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Harmonic Bloch space on the real hyperbolic ball

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Abstract
We study the Bloch and the little Bloch spaces of harmonic functions on the real hyperbolic ball. We show that the Bergman projections from $L^\infty(\mathbb{B})$ to $\mathcal{B}$, and from $C_0(\mathbb{B})$ to $\mathcal{B}_0$ are onto. We verify that the dual space of the hyperbolic harmonic Bergman space $\mathcal{B}^1_\alpha$ is $\mathcal{B}$ and its predual is $\mathcal{B}_0$. Finally, we obtain atomic decompositions of Bloch functions as series of Bergman reproducing kernels.

Keywords Real hyperbolic ball · Hyperbolic harmonic function · Bloch space · Bergman projection · Atomic decomposition

Mathematics Subject Classification 31C05 · 46E22

1 Introduction

The Bloch space of *holomorphic* functions on the unit disk or the complex unit ball have been extensively studied (see [24, Chapter 3]). Some well-known properties are the following: The Bergman projection maps L^∞ boundedly *onto* the Bloch space; the dual of the weighted Bergman space B^1_α ($\alpha > -1$) is the Bloch space, and its predual is the little Bloch space; Bloch functions admit atomic decomposition, that is, they can be represented as series of Bergman reproducing kernels. It is also well known that analogous results hold for the Bloch space of *harmonic* functions on the real unit ball (see [4, 10, 12, 21]). The purpose of this paper is to consider the Bloch space of invariant harmonic functions on the real hyperbolic ball and verify that the above properties also hold in this setting.

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For $n \geq 2$ and $x, y \in \mathbb{R}^n$, let $\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n$ be the Euclidean inner product, and $|x| = \sqrt{\langle x, x\rangle}$ the corresponding norm. Let $\mathbb{B} = \mathbb{B}_n = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball, and $\mathbb{S} = \partial\mathbb{B}$ the unit sphere.

The hyperbolic ball is $\mathbb{B}$ equipped with the hyperbolic metric

$$ds^2 = \frac{4}{(1-|x|^2)^2}\sum_{i=1}^{n} dx_i^2.$$

For a C^2 function f, the hyperbolic (invariant) Laplacian Δ_h is defined by

$$\Delta_h f(a) = \Delta(f \circ \varphi_a)(0) \qquad (a \in \mathbb{B}),$$

where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ is the Euclidean Laplacian and φ_a is the involutory Möbius transformation given in (2.2) that exchanges a and 0. Up to a factor $1/4$, Δ_h is the Laplace–Beltrami operator associated with the hyperbolic metric. A straightforward calculation shows

$$\Delta_h f(a) = (1-|a|^2)^2 \Delta f(a) + 2(n-2)(1-|a|^2)\langle a, \nabla f(a)\rangle,$$

where $\nabla = \big(\partial/\partial x_1, \ldots, \partial/\partial x_n\big)$ is the Euclidean gradient. We refer the reader to [18, Chapter 3] for details.

A C^2 function $f : \mathbb{B} \to \mathbb{C}$ is called hyperbolic (invariant) harmonic or $\mathcal{H}$-harmonic on $\mathbb{B}$ if $\Delta_h f(x) = 0$ for all $x \in \mathbb{B}$. We denote by $\mathcal{H}(\mathbb{B})$ the space of all $\mathcal{H}$-harmonic functions equipped with the topology of uniform convergence on compact subsets.

Let ν be the Lebesgue measure on $\mathbb{B}$ normalized, so that $\nu(\mathbb{B}) = 1$, and for $\alpha > -1$, let $d\nu_\alpha(x) = (1-|x|^2)^\alpha d\nu(x)$. For $0 < p < \infty$, denote the Lebesgue classes with respect to $d\nu_\alpha$ by $L^p_\alpha(\mathbb{B})$. The $\mathcal{H}$-harmonic weighted Bergman space $\mathcal{B}^p_\alpha$ is the subspace $L^p_\alpha(\mathbb{B}) \cap \mathcal{H}(\mathbb{B})$. When $p = 2$, $\mathcal{B}^2_\alpha$ is a reproducing kernel Hilbert space, and for each $x \in \mathbb{B}$, there exists $\mathcal{R}_\alpha(x, \cdot) \in \mathcal{B}^2_\alpha$, such that

$$f(x) = \int_{\mathbb{B}} f(y)\overline{\mathcal{R}_\alpha(x, y)}\, d\nu_\alpha(y) \qquad (f \in \mathcal{B}^2_\alpha). \tag{1.1}$$

The reproducing kernel $\mathcal{R}_\alpha$ is real-valued and the conjugation above can be deleted. $\mathcal{R}_\alpha(x, y) = \mathcal{R}_\alpha(y, x)$, and so, $\mathcal{R}_\alpha$ is $\mathcal{H}$-harmonic as a function of each variable. We refer the reader to [17] and [18, Chapter 10] for details.

For $\alpha > -1$, the Bergman projection operator P_α is defined by

$$P_\alpha \phi(x) = \int_{\mathbb{B}} \mathcal{R}_\alpha(x, y)\phi(y)\, d\nu_\alpha(y),$$

for $\phi \in L^1_\alpha$. In [22], estimates for the reproducing kernels have been obtained, and it is shown that when $1 \leq p < \infty$, $P_\gamma : L^p_\alpha \to \mathcal{B}^p_\alpha$ is bounded if and only $\alpha + 1 < p(\gamma + 1)$.

The purpose of this paper is to consider the $p=\infty$, i.e., the Bloch space, case. For a C^1 function f, the hyperbolic gradient ∇^h is defined by

$$\nabla^h f(a) = -\nabla(f\circ\varphi_a)(0) = (1-|a|^2)\nabla f(a).$$

The $\mathcal{H}$-harmonic Bloch space $\mathcal{B}$ consists of all $f\in\mathcal{H}(\mathbb{B})$, such that

$$p_{\mathcal{B}}(f) = \sup_{x\in\mathbb{B}}|\nabla^h f(x)| = \sup_{x\in\mathbb{B}}(1-|x|^2)|\nabla f(x)| < \infty. \tag{1.2}$$

$p_{\mathcal{B}}$ is a seminorm and $\|f\|_{\mathcal{B}} = |f(0)|+p_{\mathcal{B}}(f)$ is a norm on $\mathcal{B}$. The little Bloch space $\mathcal{B}_0$ is the subspace consisting of functions f satisfying $\lim_{|x|\to 1^-}(1-|x|^2)|\nabla f(x)| = 0$.

The properties we state below for the $\mathcal{H}$-harmonic Bloch space $\mathcal{B}$ are similar to the holomorphic or the harmonic case. However, we would like to point out that there are differences between these and the $\mathcal{H}$-harmonic case. For example, it is well known that polynomials are dense in the holomorphic little Bloch space, and similarly, harmonic polynomials are dense in the harmonic little Bloch space. However, this is not true in the $\mathcal{H}$-harmonic case. In fact, when the dimension n is odd, there are not any non-constant $\mathcal{H}$-harmonic polynomials. Besides, some basic properties of harmonic (or holomorphic) functions do not hold for $\mathcal{H}$-harmonic functions. For example, if f is harmonic, then the partial derivative $\partial f/\partial x_i$ and the dilation $f_r(x) = f(rx)$ are also harmonic. However, neither of these are true for $\mathcal{H}$-harmonic functions. Therefore, even if the final results are similar, many proofs in the harmonic (or holomorphic) case do not directly carry over to the $\mathcal{H}$-harmonic case.

Our first result is about projections onto $\mathcal{B}$ and $\mathcal{B}_0$. Let $L^\infty(\mathbb{B})$ be the Lebesgue space of essentially bounded functions, $C(\overline{\mathbb{B}})$ be the space of functions continuous on $\overline{\mathbb{B}}$, and $C_0(\mathbb{B})$ be its subspace consisting of functions vanishing on $\partial\mathbb{B}$.

Theorem 1.1 *For every $\alpha > -1$, P_α maps $L^\infty(\mathbb{B})$ boundedly onto $\mathcal{B}$. It also maps $C(\overline{\mathbb{B}})$ and $C_0(\mathbb{B})$ boundedly onto $\mathcal{B}_0$.*

It has already been verified in [22, Theorem 1.5] that $P_\alpha : L^\infty(\mathbb{B})\to\mathcal{B}$ is bounded and the main aspect of the above theorem is the surjectivity. To achieve this, we first characterize $\mathcal{B}$ and $\mathcal{B}_0$ in terms of certain fractional differential operators that are defined in Sect. 3. These operators are compatible with $\mathcal{H}$-harmonic functions and the reproducing kernels, and to understand the properties of $\mathcal{B}$, they are more suited than ∇^h or ∇ used in (1.2).

We next consider the duality problem. For $1<p<\infty$, the dual of the hyperbolic Bergman space $\mathcal{B}^p_\alpha$ can be identified with $\mathcal{B}^{p'}_\alpha$, where $p' = p/(p-1)$ is the conjugate exponent of p ([22, Corollary 1.4]). We complete the missing $p=1$ case.

Theorem 1.2 *For $\alpha > -1$, the dual of $\mathcal{B}^1_\alpha$ can be identified with $\mathcal{B}$ under the pairing*

$$\langle f, g\rangle_\alpha = \lim_{r\to 1^-}\int_{r\mathbb{B}} f(x)g(x)\,d\nu_\alpha(x). \tag{1.3}$$

More precisely, to each $\Lambda\in(\mathcal{B}^1_\alpha)^*$, there corresponds a unique $g\in\mathcal{B}$ with $\|g\|_{\mathcal{B}}$ equivalent to $\|\Lambda\|$, such that $\Lambda(f) = \langle f, g\rangle_\alpha$.

Similarly, for every $\alpha > -1$, the dual of $\mathcal{B}_0$ can be identified with $\mathcal{B}^1_\alpha$ under the pairing (1.3).

For an unbounded $g \in \mathcal{B}$ and $f \in \mathcal{B}^1_\alpha$, the integral $\int_{\mathbb{B}} f(x)g(x)d\nu_\alpha(x)$ may not converge absolutely; however, the limit in (1.3) always exists. In the case of the *holomorphic* Bloch space on the unit ball of $\mathbb{C}^n$, $g(z) = \log 1/(1 - z_1)$ is an unbounded Bloch function. We give an example of an unbounded $\mathcal{H}$-harmonic Bloch function in Lemma 6.2.

As our final result, we prove atomic decomposition of $\mathcal{H}$-harmonic Bloch functions. Atomic decomposition of *harmonic* Bergman and Bloch functions has been first obtained in [4] (see also [3]). In the $\mathcal{H}$-harmonic case, atomic decomposition of Hardy spaces is considered in [9], and Bergman spaces in [23].

The pseudo-hyperbolic metric $\rho(a, b)$ for $a, b \in \mathbb{B}$ is given by $\rho(a, b) = |\varphi_a(b)|$. For $0 < r < 1$, let $E_r(a) = \{x \in \mathbb{B} : \rho(x, a) < r\}$ be the pseudo-hyperbolic ball with center a and radius r. A sequence $\{a_m\}$ of points of $\mathbb{B}$ is called r-separated if $\rho(a_m, a_k) \geq r$ for $m \neq k$. An r-separated sequence is called an r-lattice if $\bigcup_{m=1}^\infty E_r(a_m) = \mathbb{B}$. Let ℓ^∞ be the space of bounded sequences with norm $\|\{\lambda_m\}\|_{\ell^\infty} = \sup_{m\geq 1}|\lambda_m|$, and c_0 be the subspace consisting of sequences that converge to 0.

Theorem 1.3 *Let $\alpha > -1$. There is an $r_0 < 1/2$ depending only on n and α, such that if $\{a_m\}$ is an r-lattice with $r < r_0$, then for every $f \in \mathcal{B}$ (resp. $\mathcal{B}_0$), there exists $\{\lambda_m\} \in \ell^\infty$ (resp. c_0), such that*

$$f(x) = \sum_{m=1}^{\infty} \lambda_m \frac{\mathcal{R}_\alpha(x, a_m)}{\|\mathcal{R}_\alpha(\cdot, a_m)\|_{\mathcal{B}}} \qquad (x \in \mathbb{B}), \tag{1.4}$$

where the series converges absolutely and uniformly on compact subsets of $\mathbb{B}$ and the norm $\|\{\lambda_m\}\|_{\ell^\infty}$ is equivalent to the norm $\|f\|_{\mathcal{B}}$.

By Lemma 7.1, the norm $\|\mathcal{R}_\alpha(\cdot, a_m)\|_{\mathcal{B}}$ is equivalent to $(1 - |a_m|^2)^{-(\alpha+n)}$ and the theorem remains true if one uses the representation

$$f(x) = \sum_{m=1}^{\infty} \lambda_m (1 - |a_m|^2)^{\alpha+n}\, \mathcal{R}_\alpha(x, a_m) \qquad (x \in \mathbb{B}) \tag{1.5}$$

instead of (1.4).

2 Preliminaries

We denote positive constants whose exact values are inessential by the letter C. For two positive expressions X and Y, we write $X \lesssim Y$ to mean $X \leq CY$. If both $X \leq CY$ and $Y \leq CX$, we write $X \sim Y$.

For $x, y \in \mathbb{B}$, we define

$$[x, y] := \sqrt{1 - 2\langle x, y\rangle + |x|^2|y|^2}.$$

Clearly, $[x, y]$ is symmetric; $[x, 0] = 1$, and if $y \neq 0$, then $[x, y] = \big||y|x - y/|y|\big|$. Therefore

$$[x, y] \geq 1 - |x||y| \qquad (x, y \in \mathbb{B}). \tag{2.1}$$

Denote by $\mathcal{M}(\mathbb{B})$ the group of Möbius transformations that preserve $\mathbb{B}$. For $a \in \mathbb{B}$, the canonical Möbius transformation that exchanges a and 0 is given by

$$\varphi_a(x) = \frac{a|x-a|^2 + (1-|a|^2)(a-x)}{[x, a]^2} \qquad (x \in \mathbb{B}). \tag{2.2}$$

It is an involution and for all $x \in \mathbb{B}$ the following identity holds:

$$1 - |\varphi_a(x)|^2 = \frac{(1-|a|^2)(1-|x|^2)}{[x, a]^2}. \tag{2.3}$$

The determinant of the Jacobian matrix of φ_a satisfies ([18, Theorem 3.3.1])

$$|\det J\varphi_a(x)| = \frac{(1-|\varphi_a(x)|^2)^n}{(1-|x|^2)^n}. \tag{2.4}$$

The equality

$$[a, \varphi_a(x)] = \frac{1-|a|^2}{[x, a]} \tag{2.5}$$

follows from (2.3) and is a special case of [15, Theorem 1.1].

For $a, b \in \mathbb{B}$, the pseudo-hyperbolic metric $\rho(a, b) = |\varphi_a(b)|$ satisfies the equality

$$\rho(a, b) = \frac{|a-b|}{[a, b]}. \tag{2.6}$$

The pseudo-hyperbolic ball $E_r(a)$ is also a Euclidean ball with ([18, Theorem 2.2.2])

$$\text{center} = \frac{(1-r^2)a}{1-|a|^2r^2} \quad \text{and} \quad \text{radius} = \frac{(1-|a|^2)r}{1-|a|^2r^2}. \tag{2.7}$$

For a proof of the following lemma, see [2, Lemma 2.1 and 2.2].

Lemma 2.1 *(i) For all $a, b \in \mathbb{B}$*

$$\frac{1-\rho(a,b)}{1+\rho(a,b)} \leq \frac{1-|a|}{1-|b|} \leq \frac{1+\rho(a,b)}{1-\rho(a,b)}.$$

(ii) For all $a, b, x \in \mathbb{B}$

$$\frac{1-\rho(a,b)}{1+\rho(a,b)} \leq \frac{[x,a]}{[x,b]} \leq \frac{1+\rho(a,b)}{1-\rho(a,b)}.$$

The hyperbolic metric on $\mathbb{B}$ is given by

$$\beta(a,b) = \log \frac{1+\rho(a,b)}{1-\rho(a,b)} \qquad (a,b \in \mathbb{B}).$$

Both metrics ρ and β are Möbius invariant.

Let σ be the normalized surface measure on $\mathbb{S}$. For $f \in L^1(\mathbb{B})$, the polar coordinates formula is

$$\int_{\mathbb{B}} f\, d\nu(x) = n \int_0^1 r^{n-1} \int_{\mathbb{S}} f(r\zeta)\, d\sigma(\zeta) dr.$$

Proof of the following two estimates can be found in [13, Proposition 2.2].

Lemma 2.2 *Let $s > -1$ and $t \in \mathbb{R}$. For all $x \in \mathbb{B}$*

$$\int_{\mathbb{S}} \frac{d\sigma(\zeta)}{|x-\zeta|^{n-1+t}} \sim \int_{\mathbb{B}} \frac{(1-|y|^2)^s}{[x,y]^{n+s+t}} d\nu(y) \sim \begin{cases} \dfrac{1}{(1-|x|^2)^t}, & \text{if } t > 0; \\ 1 + \log \dfrac{1}{1-|x|^2}, & \text{if } t = 0; \\ 1, & \text{if } t < 0, \end{cases}$$

where the implied constants depend only on n, s, t.

3 Reproducing kernels and fractional differential operators

In this section, we review the properties of the reproducing kernels and define a family of differential operators D_s^t.

Denote by $H_m(\mathbb{R}^n)$ the space of all homogeneous (Euclidean) harmonic polynomials of degree m on $\mathbb{R}^n$. It is finite dimensional with $\dim H_m \sim m^{n-2}$ $(m \geq 1)$. By homogeneity, $q_m \in H_m(\mathbb{R}^n)$ is determined by its restriction to $\mathbb{S}$. This restriction is called a spherical harmonic and the space of spherical harmonics of degree m is denoted by $H_m(\mathbb{S})$. Spherical harmonics of different degrees are orthogonal in $L^2(\mathbb{S})$

$$\int_{\mathbb{S}} q_m(\zeta) q_k(\zeta)\, d\sigma(\zeta) = 0 \qquad (m \neq k, q_m \in H_m(\mathbb{S}), q_k \in H_k(\mathbb{S})). \tag{3.1}$$

For every $\eta \in \mathbb{S}$, there exists $Z_m(\eta, \cdot) \in H_m(\mathbb{S})$, called the zonal harmonic of degree m with pole η, such that for all $q_m \in H_m(\mathbb{S})$

$$q_m(\eta) = \int_{\mathbb{S}} q_m(\zeta) Z_m(\eta,\zeta)\, d\sigma(\zeta). \tag{3.2}$$

$Z_m(\cdot,\cdot)$ is real-valued, symmetric, and homogeneous of degree m in each variable. On the diagonal, $Z_m(\zeta,\zeta) = \dim H_m$, and in general, $|Z_m(\eta,\zeta)| \leq Z_m(\zeta,\zeta)$. Thus

$$|Z_m(\eta,\zeta)| \lesssim m^{n-2} \qquad (m \geq 1). \tag{3.3}$$

For details, we refer the reader to [1, Chapter 5].

For $q_m \in H_m(\mathbb{R}^n)$, the solution of the $\mathcal{H}$-harmonic Dirichlet problem on $\mathbb{B}$ with boundary data $q_m|_{\mathbb{S}}$ is given by ([18, Theorem 6.1.1])

$$g(x) = S_m(|x|)q_m(x) \qquad (x \in \overline{\mathbb{B}}). \tag{3.4}$$

That is, g is $\mathcal{H}$-harmonic on $\mathbb{B}$, continuous on $\overline{\mathbb{B}}$ and equals q_m on $\mathbb{S}$. Here, the factor $S_m(r)$ $(0 \le r \le 1)$ is given by

$$S_m(r) = \frac{F(m, 1 - \frac{1}{2}n; m + \frac{1}{2}n; r^2)}{F(m, 1 - \frac{1}{2}n; m + \frac{1}{2}n; 1)}, \tag{3.5}$$

where

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k \tag{3.6}$$

is the Gauss hypergeometric function. S_m depends also on the dimension n, but we do not write this for shortness. When the dimension n is even the hypergeometric series in (3.5) terminates and S_m is a polynomial, but this is not true in odd dimensions. $S_m(r)$ is a decreasing function of r, and is normalized so that $S_m(1) = 1$. When $m \ge 1$, the estimate

$$1 \le S_m(r) \le C m^{n/2-1} \qquad (0 \le r \le 1) \tag{3.7}$$

holds, where $C = C(n)$ is a constant depending only on n (see [17, Proposition I.6], [20, Lemma 2.6] or [22, Lemma 2.13]). When $m = 0$, $S_0(r) = 1$.

Every $\mathcal{H}$-harmonic function on $\mathbb{B}$ can be written as a series of terms of the form (3.4). More precisely, for every $f \in \mathcal{H}(\mathbb{B})$, there exists a unique sequence of polynomials $q_m \in H_m(\mathbb{R}^n)$, such that

$$f(x) = \sum_{m=0}^{\infty} S_m(|x|)q_m(x) \qquad (x \in \mathbb{B}),$$

where the series converges absolutely and uniformly on compact subsets of $\mathbb{B}$ (see [8], [11], [14], [18, Theorem 6.3.1]).

The hyperbolic (invariant) Poisson kernel and its series expansion are given by [18, Theorem 6.2.2]

$$\mathbb{P}_h(x, \zeta) = \frac{(1 - |x|^2)^{n-1}}{|x - \zeta|^{2(n-1)}} = \sum_{m=0}^{\infty} S_m(|x|) Z_m(x, \zeta) \qquad (x \in \mathbb{B}, \zeta \in \mathbb{S}). \tag{3.8}$$

For $f \in L^1(\mathbb{S})$, the Poisson integral of f is $\mathbb{P}_h[f](x) = \int_{\mathbb{S}} \mathbb{P}_h(x, \zeta) f(\zeta)\, d\sigma(\zeta)$.

For the Bergman kernels $\mathcal{R}_\alpha(x, y)$, a closed formula is not known; however, the following series expansion holds: ([17, Corollary III.5], [19, Theorem 5.3])

$$\mathcal{R}_\alpha(x, y) = \sum_{m=0}^{\infty} c_m(\alpha) S_m(|x|) S_m(|y|) Z_m(x, y) \qquad (\alpha > -1, x, y \in \mathbb{B}), \quad (3.9)$$

where the coefficients $c_m(\alpha)$ are determined by

$$\frac{1}{c_m(\alpha)} = n \int_0^1 r^{2m+n-1} S_m^2(r)(1 - r^2)^\alpha \, dr. \quad (3.10)$$

An explicit expression for the above integral is not known either. However, the estimate

$$c_m(\alpha) \sim m^{\alpha+1} \qquad (m \to \infty) \quad (3.11)$$

holds (see ([17, Theorem III.6]) from which it follows that the series in (3.9) converges absolutely and uniformly on $K \times \overline{\mathbb{B}}$ for every compact $K \subset \mathbb{B}$.

It is clear from (3.10) that $c_m(\alpha) > 0$. Using these coefficients, we define a family of fractional differential operators.

Definition 3.1 Let s, t be real numbers satisfying $s > -1$ and $s+t > -1$. If $f \in \mathcal{H}(\mathbb{B})$ has the series expansion $f(x) = \sum_{m=0}^{\infty} S_m(|x|) q_m(x)$, then define

$$D_s^t f(x) = \sum_{m=0}^{\infty} \frac{c_m(s + t)}{c_m(s)} S_m(|x|) q_m(x). \quad (3.12)$$

The operator D_s^t multiplies the m^{th} term of the series expansion of f with the coefficient $c_m(s + t)/c_m(s) \sim m^t$ by (3.11). Note that the main parameter t can take any real value as long as s is large enough. Similar types of operators are frequently used in the theory of holomorphic and harmonic Bergman spaces and act as differential operators of order t (integral if $t < 0$). Our use of D_s^t follows [6] and [5]. For $\mathcal{H}$-harmonic functions, slightly different operators with multipliers $\Gamma(m+s)/\Gamma(m+s+t)$ are used in [16] and Hardy–Littlewood inequalities are obtained.

Lemma 3.2 *For $f \in \mathcal{H}(\mathbb{B})$, the series in (3.12) converges absolutely and uniformly on compact subsets of $\mathbb{B}$, and so $D_s^t f$ is in $\mathcal{H}(\mathbb{B})$. In addition, $D_s^t : \mathcal{H}(\mathbb{B}) \to \mathcal{H}(\mathbb{B})$ is continuous when $\mathcal{H}(\mathbb{B})$ is equipped with the topology of uniform convergence on compact subsets.*

This lemma can be verified in the same way as [6, Theorems 3.1 and 3.2]. An additional factor $S_m(r)$ appears, but it can easily be handled with the estimate (3.7). The operator D_s^t is invertible with

$$D_{s+t}^{-t} D_s^t = D_s^t D_{s+t}^{-t} = \text{Id}. \quad (3.13)$$

The role of s is minor and one reason for its inclusion is to simplify the action of D_s^t on the reproducing kernel $\mathcal{R}_s$

$$D_s^t \mathcal{R}_s(x, y) = \mathcal{R}_{s+t}(x, y). \tag{3.14}$$

If $f \in \mathcal{H}(\mathbb{B})$ is integrable, then $D_s^t f$ can be written as an integral.

Lemma 3.3 *Let $s > -1$, $s + t > -1$ and $f \in L_s^1(\mathbb{B})$.*

(i) $D_s^t P_s f(x) = D_s^t \int_{\mathbb{B}} \mathcal{R}_s(x, y) f(y) d\nu_s(y) = \int_{\mathbb{B}} \mathcal{R}_{s+t}(x, y) f(y) d\nu_s(y)$.
(ii) If f is also in $\mathcal{H}(\mathbb{B})$, then $D_s^t f(x) = \int_{\mathbb{B}} \mathcal{R}_{s+t}(x, y) f(y) d\nu_s(y)$.

Proof (i) For fixed $x \in \mathbb{B}$, the series in (3.9) converges uniformly for $y \in \mathbb{B}$. Thus

$$\begin{aligned}\int_{\mathbb{B}} \mathcal{R}_s(x, y) f(y)\, d\nu_s(y) &= \sum_{m=0}^{\infty} c_m(s) S_m(|x|) \int_{\mathbb{B}} Z_m(x, y) S_m(|y|) f(y)\, d\nu_s(y) \\ &=: \sum_{m=0}^{\infty} c_m(s) S_m(|x|) q_m(x). \end{aligned}\tag{3.15}$$

The function q_m is in $H_m(\mathbb{R}^n)$ and the series in (3.15) converges absolutely and uniformly on compact subsets of $\mathbb{B}$ by (3.7), (3.11), and the inequality $|Z_m(x, y)| \lesssim |x|^m m^{n-2}$. Thus, the series in (3.15) is the (unique) series expansion of the left-hand side and by (3.12), $D_s^t \int_{\mathbb{B}} \mathcal{R}_s(x, y) f(y) d\nu_s(y) = \sum_{m=0}^{\infty} c_m(s+t) S_m(|x|) q_m(x)$. This series equals $\int_{\mathbb{B}} \mathcal{R}_{s+t}(x, y) f(y) d\nu_s(y)$ by the same reasoning.

(ii) If $f \in L_s^1 \cap \mathcal{H}(\mathbb{B})$, then $P_s f = f$, because the reproducing property in (1.1) holds also for $f \in \mathcal{B}_\alpha^1$ ([22, Lemma 7.1]).

Upper estimates of the reproducing kernels $\mathcal{R}_\alpha$ have been obtained in [22, Theorem 1.2]. Here, ∇_x means that the gradient is taken with respect to the variable x.

Lemma 3.4 *Let $\alpha > -1$. There exists a constant $C = C(n, \alpha) > 0$, such that for all $x, y \in \mathbb{B}$*

(a) $|\mathcal{R}_\alpha(x, y)| \le \dfrac{C}{[x, y]^{\alpha+n}}$;

(b) $|\nabla_x \mathcal{R}_\alpha(x, y)| \le \dfrac{C}{[x, y]^{\alpha+n+1}}$.

The following projection theorem is proved in [22, Theorem 1.1].

Lemma 3.5 *Let $1 \le p < \infty$ and $\alpha, \gamma > -1$. The operator $P_\gamma : L_\alpha^p \to \mathcal{B}_\alpha^p$ is bounded if and only if $\alpha + 1 < p(\gamma + 1)$. If this holds, $P_\gamma f = f$ for $f \in L_\alpha^p \cap \mathcal{H}(\mathbb{B})$.*

4 Elementary properties of the Bloch space

We first mention a few basic properties of $\mathcal{B}$ and $\mathcal{B}_0$. The verifications are omitted as they are straightforward and are similar to the holomorphic or the Euclidean harmonic case. The space $\mathcal{B}$ is a Banach space and $\mathcal{B}_0$ is a closed subspace of $\mathcal{B}$. The seminorm $p_{\mathcal{B}}$ is Möbius invariant, i.e., $p_{\mathcal{B}}(f \circ \psi) = p_{\mathcal{B}}(f)$ for all $\psi \in \mathcal{M}(\mathbb{B})$. For $f \in \mathcal{B}$ and $x \in \mathbb{B}$

$$|f(x) - f(0)| \leq \left| \int_0^1 \langle \nabla f(tx), x \rangle \, dt \right| \leq \frac{1}{2} p_{\mathcal{B}}(f) \beta(x, 0),$$

by the fundamental theorem of calculus. Replacing f by $f \circ \varphi_y$ and x by $\varphi_y(x)$, with the Möbius invariance of $p_{\mathcal{B}}$ and β, shows

$$|f(x) - f(y)| \leq \frac{1}{2} p_{\mathcal{B}}(f) \beta(x, y) \qquad (f \in \mathcal{B}, \ x, y \in \mathbb{B}). \tag{4.1}$$

Since $\beta(x, 0) \leq 1 + \log 1/(1 - |x|)$, we have the following pointwise bound:

$$|f(x)| \leq \|f\|_{\mathcal{B}} \Big(1 + \log \frac{1}{1 - |x|} \Big) \qquad (f \in \mathcal{B}, x \in \mathbb{B}). \tag{4.2}$$

There are various results in [7] and [9] which show that $\mathcal{H}$-harmonic functions can have different behaviors depending on whether the dimension n is odd or even. We show one more difference. Let $q_m \in H_m(\mathbb{S})$. If the dimension n is even, $S_m(|x|)$ is a polynomial and the Poisson integral $\mathbb{P}_h[q_m](x) = S_m(|x|) q_m(x)$ is an $\mathcal{H}$-harmonic polynomial. This is not true when the dimension n is odd. In fact, in this case, a non-constant polynomial cannot be $\mathcal{H}$-harmonic on $\mathbb{B}$.

Lemma 4.1 *In odd dimensions, there are no non-constant polynomials in $\mathcal{H}(\mathbb{B})$.*

Proof Suppose there exists a polynomial $p = \sum_{m=0}^{M} p_m$ of degree $M \geq 1$, where p_m is homogeneous of degree m, such that $p \in \mathcal{H}(\mathbb{B})$. By [1, Theorem 5.7], the restriction of p_m to the unit sphere $\mathbb{S}$ can be written as a sum of spherical harmonics of degree at most m. Therefore, there exist $q_m \in H_m(\mathbb{S})$, $m = 0, 1, \ldots, M$, such that for $\zeta \in \mathbb{S}$, $p(\zeta) = \sum_{m=0}^{M} q_m(\zeta)$. Since p is $\mathcal{H}$-harmonic on $\mathbb{B}$, we have $p = \mathbb{P}_h[p|_{\mathbb{S}}]$ and

$$p(r\zeta) = \sum_{m=0}^{M} S_m(r) r^m q_m(\zeta) \qquad (\zeta \in \mathbb{S}, \ 0 \leq r \leq 1). \tag{4.3}$$

Because p is non-constant, there exist $1 \leq k \leq M$ and $\eta \in \mathbb{S}$, such that $q_k(\eta) \neq 0$. For $0 \leq r \leq 1$, we compute the integral $I(r) = \int_{\mathbb{S}} p(r\zeta) Z_k(\eta, \zeta) d\sigma(\zeta)$ in two ways. First

$$\begin{aligned} I(r) = \int_{\mathbb{S}} \sum_{m=0}^{M} p_m(r\zeta) Z_k(\eta, \zeta) d\sigma(\zeta) &= \sum_{m=k}^{M} r^m \int_{\mathbb{S}} p_m(\zeta) Z_k(\eta, \zeta) d\sigma(\zeta) \\ &=: \sum_{m=k}^{M} a_m r^m, \end{aligned} \tag{4.4}$$

where in the second equality, we use the fact that $\int_{\mathbb{S}} p_m(\zeta)Z_k(\eta,\zeta)d\sigma(\zeta) = 0$ for $0 \le m \le k-1$, which follows from [1, Theorem 5.7] and (3.1). Next, if we use (4.3) with (3.1) and (3.2)

$$I(r) = \sum_{m=0}^{M} S_m(r)r^m \int_{\mathbb{S}} q_m(\zeta)Z_k(\eta,\zeta)d\sigma(\zeta) = r^k S_k(r)q_k(\eta). \tag{4.5}$$

Equating (4.4) and (4.5), we see that $S_k(r)q_k(\eta) = \sum_{m=k}^{M} a_m r^{m-k}$, and since $q_k(\eta) \neq 0$, this implies that $S_k(r)$ is a polynomial of r. This is a contradiction, because when the dimension n is odd, S_k is not a polynomial for $k \ge 1$, since its hypergeometric series do not terminate.

Lemma 4.2 *If $q_m \in H_m(\mathbb{R}^n)$, then $\mathbb{P}_h[q_m|_{\mathbb{S}}](x) = S_m(|x|)q_m(x)$ is in $\mathcal{B}_0$.*

Proof When $n = 2$, the result is clear since $S_m \equiv 1$. For $n \ge 3$, we need two elementary facts about hypergeometric series. First

$$\frac{d}{dz}F(a,b;c;z) = \frac{ab}{c}F(a+1,b+1;c+1;z), \tag{4.6}$$

and second, if $\Re\{c-a-b\} > 0$, then $F(a,b;c;z)$ converges uniformly and so is bounded on the closed disk $\{z : |z| \le 1\}$. Now, by (4.6)

$$\begin{aligned}\frac{\partial}{\partial x_i}S_m(|x|)q_m(x) &= 2x_i\frac{m(1-\frac{1}{2}n)}{m+\frac{1}{2}n}\,\frac{F(m+1,2-\frac{1}{2}n;m+\frac{1}{2}n+1;|x|^2)}{F(m,1-\frac{1}{2}n;m+\frac{1}{2}n;1)}\,q_m(x)\\&\quad + S_m(|x|)\frac{\partial}{\partial x_i}q_m(x).\end{aligned}$$

The hypergeometric function in the first term is bounded, since $\Re\{c-a-b\} = n-2 > 0$. Since the second term is also bounded, the result follows.

Lemma 4.3 *For every polynomial p and $\alpha > -1$, the projection $P_\alpha p$ is in $\mathcal{B}_0$.*

Proof We can assume that p is homogeneous. By [1, Theorem 5.7] again, p can be written in the form $p = q_m + |x|^2 q_{m-2} + \cdots + |x|^{2k}q_{m-2k}$, where $k = [m/2]$ and $q_j \in H_j(\mathbb{R}^n)$. Thus, it suffices to show that $P_\alpha(|x|^k q_j) \in \mathcal{B}_0$ for all $k \ge 0$ and $q_j \in H_j(\mathbb{R}^n)$. Integrating in polar coordinates with $y = |y|\zeta = r\zeta$, and then using the uniform convergence of the series in (3.9) along with (3.1) and (3.2), we see that $P_\alpha\big(|x|^k q_j\big)(x) = \int_{\mathbb{B}} \mathcal{R}_\alpha(x,y)|y|^k q_j(y)d\nu_\alpha(y)$ equals

$$\begin{aligned}&\sum_{m=0}^{\infty} c_m(\alpha)S_m(|x|)\int_0^1 nr^{n-1}S_m(r)r^{k+m+j}(1-r^2)^\alpha \int_{\mathbb{S}} Z_m(x,\zeta)q_j(\zeta)d\sigma(\zeta)dr\\&\quad = c_j(\alpha)S_j(|x|)q_j(x)\int_0^1 nr^{n-1}S_j(r)r^{k+2j}(1-r^2)^\alpha dr\\&\quad = CS_j(|x|)q_j(x),\end{aligned}$$

which belongs to $\mathcal{B}_0$ by Lemma 4.2.

Remark 4.4 The above proof shows also that for every polynomial p and $\alpha > -1$, $P_\alpha p \in \operatorname{span}\{S_m(|x|)q_m(x) \mid q_m \in H_m(\mathbb{R}^n), m = 0, 1, 2 \ldots\}$.

5 Projections onto the Bloch and the little Bloch space

In this section, we prove Theorem 1.1.

Lemma 5.1 *For every $\alpha > -1$, $P_\alpha : L^\infty(\mathbb{B}) \to \mathcal{B}$ is bounded. In addition, if $f \in C(\overline{\mathbb{B}})$, then $P_\alpha f \in \mathcal{B}_0$.*

Proof The estimate in Lemma 3.4(b) together with Lemma 2.2 immediately implies that P_α maps $L^\infty(\mathbb{B})$ boundedly into $\mathcal{B}$ ([22, Theorem 1.5]). Since $\mathcal{B}_0$ is closed in $\mathcal{B}$, by the Stone–Weierstrass theorem and Lemma 4.3, P_α maps $C(\overline{\mathbb{B}})$ into $\mathcal{B}_0$.

To verify the surjectivity part of Theorem 1.1, we first characterize $\mathcal{B}$ and $\mathcal{B}_0$ in terms of the differential operators D_s^t. We begin with two estimates. One is similar to Lemma 2.2, and the other to [23, Lemma 4.3], but they include an extra term $\beta(x, y)$, the hyperbolic distance between x and y.

Lemma 5.2 *Let $s > -1$ and $t > 0$.*

(i) *There exists a constant $C = C(n, s, t) > 0$, such that for all $x \in \mathbb{B}$*

$$\int_{\mathbb{B}} \frac{\beta(x, y)\,(1 - |y|^2)^s}{[x, y]^{n+s+t}}\, d\nu(y) \le \frac{C}{(1 - |x|^2)^t}.$$

(ii) *Given $\varepsilon > 0$, there exists $0 < r_\varepsilon < 1$, such that for all r with $r_\varepsilon < r < 1$ and all $x \in \mathbb{B}$*

$$\int_{\mathbb{B}\setminus E_r(x)} \frac{\beta(x, y)\,(1 - |y|^2)^s}{[x, y]^{n+s+t}}\, d\nu(y) < \frac{\varepsilon}{(1 - |x|^2)^t}.$$

Proof The proof is similar to the proof of [23, Lemma 4.3], requiring only a minor modification. For $0 \le r < 1$, let

$$I_r(x) := (1 - |x|^2)^t \int_{\mathbb{B}\setminus E_r(x)} \frac{\beta(x, y)\,(1 - |y|^2)^s}{[x, y]^{n+s+t}}\, d\nu(y),$$

where for $r = 0$, $\mathbb{B}\setminus E_0(x) = \mathbb{B}$. In the integral make the change of variable $y = \varphi_x(z)$. Note that $\varphi_x^{-1}(\mathbb{B}\setminus E_r(x)) = \mathbb{B}\setminus r\mathbb{B}$. Employing (2.3)-(2.5), and the fact that $\beta(x, \varphi_x(z)) = \beta(0, z)$ by the Möbius-invariance of β, we obtain

$$I_r(x) = \int_{\mathbb{B}\setminus r\mathbb{B}} \frac{\beta(0, z)\,(1 - |z|^2)^s}{[x, z]^{n+s-t}}\, d\nu(z).$$

Using $\beta(0, z) \leq 1 + \log 1/(1 - |z|)$ and integrating in polar coordinates yields

$$I_r(x) \leq \int_r^1 n\tau^{n-1}\Big(1 + \log \frac{1}{1-\tau}\Big)\,(1-\tau^2)^s \int_{\mathbb{S}} \frac{d\sigma(\zeta)}{|\tau x - \zeta|^{n+s-t}}\, d\tau.$$

Estimating the inner integral with Lemma 2.2 in three cases and using the inequality $1 - \tau^2|x|^2 \geq 1 - \tau^2$, we see that

$$\int_{\mathbb{S}} \frac{d\sigma(\zeta)}{|\tau x - \zeta|^{n+s-t}} \leq C g(\tau) := C \begin{cases} \dfrac{1}{(1-\tau^2)^{1+s-t}}, & \text{if } 1+s-t>0;\\ 1 + \log \dfrac{1}{1-\tau^2}, & \text{if } 1+s-t=0;\\ 1, & \text{if } 1+s-t<0, \end{cases}$$

where C depends only on n, s, t. Thus

$$I_r(x) \leq C \int_r^1 n\tau^{n-1}\Big(1 + \log \frac{1}{1-\tau}\Big)\,(1-\tau^2)^s g(\tau)\, d\tau.$$

Because $s > -1$ and $t > 0$, in all the three cases, the above integral is finite when $r = 0$. This proves both parts of the lemma.

We prove one more estimate. For future use, in the lemma below, we consider the integral over $r\mathbb{B}$ for $0 < r \leq 1$, not just $\mathbb{B}$.

Lemma 5.3 *Let $s > -1$ and $t > 0$. There exists a constant $C = C(n, s, t) > 0$, such that for all $0 < r \leq 1$ and all $f \in \mathcal{B}$*

$$(1-|x|^2)^t \left| \int_{r\mathbb{B}} \mathcal{R}_{s+t}(x, y) f(y) d\nu_s(y) \right| \leq C\|f\|_{\mathcal{B}}.$$

Proof We write

$$\int_{r\mathbb{B}} \mathcal{R}_{s+t}(x, y) f(y) d\nu_s(y) = \int_{r\mathbb{B}} \mathcal{R}_{s+t}(x, y) f(x) d\nu_s(y) + \int_{r\mathbb{B}} \mathcal{R}_{s+t}(x, y)\big(f(y) - f(x)\big) d\nu_s(y) =: h_{1,r}(x) + h_{2,r}(x).$$

Integrating in polar coordinates shows

$$h_{1,r}(x) = f(x) \int_0^r n\tau^{n-1}(1-\tau^2)^s \int_{\mathbb{S}} \mathcal{R}_{s+t}(x, \tau\zeta)\, d\sigma(\zeta)\, d\tau.$$

By the mean-value property for $\mathcal{H}$-harmonic functions [18, Corollary 4.1.3], the inner integral is $\mathcal{R}_{s+t}(x, 0)$ which equals $c_0(s+t)$ for all $x \in \mathbb{B}$ by (3.9), because $Z_m(x, 0) = 0$ for $m \geq 1$, $Z_0 \equiv 1$ and $S_0 \equiv 1$. Therefore, using also (4.2), we obtain

$$\begin{aligned} \left|h_{1,r}(x)\right| &\le c_0(s+t)|f(x)| \int_0^1 n\tau^{n-1}(1-\tau^2)^s\, d\tau = C|f(x)| \\ &\le C\|f\|_{\mathcal{B}}\Big(1+\log\frac{1}{1-|x|}\Big), \end{aligned} \tag{5.1}$$

with C depending only on n, s, t. This implies $(1-|x|^2)^t|h_{1,r}(x)| \lesssim \|f|_{\mathcal{B}}$.

Next, by (4.1) and Lemma 3.4(a)

$$|h_{2,r}(x)| \lesssim p_{\mathcal{B}}(f)\int_{r\mathbb{B}} \frac{\beta(x,y)\,d\nu_s(y)}{[x,y]^{n+s+t}} \le \|f\|_{\mathcal{B}}\int_{\mathbb{B}} \frac{\beta(x,y)\,d\nu_s(y)}{[x,y]^{n+s+t}} \lesssim \frac{\|f\|_{\mathcal{B}}}{(1-|x|^2)^t},$$

where in the last inequality, we use Lemma 5.2(i). This proves the lemma.

Proposition 5.4 *Let $s>-1$, $t>0$ and $f\in\mathcal{H}(\mathbb{B})$. Then, $f\in\mathcal{B}$ (resp. $\mathcal{B}_0$) if and only if $(1-|x|^2)^t D_s^t f(x)\in L^\infty(\mathbb{B})$ (resp. $C_0(\mathbb{B})$). Moreover*

$$\|f\|_{\mathcal{B}} \sim \|(1-|x|^2)^t D_s^t f(x)\|_{L^\infty},$$

where the implicit constants depend only on n, s, t, and are independent of f.

This proposition shows that in the definition (1.2) of the Bloch space, one can replace $(1-|x|^2)|\nabla f(x)|$ with $(1-|x|^2)^t|D_s^t f(x)|$ for any $t>0$. It is more suitable to work with $D_s^t f$, since it is $\mathcal{H}$-harmonic. It also shows that for every $s>-1$ and $t>0$, $\|(1-|x|^2)^t D_s^t f(x)\|_{L^\infty}$ is a norm on $\mathcal{B}$ equivalent to $\|f\|_{\mathcal{B}}$. In the rest of the paper, we mostly employ these norms.

Proof Suppose $f\in\mathcal{B}$. Then, $f\in L_s^1\cap\mathcal{H}(\mathbb{B})$ by (4.2), and by Lemma 3.3(ii) we have $D_s^t f(x)=\int_{\mathbb{B}}\mathcal{R}_{s+t}(x,y)f(y)d\nu_s(y)$. That $\|(1-|x|^2)^t D_s^t f(x)\|_{L^\infty}\lesssim\|f\|_{\mathcal{B}}$ follows now from Lemma 5.3.

To see the other direction, suppose $(1-|x|^2)^t D_s^t f(x)\in L^\infty(\mathbb{B})$. We claim that $P_s\big[(1-|x|^2)^t D_s^t f(x)\big]=f$. This is true since $D_s^t f\in L_{s+t}^1(\mathbb{B})\cap\mathcal{H}(\mathbb{B})$ and by Lemma 3.3(ii) and (3.13)

$$\begin{aligned} P_s\big[(1-|x|^2)^t D_s^t f(x)\big](x) &= \int_{\mathbb{B}} \mathcal{R}_s(x,y) D_s^t f(y) d\nu_{s+t}(y) \\ &= D_{s+t}^{-t}(D_s^t f)(x) = f(x). \end{aligned} \tag{5.2}$$

Thus, by Lemma 5.1, $f\in\mathcal{B}$ and $\|f\|_{\mathcal{B}}\le\|P_s\|\,\|(1-|x|^2)^t D_s^t f(x)\|_{L^\infty}$.

We now consider the $\mathcal{B}_0$ case. Let $f\in\mathcal{B}_0$. For $\varepsilon>0$, pick $r>r_\varepsilon$ where r_ε is as given in Lemma 5.2 (ii). Similar to the proof of Lemma 5.3, we write $D_s^t f(x)=\int_{\mathbb{B}}\mathcal{R}_{s+t}(x,y)f(y)d\nu_s(y)$ in the form

$$\begin{aligned} D_s^t f(x) = &\int_{\mathbb{B}} \mathcal{R}_{s+t}(x,y)f(x)d\nu_s(y) + \int_{\mathbb{B}\setminus E_r(x)} \mathcal{R}_{s+t}(x,y)\big(f(y)-f(x)\big)d\nu_s(y) \\ &+ \int_{E_r(x)} \mathcal{R}_{s+t}(x,y)\big(f(y)-f(x)\big)d\nu_s(y) =: h_1(x)+h_2(x)+h_3(x). \end{aligned}$$

We have $(1-|x|^2)^t h_1(x) \in C_0(\mathbb{B})$ by (5.1). Next, applying (4.1), Lemma 3.4a and then Lemma 5.2(ii) show that for some constant $C = C(n,s,t)$

$$|h_2(x)| \leq C p_{\mathcal{B}}(f) \int_{\mathbb{B}\setminus E_r(x)} \frac{\beta(x,y)\,(1-|y|^2)^s}{[x,y]^{n+s+t}}\, d\nu(y) < C p_{\mathcal{B}}(f) \frac{\varepsilon}{(1-|x|^2)^t}.$$

Thus, $(1-|x|^2)^t |h_2(x)| \lesssim \varepsilon$.

To estimate h_3, let $y \in E_r(x)$. By the mean-value theorem of advanced calculus

$$|f(y)-f(x)| \leq |y-x| \sup_{z\in E_r(x)} |\nabla f(z)|$$

and by (2.6), $|y-x| = \rho(x,y)[x,y] < r[x,y] \lesssim r(1-|x|^2)$, since by part (ii) of Lemma 2.1, $[x,y] \sim [x,x] = 1-|x|^2$. Therefore

$$|f(y)-f(x)| \lesssim (1-|x|^2) \sup_{z\in E_r(x)} |\nabla f(z)| \lesssim \sup_{z\in E_r(x)} (1-|z|^2)|\nabla f(z)|,$$

where the last inequality follows from Lemma 2.1(i). Hence, by Lemma 2.2

$$\begin{aligned}(1-|x|^2)^t |h_3(x)| &\lesssim \sup_{z\in E_r(x)} (1-|z|^2)|\nabla f(z)|\, (1-|x|^2)^t \int_{\mathbb{B}} \frac{d\nu_s(y)}{[x,y]^{n+s+t}} \\ &\lesssim \sup_{z\in E_r(x)} (1-|z|^2)|\nabla f(z)|.\end{aligned}$$

By (2.7), for $z \in E_r(x)$, we have $|z| \geq \dfrac{\big||x|-r\big|}{1-r|x|}$ and the right-hand side tends to 1 as $|x| \to 1^-$. Since $f \in \mathcal{B}_0$, this shows that $\lim_{|x|\to 1^-} (1-|x|^2)^t |h_3(x)| = 0$. Combining these, we conclude that $(1-|x|^2)^t D_s^t f(x) \in C_0(\mathbb{B})$.

Conversely, if $(1-|x|^2)^t D_s^t f(x) \in C_0(\mathbb{B})$, then $f \in \mathcal{B}_0$ by Lemma 5.1 and (5.2).

For emphasis, we write Eq. (5.2) as a separate lemma.

Lemma 5.5 *For all* $s > -1$, $t > 0$ *and* $f \in \mathcal{B}$, $P_s[(1-|x|^2)^t D_s^t f(x)] = f$.

Proof of the onto part of Theorem 1.1 is now immediate.

Proof of Theorem 1.1 Pick some $t > 0$. If $f \in \mathcal{B}$ (resp. $\mathcal{B}_0$), then the function $\phi(x) = (1-|x|^2)^t D_\alpha^t f(x)$ is in $L^\infty(\mathbb{B})$ (resp. $C_0(\mathbb{B})$) and $P_\alpha \phi = f$.

The next corollary follows from Theorem 1.1 and Remark 4.4. It is the $\mathcal{H}$-harmonic counterpart of the fact that (Euclidean) harmonic polynomials are dense in the harmonic little Bloch space.

Corollary 5.6 $\operatorname{span}\big\{S_m(|x|)q_m(x) \mid q_m \in H_m(\mathbb{R}^n), m = 0, 1, \ldots\big\}$ *is dense in* $\mathcal{B}_0$.

6 Duality

We first write the pairing in (1.3) as an absolutely convergent integral.

Lemma 6.1 *Let $\alpha > -1$, $f \in \mathcal{B}^1_\alpha$ and $g \in \mathcal{B}$. For every $t > 0$*

$$\langle f, g\rangle_\alpha = \lim_{r\to 1^-} \int_{r\mathbb{B}} f(x)g(x)\, d\nu_\alpha(x) = \int_{\mathbb{B}} f(x)(1-|x|^2)^t D^t_\alpha g(x)\, d\nu_\alpha(x).$$

Proof By [22, Lemma 7.1], the reproducing property in (1.1) holds also in $\mathcal{B}^1_\alpha$. Using this and the fact that $f \in \mathcal{B}^1_\alpha \subset \mathcal{B}^1_{\alpha+t}$, we see that

$$\lim_{r\to 1^-} \int_{r\mathbb{B}} f(x)g(x)\, d\nu_\alpha(x) = \lim_{r\to 1^-} \int_{r\mathbb{B}} \int_{\mathbb{B}} \mathcal{R}_{\alpha+t}(x, y) f(y)\, d\nu_{\alpha+t}(y) g(x)\, d\nu_\alpha(x),$$

which, after changing the order of the integrals (possible since for $|x| \le r$, the functions $\mathcal{R}_{\alpha+t}(x, y)$ and $g(x)$ are bounded), equals

$$\lim_{r\to 1^-} \int_{\mathbb{B}} f(y)(1-|y|^2)^t \int_{r\mathbb{B}} \mathcal{R}_{\alpha+t}(x, y) g(x)\, d\nu_\alpha(x)\, d\nu_\alpha(y).$$

The term $(1-|y|^2)^t \left|\int_{r\mathbb{B}} \mathcal{R}_{\alpha+t}(x, y) g(x) d\nu_\alpha(x)\right|$ is bounded by a constant independent of r and y by Lemma 5.3 and the fact that $\mathcal{R}_{\alpha+t}$ is symmetric. Thus, by the dominated convergence theorem, we can push the limit into the integral and obtain

$$\lim_{r\to 1^-} \int_{r\mathbb{B}} f(x)g(x) d\nu_\alpha(x) = \int_{\mathbb{B}} f(y)(1-|y|^2)^t \int_{\mathbb{B}} \mathcal{R}_{\alpha+t}(x, y) g(x) d\nu_\alpha(x) d\nu_\alpha(y).$$

This gives the desired result, since the inner integral is $D^t_\alpha g(y)$ by Lemma 3.3(ii).

Proof of Theorem 1.2 For $g \in \mathcal{B}$, define $\Lambda_g : \mathcal{B}^1_\alpha \to \mathbb{C}$ by $\Lambda_g(f) = \langle f, g\rangle_\alpha$. Pick some $t > 0$. By Lemma 6.1 and Proposition 5.4

$$|\langle f, g\rangle_\alpha| \le \|f\|_{\mathcal{B}^1_\alpha} \|(1-|x|^2)^t D^t_\alpha g\|_{L^\infty} \lesssim \|f\|_{\mathcal{B}^1_\alpha} \|g\|_{\mathcal{B}}, \tag{6.1}$$

and so, $\Lambda_g \in (\mathcal{B}^1_\alpha)^*$ and $\|\Lambda_g\| \lesssim \|g\|_{\mathcal{B}}$.

Conversely, let $\Lambda \in (\mathcal{B}^1_\alpha)^*$. Pick $\gamma > \alpha$. Then, by Lemma 3.5, $\Lambda \circ P_\gamma \in (L^1_\alpha(\mathbb{B}))^*$, and by the Riesz representation theorem, there exists $\psi \in L^\infty(\mathbb{B})$ with $\|\psi\|_{L^\infty} = \|\Lambda \circ P_\gamma\|$, such that for all $\phi \in L^1_\alpha(\mathbb{B})$

$$(\Lambda \circ P_\gamma)\phi = \int_{\mathbb{B}} \phi(y)\psi(y)\, d\nu_\alpha(y).$$

Let $f \in \mathcal{B}^1_\alpha \subset L^1_\alpha(\mathbb{B})$. Then, $P_\gamma f = f$, and so, $\Lambda(f) = (\Lambda \circ P_\gamma) f$ is given by

$$\begin{aligned}\Lambda(f) &= \int_{\mathbb{B}} P_\gamma f(y)\psi(y) d\nu_\alpha(y) = \int_{\mathbb{B}}\int_{\mathbb{B}} \mathcal{R}_\gamma(y,x) f(x) d\nu_\gamma(x)\psi(y) d\nu_\alpha(y) \\ &= \int_{\mathbb{B}} f(x) \int_{\mathbb{B}} \mathcal{R}_\gamma(y,x)\psi(y) d\nu_\alpha(y) d\nu_\gamma(x), \end{aligned} \tag{6.2}$$

where we can change the order of the integrals because $\psi \in L^\infty(\mathbb{B})$ and by Lemma 3.4(a) and Lemma 2.2, $\int_{\mathbb{B}} |\mathcal{R}_\gamma(y,x)|\, d\nu_\alpha(y) \lesssim (1-|x|^2)^{-(\gamma-\alpha)}$. Set

$$g(x) := P_\alpha \psi(x) = \int_{\mathbb{B}} \mathcal{R}_\alpha(x,y)\psi(y)\, d\nu_\alpha(y) \qquad (x \in \mathbb{B}).$$

By Theorem 1.1, g is in $\mathcal{B}$ and $\|g\|_{\mathcal{B}} \lesssim \|\psi\|_{L^\infty} = \|\Lambda \circ P_\gamma\| \lesssim \|\Lambda\|$. Further, by Lemma 3.3(i)

$$D^{\gamma-\alpha}_\alpha g(x) = D^{\gamma-\alpha}_\alpha \int_{\mathbb{B}} \mathcal{R}_\alpha(x,y)\psi(y)\, d\nu_\alpha(y) = \int_{\mathbb{B}} \mathcal{R}_\gamma(x,y)\psi(y)\, d\nu_\alpha(y).$$

Hence, by (6.2) and the symmetry of $\mathcal{R}_\gamma$

$$\Lambda(f) = \int_{\mathbb{B}} f(x)(1-|x|^2)^{\gamma-\alpha} D^{\gamma-\alpha}_\alpha g(x)\, d\nu_\alpha(x),$$

which equals $\langle f, g\rangle_\alpha = \Lambda_g(f)$ by Lemma 6.1. Thus, $\Lambda = \Lambda_g$.

To see the uniqueness of g, note that for $x_0 \in \mathbb{B}$, $\mathcal{R}_\alpha(x_0,\cdot)$ is bounded on $\mathbb{B}$, and so, belongs to $\mathcal{B}^1_\alpha$. In addition, if $g \in \mathcal{B}$, then $\mathcal{R}_\alpha(x_0,\cdot) g$ is in $L^1_\alpha(\mathbb{B})$ by (4.2). Thus

$$\begin{aligned}\Lambda_g(\mathcal{R}_\alpha(x_0,\cdot)) = \langle \mathcal{R}_\alpha(x_0,\cdot), g\rangle_\alpha &= \lim_{r\to 1^-} \int_{r\mathbb{B}} \mathcal{R}_\alpha(x_0,x) g(x) d\nu_\alpha(x) \\ &= \int_{\mathbb{B}} \mathcal{R}_\alpha(x_0,x) g(x) d\nu_\alpha(x) = g(x_0),\end{aligned} \tag{6.3}$$

by the reproducing property. Hence, if $g_1 \neq g_2$, then $\Lambda_{g_1} \neq \Lambda_{g_2}$. We conclude that to each $\Lambda \in (\mathcal{B}^1_\alpha)^*$, there corresponds a unique $g \in \mathcal{B}$ with $\|g\|_{\mathcal{B}} \sim \|\Lambda\|$ and $\Lambda = \Lambda_g$.

We next show that $\mathcal{B}^*_0$ can be identified with $\mathcal{B}^1_\alpha$ for any $\alpha > -1$. For $f \in \mathcal{B}^1_\alpha$, define $\Lambda_f : \mathcal{B}_0 \to \mathbb{C}$ by $\Lambda_f(g) = \langle f, g\rangle_\alpha$. By (6.1), $\Lambda_f \in \mathcal{B}^*_0$ and $\|\Lambda_f\| \lesssim \|f\|_{\mathcal{B}^1_\alpha}$. Suppose now that $\Lambda \in \mathcal{B}^*_0$. By Theorem 1.1, $\Lambda \circ P_\alpha \in C_0(\mathbb{B})^*$, and by the Riesz representation theorem, there exists a complex Borel measure μ on $\mathbb{B}$ with $|\mu|(\mathbb{B}) = \|\Lambda \circ P_\alpha\|$, such that for all $\phi \in C_0(\mathbb{B})$

$$(\Lambda \circ P_\alpha)\phi = \int_{\mathbb{B}} \phi(y)\, d\mu(y).$$

Pick some $t > 0$. For $g \in \mathcal{B}_0$, let $\phi(x) = (1-|x|^2)^t D^t_\alpha g(x)$. Then, $\phi \in C_0(\mathbb{B})$ with $\|\phi\|_{L^\infty} \sim \|g\|_{\mathcal{B}}$ and $P_\alpha \phi = g$ by Proposition 5.4 and Lemma 5.5. Thus

$$\Lambda(g) = (\Lambda \circ P_\alpha)\phi = \int_{\mathbb{B}} \phi(y)\, d\mu(y) = \int_{\mathbb{B}} (1-|y|^2)^t D^t_\alpha g(y)\, d\mu(y).$$

Clearly, $D^t_\alpha g \in L^1_{\alpha+t}$. Therefore, $D^t_\alpha g = P_{\alpha+t}(D^t_\alpha g)$ by the reproducing property. Inserting this into the above equation and using the symmetry of $\mathcal{R}_{\alpha+t}$ show

$$\begin{aligned}\Lambda(g) &= \int_{\mathbb{B}} (1-|y|^2)^t \int_{\mathbb{B}} \mathcal{R}_{\alpha+t}(y,x) D^t_\alpha g(x)\, d\nu_{\alpha+t}(x)\, d\mu(y) \\ &= \int_{\mathbb{B}}\int_{\mathbb{B}} \mathcal{R}_{\alpha+t}(x,y)(1-|y|^2)^t\, d\mu(y)(1-|x|^2)^t D^t_\alpha g(x)\, d\nu_\alpha(x),\end{aligned} \tag{6.4}$$

where we can change the order of the integrals, since $(1-|x|^2)^t D^t_\alpha g(x)$ is bounded, and by Lemma 3.4(a) and Lemma 2.2, $\int_{\mathbb{B}} |\mathcal{R}_{\alpha+t}(y,x)|\, d\nu_\alpha(x) \lesssim (1-|y|^2)^{-t}$. Set

$$f(x) := \int_{\mathbb{B}} \mathcal{R}_{\alpha+t}(x,y)(1-|y|^2)^t\, d\mu(y) \qquad (x \in \mathbb{B}).$$

Then, $f \in \mathcal{H}(\mathbb{B})$ and by Fubini's theorem and the estimate in the previous line

$$\|f\|_{L^1_\alpha} \le \int_{\mathbb{B}} (1-|y|^2)^t \int_{\mathbb{B}} |\mathcal{R}_{\alpha+t}(x,y)|\, d\nu_\alpha(x)\, d|\mu|(y) \lesssim |\mu|(\mathbb{B}).$$

Thus, $f \in \mathcal{B}^1_\alpha$ with $\|f\|_{\mathcal{B}^1_\alpha} \lesssim |\mu|(\mathbb{B}) = \|\Lambda \circ P_\alpha\| \lesssim \|\Lambda\|$, and it follows from (6.4) and Lemma 6.1 that $\Lambda = \Lambda_f$. Uniqueness of f can be verified in the same way as done in the previous part.

We finish this section by verifying that there exists an unbounded $\mathcal{H}$-harmonic Bloch function.

Lemma 6.2 *There exists an unbounded function in $\mathcal{B}$.*

Proof Let $e_1 = (1, 0, \ldots, 0) \in \mathbb{S}$ and $\phi(x) = (1-|x|^2)^{n-1}\mathbb{P}_h(x, e_1)$, where $\mathbb{P}_h$ is the hyperbolic Poisson kernel in (3.8). Then, $\phi \in L^\infty(\mathbb{B})$ and the Bergman projection

$$f(x) := P_0\phi(x) = \int_{\mathbb{B}} \mathcal{R}_0(x,y)\mathbb{P}_h(y,e_1)(1-|y|^2)^{n-1}\, d\nu(y)$$

is in $\mathcal{B}$ by Theorem 1.1. To see that f is unbounded, we find its series expansion. Note that by the integral representation of D^t_s in Lemma 3.3(ii) (with $s = n-1$ and $t = -(n-1)$), we have $f(x) = D^{-(n-1)}_{n-1}\mathbb{P}_h(x, e_1)$. Therefore, by the series expansion of $\mathbb{P}_h$ in (3.8)

$$f(x) = D^{-(n-1)}_{n-1}\mathbb{P}_h(x,e_1) = \sum_{m=0}^{\infty} \frac{c_m(0)}{c_m(n-1)} S_m(|x|) Z_m(x, e_1).$$

Observe that when $x = re_1$, $0 < r < 1$, all the terms in the above series are positive. We have $Z_m(re_1, e_1) = r^m Z_m(e_1, e_1) \sim r^m m^{n-2}$ $(m \geq 1)$, $S_m(r) \geq 1$ by (3.7), and $c_m(0)/c_m(n-1) \sim m^{-(n-1)}$ by (3.11). Thus

$$f(re_1) \gtrsim 1 + \sum_{m=1}^{\infty} \frac{r^m}{m},$$

and the right-hand side tends to ∞ as $r \to 1^-$.

7 Atomic decomposition

Throughout the section, we employ Proposition 5.4 and use any one of the equivalent norms $\|(1-|x|^2)^t D_\alpha^t f(x)\|_{L^\infty}$ $(\alpha > -1, t > 0)$ for the Bloch space.

Lemma 7.1 *For every $\alpha > -1$ and $a \in \mathbb{B}$, the kernel $\mathcal{R}_\alpha(\cdot, a)$ is in $\mathcal{B}_0$. In addition, there exists $C = C(n, \alpha) > 0$, such that for all $a \in \mathbb{B}$*

$$\frac{1}{C\,(1-|a|^2)^{\alpha+n}} \leq \|\mathcal{R}_\alpha(\cdot, a)\|_{\mathcal{B}} \leq \frac{C}{(1-|a|^2)^{\alpha+n}}. \tag{7.1}$$

Proof Pick $t > 0$. We have $D_\alpha^t \mathcal{R}_\alpha(x, a) = \mathcal{R}_{\alpha+t}(x, a)$ by (3.14), and for fixed $a \in \mathbb{B}$, $\mathcal{R}_{\alpha+t}(x, a)$ is bounded by Lemma 3.4(a) and the inequality $[x, a] \geq 1 - |a|$ by (2.1). Thus, $(1-|x|^2)^t D_\alpha^t \mathcal{R}_\alpha(x, a)$ is in $C_0(\mathbb{B})$, and $\mathcal{R}_\alpha(x, a) \in \mathcal{B}_0$ by Proposition 5.4. Further

$$(1-|x|^2)^t \left|\mathcal{R}_{\alpha+t}(x, a)\right| \lesssim \frac{(1-|x|^2)^t}{[x, a]^{\alpha+t+n}} \lesssim \frac{1}{(1-|a|^2)^{\alpha+n}},$$

again by $[x, a] \geq 1 - |x|$ and $[x, a] \geq 1 - |a|$, which gives the second inequality in (7.1). The first inequality follows from [22, Lemma 6.1] which shows that when $x = a$, $\mathcal{R}_{\alpha+t}(a, a) \sim 1/(1-|a|^2)^{\alpha+t+n}$.

Lemma 7.2 *Suppose $\alpha > -1$ and $\{a_m\}$ is r-separated for some $0 < r < 1$. Then, the operator $T = T_{\{a_m\},\alpha} : \ell^\infty \to \mathcal{B}$ mapping $\lambda = \{\lambda_m\}$ to*

$$T\lambda(x) = \sum_{m=1}^{\infty} \lambda_m \frac{\mathcal{R}_\alpha(x, a_m)}{\|\mathcal{R}_\alpha(\cdot, a_m)\|_{\mathcal{B}}} \qquad (x \in \mathbb{B}), \tag{7.2}$$

is bounded. The above series converges absolutely and uniformly on compact subsets of $\mathbb{B}$. In addition, if $\lambda \in c_0$, then $T\lambda \in \mathcal{B}_0$.

Proof We first verify that $\sum_{m=1}^{\infty}(1-|a_m|^2)^{\alpha+n} < \infty$. To see this, note that the balls $E_{r/2}(a_m)$ are disjoint, and for fixed r, $\nu(E_{r/2}(a_m)) \sim (1-|a_m|^2)^n$ by (2.7). Also, for

$y \in E_{r/2}(a_m)$, we have $(1-|y|^2) \sim (1-|a_m|^2)$ by Lemma 2.1(i). Thus

$$\sum_{m=1}^{\infty}(1-|a_m|^2)^{\alpha+n} \sim \sum_{m=1}^{\infty}\int_{E_{r/2}(a_m)}(1-|y|^2)^{\alpha}\,d\nu(y) \leq \int_{\mathbb{B}}(1-|y|^2)^{\alpha}\,d\nu(y)$$

which is finite. To see that the series converges absolutely and uniformly on compact subsets of $\mathbb{B}$, suppose $|x| \leq R < 1$. Then $|\mathcal{R}_\alpha(x, a_m)| \leq C$ for all m by Lemma 3.4 and (2.1). Using also Lemma 7.1, we deduce

$$\sum_{m=1}^{\infty}|\lambda_m|\frac{|\mathcal{R}_\alpha(x, a_m)|}{\|\mathcal{R}_\alpha(\cdot, a_m)\|_{\mathcal{B}}} \lesssim \|\lambda\|_{\ell^\infty}\sum_{m=1}^{\infty}(1-|a_m|^2)^{\alpha+n} < \infty.$$

Next, pick $t > 0$ and apply D_α^t to $T\lambda$. By the continuity in Lemma 3.2, we can push D_α^t past the summation in (7.2). Applying (3.14), Lemma 3.4(a) and Lemma 7.1 then show

$$\left|D_\alpha^t(T\lambda)(x)\right| \lesssim \|\lambda\|_{\ell^\infty}\sum_{m=1}^{\infty}\frac{(1-|a_m|^2)^{\alpha+n}}{[x, a_m]^{\alpha+t+n}}.$$

As is done above, $(1-|a_m|^2)^{\alpha+n} \sim \nu_\alpha(E_{r/2}(a_m))$, and $[x, y] \sim [x, a_m]$ for $y \in E_{r/2}(a_m)$ by Lemma 2.1. Therefore, using also Lemma 2.2, we obtain

$$\begin{aligned}\left|D_\alpha^t(T\lambda)(x)\right| &\lesssim \|\lambda\|_{\ell^\infty}\sum_{m=1}^{\infty}\int_{E_{r/2}(a_m)}\frac{d\nu_\alpha(y)}{[x, y]^{\alpha+t+n}} \leq \|\lambda\|_{\ell^\infty}\int_{\mathbb{B}}\frac{d\nu_\alpha(y)}{[x, y]^{\alpha+t+n}} \\ &\lesssim \frac{\|\lambda\|_{\ell^\infty}}{(1-|x|^2)^t},\end{aligned}$$

Hence, $T\lambda \in \mathcal{B}$ and $\|T\lambda\|_{\mathcal{B}} \lesssim \|\lambda\|_{\ell^\infty}$.

Finally, suppose $\lambda \in c_0$. For $\varepsilon > 0$, let M be such that $\sup_{m\geq M}|\lambda_m| < \varepsilon$. Then

$$T\lambda(x) = \sum_{m=1}^{M-1}\lambda_m\frac{\mathcal{R}_\alpha(x, a_m)}{\|\mathcal{R}_\alpha(\cdot, a_m)\|_{\mathcal{B}}} + \sum_{m=M}^{\infty}\lambda_m\frac{\mathcal{R}_\alpha(x, a_m)}{\|\mathcal{R}_\alpha(\cdot, a_m)\|_{\mathcal{B}}} =: h_1(x) + h_2(x).$$

By Lemma 7.1, h_1 is in $\mathcal{B}_0$; and $\|h_2\|_{\mathcal{B}} \lesssim \sup_{m\geq M}|\lambda_m|$ by the previous paragraph. Thus, $\limsup_{|x|\to 1^-}(1-|x|^2)^t|D_\alpha^t(T\lambda)(x)| \lesssim \varepsilon$ and $T\lambda$ is in $\mathcal{B}_0$.

Following [4], we associate with an r-lattice $\{a_m\}$ the following partition $\{E_m\}$ of $\mathbb{B}$. Let $E_1 = E_r(a_1)\setminus\bigcup_{m=2}^{\infty}E_{r/2}(a_m)$ and for $m = 2, 3, \ldots,$ inductively define

$$E_m = E_r(a_m)\setminus\left(\bigcup_{k=1}^{m-1}E_k \cup \bigcup_{k=m+1}^{\infty}E_{r/2}(a_k)\right).$$

The following properties hold: (i) $E_{r/2}(a_m) \subset E_m \subset E_r(a_m)$, (ii) the sets E_m are disjoint, and (iii) $\bigcup_{m=1}^{\infty} E_m = \mathbb{B}$.

Lemma 7.3 *Suppose $\alpha > -1$, $t > 0$, and $\{a_m\}$ is an r-lattice. If $\{E_m\}$ is the associated sequence defined above, then the operator $U = U_{\{a_m\},\alpha,t} : \mathcal{B} \to \ell^\infty$ defined by*

$$Uf = \left\{ D_\alpha^t f(a_m) \, \|\mathcal{R}_\alpha(\cdot, a_m)\|_{\mathcal{B}} \, \nu_{\alpha+t}(E_m) \right\}_{m=1}^{\infty}$$

is bounded. In addition, if $f \in \mathcal{B}_0$, then $Uf \in c_0$.

Proof Because $E_{r/2}(a_m) \subset E_m \subset E_r(a_m)$ and r is fixed, by Lemma 2.1 and (2.7), $\nu_{\alpha+t}(E_m) \sim (1 - |a_m|^2)^{\alpha+t+n}$. Combining this with Lemma 7.1 shows

$$|D_\alpha^t f(a_m)| \, \|\mathcal{R}_\alpha(\cdot, a_m)\|_{\mathcal{B}} \, \nu_{\alpha+t}(E_m) \sim (1 - |a_m|^2)^t |D_\alpha^t f(a_m)|.$$

Thus, $\|Uf\|_{\ell^\infty} \lesssim \|f\|_{\mathcal{B}}$. If $f \in \mathcal{B}_0$, then $(1 - |x|^2)^t D_\alpha^t f(x) \in C_0(\mathbb{B})$, and so, Uf is in c_0, since $\lim_{m\to\infty} |a_m| = 1$.

Proof of Theorem 1.3 Pick some $t > 0$ and define the operators $U : \mathcal{B} \to \ell^\infty$ and $T : \ell^\infty \to \mathcal{B}$ as above. We show that there exists a constant $C = C(n, \alpha, t)$, such that $\|I - TU\|_{\mathcal{B}\to\mathcal{B}} \le Cr$, where I is the identity operator. This implies that $\|I - TU\|_{\mathcal{B}\to\mathcal{B}} < 1$ when r is sufficiently small, TU is invertible and hence, T is onto. In the little Bloch case replacing $\mathcal{B}$ with $\mathcal{B}_0$ and ℓ^∞ with c_0, we obtain $T : c_0 \to \mathcal{B}_0$ is onto.

Let $f \in \mathcal{B}$. In the calculations below, we suppress constants that depend only on n, α, t, and make sure that they do not depend on r or f. Note that

$$TUf(x) = \sum_{m=1}^{\infty} D_\alpha^t f(a_m) \mathcal{R}_\alpha(x, a_m) \, \nu_{\alpha+t}(E_m)$$

and the series converges absolutely and uniformly on compact subsets of $\mathbb{B}$. By continuity, we can push D_α^t past the summation, and using (3.14), we can obtain

$$\begin{aligned} D_\alpha^t (TUf)(x) &= \sum_{m=1}^{\infty} D_\alpha^t f(a_m) \mathcal{R}_{\alpha+t}(x, a_m) \, \nu_{\alpha+t}(E_m) \\ &= \sum_{m=1}^{\infty} \int_{E_m} D_\alpha^t f(a_m) \mathcal{R}_{\alpha+t}(x, a_m) \, \nu_{\alpha+t}(y). \end{aligned}$$

Further, since $D_\alpha^t f \in L^1_{\alpha+t}(\mathbb{B})$, by the reproducing property

$$\begin{aligned} D_\alpha^t f(x) &= \int_{\mathbb{B}} \mathcal{R}_{\alpha+t}(x, y) D_\alpha^t f(y) d\nu_{\alpha+t}(y) \\ &= \sum_{m=1}^{\infty} \int_{E_m} \mathcal{R}_{\alpha+t}(x, y) D_\alpha^t f(y) d\nu_{\alpha+t}(y). \end{aligned}$$

Thus

$$\begin{aligned} D^t_\alpha(I-TU)f(x) &= \sum_{m=1}^{\infty}\int_{E_m}\big(\mathcal{R}_{\alpha+t}(x,y)-\mathcal{R}_{\alpha+t}(x,a_m)\big)D^t_\alpha f(y)\,d\nu_{\alpha+t}(y)\\ &\quad+\sum_{m=1}^{\infty}\int_{E_m}\big(D^t_\alpha f(y)-D^t_\alpha f(a_m)\big)\mathcal{R}_{\alpha+t}(x,a_m)\,d\nu_{\alpha+t}(y)\\ &=: h_1(x)+h_2(x). \end{aligned}$$

We first estimate h_1. Pick $y\in E_m$. Since $E_m\subset E_r(a_m)$, a convex set, by the mean-value inequality

$$\big|\mathcal{R}_{\alpha+t}(x,y)-\mathcal{R}_{\alpha+t}(x,a_m)\big|\le|y-a_m|\sup_{z\in E_r(a_m)}|\nabla_z\mathcal{R}_{\alpha+t}(x,z)|.$$

By (2.6), $|y-a_m|=\rho(y,a_m)[y,a_m]$ and since $\rho(y,a_m)<r<1/2$, we have $[y,a_m]\sim[y,y]=1-|y|^2$ by Lemma 2.1(ii), with the suppressed constants not depending on r. Thus, $|y-a_m|\lesssim r(1-|y|^2)$. Similarly, for all $x\in\mathbb{B}$ and $z\in E_r(a_m)$, we have $[x,z]\sim[x,a_m]\sim[x,y]$ by Lemma 2.1(ii). Hence, by Lemma 3.4(b)

$$\big|\nabla_z\mathcal{R}_{\alpha+t}(x,z)\big|\lesssim\frac{1}{[x,z]^{\alpha+t+n+1}}\sim\frac{1}{[x,y]^{\alpha+t+n+1}}.$$

We conclude that for all $x\in\mathbb{B}$ and $y\in E_m$

$$\big|\mathcal{R}_{\alpha+t}(x,y)-\mathcal{R}_{\alpha+t}(x,a_m)\big|\lesssim r\frac{1-|y|^2}{[x,y]^{\alpha+t+n+1}}\lesssim\frac{r}{[x,y]^{\alpha+t+n}},\tag{7.3}$$

where in the last inequality, we use $[x,y]\ge1-|y|$ by (2.1). Thus

$$\begin{aligned} |h_1(x)| &\lesssim r\sum_{m=1}^{\infty}\int_{E_m}\frac{(1-|y|^2)^{\alpha+t}|D^t_\alpha f(y)|}{[x,y]^{\alpha+t+n}}d\nu(y)\lesssim r\|f\|_{\mathcal{B}}\int_{\mathbb{B}}\frac{d\nu_\alpha(y)}{[x,y]^{\alpha+t+n}}\\ &\lesssim r\|f\|_{\mathcal{B}}\frac{1}{(1-|x|^2)^t}, \end{aligned}$$

where the last inequality follows from Lemma 2.2.

We next estimate h_2. Pick $y\in E_m$. By the reproducing property

$$D^t_\alpha f(y)-D^t_\alpha f(a_m)=\int_{\mathbb{B}}\big(\mathcal{R}_{\alpha+t}(y,z)-\mathcal{R}_{\alpha+t}(a_m,z)\big)D^t_\alpha f(z)\,d\nu_{\alpha+t}(z).$$

Therefore, by (7.3) with the symmetry of $\mathcal{R}_{\alpha+t}$, and Lemma 2.2

$$|D^t_\alpha f(y)-D^t_\alpha f(a_m)|\lesssim r\int_{\mathbb{B}}\frac{(1-|z|^2)^{\alpha+t}|D^t_\alpha f(z)|}{[y,z]^{\alpha+t+n}}\,d\nu(z)\lesssim r\|f\|_{\mathcal{B}}\frac{1}{(1-|y|^2)^t}.$$

Hence, using also Lemma 3.4, the fact that $[x, a_m] \sim [x, y]$ for all $x \in \mathbb{B}$ and $y \in E_m$, and finally Lemma 2.2, we deduce

$$|h_2(x)| \lesssim r\|f\|_{\mathcal{B}} \sum_{m=1}^{\infty} \int_{E_m} \frac{(1-|y|^2)^{\alpha}}{[x, a_m]^{\alpha+t+n}} d\nu(y) \sim r\|f\|_{\mathcal{B}} \sum_{m=1}^{\infty} \int_{E_m} \frac{d\nu_{\alpha}(y)}{[x, y]^{\alpha+t+n}}$$
$$= r\|f\|_{\mathcal{B}} \int_{\mathbb{B}} \frac{d\nu_{\alpha}(y)}{[x, y]^{\alpha+t+n}} \lesssim r\|f\|_{\mathcal{B}} \frac{1}{(1-|x|^2)^t}.$$

Thus, $\|(1-|x|^2)^t D_{\alpha}^t (I - TU) f(x)\|_{L^\infty} \leq Cr\|f\|_{\mathcal{B}}$. The proof is completed.

To see that the representation (1.5) can be used alternatively to (1.4), the only change needed is to replace $\|\mathcal{R}_{\alpha}(\cdot, a_m)\|_{\mathcal{B}}$ with $(1-|a_m|^2)^{-(\alpha+n)}$ in the definitions of T and U. The proofs of the Lemmas 7.2 and 7.3 become simpler; and TU and the proof of Theorem 1.3 remain the same.

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References

1. Axler, S., Bourdon, P., Ramey, W.: Harmonic Function Theory, Second Edition. Graduate Texts in Math., Vol. 137, Springer, New York (2001)
2. Choe, B.R., Koo, H., Lee, Y.J.: Positive Schatten class Toeplitz operators on the ball. Studia Math. **189**, 65–90 (2008)
3. Choe, B.R., Lee, Y.J.: Note on atomic decompositions of harmonic Bergman functions. In Complex Analysis and its Applications, OCAMI Stud., Vol. 2, Osaka Munic. Univ. Press, Osaka, 11-24 (2007)
4. Coifman, R.R., Rochberg, R.: Representation theorems for holomorphic and harmonic functions in L^p. Astérisque **77**, 12–66 (1980)
5. Doğan, Ö.F., Üreyen, A.E.: Weighted harmonic Bloch spaces on the ball. Complex Anal. Oper. Theory **12**, 1143–1177 (2018)
6. Gergün, S., Kaptanoğlu, H.T., Üreyen, A.E.: Harmonic Besov spaces on the ball. Int. J. Math. **27** no.9 1650070, 59 pp. (2016)
7. Grellier, S., Jaming, P.: Harmonic functions on the real hyperbolic ball II. Hardy-Sobolev and Lipschitz spaces. Math. Nachr. **268**, 50–73 (2004)
8. Jaming, P.: Trois problémes d'analyse harmonique. PhD thesis, Université d'Orléans (1998)
9. Jaming, P.: Harmonic functions on the real hyperbolic ball I. Boundary values and atomic decomposition of Hardy spaces. Colloq. Math. **80**, 63–82 (1999)
10. Jevtić, M., Pavlović, M.: Harmonic Bergman functions on the unit ball in $\mathbb{R}^n$. Acta Math. Hungar. **85**, 81–96 (1999)
11. Jevtić, M., Pavlović, M.: Series expansion and reproducing kernels for hyperharmonic functions. J. Math. Anal. Appl. **264**, 673–681 (2001)

12. Ligocka, E.: On the reproducing kernel for harmonic functions and the space of Bloch harmonic functions on the unit ball in $\mathbb{R}^n$. Studia Math. **87**, 23–32 (1987)
13. Liu, C.W., Shi, J.H.: Invariant mean-value property and $\mathcal{M}$-harmonicity in the unit ball of $\mathbb{R}^n$. Acta Math. Sin. **19**, 187–200 (2003)
14. Minemura, K.: Harmonic functions on real hyperbolic spaces. Hiroshima Math. J. **3**, 121–151 (1973)
15. Ren, G., Kähler, U.: Pseudohyperbolic metric and uniformly discrete sequences in the real unit ball. Acta Math. Sci. **34B**(3), 629–638 (2014)
16. Ren, G., Kähler, U., Shi, J., Liu, C.: Hardy-Littlewood inequalities for fractional derivatives of invariant harmonic functions. Complex Anal. Oper. Theory **6**, 373–396 (2012)
17. Souza Peñalosa, M.P.: Espacios de Bergman de funciones armónicas en la bola hiperbólica. Posgrado en Ciencias Matemáticas, Universidad Nacional Autónoma de México, Tesis de Doctorado (2005)
18. Stoll, M.: Harmonic and Subharmonic Function Theory on the Hyperbolic Ball. London Math. Soc. Lect. Note Series, Vol. 431, Cambridge University Press, Cambridge (2016)
19. Stoll, M.: Reproducing kernels and radial eigenfunctions for the hyperbolic Laplacian. preprint, https://www.researchgate.net/publication/304998931
20. Stoll, M.: The reproducing kernel of $\mathcal{H}^2$ and radial eigenfunctions of the hyperbolic Laplacian. Math. Scand. **124**, 81–101 (2019)
21. Stroethoff, K.: Harmonic Bergman spaces. In Holomorphic Spaces, Math. Sci. Res. Inst. Publ., Vol. 33, Cambridge University, Cambridge, pp. 51–63 (1998)
22. Üreyen, A.E.: $\mathcal{H}$-Harmonic Bergman projection on the real hyperbolic ball. J. Math. Anal. Appl. **519**, 126802 (2023)
23. Üreyen, A.E.: Harmonic Bergman spaces on the real hyperbolic ball: Atomic decomposition, interpolation and inclusion relations. Complex Anal. Oper. Theory **18**, 40 (2024)
24. Zhu, K.: Spaces of Holomorphic Functions in the Unit Ball. Graduate Texts in Math., Vol. 226, Springer, New York (2005)