non-degenerate. Indeed, let $0 \neq x \in \mathcal{A}$ and $x \perp_{BJ}^a x$. Then $\|x + \lambda x\|_a \geq \|x\|_a$ for all $\lambda \in \mathbb{C}$. For $\lambda = -1$, we get $\|x\|_a = 0$, and so $ax = 0$. Therefore $x = 0$, since a is invertible.

Moreover, for any two nonzero elements $x, y \in \mathcal{A}$, if x is orthogonal to y in the a-Birkhoff–James sense, then x and y are linearly independent. In fact, if we assume to the contrary that there exists $k \in \mathbb{C}$ such that $y = kx$, then $x \perp_{BJ}^a kx$. It follows that $x \perp_{BJ}^a x$, since a-Birkhoff–James orthogonality is homogenous. Hence $ax = 0$, and so $x = 0$, which is a contradiction.

Let $f \in \mathcal{S}(\mathcal{A})$. According to [1], the linear functional defined by

$$\varphi(z) = f\left(a^{-\frac{1}{2}} z a^{-\frac{1}{2}}\right) \quad (z \in \mathcal{A}) \tag{2.3}$$

belongs to $\mathcal{S}_a(\mathcal{A})$. Now, let $x \in \mathcal{A}$ and let $a \in \mathcal{A}$ be positive and invertible such that $ax = xa$. Then

$$\|x\|_a^2 = \sup_{\varphi \in \mathcal{S}_a(\mathcal{A})} \varphi(x^*ax) = \sup_{f \in \mathcal{S}(\mathcal{A})} f\left(a^{-\frac{1}{2}}x^*axa^{-\frac{1}{2}}\right) = \sup_{f \in \mathcal{S}(\mathcal{A})} f(x^*x) = \|x\|^2. \tag{2.4}$$

Also, note that $x^\sharp = a^{-1}x^*a$ is a-adjoint of x, and so it follows from (1.1) that

$$\|x\|_a^2 = \|xa^{-1}x^*a\|_a = \|a^{-1}x^*ax\|_a = \|a^{-1}x^*a\|_a^2.$$

Hence

$$\|x\|_a^2 = \|xx^*\|_a = \|x^*x\|_a = \|x^*\|_a^2.$$

Since there is at most one norm on a $*$-algebra making it a C^*-algebra, the following result is obtained.

Corollary 2.3 *If $\mathcal{A}$ is a commutative and unital C^*-algebra and $a \in \mathcal{A}$ is positive and invertible, then $\|\cdot\|_a$ agrees with the C^*-norm of C^*-algebra $\mathcal{A}$. In this case, the a-Birkhoff–James orthogonality and the Birkhoff–James orthogonality are equivalent on $\mathcal{A}$.*

It should be noted that $\|\cdot\|_a$ does not satisfy to the C^*-condition in noncommutative C^*-algebra, even when a is invertible. To make this clear, we present the following example.

Example 2.4 Let $\mathbb{M}_2(\mathbb{C})$ be the C^*-algebra of all 2×2 complex matrices, and let Tr be the usual trace functional on $\mathbb{M}_2(\mathbb{C})$. According to the Example 2.2 of [1], for any positive matrix $h \in \mathbb{M}_2(\mathbb{C})$, let φ_h be the positive linear functional given by

$$\varphi_h(x) := \mathrm{Tr}(hx), \quad (x \in \mathbb{M}_2(\mathbb{C})).$$

It is known that any state on $\mathbb{M}_2(\mathbb{C})$ is of the form φ_h with $\mathrm{Tr}(h) = 1$. For a positive matrix $a \in \mathbb{M}_2(\mathbb{C})$, we have

$$\mathcal{S}_a(\mathbb{M}_2(\mathbb{C})) = \{\varphi_h \ : \ h \in \mathbb{M}_2(\mathbb{C})^+ \quad \text{and} \quad \mathrm{Tr}(ha) = 1\}.$$

Now, let $a = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Then with some simple matrix computations, we conclude that

$$\mathcal{S}_a(\mathbb{M}_2(\mathbb{C})) = \{\varphi_h \ : \ h \in \mathcal{L}_a\},$$

where

$$\mathcal{L}_a := \left\{ h = \begin{bmatrix} h_{11} & h_{12} \\ \overline{h}_{12} & h_{22} \end{bmatrix} \in \mathbb{M}_2(\mathbb{C})^+ \ : \ h_{12} \in \mathbb{C}, \ h_{11}, h_{22} \geq 0 \text{ and } 2h_{11} + h_{22} = 1 \right\}.$$

Hence for $x = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$, we get

$$\|x\|_a^2 = \sup_{\varphi_h \in \mathcal{S}_a(\mathbb{M}_2(\mathbb{C}))} \varphi_h(x^*ax) = \sup_{h \in \mathcal{L}_a} \mathrm{Tr}\left(\begin{bmatrix} h_{11} & 8h_{12} \\ \overline{h}_{12} & 8h_{22} \end{bmatrix}\right)$$
$$= \sup_{2h_{11}+h_{22}=1, h_{11}, h_{22} \geq 0} h_{11} + 8h_{22} = 8.$$

But similarly, we have

$$\|x^*x\|_a^2 = \sup_{\varphi_h \in \mathcal{S}_a(\mathbb{M}_2(\mathbb{C}))} \varphi_h((x^*x)a(x^*x)) = \sup_{2h_{11}+h_{22}=1, h_{11}, h_{22} \geq 0} 2h_{11} + 16h_{22} = 16.$$

The following provide us with examples reveal that the a-Birkhoff–James orthogonality is independent from the Birkhoff–James orthogonality in unital and noncommutative C^*-algebras, even when a is positive and invertible.

Example 2.5 In the context of the same $a = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ as, and similarly to the method we applied in the previous example, let $x = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$ and $y = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ be matrices in $\mathbb{M}_2(\mathbb{C})$. Then

$$\|x\|_a^2 = \sup_{\varphi_h \in \mathcal{S}_a(\mathbb{M}_2(\mathbb{C}))} \varphi_h(x^*ax) = \sup_{h \in \mathcal{L}_a} \mathrm{Tr}(h(x^*ax))$$
$$= \sup_{h \in \mathcal{L}_a} \mathrm{Tr}\left(\begin{bmatrix} h_{11} & h_{12} \\ \overline{h}_{12} & h_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\right) = \sup_{2h_{11}+h_{22}=1, h_{11}, h_{22} \geq 0} 3h_{22} = 3.$$

Also, for every $\lambda \in \mathbb{C}$, we have

$$\|x + \lambda y\|_a^2 = \sup_{h \in \mathcal{L}_a} \varphi_h((x + \lambda y)^* a(x + \lambda y))$$
$$= \sup_{2h_{11}+h_{22}=1, h_{11}, h_{22} \geq 0} \big(3(1 + |\lambda|^2) - 2\mathrm{Re}(\lambda)\big)h_{22}$$
$$= 3(1 + |\lambda|^2) - 2\mathrm{Re}(\lambda).$$

However, for $\lambda = \frac{1}{3}$, we see that $\|x + \lambda y\|_a^2 = \frac{8}{3} < 3 = \|x\|_a^2$, which yields that $x \not\perp_{BJ}^a y$. On the other hand, it can easily be seen that $\|x\|^2 = 2$ and $\|x + \lambda y\|^2 = 2 + 2|\lambda|^2$ for all $\lambda \in \mathbb{C}$. Hence $\|x + \lambda y\|^2 = 2(1 + |\lambda|^2) \geq 2 = \|x\|$, for all $\lambda \in \mathbb{C}$. Thus $x \perp_{BJ} y$.

Now, let $x = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & -1 \end{bmatrix}$ and $y = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Then $x \perp_{BJ}^a y$, since $\langle x, y\rangle_a = x^*ay = 0$. But, for every $\lambda \in \mathbb{C}$, we have

$$\|x + \lambda y\|^2 = \left|\frac{5}{4} + 2|\lambda|^2 - \mathrm{Re}(\lambda)\right|.$$

So for $\lambda = \frac{1}{4}$, we have $\|x + \lambda y\|^2 = \frac{9}{8} < \frac{5}{4} = \|x\|^2$, and therefore $x \not\perp_{BJ} y$.

Assume that $\mathcal{A}$ is a unital and commutative C^*-algebra, $x, y \in \mathcal{A}$ and $a \in \mathcal{A}$ is positive and invertible. If $x \perp_{BJ} y$, then by Theorem 1.1, there must exist $f \in \mathcal{S}(\mathcal{A})$ such that $f(x^*x) = \|x\|^2$ and $f(x^*y) = 0$. Since $\mathcal{A}$ is commutative, by Corollary 2.3 we conclude that

$$\varphi(x^*ax) = f\left(a^{-\frac{1}{2}}x^*axa^{-\frac{1}{2}}\right) = f(x^*x) = \|x\|^2 = \|x\|_a^2,$$

and

$$\varphi(x^*ay) = f\left(a^{-\frac{1}{2}}x^*aya^{-\frac{1}{2}}\right) = f(x^*y) = 0,$$

where $\varphi \in \mathcal{S}_a(\mathcal{A})$ is defined in (2.3). This fact motivates us to obtain a similar characterization for a-Birkhoff–James orthogonality in unital C^*-algebras. More precisely, we shall present a characterization of a-Birkhoff–James orthogonality in a unital C^*-algebra $\mathcal{A}$ based on the elements of its generalized state space $\mathcal{S}_a(\mathcal{A})$. In fact, we use a simple way to obtain the next fundamental result through the standard Gelfand–Naimark representation of $\mathcal{A}$ as a concrete C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ and displayed formula (2.2). However, for completion of the subject and the convenience of the reader, we present a short proof for it. Note that this characterization is a generalization of the well-known Theorem 1.1 when we take $a = 1_{\mathcal{A}}$, and plays a fundamental role to achieve our forthcoming main results.

Theorem 2.6 *Let $\mathcal{A}$ be a unital C^*-algebra, $x, y \in \mathcal{A}$ and let a be positive and invertible element of $\mathcal{A}$. Then the following statements are equivalent*:

(1) $x \perp_{BJ}^a y$.
(2) *There is $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(x^*ax) = \|x\|_a^2$ and $\varphi(y^*ax) = 0$ ($\varphi(x^*ay) = 0$).*

Proof (1) $\Rightarrow$ (2) Let $x \perp_{BJ}^a y$ and let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a unital faithful $*$-representation of $\mathcal{A}$. Since a is invertible, it follows from (2.2) that $\pi(x), \pi(y) \in \mathcal{B}_{\pi(a)^{\frac{1}{2}}}(\mathcal{H})$, and so $\pi(x) \perp_{BJ}^{\pi(a)} \pi(y)$. Hence Theorem 2.1, concludes that there exists a sequence of $\pi(a)$-unit vectors $\{h_n\} \in \mathcal{H}$ such that

$$\lim_{n\to\infty} \|\pi(x)h_n\|_{\pi(a)} = \gamma_{\pi(a)}(\pi(x)), \tag{2.5}$$

$$\lim_{n\to\infty} \langle \pi(x)h_n, \pi(y)h_n\rangle_{\pi(a)} = 0. \tag{2.6}$$

The linear functionals $\varphi_n : \mathcal{A} \to \mathbb{C}$ defined by $\varphi_n(z) = \langle \pi(z)h_n, h_n\rangle$ belong to $\mathcal{S}_a(\mathcal{A})$ for all $n \in \mathbb{N}$ (see [1, Theorem 3.5]). Now, (2.2) and (2.5) imply that

$$\begin{aligned}\lim_{n\to\infty} \varphi_n(x^*ax) &= \lim_{n\to\infty} \langle \pi(x^*ax)h_n, h_n\rangle = \lim_{n\to\infty} \langle \pi(a)\pi(x)h_n, \pi(x)h_n\rangle \\ &= \lim_{n\to\infty} \|\pi(x)(h_n)\|_{\pi(a)}^2 = \gamma_{\pi(a)}^2(\pi(x)) = \|x\|_a^2.\end{aligned}$$

In addition, from (2.6), we infer that

$$\lim_{n\to\infty}\varphi_n(y^*ax)=\lim_{n\to\infty}\langle\pi(y^*ax)h_n,h_n\rangle=\lim_{n\to\infty}\langle\pi(a)\pi(x)h_n,\pi(y)h_n\rangle$$
$$=\lim_{n\to\infty}\langle\pi(x)h_n,\pi(y)h_n\rangle_{\pi(a)}=0.$$

Thus

$$\lim_{n\to\infty}\varphi_n(x^*ax)=\|x\|_a^2,\quad \lim_{n\to\infty}\varphi_n(y^*ax)=0. \tag{2.7}$$

In addition, by Proposition 1.4, $\mathcal{S}_a(\mathcal{A})$ is w^*-compact. So there is $\varphi\in\mathcal{S}_a(\mathcal{A})$ such that $\varphi_n\xrightarrow{w^*}\varphi$. Therefore, (2.7) implies that

$$\varphi(x^*ax)=\|x\|_a^2 \quad\text{and}\quad \varphi(y^*ax)=0.$$

(2) $\Rightarrow$ (1) Assume that there is $\varphi\in\mathcal{S}_a(\mathcal{A})$ such that $\varphi(x^*ax)=\|x\|_a^2$ and $\varphi(y^*ax)=0$. Then for each $\lambda\in\mathbb{C}$, we get

$$\begin{aligned}\|x+\lambda y\|_a^2&\geq\varphi((x+\lambda y)^*a(x+\lambda y))\\&=\varphi(x^*ax)+2\mathrm{Re}(\overline{\lambda}\varphi(y^*ax))+|\lambda|^2\varphi(y^*ay)\\&=\varphi(x^*ax)+|\lambda|^2\varphi(y^*ay)\geq\varphi(x^*ax)=\|x\|_a^2.\end{aligned}$$

Therefore $x\perp_{BJ}^a y$. □

As the first direct consequence of Theorem 2.6, it is easy to see that for given linearly independent vectors $x,y\in\mathcal{A}$, there exists a unique $\alpha\in\mathbb{C}$ such that $x\perp_{BJ}^a(\alpha x+y)$. Indeed, we take $\alpha=0$ if $x\perp_{BJ}^a y$. Now, suppose that $x\not\perp_{BJ}^a y$. Since $a\in\mathcal{A}$ is invertible, there exists $\varphi\in\mathcal{S}_a(\mathcal{A})$ such that $\varphi(x^*ax)=\|x\|_a^2$, by Proposition 1.4. Furthermore, $\varphi(x^*ay)\neq 0$, by Theorem 2.6. Let $\alpha=-\dfrac{\varphi(x^*ay)}{\varphi(x^*ax)}$. Then

$$\varphi(x^*a(\alpha x+y))=-\frac{\varphi(x^*ay)}{\varphi(x^*ax)}\varphi(x^*ax)+\varphi(x^*ay)=0.$$

Therefore $x\perp_{BJ}^a(\alpha x+y)$.

Further, the next result gives us some more examples of a-Birkhoff–James orthogonality for elements of $\mathcal{A}$ to some appropriate elements.

Corollary 2.7 *Let $\mathcal{A}$ be a unital C^*-algebra and let $a\in\mathcal{A}$ be positive and invertible. For each $x,y\in\mathcal{A}$, we have*

$$x\perp_{BJ}^a\left(\|x\|_a^2ya^{\frac{1}{2}}-ya^{-\frac{1}{2}}\langle x,x\rangle_a\right).$$

Proof For the convenience, x^*ax and x^*ay are shown with the symbols $\langle x, x\rangle_a$ and $\langle x, y\rangle_a$, respectively for all $x, y \in \mathcal{A}$. For $x = 0$ or $y = 0$, the statement is trivial. Now, assume that x, y are nonzero elements of $\mathcal{A}$. Since a is invertible, there exists $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(\langle x, x\rangle_a) = \|x\|_a^2$ by Proposition 1.4. The Cauchy–Schwarz inequality and (1.1) tell us

$$\begin{aligned}
&\left|\varphi\left(\left\langle x, \|x\|_a^2\, y\, a^{\frac{1}{2}} - y\, a^{-\frac{1}{2}}\langle x, x\rangle_a\right\rangle_a\right)\right|^2\\
&= \left|\varphi\left(\|x\|_a^2\,\langle x, y\rangle_a\, a^{\frac{1}{2}} - \langle x, y\rangle_a\, a^{-\frac{1}{2}}\langle x, x\rangle_a\right)\right|^2\\
&= |\varphi\left(\langle x, y\rangle_a(\|x\|_a^2\, a^{\frac{1}{2}} - a^{-\frac{1}{2}}\,\langle x, x\rangle_a\right)|^2\\
&\le \varphi(\langle x, y\rangle_a\langle y, x\rangle_a)\,\varphi(\|x\|_a^4\, a - 2\|x\|_a^2\,\langle x, x\rangle_a + \langle x, x\rangle_a\, a^{-1}\langle x, x\rangle_a)\\
&= \varphi(\langle x, y\rangle_a\langle y, x\rangle_a)\left(\|x\|_a^4\,\varphi(a) - 2\|x\|_a^2\,\varphi(\langle x, x\rangle_a) + \varphi((x^\sharp x)^* a(x^\sharp x))\right)\\
&\le \varphi(\langle x, y\rangle_a\langle y, x\rangle_a)\,(\|x\|_a^4 - 2\|x\|_a^4 + \|x^\sharp x\|_a^2)\\
&= \varphi(\langle x, y\rangle_a\langle y, x\rangle_a)(\|x\|_a^4 - 2\|x\|_a^4 + \|x\|_a^4) = 0.
\end{aligned}$$

It follows that $\varphi(\langle x, \|x\|_a^2\, y\, a^{\frac{1}{2}} - y\, a^{-\frac{1}{2}}\langle x, x\rangle_a\rangle_a) = 0$. Consequently, Theorem 2.6 implies that

$$x \perp_{BJ}^a \left(\|x\|_a^2 y a^{\frac{1}{2}} - y a^{-\frac{1}{2}}\langle x, x\rangle_a\right). \qquad \square$$

The a-algebraic numerical range of any element $x \in \mathcal{A}$ is defined by

$$V_a(x) = \{\varphi(ax) : \varphi \in \mathcal{S}_a(\mathcal{A})\}.$$

It has been proved in [1, Theorem 4.7] that $V_a(x)$ is a nonempty convex and compact subset of complex numbers for all $x \in \mathcal{A}^a = \mathcal{A}$, since a is invertible. An extension of the William's Theorem [19, Theorem 1] is obtained in [2, Theorem 2.14].

The following direct result of Theorem 2.6 gives us an alternative proof for this fact.

Corollary 2.8 *Let $x \in \mathcal{A}$. Then $0 \in V_a(x)$ if and only if $\|x - \lambda 1_{\mathcal{A}}\|_a \ge |\lambda|$ for all $\lambda \in \mathbb{C}$.*

Proof Since $0 \in V_a(x)$, there is $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(1_{\mathcal{A}}ax) = \varphi(ax) = 0$. Also, we have $\varphi(1_{\mathcal{A}}^* a 1_{\mathcal{A}}) = \varphi(a) = 1 = \|1_{\mathcal{A}}\|_a^2$. It follows from Theorem 2.6 that $1_{\mathcal{A}} \perp_{BJ}^a x$, which implies that $\|x - \lambda 1_{\mathcal{A}}\|_a \ge |\lambda|$ for all $\lambda \in \mathbb{C}$.

Now, if $\|\lambda 1_{\mathcal{A}} - x\|_a = \|x - \lambda 1_{\mathcal{A}}\|_a \ge |\lambda|$ for all $\lambda \in \mathbb{C}$, then $1_{\mathcal{A}} \perp_{BJ}^a x$, by the homogeneity of the Birkhoff–James orthogonality. So there is $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(ax) = 0$. Therefore $0 \in V_a(x)$. $\square$

Let $\mathcal{A}$ be a unital C^*-algebra and let $x \in \mathcal{A}$. Suppose that $\mathcal{B}$ is a subspace of $\mathcal{A}$. An element $y_0 \in \mathcal{B}$ is said to be a best approximation to x in $\mathcal{B}$ if

$$\|x - y_0\| = \mathrm{dist}(x, \mathcal{B}) := \inf\{\|x - y\| : y \in \mathcal{B}\}.$$

The problem of finding characterizations of orthogonality of an element to subspace $\mathcal{B}$ is closely related to the best approximation problems. A specific question is when is the zero vector a best approximation to x from $\mathcal{B}$? This is the same as asking when is x orthogonal to $\mathcal{B}$? Due to the Theorem 1.1, it has been proved in [5, 9] that for any elements x and y of C^*-algebra $\mathcal{A}$, 0 is a best approximation to x in $\mathcal{B} = \mathbb{C}y$ if and only if there exists $f \in \mathcal{S}(\mathcal{A})$ such that $f(x^*x) = \|x\|^2$ and $f(x^*y) = 0$. Moreover, a generalized version of this fact has been proved in [13] for any element x and for any subspace $\mathcal{B}$ of $\mathcal{A}$. As an application of Theorem 2.6, we present the following characterization of the best approximation for an element of $\mathcal{A}$ with respect to the norm $\|\cdot\|_a$. To achieve this goal, we need the following nice result from [2].

Theorem 2.9 [2, Theorem 2.13] *Let $\mathcal{A}$ be a unital C^*-algebra and let a be a positive element of $\mathcal{A}$. Let $f : a\mathcal{A}^a \to \mathbb{C}$ be a linear functional such that $f(a) = 1$ and $|f(az)| \leq \|z\|_a$ for all $z \in \mathcal{A}^a$. Then there exists $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(az) = f(az)$ for all $z \in \mathcal{A}^a$.*

Theorem 2.10 *Let $\mathcal{A}$ be a unital C^*-algebra, $a \in \mathcal{A}$ be a positive and invertible element and let $\mathcal{B}$ be a subspace of $\mathcal{A}$. Then $y_0 \in \mathcal{B}$ is a best approximation to an element $x \in \mathcal{A}$ with respect to $\|\cdot\|_a$ if and only if there exists $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that*

$$\varphi((x - y_0)^* a(x - y_0)) = \|x - y_0\|_a^2$$

and

$$\varphi(x^* ay) = \varphi(y_0^* ay) \quad (\forall y \in \mathcal{B}).$$

Proof If $\mathcal{A}$ is commutative, then the desired result immediately follows from [13, Theorem 1.1] and Corollary 2.3. Now, suppose that $\mathcal{A}$ is a noncommutative C^*-algebra and $y_0 \in \mathcal{B}$ is a best approximation to x with respect to $\|\cdot\|_a$. Since

$$\operatorname{dis}(x - y_0, \mathcal{B}) = \inf_{y' \in \mathcal{B}} \|x - y_0 - y'\|_a = \inf_{y' \in \mathcal{B}} \|x - (y_0 + y')\|_a = \operatorname{dis}(x, \mathcal{B}),$$

without loss of generality, we may assume that $y_0 = 0$.

Now, suppose that $x \in \mathcal{A}$ and there exists $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(x^*ax) = \|x\|_a^2$ and $\varphi(x^*ay) = 0$ for all $y \in \mathcal{B}$. By Theorem 2.6 and homogeneity of a-Birkhoff–James orthogonality, we conclude that $\|x - \lambda y\|_a \geq \|x\|_a$ for all $y \in \mathcal{B}$ and all $\lambda \in \mathbb{C}$. Hence

$$\|x\|_a = \inf_{y \in \mathcal{B}} \|x - y\|_a = \operatorname{dis}(x, \mathcal{B}).$$

Therefore $y_0 = 0$ is a best approximation to x in $\mathcal{B}$.

Conversely, suppose that $y_0 = 0$ is a best approximation to x in $\mathcal{B}$. Then we have $\|x\|_a \leq \|x + \lambda y\|_a$ for all $y \in \mathcal{B}$ and all $\lambda \in \mathbb{C}$. Theorem 2.6 tells us for each $y \in \mathcal{B}$ there exists $\varphi_y \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi_y(x^*ax) = \|x\|_a^2$ and $\varphi_y(x^*ay) = 0$.

Let $\mathcal{M} = \{\alpha x^*ax + \beta a + x^*ay \ : \ \alpha, \beta \in \mathbb{C}, \ y \in \mathcal{B}\}$ be a subspace of $\mathcal{A}$ generated by x^*ax, a and $x^*a\mathcal{B}$. Since a is invertible, it is known that x has a unique decomposition

$x = x_1 + ix_2$ such that x_1 and x_2 are a-selfadjoint. In fact, $x_1 = \dfrac{x + x^\sharp}{2}$ and $x_2 = \dfrac{x - x^\sharp}{2i}$. Hence

$$\begin{aligned}\mathcal{M} &= \{\alpha(x_1 + ix_2)^* a(x_1 + ix_2) + \beta a + (x_1 + ix_2)^* ay \,:\, \alpha, \beta \in \mathbb{C},\ y \in \mathcal{B}\}\\ &= \{\alpha(x_1^* a - ix_2^* a)(x_1 + ix_2) + \beta a + (x_1^* a - ix_2^* a)y \,:\, \alpha, \beta \in \mathbb{C},\ y \in \mathcal{B}\}\\ &= \{\alpha(ax_1 - iax_2)(x_1 + ix_2) + \beta a + (ax_1 - iax_2)y \,:\, \alpha, \beta \in \mathbb{C},\ y \in \mathcal{B}\}\\ &= \{a\,(\alpha(x_1 - ix_2)(x_1 + ix_2) + \beta 1_{\mathcal{A}} + (x_1 - ix_2)y) \,:\, \alpha, \beta \in \mathbb{C},\ y \in \mathcal{B}\}.\end{aligned}$$

Define the mapping $\psi : \mathcal{M} \to \mathbb{C}$ by

$$\psi(\alpha x^* ax + \beta a + x^* ay) = \alpha \|x\|_a^2 + \beta.$$

Clearly ψ is a linear mapping. To show that ψ is well defined, it is enough to prove that if $\alpha x^* ax + \beta a + x^* ay = 0$, then $\psi(\alpha x^* ax + \beta a + x^* ay) = 0$. Note that for each $\alpha, \beta \in \mathbb{C}$ and any $y \in \mathcal{B}$, we have $\varphi_y(\alpha x^* ax + \beta a + x^* ay) = \alpha\|x\|_a^2 + \beta$, since $\varphi_y(a) = 1$, $\varphi_y(x^* ax) = \|x\|_a^2$ and $\varphi_y(x^* ay) = 0$. Now, let $u(\alpha, \beta, y) = \alpha(x_1 - ix_2)(x_1 + ix_2) + \beta 1_{\mathcal{A}} + (x_1 - ix_2)y$ for all $\alpha, \beta \in \mathbb{C}$ and all $y \in \mathcal{B}$. Then by the Cauchy-Schwartz inequality, we get

$$\begin{aligned}&|\psi(\alpha x^* ax + \beta a + x^* ay)| = |\,\alpha\,\|x\|_a^2 + \beta\,| = |\varphi_y(\alpha x^* ax + \beta a + x^* ay)|\\ &\quad= |\varphi_y(au(\alpha, \beta, y))| = |\varphi_y(a^{\frac{1}{2}} a^{\frac{1}{2}} u(\alpha, \beta, y))|\\ &\quad\le \sqrt{\varphi_y(a)}\sqrt{\varphi_y(u(\alpha, \beta, y)^* a\, u(\alpha, \beta, y))}\\ &\quad= \sqrt{\varphi_y(u(\alpha, \beta, y)^* a\, u(\alpha, \beta, y))}\\ &\quad\le \|u(\alpha, \beta, y)\| = \|\alpha(x_1 - ix_2)(x_1 + ix_2) + \beta 1_{\mathcal{A}} + (x_1 - ix_2)y\|_a.\end{aligned} \tag{2.8}$$

If $\alpha x^* ax + \beta a + x^* ay = 0$, then $a(\alpha(x_1 - ix_2)(x_1 + ix_2) + \beta 1_{\mathcal{A}} + (x_1 - ix_2)y) = 0$, and so $\|\alpha(x_1 - ix_2)(x_1 + ix_2) + \beta 1_{\mathcal{A}} + (x_1 - ix_2)y\|_a = 0$. Thus $\psi(\alpha x^* ax + \beta a + x^* ay) = \psi(0) = 0$, by (2.8).

Define $\mathcal{N} : a\mathcal{A} \to [0, \infty)$ by $\mathcal{N}(az) = \|z\|_a$ for all $z \in \mathcal{A}$ and note that $\mathcal{N}$ is a norm on $a\mathcal{A}$. Moreover, (2.8) follows that

$$\begin{aligned}|\psi(\alpha x^* ax + \beta a + x^* ay)| &\le \mathcal{N}(a(\alpha(x_1 - ix_2)(x_1 + ix_2) + \beta 1_{\mathcal{A}} + (x_1 - ix_2)y))\\ &= \mathcal{N}(\alpha x^* ax + \beta a + x^* ay).\end{aligned}$$

Hence $\|\psi\| \le 1$ with respect to the norm $\mathcal{N}$, and therefore $\psi : (\mathcal{M}, \mathcal{N}(\cdot)) \subseteq a\mathcal{A} \to \mathbb{C}$ is a bounded linear functional. The Hahn–Banach Theorem tells us ψ can be extend to a linear functional $f : a\mathcal{A} \to \mathbb{C}$ such that $\|f\| = \|\psi\| \le 1$, $f|_{(\mathcal{M}, \mathcal{N}(\cdot))} = \psi$ and $f(a) = 1$. In addition,

$$|f(az)| \le \|f\|\,\mathcal{N}(az) \le \mathcal{N}(az) = \|z\|_a \quad (\forall z \in \mathcal{A}).$$

Taking the above considerations into account, by Theorem 2.9 one can find $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(az) = f(az)$ for all $z \in \mathcal{A}$. Therefore, there exists $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that

$$\begin{aligned}\varphi(x^*ax) &= \varphi((x_1+ix_2)^*a(x_1+ix_2)) = \varphi(a(x_1-ix_2)(x_1+ix_2)) \\ &= f(a(x_1-ix_2)(x_1+ix_2)) = f((x_1+ix_2)^*a(x_1+ix_2)) \\ &= f(x^*ax) = \psi(x^*ax) = \|x\|_a^2,\end{aligned}$$

and

$$\begin{aligned}\varphi(x^*ay) &= \varphi((x_1+ix_2)^*ay) = \varphi(a(x_1-ix_2)y) = f(a(x_1-ix_2)y) \\ &= f((x_1+ix_2)^*ay) = f(x^*ay) = \psi(x^*ay) = 0 \quad (\forall y \in \mathcal{B}).\end{aligned}$$ □

As a direct consequence of Theorem 2.10, we get the following characterization of a-Birkhoff–James orthogonality to a subspace in a unital C^*-algebra.

Corollary 2.11 *Let $\mathcal{B}$ be a subspace of a unital C^*-algebra $\mathcal{A}$ and let x be an element of $\mathcal{A}$. Then x is a-Birkhoff–James orthogonal to $\mathcal{B}$ if and only if there is $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(x^*ax) = \|x\|_a^2$ and $\varphi(x^*ay) = 0$ for all $y \in \mathcal{B}$.*

The next result present a generalization of the well-known distance formula which obtained by Williams in [19].

Corollary 2.12 *Let $\mathcal{A}$ be a unital C^*-algebra, $a \in \mathcal{A}$ be a positive and invertible element and let $x \in \mathcal{A}$. Then*

$$\mathrm{dist}^2(x, \mathbb{C}1_{\mathcal{A}}) = \min_{\lambda \in \mathbb{C}} \|x - \lambda 1_{\mathcal{A}}\|_a^2 = \max\{\varphi(x^*ax) - |\varphi(ax)|^2 \,:\, \varphi \in \mathcal{S}_a(\mathcal{A})\}.$$

Proof Let $\alpha \in \mathbb{C}$ be such that $\|x - \alpha 1_{\mathcal{A}}\|_a = \mathrm{dist}(x, \mathbb{C}1_{\mathcal{A}})$. For any $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(ax) = \alpha$, we have

$$\varphi(x^*ax) - |\varphi(ax)|^2 = \varphi((x - \alpha 1_{\mathcal{A}})^*a(x - \alpha 1_{\mathcal{A}})) \le \|x - \alpha 1_{\mathcal{A}}\|_a^2 = \mathrm{dist}^2(x, \mathbb{C}1_{\mathcal{A}}).$$

Hence

$$\max\{\varphi(x^*ax) - |\varphi(ax)|^2 \,:\, \varphi \in \mathcal{S}_a(\mathcal{A})\} \le \mathrm{dist}^2(x, \mathbb{C}1_{\mathcal{A}}) = \min_{\lambda \in \mathbb{C}} \|x - \lambda 1_{\mathcal{A}}\|_a^2.$$

On the other hand, by Theorem 2.10, there is $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that

$$\varphi((x - \alpha 1_{\mathcal{A}})^*a(x - \alpha 1_{\mathcal{A}})) = \|x - \alpha 1_{\mathcal{A}}\|_a^2 \quad \text{and} \quad \varphi(ax) = \alpha.$$

Therefore $\varphi(x^*ax) - |\varphi(ax)|^2 = \mathrm{dist}^2(x, \mathbb{C}1_{\mathcal{A}})$, and so

$$\max\{\varphi(x^*ax) - |\varphi(ax)|^2 \,:\, \varphi \in \mathcal{S}_a(\mathcal{A})\} \ge \mathrm{dist}^2(x, \mathbb{C}1_{\mathcal{A}}).$$ □

3 Strong *a*-Birkhoff–James orthogonality in C^*-algebras

Our main goal in this section is to introduce and study the notion of strong Birkhoff–James orthogonality with respect to the norm $\|\cdot\|_a$ in unital C^*-algebras. It should be noted that what is obtained in this section is an extension and modification of some results of [4, 6]. We start this section with introducing the concept of strong a-Birkhoff–James orthogonality.

Definition 3.1 Let $\mathcal{A}$ be a unital C^*-algebra and $a \in \mathcal{A}$ be a positive and invertible element. An element $x \in \mathcal{A}$ is said to be strongly a-Birkhoff–James orthogonal to an element $y \in \mathcal{A}$, in short $x \perp^a_{S-BJ} y$, if

$$\|x + yb\|_a \geq \|x\|_a \quad (\forall b \in \mathcal{A}).$$

Obviously, $x \perp^a_{S-BJ} y$ implies $x \perp^a_{BJ} y$ for all $x, y \in \mathcal{A}$. So for every $x, y \in \mathcal{A}$, we obtain:

$$\langle x, y\rangle_a := x^*ay = 0 \Rightarrow x \perp^a_{S-BJ} y \Rightarrow x \perp^a_{BJ} y. \tag{3.1}$$

Indeed, if $\langle x, y\rangle_a = 0$, then for each $b \in \mathcal{A}$, we have

$$\begin{aligned}
\|x + yb\|_a^2 &= \sup_{\varphi \in \mathcal{S}_a(\mathcal{A})} \varphi((x + yb)^*a(x + yb)) \\
&= \sup_{\varphi \in \mathcal{S}_a(\mathcal{A})} \big(\varphi(\langle x, x\rangle_a) + 2\mathrm{Re}\varphi(\langle x, y\rangle_a b) + \varphi(\langle yb, yb\rangle_a)\big) \\
&= \sup_{\varphi \in \mathcal{S}_a(\mathcal{A})} \big(\varphi(\langle x, x\rangle_a) + \varphi(\langle yb, yb\rangle_a)\big) \geq \sup_{\varphi \in \mathcal{S}_a(\mathcal{A})} \varphi(\langle x, x\rangle_a) = \|x\|_a^2.
\end{aligned}$$

Also, note that

$$x \perp^a_{S-BJ} y \Leftrightarrow x \perp^a_{BJ} yb \quad (\forall b \in \mathcal{A}). \tag{3.2}$$

The converses in (3.1) do not hold in general. The following example explains this fact.

Example 3.2 Let $a = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. If $x = I_2$ and $y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then for every $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
\|x + \lambda y\|_a^2 &= \sup_{h \in \mathcal{L}_a} \varphi_h((x + \lambda y)^*a(x + \lambda y)) \\
&= \sup_{h \in \mathcal{L}_a} \big((2 + |\lambda|^2)h_{11} + 2\mathrm{Re}((2\overline{\lambda} + \lambda)h_{12}) + (2|\lambda|^2 + 1)h_{22}\big) \\
&\geq 1 + \frac{|\lambda|^2}{2} \geq 1 = \|x\|_a^2,
\end{aligned}$$

since $h_0 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{L}_a$. Hence $x \perp^a_{BJ} y$. But, we may easily check that $x \not\perp^a_{S-BJ} y$. To this end, note that for $b = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{4} \end{bmatrix}$, we get

$$\begin{aligned}
\|x + yb\|_a^2 &= \sup_{h \in \mathcal{L}_a} \mathrm{Tr}(h(x+yb)^* a(x+yb)) \\
&= \sup_{h \in \mathcal{L}_a} \mathrm{Tr}\left(\begin{bmatrix} h_{11} & h_{12} \\ \overline{h}_{12} & h_{22} \end{bmatrix} \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{8} \end{bmatrix} \right) \\
&= \sup_{2h_{11}+h_{22}=1, h_{11}, h_{22} \geq 0} \left(\frac{3}{4} h_{11} + \frac{3}{8} h_{22} \right) = \frac{3}{8} < 1 = \|x\|_a^2.
\end{aligned}$$

Now, let $x = I_2$ and $y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. If $b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in \mathbb{M}_2(\mathbb{C})$ is arbitrary, then

$$\begin{aligned}
\|x + yb\|_a^2 &= \sup_{h \in \mathcal{L}_a} \varphi_h((x+yb)^* a(x+yb)) \\
&= \sup_{h \in \mathcal{L}_a} \left((2 + |b_{11}|^2) h_{11} + 2\mathrm{Re}(b_{11}(1 + \overline{b}_{12}) h_{12}) + |1 + b_{12}|^2 h_{22} \right) \\
&\geq 1 + \frac{|b_{11}|^2}{2} \geq 1 = \|x\|_a^2.
\end{aligned}$$

So $x \perp^a_{S-BJ} y$, while clearly, $\langle x, y \rangle_a = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq 0$.

Our next result gives us a characterization of strong a-Birkhoff–James orthogonality based on elements of generalized state space $\mathcal{S}_a(\mathcal{A})$ of unital C^*-algebra $\mathcal{A}$. Actually, this result extend Theorem 2.5 of [4] for the norm $\|\cdot\|_a$ on $\mathcal{A}$.

Theorem 3.3 *Let $\mathcal{A}$ be a unital C^*-algebra, $x, y \in \mathcal{A}$ and let $a \in \mathcal{A}$ be a positive and invertible element. Then the following statements are equivalent*:

(1) $x \perp^a_{S-BJ} y$;
(2) $x \perp^a_{BJ} y\langle y, x \rangle_a$;
(3) *There is $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(x^* a x) = \|x\|_a^2$ and $\varphi(\langle x, y \rangle_a \langle y, x \rangle_a) = 0$;*
(4) *There is $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(x^* a x) = \|x\|_a^2$ and $\varphi(\langle x, y \rangle_a b) = 0$, for all $b \in \mathcal{A}$.*

Proof (1) $\Rightarrow$ (2) If $x \perp^a_{S-BJ} y$, then $x \perp^a_{BJ} yb$ for all $b \in \mathcal{A}$, by (3.2). Now, let $b = \langle y, x \rangle_a$. Then $x \perp^a_{BJ} y\langle y, x \rangle_a$.

(2) $\Rightarrow$ (3) If $x \perp^a_{BJ} y\langle y, x \rangle_a$, then it follows from Theorem 2.6 that there is $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(x^* a x) = \|x\|_a^2$ and $\varphi(\langle x, y \rangle_a \langle y, x \rangle_a) = \varphi(\langle x, y\langle y, x \rangle_a \rangle_a) = 0$.

(3) $\Rightarrow$ (4) If there exists $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(x^* a x) = \|x\|_a^2$ and $\varphi(\langle x, y \rangle_a \langle y, x \rangle_a) = 0$, then by the Cauchy–Schwarz inequality, we have

$$|\varphi((x^* a y) b)|^2 \leq \varphi((x^* a y)(y^* a x))\, \varphi(b^* b) = 0, \quad (\forall b \in \mathcal{A}),$$

which follows that $\varphi(\langle x, y\rangle_a b) = 0$ for all $b \in \mathcal{A}$.

(4) $\Rightarrow$ (1) It follows directly from Theorem 2.6 and the definition of strong a-Birkhoff–James orthogonality. $\square$

Proposition 3.4 *Let $\mathcal{A}$ be a unital C^*-algebra, $x, y \in \mathcal{A}$ and let $a \in \mathcal{A}$ be a positive and invertible element. If $\langle x, y\rangle_a \geq 0$, then*

$$x \perp^a_{S-BJ} y \Leftrightarrow x \perp^a_{BJ} y.$$

Proof Assume that $x \perp^a_{BJ} y$. By Theorem 2.6, there exists $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(\langle x, x\rangle_a) = \|x\|_a^2$ and $\varphi(\langle x, y\rangle_a) = 0$. Since $\langle x, y\rangle_a \geq 0$, by the Cauchy–Schwarz inequality, for every $b \in \mathcal{A}$, we get

$$\begin{aligned} |\varphi(\langle x, y\rangle_a b)|^2 &= |\varphi(\langle x, y\rangle_a^{\frac{1}{2}} \langle x, y\rangle_a^{\frac{1}{2}} b)|^2 \\ &\leq \varphi(\langle x, y\rangle_a^{\frac{1}{2}} \langle x, y\rangle_a^{\frac{1}{2}})\, \varphi(b^* \langle x, y\rangle_a^{\frac{1}{2}} \langle x, y\rangle_a^{\frac{1}{2}} b) \\ &\leq \varphi(\langle x, y\rangle_a)\, \varphi(b^* \langle x, y\rangle_a b) = 0. \end{aligned}$$

Thus $\varphi(\langle x, y\rangle_a b) = 0$ for all $b \in \mathcal{A}$. Therefore, Theorem 3.3 shows that $x \perp^a_{S-BJ} y$. $\square$

Theorem 3.5 *Let $\mathcal{A}$ be a unital C^*-algebra and let $a \in \mathcal{A}$ be positive and invertible. If*

$$x \perp^a_{S-BJ} y \Leftrightarrow x \perp^a_{BJ} y \quad (\forall x, y \in \mathcal{A}),$$

then the C^*-algebra $\mathcal{A}$ is commutative.

Proof First, note that $\mathcal{A}^a = \mathcal{A}$, since a is invertible. We shall show that for every $x, b \in \mathcal{A}$ there is a scalar $0 \neq \alpha \in \mathbb{C}$ such that

$$xb \perp^a_{S-BJ} (xb^2 + \alpha xb). \tag{3.3}$$

If $xb = 0$, obviously (3.3) holds. Now, let x be an element of $\mathcal{A}$ such that $xb \neq 0$. Then $xb \not\perp^a_{BJ} x$. Indeed, if $xb \perp^a_{BJ} x$, then $xb \perp^a_{S-BJ} x$, by the assumption and thus $xb \perp^a_{BJ} xb$, by (3.2). It follows that $xb = 0$, which is not possible.

Moreover, by the definition of $\|\cdot\|_a$ and invertibility of a, there is $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(\langle xb, xb\rangle_a) = \|xb\|_a^2$. Hence by Theorem 2.6, we conclude that $\varphi(\langle xb, x\rangle_a) \neq 0$. Now, take $\alpha = \dfrac{-\|xb\|_a}{\varphi(\langle xb, x\rangle_a)}$. Thus

$$\varphi(\langle xb, xb + \alpha x\rangle_a) = \|xb\|_a^2 - \frac{\|xb\|_a^2}{\varphi(\langle xb, x\rangle_a)}\varphi(\langle xb, x\rangle_a) = 0.$$

The assumption and the Theorem 2.6 yields that $xb \perp^a_{S-BJ} (xb+\alpha x)$. Hence $xb \perp^a_{BJ} (xb^2 + \alpha xb)$, by (3.2), and so

$$xb \perp^a_{S-BJ} (xb^2 + \alpha xb),$$

by the hypothesis.

If $\mathcal{A}$ is not commutative, there will a nonzero $b \in \mathcal{A}$ with $b^2 = 0$ (see [12], p. 68). By (3.3), for $x = b^*$ there is a scalar $\alpha \neq 0$ such that $xb \perp^a_{S-BJ} \alpha xb$. Hence $b^*b = xb = 0$, and so $b = 0$. This contradiction shows that $\mathcal{A}$ is commutative. □

The next two results are direct consequences of Theorem 1.3, Corollary 2.3 and Theorem 3.5.

Corollary 3.6 *Let $\mathcal{A}$ be a unital C^*-algebra and let $a \in \mathcal{A}$ be positive and invertible. The following statements are equivalent*:

(1) *For all $x, y \in \mathcal{A}$, $x \perp^a_{BJ} y$ if and only if $\langle x, y\rangle_a = 0$;*
(2) *For all $x, y \in \mathcal{A}$, $x \perp^a_{BJ} y$ if and only if $x \perp^a_{S-BJ} y$;*
(3) *$\mathcal{A}$ is isomorphic to $\mathbb{C}$.*

Corollary 3.7 *Let $\mathcal{A}$ be a unital C^*-algebra and let $a \in \mathcal{A}$ be positive and invertible. If*

$$x \perp^a_{S-BJ} y \Leftrightarrow \langle x, y\rangle_a = 0 \quad (\forall x, y \in \mathcal{A}),$$

then $\mathcal{Z}(\mathcal{A}) \cong \mathbb{C}1_{\mathcal{A}}$.

Let $\mathcal{A}$ be a unital C^*-algebra and $a \in \mathcal{A}$ be positive and invertible. If $z \in \mathcal{A}$ is a noninvertible element of $\mathcal{A}$, then zz^*a is not invertible, and so $0 \in \sigma(zz^*a) = \sigma_a(zz^*a) \subseteq V_a(zz^*a)$, by [15, Remark 2.13 and Corollary 3.9]. Hence there exists $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(\langle 1_{\mathcal{A}}, z\rangle_a \langle z, 1_{\mathcal{A}}\rangle_a) = \varphi(azz^*a) = 0$. Also, we have $\varphi(1^*_{\mathcal{A}} a 1_{\mathcal{A}}) = \varphi(a) = 1 = \|1_{\mathcal{A}}\|^2_a$. Consequently, Theorem 3.3 implies that

$$1_{\mathcal{A}} \perp^a_{S-BJ} z, \quad \text{and so} \quad 1_{\mathcal{A}} \perp^a_{BJ} z. \tag{3.4}$$

It has been shown in [7] that the left-additivity (right-additivity) of the (strong) Birkhoff–James orthogonality on a unital C^*-algebra implies that $\mathcal{A}$ is isomorphic to $\mathbb{C}1_{\mathcal{A}}$. As a final result of this section, we will prove that if the (strong) a-Birkhoff–James orthogonality is right-additive on a unital C^*-algebra $\mathcal{A}$, then the center of $\mathcal{A}$ is trivial; i.e., $\mathcal{Z}(\mathcal{A}) \cong \mathbb{C}1_{\mathcal{A}}$.

Theorem 3.8 *Let $\mathcal{A}$ be a unital C^*-algebra and let a be a positive and invertible element of $\mathcal{A}$. If* (*strong*) *a-Birkhoff–James orthogonality is right-additive on $\mathcal{A}$, then $\mathcal{Z}(\mathcal{A}) \cong \mathbb{C}1_{\mathcal{A}}$.*

Proof First, assume that $\mathcal{A}$ is commutative. Then $\mathcal{Z}(\mathcal{A}) = \mathcal{A} \cong \mathbb{C}1_{\mathcal{A}}$, by Corollary 2.3 and [7, Remark 2.8]. Now, suppose that $\mathcal{A}$ is noncommutative and $x \in \mathcal{A}$ is a noninvertible element of $\mathcal{A}$. Then by (3.4), $1_{\mathcal{A}} \perp^a_{BJ} x^*x$. If we assume that $ax = xa$, then (2.4) follows that $\|x^*x\|_a = \|x^*x\|$. Hence $\|x^*x\|_a 1_{\mathcal{A}} - x^*x$ is not invertible, since

$$\|x^*x\|_a = \|x^*x\| \in \sigma(x^*x).$$

Thus $1_{\mathcal{A}} \perp^a_{BJ} (\|x^*x\|_a 1_{\mathcal{A}} - x^*x)$. The right-additivity of a-Birkhoff–James orthogonality follows that $1_{\mathcal{A}} \perp^a_{BJ} \|x^*x\|_a 1_{\mathcal{A}}$. So $\|x^*x\|_a = \|x\|^2_a = 0$, because of the non-degeneracy of a-Birkhoff–James orthogonality. Hence $x = 0$. Therefore we have proved that every nonzero element of C^*-subalgebra $\mathcal{Z}(\mathcal{A})$ is invertible, and so $\mathcal{Z}(\mathcal{A}) \cong \mathbb{C}1_{\mathcal{A}}$ by the Gelfand–Mazur Theorem. A similar argument works for strong a-Birkhoff–James orthogonality. □

Remark 3.9 Suppose that a-Birkhoff–James orthogonality is left-additive in unital C^*-algebra $\mathcal{A}$ and let $x \in \mathcal{A}$ be positive and noninvertible such that $xa = ax$. Then the C^*-subalgebra, $\mathcal{B} := C^*(1_{\mathcal{A}}, a, x)$, generated by $1_{\mathcal{A}}$, a and x is commutative. According to the Corollary 2.3, Birkhoff–James orthogonality is left-additive on $\mathcal{B}$. Hence $x = 0$, by [7, Remark 2.8]. It follows that every nonzero element of $\mathcal{Z}(\mathcal{A})$ is invertible, and so $\mathcal{Z}(\mathcal{A})$ is trivial. It should be noted that the same proof works for right-additivity of a-Birkhoff–James orthogonality. However, a different approach is presented to study right-additivity in the previous Theorem.

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References

1. Abdellatif, B., Mabrouk, M.: a-numerical range on C^*-algebras. Positivity **25**(4), 1489–1510 (2021)
2. Alahmari, A., Mabrouk, M., Zamani, A.: Further results on the a-numerical range in C^*-algebras. Banach J. Math. Anal. **16**(2), Paper No. 25 (2022)
3. Ando, T., Davis, C., Jain, T., Kittaneh, F., Moslehian, M.S., Spitkovsky Ilya, M.: Rajendra Bhatia and his mathematical achievements. Adv. Oper. Theory **5**(3), 850–863 (2020)
4. Arambašić, L., Rajić, R.: A strong version of the Birkhoff–James orthogonality in Hilbert C^*-modules. Ann. Funct. Anal. **5**(1), 109–120 (2014)
5. Arambašić, L., Rajić, R.: The Birkhoff–James orthogonality in Hilbert C^*-modules. Linear Algebra Appl. **437**(7), 1913–1929 (2012)
6. Arambašić, L., Rajić, R.: On three concepts of orthogonality in Hilbert C^*-modules. Linear Multilinear Algebra **63**(7), 1485–1500 (2015)
7. Arambašić, L., Rajić, R.: On symmetry of the (strong) Birkhoff–James orthogonality in Hilbert C^*-modules. Ann. Funct. Anal. **7**(1), 17–23 (2016)
8. Arias, M.L., Corach, G., Gonzalez, M.G.: Partial isometries in semi-Hilbertian spaces. Linear Algebra Appl. **428**(7), 1460–1475 (2008)

9. Bhattacharyya, T., Grover, P.: Characterization of Birkhoff–James orthogonality. J. Math. Anal. Appl. **407**(2), 350–358 (2013)
10. Bhatia, R., Šemrl, P.: Orthogonality of matrices and some distance problems. Special issue celebrating the 60th birthday of Ludwig Elsner. Linear Algebra Appl. **287**(1–3), 77–85 (1999)
11. Birkhoff, G.: Orthogonality in linear metric spaces. Duke Math. J. **1**(2), 169–172 (1935)
12. Dixmier, J.: C^*-Algebras, vol. 15. North-Holland Publishing, Amsterdam (1977)
13. Grover, P., Singla, S.: Best approximations, distance formulas and orthogonality in C^*-algebras. J. Ramanujan Math. Soc. **36**(1), 85–91 (2021)
14. James, R.C.: Orthogonality and linear functionals in normed linear spaces. Trans. Am. Math. Soc. **61**, 265–292 (1947)
15. Mabrouk, M., Zamani, A.: A-spectral permanence property for C^*-algebras. Mediterr. J. Math. **21**, 26 (2024). https://doi.org/10.1007/s00009-023-02567-z
16. Moslehian, M.S., Zamani, A.: Characterizations of operator Birkhoff–James orthogonality. Can. Math. Bull. **60**(4), 816–829 (2017)
17. Murphy, G.J.: C^*-Algebras and Operator Theory. Academic Press, Inc., Boston (1990). ISBN:0-12-511360-9
18. Paul, K.: Translatable radii of an operator in the direction of another operator. Sci. Math. **2**(1), 119–122 (1999)
19. Williams, J.P.: Finite operators. Proc. Am. Math. Soc. **26**, 129–136 (1970)
20. Zamani, A.: Birkhoff–James orthogonality of operators in semi-Hilbertian spaces and its applications. Ann. Funct. Anal. **10**(3), 433–445 (2019)