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Regularity results for classes of Hilbert C*-modules with respect to special bounded modular functionals

Michael Frank[1]

Abstract

Considering the deeper reasons of the appearance of a remarkable counterexample by Kaad and Skeide (J Operat Theory 89(2):343–348, 2023) we consider situations in which two Hilbert C*-modules $M \subset N$ with $M^\perp = \{0\}$ over a fixed C*-algebra A of coefficients cannot be separated by a non-trivial bounded A-linear functional $r_0 : N \to A$ vanishing on M. In other words, the uniqueness of extensions of the zero functional from M to N is focussed. We show this uniqueness of extension for any such pairs of Hilbert C*-modules over W*-algebras, over monotone complete C*-algebras and over compact C*-algebras. Moreover, uniqueness of extension takes place also for any one-sided maximal modular ideal of any C*-algebra. Such a non-zero separating bounded A-linear functional r_0 exist for a given pair of full Hilbert C*-modules $M \subseteq N$ over a given C*-algebra A iff there exists a bounded A-linear non-adjointable operator $T_0 : N \to N$, such that the kernel of T_0 is not biorthogonally closed w.r.t. N and contains M. This is a new perspective on properties of bounded modular operators that might appear in Hilbert C*-module theory. By the way, we find a correct proof of Lemma 2.4 of Frank (Int J Math 13:1–19, 2002) in the case of monotone complete and compact C*-algebras, but find it not valid in certain particular cases.

Keywords Hilbert C*-modules · Monotone complete C*-algebras · W*-algebras · Compact C*-algebras · Extensions of zero modular functionals · Kernels of non-selfadjoint module operators · (Left) multiplier algebras

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✉ Michael Frank
michael.frank@htwk-leipzig.de

1   Fakultät Informatik und Medien, Hochschule für Technik, Wirtschaft und Kultur (HTWK) Leipzig, PF 301166, 04251 Leipzig, Germany

1 Introduction

The theory of Hilbert C*-modules exists for about 70 resp. 50 years since the famous works by I. Kaplansky, resp. by W. L. Paschke and by M. A. Rieffel. Nevertheless, a new problem has been discovered which forces to review parts of the theory. Some years ago Shalit and Solel investigated Hilbert subproduct systems of Hilbert product systems, cf. [41, 42]. One core question has been whether there might exist special bounded modular functionals on pairs of Hilbert C*-modules into the C*-algebra of their coefficients: Let $M \subset N$ be two Hilbert A-modules over a given C*-algebra A, such that the orthogonal complement of M w.r.t. N equals $\{0\}$. Does there exist a non-trivial modular extension r_0 of the zero map from M to A to N? This question is the analogue of the categorical separation problem for similar pairs of linear spaces of quite different kinds by (sorts of) bounded linear functionals. We refer to Kaad and Skeide [24] and to Manuilov [33] for initial investigations.

Considering Hilbert spaces and their Hilbert subspaces as a class of examples, there seems to be not any such problem. To indicate one more complicated background of the problem consider maximal one-sided (say, right) norm-closed ideals D of unital C*-algebras A. Both A and $D \subset A$ can be considered as (right) Hilbert A-modules inducing the necessary algebraic structures from the C*-structures of A in the usual way. Intuitively, for C*-algebras A without finite part, $aD = 0$ for some $a \in A$ should force $a = 0$, so the zero functional on D would have only the zero modular functional on A as its continuation, or alternatively in the opposite case, D is even a direct orthogonal summand of A as for matrix algebras or for atomic carrier projections from an occasionally existing finite part of A. However, a proof even for modular maximal right ideals D of A will need a deep dive into C*-theory as will be shown in the last section. Note, that the example of $A = C([0, 1])$ and $D = C_0((0, 1])$ of all continuous functions on the unit interval and of the subset of functions vanishing at zero would yield the Banach algebra of all bounded continuous functions $D' = C_b((0, 1]) \supset A$ on the half-open unit interval $(0, 1]$ as the dual Banach A-module D' of D. Therefore, the problem of the extension of the zero functional on D reaches beyond the hosting selfdual Hilbert A-module A in this case. Moreover, D coincides with its bidual Banach A-module D'' in this case, another special situation. However, turning to arbitrary norm-closed ideals of C*-algebras one has to struggle with the deficiencies of noncommutative topology.

Example 1.1 Let $X = [0, 1]$ be the unit interval of real numbers equipped with the usual metric topology arising from the absolute value of the difference of two numbers. Consider the set J of all bounded Borel functions f on X, such that the set $\{x \in X : f(x) \neq 0\}$ is meager. The set J is a two-sided norm-closed ideal in the monotone σ-complete C*-algebra of all bounded Borel functions on X with its supremum norm. The C*-algebra $A = D(X)$ constructed as the quotient algebra of the latter by J is known as the Dixmier algebra on X. It is a monotone complete C*-algebra (hence, AW*) and, as a commutative AW*-algebra, also injective, in fact the injective envelope $I(C(X))$ of the C*-algebra C(X), cf. [9, 17, 20, 40].

Consider the Hilbert A-module $M = l_2(A)$ of all sequences of elements $a_i \in A$, such that the series $\sum_i |a_i|^2$ converges in norm. This module M is countably generated

over A by its the orthonormal basis $\{e_i = (0, \ldots, 0, 1_A^{(i)}, 0, \ldots)\}$. The A-dual Banach A-module $l_2(A)'$ of the Hilbert A-module $l_2(A)$ can be identified with the set of sequences of elements $a_i \in A$, such that the sequence of partial sums $\{\sum_{i=1}^{k} |a_i|^2\}_k$ is bounded by a sequence-specific constant from above. The norm of each sequence is derived from the respective least upper bound. Note, that these series are well-defined since A is monotone complete, the respective sequence of partial sums $\{\sum_{i=1}^{k} |a_i|^2\}_k$ is norm-bounded, positive and monotone increasing in A, what makes the supremum exist as a positive element of A. Therefore, one can define an A-valued inner product on $l_2(A)'$ setting $\langle \{a_i\}_i, \{a_i\}_i \rangle := \sum_i |a_i|^2$, applying the polarization formula to count the values $\langle x, y \rangle = \frac{1}{4} \sum_{k=0}^{3} \mathbf{i}^k \langle x + \mathbf{i}^k y, x + \mathbf{i}^k y \rangle$ for $x, y \in l_2(A)'$. The Banach A-module $N = l_2(A)'$ turns into a self-dual Hilbert A-module with an isometrically embedded copy of $M = l_2(A)$. Note, that for any element $f = (a_1, \ldots, a_i, \ldots)$ of $l_2(A)'$ one has $f(e_i) = a_i$ for any index i. Also, the orthogonal complement of $l_2(A)$ in $l_2(A)'$ is $\{0\}$. Consequently, every bounded A-linear functional f_0 on $l_2(A)$ that vanishes on $l_2(A) \subset l_2(A)'$ should be represented by the zero sequence in $l_2(A)'$ and, therefore, has an A-valued inner product value 0_A, i.e., f_0 has to be the zero functional.

This shows that the arguments given by Manuilov in Lemma 11 and Corollary 12 of [33] need a thorough revision. The example can be repeated for any compact Hausdorff space X and the injective envelope $I(C(X))$, cf. [17], as well for any Hilbert A-module M and its A-dual Banach A-module $N = M'$ over monotone complete C*-algebras A, cf. [22].

The question of non-trivial modular extensions of the zero functional on the named class of pairs of Hilbert C*-modules is closely related to the problem, whether there are Hilbert C*-modules and bounded C*-linear operators between them, the kernels of which are not biorthogonally complemented, or not. The latter question was investigated by Kaad and Skeide in [24] in 2021 giving a class of examples with this new phenomenon. Also, Manuilov considered this circle of problems in [33]. Therefore, we cannot hope simply to extend our understanding of the Hilbert space situation and of the maximal norm-closed ideal type examples to, for example, all Hilbert C*-modules over monotone complete C*-algebras. We need a separate investigation which is done for Hilbert C*-modules over W*-algebras, over monotone complete C*-algebras and over compact C*-algebras in the present paper. By the way, we find a correct proof of [15, Lemma 2.4] in the monotone complete and compact C*-case, but disproved it for certain C*-algebras, cf. [24]. Generally speaking, for these classes of Hilbert C*-modules the discussed situation is pretty much similar to that one of the class of Hilbert spaces. In the present paper we do not cite any fact from [15] to avoid any influence from the unproven [15, Lemma 2.4] on the present explanations.

2 Some basic definitions and facts

Our basic references for facts on Hilbert C*-modules are [3, 6, 11, 12, 28, 29, 32, 35, 36, 44] and others. To start with, we give some basic definitions and arguments to introduce to the circle of problems treated. The basic structures are Hilbert C*-modules, i.e., (non-unital, in general) C*-algebras A and (right, Banach) A-modules

M, such that there exist an A-valued inner product $\langle ., . \rangle : M \times M \to A$ compatible with both the complex structures on A and on M, such that

1. $\langle z, xa + y \rangle = \langle z, x \rangle a + \langle z, y \rangle$ for any $a \in A$, $x, y, z \in M$,
2. $\langle x, y \rangle = \langle y, x \rangle^*$ for any $x, y \in M$,
3. $\langle x, x \rangle \geq 0$ for any $x \in M$,
4. $\langle x, x \rangle = 0$ iff $x = 0$ in M.

That is, $\langle ., . \rangle$ is a conjugate A-bilinear mapping. We treat only modules that are complete with respect to the derived norm $\|x\| := \|\langle x, x \rangle\|_A^{1/2}$ for $x \in M$. But, this is not precisely enough. We have to consider Hilbert C*-modules always as pairs of a module and of its C*-valued inner product, cf. [14] and Lemma 3.1 below.

Standard examples are Hilbert spaces over $\mathbb{C}$ or norm-closed right ideals of C*-algebras A. Often the free projective modules A^n of all n-tuples of elements of A or the standard countably generated A-module $l_2(A)$ of all sequences of elements of A for which the respective inner product series converge in norm are considered, cf. the Swan–Serre and Kasparov theorems. Generally speaking, full Hilbert C*-modules M over some C*-algebra A are at the same time (left) Banach modules over their C*-algebra $K_A(M)$ of "compact" module operators over them. Moreover, this relation is symmetric since there exists a $K_A(M)$-valued inner product on M inducing the same norm, and $K_{K_A(M)}(M)$ equals to A. This leads to (strong) Morita equivalence of C*-algebras.

However, Hilbert C*-modules in particular examples can be much more different in usually supposed "good" properties. Common knowledge is the sometimes missing self-duality of Hilbert C*-modules and the possible non-adjointability of some bounded module operators. Less known is the possible existence of two (or more) C*-valued inner products on certain Hilbert C*-modules inducing equivalent norms on them, but which are not unitarily equivalent or similar via a bounded adjointable (or even modular) invertible operator, what may turn the properties of a bounded module operator to be "compact" or to be adjointable into a relative property. Thinking further, modularly generating sets of elements are also affected in their possible property to be a standard modular frame by this effect (cf. [16, Cor. 6.6]), but there exist quite regularly Hilbert C*-modules that do not admit even this weak form of a "nice" generating set, not talking about kinds of orthonormal bases, cf. [2, 29]. So, the point is to identify classes of "good" Hilbert C*-modules. Candidates are the class of Hilbert C*-modules over compact C*-algebras, i.e., C*-algebras that admit a faithful ∗-representation in some C*-algebra of (all) compact operators on a Hilbert space. Another good choice are the classes of Hilbert C*-modules over von Neumann (i.e., W*-)algebras or over monotone complete C*-algebras. We are going to obtain further facts for these classes adding more evidence.

The following fact is non-obvious, but very useful for insights: Let N be a Hilbert C*-module over a C*-algebra A. Denote by $\langle N, N \rangle$ the norm-closed A-linear hull of all A-valued inner product values of elements of N in A. Moreover, the sets N and AN coincide and any element $n \in N$ can be represented as $n = ax$ for certain elements $a \in A$, $x \in N$ by the Cohen–Hewitt factorization theorem. This fact has been mentioned by several authors in the context of Banach C*-modules and in more general contexts, cf. e.g. [38, Thm. 4.1] or [23, Thm. 32.22], also [4, Thm. II.5.3.7].

This is one reason why Hilbert A-modules N can be considered as Hilbert $\mathbb{M}(A)$-modules over the multiplier algebras $\mathbb{M}(A)$ of A, similarly as Banach A-modules over the (right) multiplier algebra $\mathbb{RM}(A)$ of A. Recalling Morita equivalence and the symmetry of both the module actions, N is also a Hilbert $\mathrm{End}_A^*(N)$-module over the C*-algebra of all bounded adjointable operators $\mathrm{End}_A^*(N) = \mathbb{M}(K_A(N))$, as well as a Banach $\mathrm{End}_A(N)$-module over the Banach algebra $\mathrm{End}_A(M) = \mathbb{LM}(K_A(N))$, cf. [18, 26, 30].

Note, that we consider A-dual Banach A-modules N' of Hilbert A-modules N as right A-modules, too, defining ra for (right A-linear elements) $r \in N'$ and $a \in A$ by $(ra)(x) := a^* \cdot r(x)$ for any $x \in N$, cf. [35, p. 450]. Whenever A-valued inner products can be extended from N to N' this simplifies the isometric embedding of N into N' treating elements of both these modules in the same way in formulae.

3 Extension of C*-linear functionals: the monotone complete C*-case

In this section we investigate the question whether the zero bounded C*-linear maps of Hilbert C*-submodules over monotone complete C*-algebras A to their C*-algebra of coefficients could be continued by a non-zero bounded C*-linear functional on the hosting Hilbert C*-module over the same monotone complete C*-algebra in case the orthogonal complement of the submodule is trivial. We treat the cases of W*-algebras and of monotone complete C*-algebras as C*-algebras simultaneously despite of the different spheres of application and partially different techniques. This approach should help readers without deeper knowledge on non-W*, monotone complete C*-algebras to understand the arguments.

We need some intrinsic characterization of selfduality of Hilbert C*-modules over monotone complete C*-algebras. In [12] some kind of order type convergence in such Hilbert C*-modules has been introduced based on order convergence in monotone complete C*-algebras (cf. [25], [21, Section 1]). Note, that order convergence in monotone complete C*-algebras (denoted by LIM in the sequel) is not supported by any locally convex Hausdorff topology that preserves all algebraic C*-algebra structures, generally speaking (cf. [10], [12, Remark on p. 67]). But, for W*-algebras the w*-topology supports order convergence. Let A be a monotone complete C*-algebra, M be a Hilbert A-module and I be a net for indexing. A norm-bounded set $\{x_\alpha : \alpha \in I\}$ of elements of M is fundamental in the sense of τ_2^o-convergence iff the limits $\mathrm{LIM}\{\langle y, x_\alpha - x_\beta \rangle : \alpha \in I\}$ exist for every $\beta \in I$, any $y \in M$, and the limits $\mathrm{LIM}\{\mathrm{LIM}\{\langle y, x_\alpha - x_\beta \rangle : \alpha \in I\} : \beta \in I\}$ exist for any $y \in M$, too, and equal to zero. Such a set has the τ_2^o-limit $x \in M$ iff the limits $\mathrm{LIM}\{\langle y, x_\alpha - x \rangle : \alpha \in I\}$ exist for any $y \in M$ and equal to zero. This τ_2^o-convergence respects the module structures and preserves norm-bounded balls, cf. [12, Def. 2.3, Lemma 2.4]. Considering the W*-case a Hilbert W*-module M is self-dual iff its unit ball is complete with respect to the topology generated by the set $\{f(\langle x, . \rangle) : f \in A_*, x \in M\}$. This set generates the w*-topology on the A-dual Hilbert A-module M' of M, cf. [35, Prop. 3.8, Remark 3.9], [11, Def. 3.1, Thm. 3.2].

Lemma 3.1 *Let A be a C*-algebra and $M \subseteq N$ be two full Hilbert A-modules. Suppose that $M \subseteq N$ has the orthogonal complement $M_N^\perp = \{0\}$ with respect to N. Then*:

1. *Two different elements $n_1, n_2 \in N$ restricted to $M \subseteq N$ realize pairwise distinct bounded A-linear functionals $\langle n_1, . \rangle$, $\langle n_2, . \rangle$ on M.*
2. *In case of the existence of a non-zero bounded A-linear functional $r_0 : N \to A$, such that r_0 vanishes on M, the entire module $\{r_0 a : a \in A\}$ as well as its norm-closure in N' represents the zero functional on M. So r_0 cannot be represented by an element of N via the A-valued inner product on N by supposition.*
3. *Let r_0 as in* (ii). *The Hilbert A-module N admits (at least) two A-valued inner products inducing equivalent norms, whose reductions to M coincide. The more, these two A-valued inner products are not equivalent, i.e. $\langle T(.), . \rangle^{(1)} \not\equiv \langle ., . \rangle^{(2)}$ for any positive w.r.t. $\langle ., . \rangle^{(1)}$, bounded bijective A-linear operator T on N.*
4. *Given the situation at* (iii), *the notions of bounded module operators on N to be "compact" or to be adjointable w.r.t. one of these two named A-valued inner products depend on the choice of the A-valued inner product on N with identical reductions to M.*

Proof To see the assertion (i) select $n_1, n_2 \in N$ and consider the difference $n_1 - n_2$. Suppose, both n_1 and n_2 induce the same bounded A-linear map from M to A taking $\langle n_i, . \rangle$ and reducing it to $M \subseteq N$. Then $n_1 - n_2 = 0$ by supposition, i.e., $n_1 = n_2$ in N follows.

In case there exists an element $a \in A$, such that $r_0 a \in N$ and $r_0 a \neq 0$ we would have a non-zero element of N giving a zero A-valued functional on M via $\langle r_0 a, . \rangle$, a contradiction to the supposition.

Now, set $\langle ., . \rangle^{(2)} := \langle ., . \rangle^{(1)} + r_0^*(.) r_0(.)$. Obviously, it is an A-valued inner product on N whose reduction to M gives the initial A-valued inner product back. By general operator inequalities [35, Prop. 2.6], we have

$$\langle n, n \rangle^{(1)} \leq \langle n, n \rangle^{(2)} \leq (1 + \|r_0\|^2) \langle n, n \rangle^{(1)}.$$

This shows the equivalence of the induced norms on N. Now, suppose $\langle T(.), . \rangle^{(1)} \equiv \langle ., . \rangle^{(2)}$ for some positive w.r.t. $\langle ., . \rangle^{(1)}$, bounded bijective A-linear operator T on N. (This is equivalent to the existence of some adjointable w.r.t. $\langle ., . \rangle^{(1)}$, bounded bijective operator S on N with $\langle S(.), S(.) \rangle^{(1)} \equiv \langle ., . \rangle^{(2)}$.) We obtain the functional equality $\langle (T - \mathrm{id}_N)(n), . \rangle^{(1)} = r_0^*(n) r_0(.)$ for any $n \in N$. Therefore, the bounded A-linear functional $r_0^*(n) r_0(.) \in N'$ is represented by a non-zero element of N for any fixed $n \in N$. This contradicts assertion (ii) since there are non-zero functionals in this set $\{r_0^*(n) r_0(.); n \in N\}$ by supposition.

The last assertion follows from [14, Thm. 3.7, Prop. 5.3]. □

Lemma 3.2 *Let A be a monotone complete C*-algebra and $M \subseteq N$ be two Hilbert A-modules. Suppose that $M \subseteq N$ has the orthogonal complement $M_N^\perp = \{0\}$ with respect to N. Then*:

1. *The C*-algebras $\langle M, M \rangle$ and $\langle N, N \rangle$ in A have the same central carrier projection $p \in A$, i.e., both their annihilators with respect to A equal to $(1 - p)A$ with*

$p \in Z(A)$. *Obviously, both* $\langle M, M \rangle$ *and* $\langle N, N \rangle$ *are two-sided norm-closed ideals of the monotone complete C*-algebra* pA, *as well as* $\langle M, M \rangle$ *is a two-sided ideal of* $\langle N, N \rangle \subseteq pA$.

2. *The multiplier C*-algebras of their centers equal to* $pZ(A)$, *so the unitizations of* $\langle M, M \rangle$ *and of* $\langle N, N \rangle$ *inside* pA *share the identity element* p *of* $pZ(A) \subseteq pA$ *as their respective identities.*

3. *The multiplier C*-algebras of both* $\langle M, M \rangle$ *and* $\langle N, N \rangle$ *equal to* pA, *so the centers of their multiplier C*-algebras both equal to* $pZ(A)$, *too.* (*In general,* $Z(\mathbb{M}(A))$ *can be larger than* $\mathbb{M}(Z(A))$.) *So the two C*-algebras* $\langle M, M \rangle$ *and* $\langle N, N \rangle$ *share the identity element* $p \in pZ(A)$ *of their unitizations even in this sense.*

4. *The centers of the C*-algebras* $\text{End}_A^*(M)$ *and* $\text{End}_A^*(N)$ *of all* (*bounded*) *adjointable A-linear operators on M and on N, respectively, are isometrically* $*$*-isomorphic to* $pZ(A) \subseteq pA$.

Proof By [40, Prop. 8.2.2] every subset S of an AW*-algebra A admits a right and a left annihilator set of the form $p_r A$ and $A p_l$ where $p_l, p_r \in A$ are orthogonal projections. If S is a two-sided (norm-closed) ideal of A, then $p_l = p_r$ is a central projection in A. Whenever the central supports $(1 - p_M)$ and $(1 - p_N)$ of $\langle M, M \rangle$ and of $\langle N, N \rangle$, respectively, would be different, the subset $(p_M - p_N)A$ would act non-trivially on N and trivially on M. So, the subset $(p_M - p_N)N$ would be orthogonal to $M \subseteq N$, but it has to consist of the zero element only by supposition. This is only possible, if $p_M = p_N$. The set $\langle M, M \rangle$ is obviously a subset and a C*-subalgebra of the C*-algebra $\langle N, N \rangle \subseteq pA$, so it is a two-sided ideal in $\langle N, N \rangle$ because it is a two-sided ideal of pA.

To derive the next two statements we make use of [37, Theorem]: Let B be a C*-subalgebra of an AW*-algebra A with zero annihilator of B in A. Then the set of two-sided multipliers of B in A is isometrically $*$-isomorphic to the set of double centralizers of B (i.e., the set $\mathbb{M}(B)$ of multipliers of B in its enveloping von Neumann algebra $B'' \equiv B^{**}$) via an isomorphism that extends the identity map on B. Assertion (ii) follows if we consider the commutative (monotone complete) AW*-algebra $pZ(A)$, where $Z(\langle M, M \rangle) \subseteq Z(\langle N, N \rangle) \subseteq pZ(A)$. Commutativity and zero annihilators give $\mathbb{M}(Z(\langle M, M \rangle)) = \mathbb{M}(Z(\langle N, N \rangle)) = pZ(A)$ by the cited theorem. To obtain assertion (iii) we replace the centers by the entire respective C*-algebras. So, $\mathbb{M}(\langle M, M \rangle) = \mathbb{M}(\langle N, N \rangle) = pA$ by the ideal properties, and therefore, $Z(\mathbb{M}(\langle M, M \rangle)) = Z(\mathbb{M}(\langle N, N \rangle)) = pZ(A)$.

By results by Green [18, Lemma 16] and by Kasparov [26, Th. 1] we can identify the set of all bounded adjointable (module) maps on a given Hilbert C*-module isometrically $*$-isomorphically with the multiplier algebra of the C*-algebra of all "compact" module operators on it. Obviously, the set $\{x \cdot \text{id}_M : x \in Z(M(A))\}$ is contained in the center of $\text{End}_A^*(M)$. Considering M as an $\langle M, M \rangle$-$K_A(M)$ equivalence bimodule (cf. [6, Section 1]) we can interchange the roles of $\langle M, M \rangle$ and of $K_A(M)$ as primary and secondary C*-algebras of coefficients of a respectively (double-)full Hilbert C*-bimodule. Hence, the center of $\text{End}_A^*(M)$ is canonically contained in the center of pA by (ii) and (iii) above. So, they have to coincide with the center of pA, $pZ(A)$. The argument for N is similar. The centers of $\text{End}_A^*(M)$ and of $\text{End}_A^*(N)$ turn out to be isometrically *-isomorphic to $pZ(A)$, and to each other. $\qquad\square$

The following theorem has parallels in the result [33, Thm. 13] by Manuilov which was stated for the commutative W*-case:

Theorem 3.3 *Let A be a monotone complete C*-algebra and $M \subseteq N$ be two Hilbert A-modules. Suppose that $M \subseteq N$ has the orthogonal complement $M_N^\perp = \{0\}$ with respect to N. Then the selfdual Hilbert A-module M' admits an isometric embedding as a Hilbert A-submodule of the selfdual Hilbert A-module N', which extends the given isometric embedding of M in N in the same way as M is isometrically embedded in M'. The embedded copy of M' in N' is an orthogonal direct summand.*

Proof We need a type of order convergent nets in M to give an intrinsic characterization of selfduality for Hilbert C*-modules over monotone C*-algebras. By [30, Lemma 3.7], for Hilbert C*-modules M over monotone complete C*-algebras A the A-valued inner product on M can be continued to an A-valued inner product on the dual Banach A-module M', such that M is isometrically embedded in M' by the map $x \in M \to \langle x, . \rangle \in M'$ preserving the inner product values, $r(x) = \langle r, x \rangle$ for any $x \in M \subseteq M'$, $r \in M'$, and $\langle r, r \rangle = \sup\{r(x)^* r(x) : x \in M, \|x\| \leq 1\}$. By [12, Thm. 4.1] a Hilbert A-module over a monotone complete C*-algebra A is self-dual iff the unit ball of M is complete with respect to τ_2^o-convergence.

Denote by $M^\dagger$ the τ_2^o-completed canonical copy of M in M', which is also isometrically embedded in M' by construction. It has to be self-dual, and so either $M^\dagger = M'$ or $(M^\dagger)^\perp \neq \{0\}$ in M' since self-dual Hilbert C*-submodules are always direct summands. The latter would force $M^\perp \neq \{0\}$ in M', a contradiction to the definition of M'. In particular, M' does not contain any non-trivial elements perpendicular to the submodule M.

Consider the isometric embedding of M into N, which can be seen as an isometric embedding of M into the self-dual Hilbert A-module N' via the canonical isometric embedding of N into its A-dual N'. By definition isometric module embeddings preserve the module structure and the norm of each element. By [5, 27] surjective module isometries of Hilbert C*-modules preserve the C*-inner product values, cf. [13, Thm. 5] and [43, Thm. 1.1]. Since N' is a self-dual Hilbert A-module the biorthogonal complement $M^{\perp\perp}$ of M embedded in N' is also a self-dual Hilbert A-submodule and direct orthogonal summand of N'. The more, $M_{M^{\perp\perp}}^\perp = \{0\}$ by construction.

So for the isometric embedding of M into N' we can repeat the process of τ_2^o-completion canonically restricting to elements of its biorthogonal completion $M^{\perp\perp}$ with respect to N', without knowing the nature of its orthogonal completion $M^\perp$ with respect to N'. For the τ_2^o-completion of M in $M^{\perp\perp} \subseteq N'$ we obtain an isometric embedding of $M^\dagger$ into N'. It has to be self-dual by [12, Thm. 4.1] and it has to coincide with $M_{N'}^{\perp\perp}$, and so either $M^\dagger = N'$ or $(M^\dagger)^\perp \neq \{0\}$ in N' since self-dual Hilbert C*-submodules are always direct summands. Also, $M^\dagger$ is isometrically isomorphic to M' as a Hilbert A-module. We denote the orthogonal, positive projection of N' onto the isometrically embedded copy of M' by P.

Consider the elements $y \in N$ as A-linear functionals $\langle y, . \rangle_N$ on the Hilbert A-submodule M. Two such elements y_1, y_2 induce the same bounded A-linear map $\langle y_1, . \rangle \equiv \langle y_2, . \rangle$ on M if and only if their difference is the zero element of N since $M_N^\perp = \{0\}$ by supposition. So, the elements of N can be identified with elements of

$M' = M^\dagger \subseteq N'$ injectively. Moreover, any element of $N \setminus M$ acts in another way on M as any element of M.   □

In the following, we shall show that $P = \mathrm{id}_{N'}$.

Lemma 3.4 *Let A be a monotone complete C*-algebra and $M \subseteq N$ be two Hilbert A-modules. Suppose that $M \subseteq N$ has the orthogonal complement $M_N^\perp = \{0\}$ with respect to N. Then any orthogonal, positive projection $P : N' \to N'$ with $P \neq \mathrm{id}_N$ and $M' \subseteq P(N')$ is not an element of the multiplier C*-algebra $\mathrm{End}_A^*(N)$ of the C*-algebra of "compact" module operators $K_A(N)$ of the isometrically embedded in N' copy of N, i.e., any such $P \neq \mathrm{id}_N$ with $M' \subseteq P(N')$ is not a bounded adjointable (module) operator on N.*

Proof Again, by results by Green [18, Lemma 16] and by Kasparov [26, Th. 1] we can identify the set of all bounded adjointable (module) maps on a given Hilbert C*-module isometrically with the multiplier algebra of the C*-algebra of all "compact" module operators on it. So, if P would be a non-one two-sided multiplier of the C*-algebra $K_A(N)$ then P would belong to $\mathbb{M}(K_A(N)) = \mathrm{End}_A^*(N)$. That is, $P(N) = M$ would have a non-trivial orthogonal complement $(\mathrm{id}_N - P)(N)$ in N, a contradiction to the supposition.   □

Proposition 3.5 *Let A be a monotone complete C*-algebra and $M \subseteq N$ be two Hilbert A-modules. Suppose that $M \subseteq N$ has the orthogonal complement $M_N^\perp = \{0\}$ with respect to N. The four operator norms $\|T\|_M$, $\|T\|_N$, $\|T\|_{M'}$ and $\|T\|_{N'}$ coincide for any "compact" operator $T \in K_A(M)$ realized on resp. M, N, M' and N'. As a consequence, the given isometric modular embedding $M \subseteq N$ gives rise to an isometric ∗-representation of $K_A(M)$ in $K_A(N)$ on the level of "compact" modular operators on M and on N, respectively.*

Proof One knows M is a Hilbert A-submodule of N, of M' and of N', resp., via canonical isometric modular embeddings. Only M' and N' are a priori self-dual. So, any elementary "compact" operator $\theta_{x,y}(\cdot) = y\langle x, .\rangle$ with $x, y \in M$ can be uniquely continued to a "compact" operator on its self-dual hosting Hilbert A-module with the same operator norm on the self-dual hosting Hilbert A-module, cf. [35, Prop. 3.6] for the W*-case and [12, Cor. 6.3] for the monotone complete C*-case. Of course, this continuation can be considered as $\theta_{x,y}$ again, with x, y of the isometrically embedded copy of M, otherwise the operator norm would increase. So we have the operator norm equalities $\|\theta_{x,y}\|_M = \|\theta_{x,y}\|_{M'}$, $\|\theta_{x,y}\|_M = \|\theta_{x,y}\|_{N'}$. For analogous reasons, $\|\theta_{x,y}\|_N = \|\theta_{x,y}\|_{N'}$ for the particular case of the canonical isometric modular embedding of N into N'. Consequently, $\|\theta_{x,y}\|_M = \|\theta_{x,y}\|_N$ for any $x, y \in M$ and for the given isometric modular embedding $M \subset N$ fulfilling the supposition.

Since the elementary "compact" operators in $K_A(M)$ can be isometrically identified with elementary "compact" operators in $K_A(N)$ which admit the two generating elements from $M \subset N$ we can continue this isometric identification to finite sums $T := \sum_i \lambda_i \theta_{x_i, y_i}$ with elements $\{x_i, y_i\} \in M \subset N$ and complex numbers $\{\lambda_i\}$. Indeed,

for the chain of isometric modular embeddings $M \subset N \subset N' = M'$ we have

$$
\begin{aligned}
\sup_{z \in N, \|z\| \leq 1} \|T(z)\|_N &= \sup_{z \in N', \|z\| \leq 1} \|T(z)\|_{N'} \\
&= \sup_{z \in M', \|z\| \leq 1} \|T(z)\|_{M'} \\
&= \sup_{z \in M, \|z\| \leq 1} \|T(z)\|_M.
\end{aligned}
$$

Further, we obtain any "compact" element $T \in K_A(M)$ and of $K_A(N)$ as the norm-limit of sequences of such finite sums, which can be step by step isometrically identified with resp. "compact" operators on M and on N. Because of the analogous algebraic structures in $K_A(M)$ and in $K_A(N)$ and because of the demonstrated isometric identifications, we get an isometric $*$-representation of $K_A(M)$ in $K_A(N)$. □

Proposition 3.6 *Let A be a monotone complete C*-algebra and $M \subseteq N$ be two Hilbert A-modules. Suppose that $M \subseteq N$ has the orthogonal complement $M_N^\perp = \{0\}$ with respect to N. Then $End_A^*(N)$ does not contain any non-zero element T such that T is perpendicular to $K_A(M) \subseteq K_A(N)$.*

Proof Suppose, $\theta_{x,y} T = 0$ on N for any $x, y \in M$, i.e., $y \langle x, T(z) \rangle = 0$ for any $z \in N$, any $x, y \in M$. Since $y \in M$ is arbitrary and Lemma 3.2,(i) holds this is equivalent to the condition $\langle x, T(z) \rangle = 0$ for any $x \in M$, any $z \in N$. By supposition this means $T(z) = 0$ for any $z \in N$. Therefore, $T = 0$. If one investigates the opposite multiplication order $T\theta_{x,y} = 0$ we can apply the involution on $End_A^*(N)$ and reduce the problem to the one treated. □

Proposition 3.7 *Let A be a monotone complete C*-algebra and $M \subseteq N$ be two Hilbert A-modules. Suppose that $M \subseteq N$ has the orthogonal complement $M_N^\perp = \{0\}$ with respect to N. Then the identity operator on M extends to the identity operator on N. This is its unique extension as a bounded modular adjointable operator which preserves the norm.*

Proof Consider the isometric modular embeddings $M \subset N$ as given, $N \subseteq N'$ and $M \subseteq M'$ by [35, Thm. 3.2] and $M' \subseteq N'$ as described in Theorem 3.3. Since every bounded module operator on the smaller Hilbert A-module of each of the last three pairings admits a unique extension to a bounded module operator on the larger Hilbert A-module preserving its norm by [35, Cor. 3.7], we can extend the identity operator on M to a unique bounded module operator on N which is an orthogonal positive projection Q on N. Obviously, $\|Q\| = \|id_N\| = 1$, so by the uniqueness of the extensions $Q = P \in End_A^*(N')$ with P from the end of the proof of Theorem 3.3. However, $P \notin End_A^*(N)$ by Lemma 3.4 in case it is not the identity of $End_A^*(N)$. Therefore, $Q = P = id_N$ is the only possible alternative according to Theorem 3.3, Proposition 3.6. In other words, $End_A^*(M)$ and $End_A^*(N)$ share their identity operators in this canonical setting. Moreover, by [35, Thm. 3.2] and [35, Cor. 3.7] we have $id_M = id_{M'}$ and $id_N = id_{N'}$ for the respective identity operators for the canonical isometric modular embeddings. Therefore, $id_{M'} = id_{N'}$ and $P = id_N$.

By [37, Thm.] and Lemma 3.4 any non-degenerated isometric $*$-representation of $K_A(N)$ is at the same time a non-degenerated isometric $*$-representation of $K_A(M)$, and hence, an isometric $*$-representation of their respective multiplier algebras $\text{End}_A^*(M)$ and $\text{End}_A^*(N)$ which share their identity elements. So, this picture is highly stable. The derived facts of representation theory support the conclusion. □

We arrive at a central result, a particular case of which was published by Manuilov as [33, Thm. 9] for commutative W*-algebras and as [33, Thm. 10] for type I von Neumann algebras..

Theorem 3.8 *Let A be a monotone complete C*-algebra and $M \subseteq N$ be two Hilbert A-modules. Suppose that $M \subseteq N$ has the orthogonal complement $M_N^\perp = \{0\}$ with respect to N. Then the selfdual Hilbert A-module M' admits an isometric embedding as a Hilbert A-submodule of the selfdual Hilbert A-module N', such that M' coincides with N' preserving the given isometric inclusions. In particular, there does not exist any bounded A-linear functional $r_0 : N \to A$, such that r_0 vanishes on M, but which is not the zero functional on N.*

Proof By Theorem 3.3 the isometric embedding of M into N can be continued to an isometric embedding of M' into N' as a direct orthogonal summand. Since $M^\perp = \{0\}$ Lemma 3.4 and Proposition 3.7 imply the isometric modular isomorphism $M' = N'$. By the definition of M' the (bounded A-linear) zero functional on M with values in A has only the zero functional from N to A as its continuation. □

The following fact is not true for any C*-algebra A and any Hilbert A-module N, what makes it remarkable.

Corollary 3.9 *Let A be a monotone complete C*-algebra and $M \subseteq N$ be two Hilbert A-modules. Suppose that $M \subseteq N$ has the orthogonal complement $M_N^\perp = \{0\}$ with respect to N. Then the A-dual Banach A-modules of N and of M isometrically coincide as Banach A-modules. In particular, for a given Hilbert A-module N this holds for any (smaller-equal) Hilbert A-submodule M with $M_N^\perp = \{0\}$.*

As a central result we got that any isometric embedding of a Hilbert A-module M into another Hilbert A-module N with $M^\perp = \{0\}$ continues to an isometric coincidence of the A-dual Banach A-modules M' and N'. Therefore, the following corollary excludes the existence of examples in the described context like those given by Kaad and Skeide in [24].

Corollary 3.10 *Let A be a monotone complete C*-algebra and $M \subseteq N$ be two Hilbert A-modules. Suppose that $M \subseteq N$ has the orthogonal complement $M_N^\perp = \{0\}$ with respect to N. The for any element $n \in N \setminus M$ the equality $\|n\|_N = \|n\|_{M'}$ holds for these two norms of it.*

Remark 3.11 Let A be a monotone complete C*-algebra and $M \subseteq N$ be two Hilbert A-modules. Suppose that $M \subseteq N$ has the orthogonal complement $M_N^\perp = \{0\}$ with respect to N. Then by the unique extension theorem by Paschke [35, Cor. 3.7] any bounded (adjointable) operator on M or on N, respectively, has a unique equal-norm extension to a bounded adjointable operator on the A-dual Hilbert A-modules

$N' = M'$. However, in general the C*-algebras of "compact" modular operators $K_A(N)$ and $K_A(M)$ are not a pair of a C*-algebra and one of its norm-closed two-sided ideals if considered as C*-subalgebras of $\mathrm{End}_A(N') = \mathrm{End}_A(M')$. Moreover, bounded (adjointable) module operators on M extended to the monotone complete C*-algebra $\mathrm{End}_A(N') = \mathrm{End}_A(M')$ and, afterwards, reduced to bounded module operators with domain N might not preserve N. Consequently, if for pairs $I \subseteq J$ of a two-sided norm-closed ideal I in a C*-algebra J we canonically have $\mathbb{M}(J) \subseteq \mathbb{M}(I)$ (cf. local multiplier algebra definition for C*-algebras, [1, Prop. 1.2.20, Def. 2.3.1, Prop. 2.3.4]), we do not know anything definite on the interrelation of the multiplier C*-algebras $\mathrm{End}_A^*(M)$ and $\mathrm{End}_A^*(N)$ except that they share $K_A(M) \oplus Z(\mathbb{M}(K_A(M))\mathrm{id}_N$ by Lemma 3.2. However, in case N is selfdual we have an isometric $*$-representation of $\mathrm{End}_A^*(M)$ in $\mathrm{End}_A(N')$ as an order-dense C*-subalgebra with the same identity operator and center.

4 Special bounded modular functionals and kernels of bounded modular operators

For bounded adjointable operators $T : M \to N$ between Hilbert A-modules M, N over a C*-algebra A there are some simple facts characterizing basic situations. Obviously, T is A-linear, the kernel $\ker(T)$ and the set $T^*(N)$ are orthogonal to each other in M and norm-closed.

Lemma 4.1 *Let M, N be two Hilbert A-modules over a C*-algebra A, and let $T : M \to N$ be a bounded adjointable operator. Then*:

1. *The kernel $\ker(T)$ of T is biorthogonally complemented.*
2. *There does not exist any non-zero element of M orthogonal to both the sets $\mathrm{Ker}(T)$ and $T^*(N)^{\perp\perp}$.*
3. *The direct orthogonal sum of $\mathrm{Ker}(T)$ and $(T^*(N))^{\perp\perp}$ might not be equal to M.*

Proof To show (i), suppose there exists an element $x \in \ker(T)^{\perp\perp}$, such that $T(x) \neq 0$. Then x is orthogonal to $T^*(N)^{\perp}$, i.e., $0 = \langle x, T^*(y) \rangle = \langle T(x), y \rangle$ for any $y \in N$. Therefore, $T(x) = 0$, a contradiction. To derive (ii), the argument is the same: Any element $x \in M$ orthogonal to $T^*(N)$ belongs to $\ker(T)^{\perp\perp}$, so there is no element of M orthogonal to both $\ker(T) = \ker(T)^{\perp\perp}$ and $(T^*(N))^{\perp\perp}$, as well as to their direct orthogonal sum in M.

To show the possible non-coincidence of M and of the direct orthogonal sum of $\mathrm{Ker}(T)$ and $T^*(N)^{\perp\perp}$ for a certain bounded adjointable operator between Hilbert C*-modules consider $A = C([0, 1])$ as a Hilbert A-module over itself, and the element $f_0 \in A$, such that f_0 equals to zero on $[0, \frac{1}{2}]$ and f_0 is strongly positive on $(\frac{1}{2}, 1]$. Then the multiplication operator T of A by f_0 has the property $\mathrm{Ker}(T) \oplus (T^*(N))^{\perp\perp} \neq M$. □

Proposition 4.2 *Let N be a Hilbert A-module over a C*-algebra A. Suppose, there exists a bounded module operator $T_0 : N \to N$ the kernel of which is not biorthogonally complemented. Then the biorthogonal complement of the kernel of T_0 with*

respect to N admits a non-zero bounded A-linear functional $r_0 : \mathrm{Ker}(T_0)^{\perp\perp} \to A$, such that r_0 is the zero functional on $\mathrm{Ker}(T_0)$. The map r_0 can be extended to a bounded A-linear functional on N.

Proof In the situation given $\mathrm{Ker}(T_0)^{\perp\perp}$ contains an element $x_0 \neq 0$ that does not belong to $\mathrm{Ker}(T_0)$ and for which $T_0(x_0) \neq 0$. Consequently, the bounded A-linear functional $r_0(.) := \langle T_0(x_0), T_0(.) \rangle : N \to A$ maps $\mathrm{Ker}(T_0)$ to zero, but is unequal to zero on $\mathrm{Ker}(T_0)^{\perp\perp}$ since $\langle T_0(x_0), T_0(x_0) \rangle > 0$ by assumption. Obviously, the defined map r_0 can be applied to any element of N. □

Proposition 4.3 *Let $M \subset N$ be a pair of Hilbert A-modules over a C*-algebra A, such that $M^{\perp} = \{0\}$ and N is full. Suppose, there exists a non-trivial bounded module functional $r_0 : N \to A$ the kernel of which contains M. Then N admits a bounded non-adjointable module operator $T_0 : N \to N$, such that its kernel is not biorthogonally complemented and contains M.*

Proof Note, that any C*-algebra A can be considered as a Hilbert A-module over itself, so any bounded A-linear functional $r : M \to A$ can be considered as a bounded A-linear module map. Thus, any pair of non-self-dual Hilbert A-modules M, N with $M \subset N$, $M^{\perp} = \{0\}$ and an existing non-zero bounded A-linear functional $r_0 : N \to A$ vanishing on M gives rise to $\mathrm{Ker}(r_0) \neq \mathrm{Ker}(r_0)^{\perp\perp}$ inside N. Consider the set of operators $\{T_{0,x,a} : T_{0,x,a}(\cdot) = x(a \cdot r_0(\cdot))$ with $x \in N, a \in A\}$. The set $\{a \cdot r_0(z) : z \in N, a \in A\}$ forms a two-sided ideal in A by the right A-linearity of r_0, by the right A-module property of N, by the free choice of $a \in A$ and by the supposition $\langle N, N \rangle = A$. In case $\langle x, x \rangle$ would be orthogonal to the set $\{a \cdot r_0(z) : z \in N, a \in A\}$ for any $x \in N$ the two-sided ideal $\{a \cdot r_0(z) : z \in N, a \in A\}$ would have to consist only of the zero element of $A = \langle N, N \rangle$, a contradiction to the non-triviality of r_0. Therefore, the set $\{T_{0,x,a} : T_{0,x,a}(\cdot) = x(a \cdot r_0(\cdot))$ with $x \in N, a \in A\}$ contains at least one non-zero bounded non-adjointable module operator T_0 on N with a not biorthogonally complemented kernel containing $\mathrm{Ker}(r_0) \supseteq M$. □

Therefore, for a given C*-algebra A and a given full Hilbert A-module N the problem of the existence of bounded A-linear functionals r_0 possessing a not biorthogonally closed kernel that admits a trivial orthogonal complement is tightly connected to the problem of the existence of bounded A-linear operators T_0 on N possessing a not biorthogonally closed kernel that admits a trivial orthogonal complement.

Theorem 4.4 (Cf. [15, Lemma 2.4]) *The kernel of any bounded A-linear operator between two Hilbert A-modules over a monotone complete C*-algebra A is biorthogonally complemented.*

The proof is a combination of Theorem 3.8 and Proposition 4.2. In total, we finally found a correct proof of [15, Lemma 2.4] for monotone complete C*-algebras and for compact C*-algebras. The statement [15, Lemma 2.4] is false in the general C*-case by [24].

5 Extension of C*-linear functionals: the compact C*-case

Among the C*-algebras A for which the category of Hilbert C*-modules over them admits most of the positive properties one needs for easy applications, is the class of compact C*-algebras, i.e., those that admit a faithful *-representation in some C*-algebra of compact operators on a suitable Hilbert space. A similar result from a slightly different point of view has been obtained by Manuilov, cf. [33, Thm. 10].

Theorem 5.1 *Let A be a compact C*-algebra and $M \subseteq N$ be two Hilbert A-modules. Suppose that $M \subseteq N$ has the orthogonal complement $M_N^\perp = \{0\}$ with respect to N. Then the bounded A-linear zero functional of M to A admits a unique extension to a bounded A-linear functional of N to A, such that it vanishes on M—the zero functional, because $M = N$. The more, the kernel of every bounded A-linear operator mapping it to another Hilbert A-module is biorthogonally complemented in it.*

Proof For a compact C*-algebra A and any Hilbert A-module over it we have two nice properties: (i) There exists an orthogonal basis of it, i.e., a generating set $\{x_\alpha : \alpha \in I\}$ in it such that $\langle x_\alpha, x_\beta \rangle = 0$ for any $\alpha \neq \beta$ in I and $\langle x_\alpha, x_\alpha \rangle = p_\alpha$ with p_α an non-zero atomic projection in A (cf. [3, Thm. 2, Thm. 4]); (ii) the Hilbert A-module is an orthogonal direct summand whenever it is isometrically embedded into another Hilbert A-module as a Hilbert A-submodule (cf. [19, 31]). Therefore, M is a direct orthogonal summand of N with trivial orthogonal complement, i.e., M and N coincide. This forces the isometric coincidence of their A-dual Banach A-modules.

The kernel of any bounded A-linear operator on the Hilbert A-module under consideration is a Hilbert A-submodule of it, and hence, an orthogonal direct summand of it (cf. [31]). In particular, it coincides with its biorthogonal complement.   □

6 The situations for one-sided maximal modular ideals of general C*-algebras and for other norm-closed ideals

Let us return to the class of examples of right norm-closed maximal ideals D of C*-algebras A. We would like to consider the class of modular right maximal ideals $D \subset A$, i.e., of such ideals for which there exists an element $u_D \in A$, such that $(a - u_D a) \in D$ for any $a \in A$. They are automatically norm-closed, cf. [34, Thm. 1.3.1]. For C*-algebras even more is known: every maximal right ideal of a C*-algebra is norm-closed [7, Cor. 3.6], every maximal right ideal of a Fréchet algebra is modular [8, Thm. 2.2.42, Prop. 4.10.23], and in the commutative C*-case the codimension is 1 ([7, Thm. 2.4(i)], Gel'fand–Mazur theorem).

Theorem 6.1 *Let A be a C*-algebra and D be a right modular maximal ideal in A. Consider them as Hilbert A-modules in the usual way transferring the algebraic properties of A to modular and A-valued inner product structures on both A and $D \subset A$. Then the zero functional on the Hilbert A-submodule D has only the zero modular functional on its biorthogonal complement $D^{\perp\perp}$ in the hosting Hilbert A-module A as its continuation.*

Remark 6.2 In case the C*-algebra A contains a finite-dimensional (matrix) block and the left annihilator projection $p \in A^{**}$ of D is an atomic projection in this finite-dimensional block of $A \subseteq A^{**}$, then $D^{\perp\perp} = (1 - p)A \subset A$ (but of A^{**}). Otherwise, the left annihilator projection of D in A^{**} is not an element of A, and $D^{\perp\perp} = A$.

Proof By [34, Thm. 5.3.5] there exists a bijection between the set of pure states ρ on A and the set of all modular maximal right ideals N_ρ of A, where $N_\rho = \{a \in A : \rho(aa^*) = 0\}$. The inverse mapping is induced by a factorization of A by a modular maximal right ideal D resulting in a one-dimensional $*$-representation of A with a cyclic vector that induces the pure state. Considering the universal $*$-representation π_u of A on a Hilbert space H_u as defined in [39, 3.7.6], these one-dimensional $*$-representations of A are direct summands of π_u. Therefore, the bicommutant of $\pi_u(A)$, a von Neumann algebra, contains atomic projections p_ρ in its type I part that project onto these related one-dimensional $*$-representation spaces and realize any pure state ρ on A that way. The more, the enveloping von Neumann algebra $\pi_u(A)''$ of A is isomorphic, as a Banach space, to the second dual Banach space A^{**} of A, cf. [39, Thm. 3.7.8]. That is, we have a one-to-one relation between the set of all modular maximal right ideals N_ρ of A and atomic projections p_ρ of $A^{**} = \pi_u(A)''$ as $\pi_u(N_\rho)^{\perp\perp} = (1 - p_\rho)A^{**}$ in the sense of taking the left annihilator of $\pi_u(N_\rho)$ and then the right annihilator.

So the initial problem of the assertion above translates into the question of the nature of the intersection of $\pi_u(A)p_\rho$ with $A^{**}p_\rho$. Since $p_\rho A^{**} p_\rho = \mathbb{C}$, either $p_\rho \in \pi_u(A)$, i.e., in the (existing in this case) matrix part of A, and $p_\rho \pi_u(D)$ as a Hilbert A-submodule of $p_\rho \pi_u(A)$ is a direct orthogonal summand of $p_\rho \pi_u(A)$, or $\pi_u(A)p_\rho \cap A^{**}p_\rho = \{0\}$. In the first case $p_\rho \pi_u(D)$ is self-dual, so it contains already all modular continuations of the zero functional to its biorthogonal complement in $p_\rho \pi_u(A)$. In the second case we end up with the zero element as the unique continuation. So the result follows. $\qquad\square$

Proposition 6.3 *Let A be a commutative C*-algebra and $I \subseteq J$ be two essential norm-closed ideals of A. Then*:

1. *The left and the right annihilator sets of I w.r.t. J and of I or J w.r.t. A consist only of the zero element.*
2. *The multiplier algebra of J is isometrically $*$-algebraically represented in the multiplier algebra of I, i.e., $\mathbb{M}(J) \subseteq \mathbb{M}(I)$.*
3. *Suppose, I and J are considered as (right) Hilbert A-modules. Then any bounded A-linear functional from J to A which is supposed to be the zero functional on I equals to zero on J.*

Proof Start with the basic situation of two essential norm-closed ideals $I \subseteq J$ of A. Since norm-closed ideals of a commutative C*-algebra are C*-algebras themselves, any one-sided annihilator of I or J forces its adjoint to be an one-sided annihilator of I or J from the other side, simply by commutativity of A. The property of I and of J to be essential norm-closed ideals of A yields one-sided annihilators of them w.r.t. J and/or to A to be the set $\{0_A\}$. By commutativity of A any one-sided multiplier of I or J is in fact a two-sided multiplier.

Moreover, one has $\mathbb{M}(J) \subseteq \mathbb{M}(I)$ in the sense of an injective $*$-algebraical inclusion of C*-algebras, cf. [1, Prop. 1.2.20, Def. 2.3.1, Prop. 2.3.4]. Any bounded A-linear functional $r : J \rightarrow A$ can be described as the multiplication by a fixed element $n_r \in \mathbb{M}(J)$ since the A-dual Banach A-module of J can be represented as $\mathbb{M}(J)$ in that way. Therefore, n_r can be seen as a multiplier of I, too. So, any bounded A-linear functional r on J vanishing on I leads to an annihilator of I w.r.t. J. And this can be only the zero element of A by supposition. $\square$

We cannot transfer the proof to the non-commutative situation straightforwardly, since the A-dual Banach A-module I' of I has to be identified with the right multiplier algebra $\mathbb{RM}(I)$ of I which might be larger than $\mathbb{M}(I)$ and, therefore, might be not invariant w.r.t. involution, i.e., might be not a $*$-algebra.

The search for counterexamples of special C*-functionals in the class of monotone complete C*-algebras A, setting $N = A$, and right norm-closed ideals D in A, setting $M = D$, is not successful. Basically, the left annihilator of D is always of the form Ap with $p \in A$, a fact from AW*-algebra theory (cf. [40, Prop. 8.2.2]). And additionally, D is order-dense in $(1 - p)A$. Moreover, $(1 - p)A$ and A are right self-dual Hilbert A-modules. So we can formulate a corollary.

Corollary 6.4 *Let A be a monotone complete C*-algebra and D be a right, norm-closed ideal of A which is order dense in A. Then there does not exist any non-zero bounded A-linear functional $r_0 : A \rightarrow A$, such that $r_0(D) = \{0\}$.*

Set $N = A$ and $M = D$, recall $D^\perp = \{0\}$ and apply Theorem 3.8. We get the desired result. Going further we consider a pair of two ideals in a monotone complete C*-algebra.

Corollary 6.5 *Let A be a monotone complete C*-algebra, let I and J be two right norm-closed ideals of A, such that $I \subset J \subseteq A$, both with the orthogonal complement pA, $p = p^2 \geq 0$ in A. Switching to a consideration of I and J as Hilbert A-submodules of A, the orthogonal complement of I with respect to J is equal to $\{0\}$. Then the right ideals $I \subset J$ have the same left carrier projection $(1 - p) \in A$, and as Hilbert A-submodules of A they fulfil $I' = J' = (1 - p)A$. In particular, I and J are order dense in $(1 - p)A$, or equivalently in the picture of Hilbert A-modules, τ_2^o-dense. So, there does not exist any bounded A-linear functional $r_0 : J \rightarrow A$ such that r_0 vanishes on I, but which is not the zero functional on J.*

Proof Consider the two ideals as (right) Hilbert A-modules $M = I$ and $N = J$ in A. The right annihilator of I and of J w.r.t. A equals to zero since they are right ideals of A and right A-modules. Suppose, the left annihilator of I w.r.t. A is pA and of J w.r.t. A is qA. Since $I \subset J$ we have $p \geq q$. Switch to the picture of a Hilbert A-modules $M \subset N$. In case $p > q$ the subset $(p - q)N$ is non-zero and orthogonal to M, a contradiction to the supposition. So $p = q$. By Theorem 3.8 $M' = N' = (1 - p)A$ as Hilbert A-modules, and the uniqueness of the continuation of the zero functional from I to J follows. $\square$

Summing up, we found results for the zero functional continuation problem for pairs of a Hilbert C*-submodule with trivial orthogonal complement in another Hilbert C*-module, with quite different roots. So the research about this problem has to be continued.

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