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Positive solutions for a class of biharmonic problems: existence, nonexistence and multiplicity

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Abstract
The main objective of this article is to consider a biharmonic problem with Navier boundary conditions. Among others, some criteria for the existence, multiplicity and nonexistence of positive solutions are established by employed fixed point theorems in a cone. In addition, we not only consider the sublinear case, but also we will study the case of appropriate combinations of superlinearity and sublinearity.

Keywords Biharmonic equation · Navier boundary conditions · Positive solution · Fixed point theorem · Existence · Nonexistence and multiplicity

Mathematics Subject Classification 35J60 · 35J40

1 Introduction and main results

Biharmonic elliptic equations with various boundary conditions come from the study of traveling waves in suspension bridges [5] and static deflection of a bending beam [16], and have attracted the interest of many researchers. Some classical methods have been widely used to study biharmonic elliptic equations: the Pohozaev identities and decay estimates, see Guo-Liu [12] and Guo-Wei [13]; comparison principles, see Cosner-Schaefer [6] and Mareno [22]; degree argument, see Tarantello [28]; perturbation theory, see Wang-Shen [29]; bifurcation theory, see Lazer-McKenna [19]; the method of upper and lower solutions, see Ferrero-Warnault [10] and Pao [25]; computational methods for numerical solutions, see Pao [26] and Pao-Lu [27]; phase space analysis, see Chang-Chen [4], Díaz-Lazzo-Schmidt [9]; fixed point theorems, see Kusano-Naito-Swanso [18]; variational method, see Micheletti-Pistoia [23, 24],

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Xu-Zhang [31], Zhang [33], Zhou-Wu [36] and Ye-Tang [32]; Morse index, see Li-Zhang [35], Davila-Dupaigne-Wang-Wei [7], Khenissy [17] and Wei-Ye [30], and the moving-plane method, see Lin [20] and Guo-Huang-Zhou [14].

We recall some recent results of Abid-Baraket [1], Guo-Wei-Zhou [15], Arioli-Gazzola-Grunau-Mitidieri [3] and Liu-Wang [21]. In [1], Abid-Baraket applied the maximum principle to analyze the existence of singular solution to the following biharmonic elliptic problem

$$\begin{cases} \Delta^2 u = u^p \ \text{ in } \Omega, \\ u = \Delta u = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

Recently, Guo-Wei-Zhou [15] employed the entire radial solutions of a equation with supercritical exponent and the Kelvin's transformation to obtain positive singular radial entire solutions of the biharmonic equation with subcritical exponent. Then, they constructed solutions with a prescribed singular set for problem (1.1) by using the expansions of such singular radial solutions at the singular point 0.

In [3], Arioli-Gazzola-Grunau-Mitidieri studied the boundary value problem

$$\begin{cases} \Delta^2 u = \lambda e^u \ \text{ in } \Omega, \\ u = \frac{\partial u}{\partial \mathbf{n}} = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\lambda \geq 0$ is a parameter, Ω is the unit ball in $\mathbb{R}^n$ $(n \geq 5)$ and $\frac{\partial u}{\partial \mathbf{n}}$ denotes the differentiation with respect to the exterior unit normal. They proved the existence of singular solutions for problem (1.2) by means of computer assistance when $5 \leq n \leq 16$.

In [21], Liu-Wang employed a variant version of Mountain Pass Theorem to study the existence and nonexistence of positive solution to the biharmonic problem

$$\begin{cases} \Delta^2 u = f(x, u) \ \text{ in } \Omega, \\ u = \Delta u = 0 \ \text{ on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ $(n > 4)$, and f satisfies

(C_1) $f(x,t) \in C(\bar{\Omega} \times \mathbb{R})$; $f(x,0) = 0$, $\forall x \in \bar{\Omega}$; $f(x,t) \geq 0$, $\forall t \geq 0$, $x \in \bar{\Omega}$ and $f(x,t) \equiv 0$, $\forall t \leq 0$, $x \in \bar{\Omega}$;

(C_2) $\lim\limits_{t\to 0} \frac{f(x,t)}{t} = p(x)$, $\lim\limits_{t\to +\infty} \frac{f(x,t)}{t} = l$ $(0 < l \leq +\infty)$ uniformly in a.e. $x \in \Omega$ where $|p(x)|_\infty < \Lambda_1$, Λ_1 is the first eigenvalue of $(\Delta^2, H^2(\Omega) \cap H_0^1(\Omega))$;

(C_3) for a.e. $x \in \Omega$, $\lim\limits_{t\to 0} \frac{f(x,t)}{t}$ is nondecreasing with respect to $t > 0$.

However, to our best knowledge, in the literature, there are almost no papers using the fixed point theory in cons for completely continuous operators to study the existence, nonexistence and multiplicity of positive solutions for analogous biharmonic elliptic problems. More precisely, the study is still open for the Navier boundary value problem

$$\begin{cases} \Delta^2 u = \lambda f(x, u) \ \text{ in } \Omega, \\ u = \Delta u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\lambda \neq 0$ is a parameter, Ω is a smooth bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, and the nonlinearity f satisfies:

$(\mathbf{f_1})$ $f \in C(\bar{\Omega} \times [0, +\infty), [0, +\infty))$.

If $\lambda = 1$ and $f(x, u) = u^p$, then problem (1.3) reduces to the problem studied by Abid-Baraket [1] and Guo-Wei-Zhou [15].

Let

$$f^0 := \lim_{u \to 0^+} \frac{f(x, u)}{u} \text{ uniformly for } x \in \bar{\Omega};$$
$$f^\infty := \lim_{u \to +\infty} \frac{f(x, u)}{u} \text{ uniformly for } x \in \bar{\Omega};$$
$$f_\infty := \varliminf_{u \to +\infty} f(x, u) \text{ uniformly for } x \in \bar{\Omega}.$$

The main results of this paper are the following theorems.

Theorem 1.1 *Under condition* $(\mathbf{f_1})$*, if* $f^0 = 0$, $f^\infty = 0$ *and* $f_\infty \in (0, +\infty]$*, then, for any given* $\tau > 0$*, there exists* $\xi > 0$ *so that, for* $\lambda > \xi$*, problem (1.3) admits at least two positive solutions* $u_\lambda^{(1)}(x)$, $u_\lambda^{(2)}(x)$ *and* $\max_{x \in \bar{\Omega}} u_\lambda^{(1)}(x) > \tau$.

Remark 1.2 One of the contributions of Theorem 1.1 is to use a simpler method, i.e. index theory of fixed points on cones to prove the multiplicity of positive solutions for biharmonic problems.

Remark 1.3 The approach used in Theorem 1.1 is completely different from those used in Abid-Baraket [1], Guo-Wei-Zhou [15], Arioli-Gazzola-Grunau-Mitidieri [3], Liu-Wang [21] and other related papers. In particular, comparing with Liu-Wang [21], the main difficulties of Theorem 1.1 lie in three main directions:

(1) $\lambda > 0$ is considered;
(2) multiple positive solutions are obtained;
(3) in the proof process, we do not need the monotonicity condition (C_3).

Theorem 1.4 *Under condition* $(\mathbf{f_1})$*,* (i) *if* $0 < f^0 < +\infty$*, then there are* $l_0 > 0$ *and* $\lambda_0 > 0$ *such that, for every* $0 < r < l_0$*, problem (1.3) admits a positive solution* u_r *satisfying* $\|u_r\|_C = r$ *associated with*

$$\lambda = \lambda_r \in (0, \lambda_0]. \tag{1.4}$$

(ii) *if* $f^0 = +\infty$*, then there are* $l^* > 0$ *and* $\lambda^* > 0$ *such that, for any* $0 < r^* < l^*$*, problem (1.3) admits a positive solution* u_{r^*} *satisfying* $\|u_{r^*}\|_C = r^*$ *for any*

$$\lambda = \lambda_{r^*} \in (0, \lambda^*].$$

(iii) *if* $f^0 < +\infty$ *and* $f^\infty < +\infty$*, then there exists* $\underline{\lambda} > 0$ *such that problem (1.3) admits no positive solutions for* $\lambda \in (\underline{\lambda}, \infty)$.

Corollary 1.5 *Under condition* $(\mathbf{f_1})$*, if* $f^0 = 0$ *and* $f^\infty = 0$*, then problem (1.3) admits no positive solution for sufficiently large* λ.

Next, in Theorems 1.6–1.9 and Theorem 1.11, we will employ some techniques different from that used in Theorem 1.1 to prove some existence and multiplicity results. Conclusions to be demonstrated in Theorems 1.6–1.9 and Theorem 1.11 are true for any positive parameter λ. We hence may suppose that $\lambda = 1$ in problem (1.3) for simplicity.

We introduce the following notations.

$$f^{\gamma} = \limsup_{u\to\gamma} \max_{x\in\bar{\Omega}} \frac{f(x,u)}{u^{\alpha}}, \quad f_{\gamma} = \liminf_{u\to\gamma} \min_{x\in\bar{\Omega}} \frac{f(x,u)}{u^{\beta}},$$

where γ denotes 0^+ or $+\infty$, $\alpha, \beta \in (0, +\infty)$.

We consider the following three cases for $\alpha, \beta \in (0, +\infty)$:

$$\alpha = 1; 0 < \alpha, \beta < 1 \text{ and } \alpha > 1.$$

Case $\alpha = 1$ is treated in Theorems 1.6–1.7.

Theorem 1.6 *Under condition* $(\mathbf{f_1})$*, if* $f^0 = 0$ *or* $f^{\infty} = 0$*, and there exist* $\eta > 0$ *and* $l > 0$ *so that* $u \geq \eta$ *and* $x \in \bar{\Omega}$ *implies*

$$f(x,u) \geq l, \tag{1.5}$$

then problem (1.3) possesses at least one positive solution.

Theorem 1.7 *Under condition* $(\mathbf{f_1})$*, if* $f^0 = 0$ *and* $f^{\infty} = 0$*, and there exist* $\eta > 0$ *and* $l > 0$ *such that* $u \geq \eta$ *and* $x \in \bar{\Omega}$ *implies (1.5) holds, then problem (1.3) possesses at least two positive solutions* u^* *and* u^{**} *with*

$$0 < \|u^*\|_C < \eta < \|u^{**}\|_C,$$

where $\|\cdot\|_C$ *denotes the norm of real Banach space* $C(\bar{\Omega})$*.*

Theorems 1.8–1.9 deal with the case $0 < \alpha < 1$ and $0 < \beta < 1$.

Theorem 1.8 *Under condition* $(\mathbf{f_1})$*, if*

$$f_0 = \infty \text{ and } f^{\infty} = 0,$$

then problem (1.3) possesses at least one positive solution.

Theorem 1.9 *Under condition* $(\mathbf{f_1})$*, if* $f^{\infty} = 0$ *and there exist* $\eta > 0$ *and* $l > 0$ *such that* $u \geq \eta$ *and* $x \in \bar{\Omega}$ *implies (1.5) holds, then problem (1.3) possesses at least one positive solution.*

Remark 1.10 The method applied in the proof of Theorems 1.6–1.7 is invalid when we employ it to demonstrate Theorems 1.8–1.9 for the case $0 < \alpha < 1$ and $0 < \beta < 1$. Therefore we need to introduce a different technique to verify Theorems 1.8–1.9.

Next we study the case $\alpha > 1$ in Theorem 1.11.

Theorem 1.11 *Under condition* $(\mathbf{f_1})$*, if* $f^0 = 0$ *and there exist* $\eta > 0$ *and* $l > 0$ *such that* $u \geq \eta$ *and* $x \in \bar{\Omega}$ *implies (1.5) holds, then problem (1.3) possesses at least one positive solution.*

Remark 1.12 The fixed point index theorem on cones is valid in the proof of Theorem 1.11, but the approach used in the proof of Theorems 1.6–1.9 is invalid.

The rest of the paper is organized as follows. In Sect. 2, we first apply an idea from Guo-Huang-Zhou [14] to transfer problem (1.3) into a second-order elliptic system. Then, we obtain the Green's function of problem (1.3) by means of the Green's function of the corresponding second-order elliptic boundary value problem. Consequently we get the expression of the solution for problem (1.3). In Sect. 3, we apply index theory of fixed points for completely continuous operators to study the existence, nonexistence and multiplicity of positive solutions to problem (1.3). Sections 4–6 are, respectively, devoted to the study of existence and multiplicity of positive solutions to problem (1.3) under the case $\lambda = 1$.

2 Second-order elliptic system

In this section, we first apply an idea from Guo-Huang-Zhou [14] to transfer problem (1.3) into a second order elliptic system. Then, we obtain the Green's function of problem (1.3) by means of the Green's function of the corresponding second order elliptic boundary value problem. Consequently, we get the expression of the solution for problem (1.3).

Let $-\Delta u = \omega$. Then, we can transfer the biharmonic problem (1.3) into the following second order elliptic system

$$\begin{cases} -\Delta u = \omega \ \text{ in } \Omega, \\ -\Delta \omega = \lambda f(x, u) \ \text{ in } \Omega, \\ u = 0 = \omega \ \text{ on } \partial\Omega. \end{cases} \tag{2.1}$$

It follows from (2.1) that

$$u(x) = \int_{\bar{\Omega}} G^*(x, y)\omega(y)\mathrm{d}y, \tag{2.2}$$

$$\omega(x) = \lambda \int_{\bar{\Omega}} G^*(x, y) f(y, u(y))\mathrm{d}y, \tag{2.3}$$

where $G^*(x, y)$ is the Green's function of $-\Delta$ on Ω, which verifies

$$0 \leq G^*(x, y) \leq C|x - y|^{2-n},$$

where $n \geq 3$, the constant C depends only on Ω.

Moreover, for $x, y \in \Omega$, $x \neq y$, one finds that

$$0 \le G^*(x,y) \le \frac{1}{4\pi|x-y|}, \quad n=3,$$
$$0 \le G^*(x,y) \le \frac{1}{2\pi}\ln\frac{d}{|x-y|}, \quad n=2,$$

where d denotes the diameter of Ω.

Since, for $x, y \in \bar{\Omega} \subset \mathbb{R}^n$ $(n \ge 2)$, $G^*(x,y)$ is nonnegative, continuous (when $x \neq y$) and symmetric, there must be two points x_0 and y_0 with $x_0 \neq y_0$, which are interior points of Ω, such that

$$G^*(x_0,y_0) = G^*(y_0,x_0) > 0.$$

Thus, there are $\tau_1, \tau_2, \tau_3 > 0$ and two disjoint small closed balls $B_1, B_2, B_3 \in \Omega$ such that

$$\begin{cases} G^*(x,y) \ge \tau_1, & \forall (x,y) \in (B_1 \times B_2) \cup (B_2 \times B_1), \\ G^*(y,z) \ge \tau_2, & \forall (y,z) \in (B_2 \times B_3) \cup (B_3 \times B_2), \\ G^*(x,z) \ge \tau_3, & \forall (x,z) \in (B_1 \times B_3) \cup (B_3 \times B_1), \end{cases} \tag{2.4}$$

where

$$B_1 = \{x \in \Omega : |x - x_0| \le \delta\},$$
$$B_2 = \{x \in \Omega : |x - y_0| \le \delta\},$$
$$B_3 = \{x \in \Omega : |x - z_0| \le \delta\}.$$

It is easy to see that

$$\text{mes} B_1 = \text{mes} B_2 = \text{mes} B_3.$$

On the other hand, from (2.2) and (2.3), we have

$$\begin{aligned} u(x) &= -\int_{\bar{\Omega}} G^*(x,y)\omega(y)\mathrm{d}y \\ &= \lambda \int_{\bar{\Omega}} G^*(x,y) \int_{\bar{\Omega}} G^*(y,z) f(z,u(z))\mathrm{d}z\mathrm{d}y \\ &= \lambda \int_{\bar{\Omega}} \int_{\bar{\Omega}} G^*(x,y) G^*(y,z) f(z,u(z))\mathrm{d}z\mathrm{d}y \\ &= \lambda \int_{\bar{\Omega}} G(x,z) f(z,u(z))\mathrm{d}z, \end{aligned} \tag{2.5}$$

where

$$G(x,z) = \int_{\bar{\Omega}} G^*(x,y) G^*(y,z)\mathrm{d}y. \tag{2.6}$$

Thus, we give the expression of Green's function for problem (1.3). Obviously, $G(x,z) = G(z,x)$ and $G(x,z) \ge 0$ for $x, z \in \bar{\Omega}$.

3 Proof of Theorems 1.1 and 1.4

In this section, we first consider the multiplicity of positive solutions for problem (1.3) by using the fixed point index in a cone, which is used in Zhang [34].

Lemma 3.1 ([8]) *Let E be a real Banach space and K be a cone in E. For $r > 0$, define $K_r = \{x \in K : \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{x \in K : \|x\| = r\}$.*

(i) If $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$.

(ii) If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 1$.

Let $E = C(\bar{\Omega})$ be the real Banach space with supremum norm $\|\cdot\|_C$, and define a cone K in E as

$$K = \{u : u \in E,\ u(x) \geq 0,\ x \in \bar{\Omega}\}. \tag{3.1}$$

For $\varrho > 0$, we also define

$$D_\varrho = \{u : u \in E,\ \|u\|_C < \varrho\},$$

and

$$\partial K_\varrho = K \cap \partial D_\varrho = \{u \in K : \|u\|_C = \varrho\}.$$

For $u \in K$, we define $T_\lambda : K \to E$ to be

$$T_\lambda u(x) = \lambda \int_{\bar{\Omega}} G(x, y) f(y, u(y)) \mathrm{d}y, \tag{3.2}$$

where $G(x, y)$ is defined in (2.6).

When $(\mathbf{f_1})$ hold, it is well known that $T_\lambda : K \to E$ is completely continuous.

Proof of Theorem 1.1. For any given $\tau > 0$, it follows from $f_\infty \in (0, +\infty]$ that there exist $\eta > 0$ and $l > \tau$ such that

$$f(x, u) \geq \eta,\ \forall x \in \bar{\Omega},\ \ u \geq l. \tag{3.3}$$

Letting $\xi = (\tau_1 \tau_2 \eta (\mathrm{mes} B_2)(\mathrm{mes} B_3))^{-1} l$, then for $0 < \lambda < \xi$, one can prove that $\hat{T}_\lambda : K \to K$ is completely continuous.

Considering $f^0 = 0$, there exists $0 < r < l$ such that

$$f(x, u) \leq \varepsilon_1 u,\ \forall x \in \bar{\Omega},\ 0 \leq u \leq r, \tag{3.4}$$

where $\varepsilon_1 > 0$ satisfies

$$\lambda \varepsilon_1 \|\phi_1\|_C \|\phi_2\|_C < 1, \tag{3.5}$$

and for $i \in \{1, 2\}$, $\phi_i \in C^2(\bar{\Omega})$ gratify

$$\begin{cases} -\Delta\phi_i = 1 \text{ in } \Omega, \\ \phi_i = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.6}$$

So, for $u \in K \cap \partial D_r$, we have from (2.5), (2.6), (3.2), (3.4) and (3.5) that

$$\begin{aligned} \|T_\lambda u\|_C &= \max_{x\in\bar{\Omega}} \lambda \int_{\bar{\Omega}} G(x, y) f(y, u(y))\mathrm{d}y \\ &= \max_{x\in\bar{\Omega}} \lambda \int_{\bar{\Omega}} \int_{\bar{\Omega}} G^*(x, z)G^*(z, y) f(y, u(y))\mathrm{d}z\mathrm{d}y \\ &= \max_{x\in\bar{\Omega}} \lambda \int_{\bar{\Omega}} G^*(x, z)\mathrm{d}z \int_{\bar{\Omega}} G^*(z, y) f(y, u(y))\mathrm{d}y \\ &\le \lambda\varepsilon_1 \|u\|_C \int_{\bar{\Omega}} G^*(x, z)\mathrm{d}z \int_{\bar{\Omega}} G^*(z, y)\mathrm{d}y \\ &\le \lambda\varepsilon_1 \|u\|_C \|\phi_1\|_C \|\phi_2\|_C \\ &< \|u\|_C. \end{aligned}$$

It hence follows from (ii) of Lemma 3.1 that

$$i(T_\lambda, K_r, K) = 1. \tag{3.7}$$

Now turning to $f^\infty = 0$, there exists $\sigma > 0$ so that

$$f(x, u) \le \varepsilon_2 u, \quad \forall x \in \bar{\Omega}, \quad u > \sigma,$$

where $\varepsilon_2 > 0$ satisfies

$$2\lambda\varepsilon_2 \|\phi_1\|_C \|\phi_2\|_C \le 1. \tag{3.8}$$

We hence have

$$0 \le f(x, u) \le \varepsilon_2 u + \mathfrak{M}_\sigma, \quad \forall x \in \bar{\Omega}, \quad u \ge 0, \tag{3.9}$$

where

$$\mathfrak{M}_\sigma = \max_{x\in\bar{\Omega},\ 0\le u\le\sigma} f(x, u) + 1 > 0.$$

Let

$$R > \max\left\{l, 2\lambda\mathfrak{M}_\sigma \|\phi_1\|_C \|\phi_2\|_C\right\}. \tag{3.10}$$

Thus, for $u \in K \cap \partial D_R$, we derive from (2.5), (2.6), (3.2), (3.8), (3.9) and (3.10) that

$$\begin{aligned}\|T_\lambda u\|_C &= \max_{x\in\bar\Omega} \lambda \int_{\bar\Omega} G(x,y) f(y,u(y))\mathrm{d}y \\ &= \max_{x\in\bar\Omega} \lambda \int_{\bar\Omega}\int_{\bar\Omega} G^*(x,z)G^*(z,y) f(y,u(y))\mathrm{d}z\mathrm{d}y \\ &= \max_{x\in\bar\Omega} \lambda \int_{\bar\Omega} G^*(x,z)\mathrm{d}z \int_{\bar\Omega} G^*(z,y) f(y,u(y))\mathrm{d}y \\ &\le \lambda(\varepsilon_2\|u\|_C + \mathfrak{M}_\sigma)\int_{\bar\Omega} G^*(x,z)\mathrm{d}z \int_{\bar\Omega} G^*(z,y)\mathrm{d}y \\ &\le \lambda(\varepsilon_2\|u\|_C + \mathfrak{M}_\sigma)\|\phi_1\|_C\|\phi_2\|_C \\ &< \frac{\|u\|_C}{2} + \frac{R}{2} = \|u\|_C.\end{aligned}$$

It hence follows from (ii) of Lemma 3.1 that

$$i(T_\lambda, K_R, K) = 1. \tag{3.11}$$

On the other hand, for $u \in \bar K_l^R = \{u \in K : \|u\|_C \le R, \min_{x\in B_3} u(x) \ge l\}$, (2.5), (2.6), (3.2), (3.8), (3.9) and (3.10) yield that

$$\|T_\lambda u\|_C < R.$$

Furthermore, for $u \in \bar K_l^R$, from (2.4) (2.5), (2.6), (3.2), and (3.3), we obtain that

$$\begin{aligned}\min_{x\in B_1}(T_\lambda u)(x) &= \lambda \min_{x\in B_1} \int_{\bar\Omega} G(x,y) f(y,u(y))\mathrm{d}y \\ &\ge \lambda \min_{x\in B_1} \int_{B_2}\int_{B_3} G^*(x,z)G^*(z,y) f(y,u(y))\mathrm{d}z\mathrm{d}y \\ &\ge \lambda\tau_1\tau_2\eta(\mathrm{mes}B_2)(\mathrm{mes}B_3) \\ &> \xi\tau_1\tau_2\eta(\mathrm{mes}B_2)(\mathrm{mes}B_3) \\ &= l.\end{aligned}$$

Letting $u_0 \equiv \frac{l+R}{2}$ and $H(t,u) = (1-t)T_\lambda u + tu_0$, then $H : [0,1]\times \bar K_l^R \to K$ is completely continuous, and from the analysis above, we obtain for $(t,u) \in [0,1]\times \bar K_l^R$

$$H(t,u) \in K_l^R. \tag{3.12}$$

Therefore, for $t \in [0,1]$, $u \in \partial K_l^R$, we have $H(t,u) \ne u$. Hence, by the normality property and the homotopy invariance property of the fixed point index, we obtain

$$i(T_\lambda, K_l^R, K) = i(u_0, K_l^R, K) = 1. \tag{3.13}$$

Consequently, by the solution property of the fixed point index, T_λ admits a fixed point $u_\lambda^{(1)}$ with $u_\lambda^{(1)} \in K_l^R$, and

$$\max_{x\in\bar\Omega} u_\lambda^{(1)}(x) \ge \min_{x\in B_3} u_\lambda^{(1)}(x) > l > \tau.$$

On the other hand, it follows from (3.7), (3.11) and (3.13) together with the additivity of the fixed point index that

$$\begin{aligned} & i(T_\lambda, K_R \backslash (\bar{K}_r \cup \bar{K}_l^R), K) \\ & = i(T_\lambda, K_R, K) - i(T_\lambda, K_l^R, K) - i(T_\lambda, K_r, K) \\ & = 1 - 1 - 1 = -1. \end{aligned} \tag{3.14}$$

According to the solution property of the fixed point index, T_λ so possesses a fixed point $u_\lambda^{(2)}$ with $u_\lambda^{(2)} \in K_R \backslash (\bar{K}_r \cup \bar{K}_l^R)$. It is easy to see that $u_\lambda^{(1)} \neq u_\lambda^{(2)}$. This finishes the proof of Theorem 1.1. □

Next, we will prove the existence and nonexistence of positive solution to problem (1.3). To this goal, we need to state one well-known result of the fixed point index on cones for completely continuous operators, which is the base of our approaches.

Lemma 3.2 (Corollary 2.3.1, Guo and Lakshmikantham [11]) *Let K be a cone in a real Banach space E and let Ω be a bounded open set of E. Assume that the operator $A : K \cap \bar{\Omega} \to K$ is completely continuous. If there exists a $u_0 > 0$ such that*

$$u - Au \neq tu_0, \quad \forall u \in K \cap \partial\Omega, \quad t \geq 0,$$

then

$$i(A, K \cap \Omega, K) = 0.$$

Proof of Theorem 1.4. It is well known that problem (1.3) is equivalent to the following nonlinear integral equation

$$u(x) = \lambda \int_{\bar{\Omega}} G(x, y) f(y, u(y)) \mathrm{d}y, \tag{3.15}$$

where $G(x, y)$ is defined in (2.6).

Consider the operator

$$\hat{T} u(x) = \int_{\bar{\Omega}} G(x, y) f(y, u(y)) \mathrm{d}y. \tag{3.16}$$

Since $G(x, y)$ and $f(x, u)$ are nonnegative, it is easy to see that $\hat{T} : K \to K$ is completely continuous.

Part (i). It follows from $0 < f^0 < +\infty$ that there exist $0 < l_1 < l_2$ and $\mu > 0$ such that

$$l_1 u < f(x, u) < l_2 u \quad (\forall x \in \bar{\Omega}, \quad 0 \leq u \leq \mu). \tag{3.17}$$

Let $\lambda_0 = (l_1 \tau_1 \tau_2 \mathrm{mes} B_2 \mathrm{mes} B_3)^{-1}$ and $l_0 = \mu$. We now demonstrate that λ_0 and l_0 are the numbers to be required.

On one hand, for $u \in K \cap \partial D_r$, we have

$$0 \le u(x) \le r < l_0 = \mu, \quad x \in \bar{\Omega}.$$

On the other hand, we may suppose that

$$u - \lambda_0 \hat{T} u \ne 0 \quad (\forall u \in K \cap \partial D_r). \tag{3.18}$$

If not, then there is $u_r \in K \cap \partial D_r$ such that $\lambda_0 \hat{T} u_r = u_r$ and so (1.4) already holds for $\lambda_r = \lambda_0$.

Define $\psi(x) \equiv 1$ for $x \in \bar{\Omega}$. Then, $\psi \in K$ gratifying $\|\psi\|_C \equiv 1$. We now demonstrate that

$$u - \lambda_0 \hat{T} u \ne \zeta \psi \quad (\forall u \in K \cap \partial D_r,\ \zeta \ge 0). \tag{3.19}$$

Assume that there are $u_1 \in K \cap \partial D_r$ and $\zeta_1 \ge 0$ such that $u_1 - \lambda_0 \hat{T} u_1 = \zeta_1 \psi$, then (3.18) indicates that $\zeta_1 > 0$, and $u_1 = \zeta_1 \psi + \lambda_0 \hat{T} u_1 \ge \zeta_1 \psi$. Let $\zeta^* = \sup\{\zeta | u_1 \ge \zeta \psi\}$. Then $\zeta_1 \le \zeta^* < +\infty$ and $u_1 \ge \zeta^* \psi$. Therefore,

$$\zeta^* = \zeta^* \|\psi\|_C \le \|u_1\|_C = r. \tag{3.20}$$

Consequently, for any $x \in B_1$, we derive from (2.4), (2.5), (2.6), (3.16), and (3.17), that

$$\begin{aligned}
u_1(x) &= \lambda_0 \int_{\bar{\Omega}} G(x, y) f(y, u_1(y)) \mathrm{d}y + \zeta_1 \psi(x) \\
&= \lambda_0 \int_{\bar{\Omega}} \int_{\bar{\Omega}} G^*(x, z) G^* z, y f(y, u_1(y)) \mathrm{d}z \mathrm{d}y + \zeta_1 \psi(x) \\
&\ge \lambda_0 \int_{\bar{\Omega}} \int_{\bar{\Omega}} G^*(x, z) G^* z, y l_1 u_1(y) \mathrm{d}z \mathrm{d}y + \zeta_1 \psi(x) \\
&\ge \lambda_0 \int_{\bar{\Omega}} \int_{\bar{\Omega}} G^*(x, z) G^* z, y l_1 \zeta^* \psi(z) \mathrm{d}z \mathrm{d}y + \zeta_1 \psi(x) \\
&\ge \lambda_0 l_1 \zeta^* \int_{B_2} G^*(x, z) \mathrm{d}z \int_{B_3} G^* z, y \mathrm{d}y + \zeta_1 \psi(x) \\
&\ge \lambda_0 l_1 \zeta^* \tau_1 \tau_2 \mathrm{mes} B_2 \mathrm{mes} B_3 + \zeta_1 \psi(x) \\
&= \zeta^* + \zeta_1 \psi(x) \\
&= (\zeta^* + \zeta_1) \psi(x),
\end{aligned}$$

which indicates that $u_1(x) \ge (\zeta^* + \zeta_1)\psi(x)$ for $x \in B_1$. This contradicts the definition of ζ^*. Thus, (3.19) holds and hence it yields from Lemma 3.2 that

$$i(\lambda_0 \hat{T}, K \cap D_r, K) = 0. \tag{3.21}$$

It is widely known that

$$i(\theta, K \cap D_r, K) = 1, \tag{3.22}$$

where θ denotes the zero operator.

Therefore, it derives from (3.21) and (3.22) and the homotopy invariance that there are $u_r \in K \cap \partial D_r$ and $0 < \nu_r < 1$ so that $\nu_r \lambda_0 \hat{T} u_r = u_r$, which indicates that

$$0 < \lambda_r = \lambda_0 \nu_r < \lambda_0.$$

This gives the proof of (1.4).

Part (ii). If $f^0 = +\infty$, then there are $l_3 > 0$ and $\mu^* > 0$ so that

$$f(x,u) > l_3 u, \quad \forall x \in \bar{\Omega}, \quad 0 \le u \le \mu^*.$$

Next, we verify that $l^* = \mu^*$ and $\lambda_* = (l_3 \tau_1 \tau_2 \mathrm{mes} B_2 \mathrm{mes} B_3)^{-1}$ are required. Thus, for $u \in K \cap \partial D_{r^*}$, we derive that

$$0 \le u(x) \le r^* < l^* = \mu^*, \quad x \in \bar{\Omega}.$$

Similar to the proof of (i), replacing (3.18), one can suppose that

$$u - \lambda_* \hat{T} u \ne 0 \ \ (\forall u \in K \cap \partial D_{r^*}),$$

and replacing (3.19) we can demonstrate that

$$u - \lambda_* \hat{T} u \ne \zeta \psi \ \ (\forall u \in K \cap \partial D_{r^*}, \ \zeta \ge 0).$$

It follows from Lemma 3.2 that $i(\lambda_* \hat{T}, K \cap D_{r^*}, K) = 0$. Seeing that $i(\theta, K \cap D_r, K) = 1$, one can easily demonstrate that there are $u_{r^*} \in K \cap \partial D_{r^*}$ and $0 < \nu_{r^*} < 1$ so that $\nu_{r^*} \lambda_* \hat{T} u_{r^*} = u_{r^*}$. So, Theorem 1.4 (ii) holds for $\lambda_{r^*} = \lambda_* \nu_{r^*} < \lambda_*$.

Part (iii). If $f^0 < \infty$ and $f^\infty < \infty$, then there are positive numbers $\eta_1 > 0, \ \eta_2 > 0, \ h_1 > 0$ and $h_2 > 0$ so that $h_1 < h_2$ and for $x \in \bar{\Omega}, \ 0 < u \le h_1$, we derive that

$$f(x,u) \le \eta_1 u, \tag{3.23}$$

and for $x \in \bar{\Omega}, \ u \ge h_2$, we derive that

$$f(x,u) \le \eta_2 u. \tag{3.24}$$

Set

$$\eta^* = \max\left\{\eta_1, \eta_2, \ \max\left\{\frac{f(x,u)}{u} : x \in \bar{\Omega}, \ h_1 \le u \le h_2\right\}\right\} > 0.$$

Then, we derive that

$$f(x,u) \le \eta^* u, \ x \in \bar{\Omega}, \ u \in [0,\infty). \tag{3.25}$$

Assume that $v \in K$ is a positive solution to problem (1.3). We will demonstrate that this leads to a contradiction for $\lambda < \underline{\lambda} = (\eta^* \|\phi_1\|_C \|\phi_2\|_C)^{-1}$.

In fact, for $\lambda < \underline{\lambda}$, we derive from (2.5), (2.6), (3.2), and (3.25) that

$$\begin{aligned}\|T_\lambda v\|_C &= \max_{x\in\bar{\Omega}} \lambda \int_{\bar{\Omega}} G(x,y) f(y,v(y))\mathrm{d}y\\ &= \max_{x\in\bar{\Omega}} \lambda \int_{\bar{\Omega}}\int_{\bar{\Omega}} G^*(x,z)G^*(z,y) f(y,v(y))\mathrm{d}z\mathrm{d}y\\ &= \max_{x\in\bar{\Omega}} \lambda \int_{\bar{\Omega}} G^*(x,z)\mathrm{d}z \int_{\bar{\Omega}} G^*(z,y) f(y,v(y))\mathrm{d}y\\ &\le \lambda\eta^*\|v\|_C \int_{\bar{\Omega}} G^*(x,z)\mathrm{d}z \int_{\bar{\Omega}} G^*(z,y)\mathrm{d}y\\ &\le \lambda\eta^*\|v\|_C\|\phi_1\|_C\|\phi_2\|_C\\ &< \underline{\lambda}\eta^*\|v\|_C\|\phi_1\|_C\|\phi_2\|_C\\ &= \|v\|_C,\end{aligned}$$

which indicates that

$$\|v\|_C = \|T_\lambda v\|_C < \|v\|_C.$$

This is a contradiction. □

Proof of Corollary 1.5. The proof of Corollary 1.1 is a direct consequence of the proof for Theorem 1.4 (iii). Under the conditions of Corollary 1.5, we can obtain the intervals of λ so that problem (1.3) admits no positive solutions. □

Remark 3.3 If we consider the following Navier boundary value problem

$$\begin{cases}\lambda\Delta^2 u = f(x,u) \ \text{ in } \Omega,\\ u = \Delta u = 0 \ \text{ on } \partial\Omega,\end{cases} \tag{3.26}$$

where $\lambda \neq 0$ is a parameter, Ω is a bounded domain in $\mathbb{R}^n$ ($n \ge 2$), and the nonlinearity f satisfies ($\mathbf{f_1}$), then we have the following conclusions.

Theorem 3.4 *Under condition* ($\mathbf{f_1}$) *holds, if* $0 < f^0 < +\infty$, *then there exists* $l_0 > 0$ *such that, for every* $0 < r < l_0$, *problem (3.26) admits a positive solution* u_r *satisfying* $\|u_r\|_C = r$ *associated with*

$$\lambda = \lambda_r \in [\lambda_0, \bar{\lambda}_0],$$

where λ_0 *and* $\bar{\lambda}_0$ *are two positive finite numbers.*

Proof The proof is similar to that of Theorem 1.4 (i). We hence omit it here. □

4 Proof of Theorem 1.6 and Theorem 1.7

In this section, we will prove Theorem 1.6 and Theorem 1.7. To achieve this goal, we first state a well-known result of the fixed point, which is the base of our approaches.

Lemma 4.1 (Theorem 2.3.3, Guo and Lakshmikantham [11]) *Let* Ω_1 *and* Ω_2 *be two bounded open sets in a real Banach space* E *such that* $\theta \in \Omega_1$ *and* $\bar{\Omega}_1 \subset \Omega_2$. *Let* P

be a cone in E and let operator $A : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ *be completely continuous. Suppose that one of the following two conditions*

(a) $Ax \ngeq x, \forall\ x \in P \cap \partial\Omega_1$ *and* $Ax \nleq x, \forall\, x \in P \cap \partial\Omega_2$

and

(b) $Ax \nleq x, \forall\, x \in P \cap \partial\Omega_1$ *and* $Ax \ngeq x, \forall\ x \in P \cap \partial\Omega_2$

is satisfied. Then, A has at least one fixed point in $P \cap (\Omega_2 \backslash \bar{\Omega}_1)$.

Remark 4.2 It is clear to see that the fixed point of A in Lemmas 4.1 can not reach the boundary of Ω_1 and Ω_2.

Proof of Theorem 1.6. It is well known that problem (1.3) is equivalent to the following nonlinear integral equation

$$u(x) = \int_{\bar{\Omega}} G(x, y) f(y, u(y))\mathrm{d}y \tag{4.1}$$

when $\lambda = 1$, where $G(x, y)$ is defined in (2.6).

Consider the operator

$$T_1 u(x) = \int_{\bar{\Omega}} G(x, y) f(y, (u(y))\mathrm{d}y, \tag{4.2}$$

It is generally known that T_1 maps K into K is a completely continuous operator.

Case (1), $f^0 = 0$.

Considering $f^0 = 0$, there exists $r > 0$ such that

$$f(x, u) \le \varepsilon r,\ \ \forall x \in \bar{\Omega},\ \ 0 \le u \le r, \tag{4.3}$$

where $\varepsilon > 0$ satisfy $\varepsilon \|\phi_1\|_C \|\phi_2\|_C < 1$, and ϕ_i are defined in (3.6) for $i \in \{1, 2\}$.

We can prove that

$$Au \ngeq u,\ \ u \in K,\ \ \|u\|_C = r. \tag{4.4}$$

In fact, if there exists $u_1 \in K \cap \partial D_r$ such that $T_1 u_1 \ge u_1$, then from (2.5), (2.6), (3.6), (4.2) and (4.3) we have

$$\begin{aligned} 0 \le u_1(x) \le\ & T_1 u_1(x) \\ &= \int_{\bar{\Omega}} G(x, y) f(y, u_1(y))\mathrm{d}y \\ &= \int_{\bar{\Omega}} \int_{\bar{\Omega}} G^*(x, z) G^*(z, y) f(y, u_1(y))\mathrm{d}z\mathrm{d}y \\ &= \int_{\bar{\Omega}} G^*(x, z)\mathrm{d}z \int_{\bar{\Omega}} G^*(z, y) f(y, u_1(y))\mathrm{d}y \\ &\le \varepsilon r \int_{\bar{\Omega}} G^*(x, z)\mathrm{d}z \int_{\bar{\Omega}} G^*(z, y)\mathrm{d}y \\ &\le \varepsilon r \|\phi_1\|_C \|\phi_2\|_C \\ &< r = \|u_1\|_C. \end{aligned}$$

This leads to $\|u_1\|_C < \|u_1\|_C$, which is a contraction. It so follows that (4.4) holds.

Case (2), $f^\infty = 0$.

Considering $f^\infty = 0$, there exists $\bar{R} > r > 0$ such that

$$f(x,u) \le \bar{\varepsilon} u,\ \forall x \in \bar{\Omega},\ u \ge \bar{R},$$

where $\bar{\varepsilon} > 0$ satisfies $\frac{1}{2}\bar{\varepsilon}\|\phi_1\|_C\|\phi_2\|_C < 1$.

We hence have

$$0 \le f(x,u) \le \bar{\varepsilon} u + \mathfrak{M},\ \ \forall x \in \bar{\Omega},\ \ u \ge 0, \tag{4.5}$$

where

$$\mathfrak{M} = \max_{x\in\bar{\Omega},\ 0\le u\le \bar{R}} f(x,u) + 1 > 0.$$

Let

$$R > \max\left\{\bar{R}, 2\mathfrak{M}\|\phi_1\|_C\|\phi_2\|_C\right\}. \tag{4.6}$$

Then, one can prove that

$$T_1 u \not\geq u,\ \ u \in K,\ \ \|u\|_C = R. \tag{4.7}$$

In fact, if there exists $u_2 \in K$ with $\|u_2\|_C = R$ so that $T_1 u_2 \ge u_2$, then it follows from (2.5), (2.6), (3.6), (4.5) and (4.6) that

$$\begin{aligned} 0 \le u_2(x) &\le T_1 u_2(x) \\ &= \int_{\bar{\Omega}} G(x,y) f(y,u_2(y))\mathrm{d}y \\ &= \int_{\bar{\Omega}}\int_{\bar{\Omega}} G^*(x,z)G^*(z,y) f(y,u_2(y))\mathrm{d}z\mathrm{d}y \\ &= \int_{\bar{\Omega}} G^*(x,z)\mathrm{d}z \int_{\bar{\Omega}} G^*(z,y) f(y,u_2(y))\mathrm{d}y \\ &\le \varepsilon(\bar{\varepsilon}\|u_2\|_C + \mathfrak{M}) \int_{\bar{\Omega}} G^*(x,z)\mathrm{d}z \int_{\bar{\Omega}} G^*(z,y)\mathrm{d}y \\ &\le (\bar{\varepsilon}\|u_2\|_C + \mathfrak{M})\|\phi_1\|_C\|\phi_2\|_C \\ &< \frac{\|u_2\|_C}{2} + \frac{R}{2} = \|u_2\|_C, \end{aligned}$$

which leads to $\|u_2\|_C < \|u_2\|_C$. This is a contraction. It so follows that (4.7) holds.

Next, we demonstrate that

$$T_1 u \not\leq u,\ \ u \in K,\ \ \|u\|_C = \eta, \tag{4.8}$$

where $\bar{R} < \eta < R$.

In reality, if there is $u_0 \in K$ with $\|u_0\|_C = \eta$ so that $T_1 u_0 \le u_0$.

It so follows from (1.5), when a η is fixed, there is a $l > 0$ so that

$$f(x,u_1) \ge l > \frac{\eta}{\tau_1\tau_2 \mathrm{mes} B_2 \mathrm{mes} B_3},\ \forall x \in \bar{\Omega},\ u \ge \eta. \tag{4.9}$$

We notice that $u \geq \eta$ implies that $\|u\|_C \geq \eta$. Therefore, for $u \in \partial K_\eta$, we derive from (2.4), (2.5), (2.6), (3.2) and (4.9) that

$$\begin{aligned} x \in B_1 \Longrightarrow u_0(x) &\geq T_1 u_0(x) \\ &= \int_{\bar{\Omega}} G(x, y) f(y, u_0(y)) \mathrm{d}y \\ &\geq \int_{B_2} \int_{B_3} G^*(x, z) G^*(z, y) f(y, u_0(y)) \mathrm{d}z \mathrm{d}y \\ &\geq \tau_1 \tau_2 l (\mathrm{mes} B_2)(\mathrm{mes} B_3) \\ &> \eta. \end{aligned}$$

This is a contraction. It so follows that (4.8) holds.

Applying Lemma 4.1 to (4.4) and (4.8) or (4.7) and (4.8) yields that operator T_1 possesses one fixed point u with $u \in K_{r,\eta}$ or $u \in K_{\eta,R}$. It hence follows that problem (1.3) admits at least one positive solutions u with $r < \|u\|_C < \eta$ or $\eta < \|u\|_C < R$. This gives the proof of Theorem 1.6. □

Proof of Theorem 1.7. Take $0 < \eta_1 < \eta < \eta_2$. When $f^0 = 0$, similar to the proof of (4.4), one can demonstrate that

$$T_1 u \not\geq u, \quad u \in K, \quad \|u\|_C = \eta_1. \tag{4.10}$$

When $f^\infty = 0$, similar to the proof of (4.7), we derive that

$$T_1 u \not\geq u, \quad u \in K, \quad \|u\|_C = \eta_2. \tag{4.11}$$

Under condition (1.5), similar to the proof of (4.8), one can prove that

$$T_1 u \not\leq u, \quad u \in K, \quad \|u\|_C = \eta. \tag{4.12}$$

Therefore, from (4.10), (4.11) and (4.12), Lemma 4.1 yields that T_1 possesses two fixed point u^*, u^{**} gratifying that $u^* \in K_{\eta_1,\eta}$, $u^{**} \in K_{\eta,\eta_2}$. It so follows that problem (1.3) possesses at least two positive solutions u^*, u^{**} gratifying that

$$0 < \|u^*\|_C < \eta < \|u^{**}\|_C.$$

This completes the proof of Theorem 1.7. □

5 Proof of Theorems 1.8 and 1.9

In this section, we will employ the following fixed point theorems on cones to prove Theorem 1.8 and Theorem 1.9 for the case $0 < \alpha < 1$ and $0 < \beta < 1$.

Lemma 5.1 (See [2], Theorem 12.3) *Let P be a cone in a real Banach space E. Assume Ω_1, Ω_2 are bounded open sets in E with $\theta \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$. If*

$$A : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$$

is completely continuous such that either

(a) *there exists a* $u_0 > 0$ *such that* $u - Au \neq tu_0, \forall u \in P \cap \partial\Omega_2, t \geq 0$; $Au \neq \mu u, \forall u \in P \cap \partial\Omega_1, \mu \geq 1$, *or*

(b) *there exists a* $u_0 > 0$ *such that* $u - Au \neq tu_0, \forall u \in P \cap \partial\Omega_1, t \geq 0$; $Au \neq \mu u, \forall u \in P \cap \partial\Omega_2, \mu \geq 1$.

Then A *has at least one fixed point in* $P \cap (\Omega_2 \backslash \bar{\Omega}_1)$.

Remark 5.2 Obviously, the fixed point of A in Lemmas 5.1 can not reach the boundary of Ω_1 and Ω_2.

Proof of Theorem 1.8. We assume that there is $r_1 > 0$ so that

$$u - T_1 u \neq \theta, \ \forall u \in K, \ 0 < \|u\|_C \leq r_1. \tag{5.1}$$

If not, then there is $u_{r_1} \in K \cap \partial D_{r_1}$ so that

$$T_1 u_{r_1} = u_{r_1}.$$

Considering $f_0 = \infty$, there is $\sigma > 0$ and $r_2 > 0$ so that

$$f(x, u) \geq \sigma u^\beta \ \ (\forall x \in \bar{\Omega}, \ \ 0 \leq u \leq r_2). \tag{5.2}$$

Let $\psi(x) \equiv 1$ for $x \in \bar{\Omega}$. Then $\psi \in K$ with $\|\psi\|_C \equiv 1$. Next, we demonstrate that

$$u - T_1 u \neq \zeta\psi \ \ (\forall u \in K \cap \partial D_r, \ \zeta \geq 0), \tag{5.3}$$

where

$$0 < r < \min\{r_1, r_2, (\tau\sigma \mathrm{mes} B_\delta)^{\frac{1}{1-\beta}}\}.$$

In reality, if there are $u_1 \in K \cap \partial D_r$ and $\zeta_1 \geq 0$ such that $u_1 - T_1 u_1 = \zeta_1 \psi$, then (5.1) indicates that $\zeta_1 > 0$. But, $u_1 = \zeta_1\psi + T_1 u_1 \geq \zeta_1 \psi$. Set

$$\zeta^* = \sup\{\zeta | u_1 \geq \zeta\psi\}.$$

Thus, we have $\zeta_1 < \zeta^* < +\infty$ and $u_1 \geq \zeta^*\psi$. So,

$$\zeta^* = \zeta^*\|\psi\|_C \leq \|u_1\|_C = r \leq (\tau_1\tau_2\sigma \mathrm{mes} B_2 \mathrm{mes} B_3)^{\frac{1}{1-\beta}}. \tag{5.4}$$

Therefore, for any $x \in B_1$, we follow from (2.4), (2.5), (2.6), (3.2), (5.2) and (5.4) that

$$\begin{aligned} u_1(x) &= \int_{\bar{\Omega}} G(x, y) f(y, u_1(y)) \mathrm{d}y + \zeta_1\psi(x) \\ &\geq \int_{\bar{\Omega}} G(x, y)\sigma u_1^\beta(y)\mathrm{d}y + \zeta_1\psi(x) \\ &\geq \int_{\bar{\Omega}} G(x, y)\sigma(\zeta^*\psi(y))^\beta \mathrm{d}y + \zeta_1\psi(x) \\ &\geq \sigma(\zeta^*)^\beta \int_{B_2} G^*(x, z)\mathrm{d}z \int_{B_3} G^*(z, y)\mathrm{d}y + \zeta_1\psi(x) \\ &\geq \sigma(\zeta^*)^\beta \tau_1\tau_2 \mathrm{mes} B_2 \mathrm{mes} B_3 + \zeta_1\psi(x) \\ &\geq \zeta^* + \zeta_1\psi(x) \\ &= (\zeta^* + \zeta_1)\psi(x). \end{aligned}$$

This contradicts the definition of ζ^*. So (5.3) holds.

Next, turning to $f^\infty = 0$, then there are $l > 0$ and $r_2 > 0$ so that

$$f(x, u) \leq l u^\alpha(x), \ \forall u \geq r_2.$$

Let

$$L = \max_{x \in \bar{\Omega}, 0 \leq u \leq r_2} f(x, u).$$

Then we derive that

$$f(x, u) \leq l\|u\|_C^\alpha + L, \ \forall x \in \bar{\Omega}, u \in [0, +\infty). \tag{5.5}$$

Take R be large enough ($R > r$) such that

$$\frac{L\|\phi_1\|_C\|\phi_2\|_C}{R} + \frac{l\|\phi_1\|_C\|\phi_2\|_C}{R^{1-\alpha}} < 1. \tag{5.6}$$

Now, we are going to prove that

$$\forall u \in K \cap \partial D_R, \ \mu \geq 1 \Rightarrow T_1 u \neq \mu u. \tag{5.7}$$

In fact, if there are $u_2 \in K \cap \partial D_R$ and $\mu_0 \geq 1$ such that $T_1 u_2 = \mu_0 u_2$, then it follows from (2.5), (2.6), (3.2), (3.6) and (5.5) that

$$\begin{aligned} \mu_0 u_2(x) &= \int_{\bar{\Omega}} G(x, y) f(y, u_2(y)) \mathrm{d}y \\ &= \int_{\bar{\Omega}} \int_{\bar{\Omega}} G^*(x, z) G^* z, y f(y, u_2(y)) \mathrm{d}z \mathrm{d}y \\ &\leq \int_{\bar{\Omega}} \int_{\bar{\Omega}} G^*(x, z) G^*(z, y)(L + l u^\alpha(z)) \mathrm{d}z \mathrm{d}y \\ &\leq (L + l\|u\|^\alpha) \int_{\bar{\Omega}} \int_{\bar{\Omega}} G^*(x, z) G^*(z, y) \mathrm{d}z \mathrm{d}y \\ &\leq (L + l\|u\|^\alpha)\|\phi_1\|_C\|\phi_2\|_C. \end{aligned} \tag{5.8}$$

Thus it follows from (5.8) that

$$\begin{aligned} \mu_0 R = \mu_0\|u_2\|_C &\leq (L + l\|u\|^\alpha)\|\phi_1\|_C\|\phi_2\|_C \\ &\leq (L + l R^\alpha)\|\phi_1\|_C\|\phi_2\|_C. \end{aligned}$$

It hence derives from (5.6) that

$$\mu_0 \leq \frac{L\|\phi_1\|_C\|\phi_2\|_C}{R} + \frac{l\|\phi_1\|_C\|\phi_2\|_C}{R^{1-\alpha}} < 1.$$

This contradicts $\mu_0 \geq 1$, which indicates that (5.7) holds.

Therefore, according to (b) of Lemma 5.1, it yields from (5.3) and (5.7) that operator T_1 possesses a fixed point u in $K \cap (D_R \backslash \bar{D}_r)$ with $r < \|u\|_C < R$. This follows that problem (1.3) has at least one positive solution u with $r < \|u\|_C < R$, and so the proof of Theorem 1.8 is completed. □

Proof of Theorem 1.9. Take $0 < \eta < \eta_1$. When $f^\infty = 0$, similar to the proof of (5.7), we can prove that

$$\forall u \in K \cap \partial D_{\eta_1},\ \mu \geq 1 \Rightarrow T_1 u \neq \mu u. \tag{5.9}$$

Let $\psi(x) \equiv 1$ for $x \in \bar{\Omega}$. Then, $\psi \in K$ with $\|\psi\|_C \equiv 1$. Next, we can prove that

$$u - T_1 u \neq \zeta\psi \ \ (\forall u \in K \cap \partial D_\eta,\ \zeta \geq 0). \tag{5.10}$$

In reality, if there are $u_2 \in K \cap \partial D_\eta$ and $\zeta_2 \geq 0$ such that $u_2 - T_1 u_2 = \zeta_2 \psi$, then (5.10) indicates that $\zeta_2 > 0$. But, $u_2 = \zeta_2\psi + T_1 u_2 \geq \zeta_2\psi$. Set

$$\zeta^{**} = \sup\{\zeta^* | u_2 \geq \zeta^*\psi\}.$$

Then we derive $\zeta_2 < \zeta^{**} < +\infty$ and $u_2 \geq \zeta^{**}\psi$. We so have

$$\zeta^{**} = \zeta^{**}\|\psi\|_C \leq \|u_2\|_C = \eta \leq (\tau_1\tau_2\sigma \mathrm{mes} B_2 \mathrm{mes} B_3)^{\frac{1}{1-\beta}}. \tag{5.11}$$

On the other hand, it follows from (1.5), when a η is fixed, there exists a $l > 0$ such that

$$f(x, u_2) \geq l > \frac{\eta}{\tau_1\tau_2 \mathrm{mes} B_2 \mathrm{mes} B_3},\ \forall x \in \bar{\Omega},\ u \geq \eta. \tag{5.12}$$

Thus, for any $x \in B_1$ and $u \in \partial K_\eta$, we follow from (2.4), (2.5), (2.6), (3.2), (5.11) and (5.12) that

$$\begin{aligned} u_2(x) &= \int_{\bar{\Omega}} G(x, y) f(y, u_2(y)) \mathrm{d}y + \zeta_2\psi(x) \\ &= \int_{\bar{\Omega}}\int_{\bar{\Omega}} G^*(x, z) G^*(z, y) f(y, u_2(y)) \mathrm{d}z\mathrm{d}y + \zeta_2\psi(x) \\ &\geq l \int_{B_2}\int_{B_3} G^*(x, zy) G^*(z, y) \mathrm{d}z\mathrm{d}y + \zeta_2\psi(x) \\ &\geq l\tau_1\tau_2 \mathrm{mes} B_2 \mathrm{mes} B_3 + \zeta_2\psi(x) \\ &> \eta + \zeta_2\psi(x) \\ &\geq \zeta^{**} + \zeta_2\psi(x) \\ &= (\zeta^{**} + \zeta_2)\psi(x). \end{aligned}$$

This contradicts the definition of ζ^{**}. Therefore (5.10) holds. This completes the proof of Theorem 1.9. □

6 Proof of Theorem 1.11

In this section, we intend to apply the following fixed point theorem on cones to demonstrate Theorem 1.11 for the case $\alpha > 1$.

Lemma 6.1 (Theorem 2.3.4, Guo-Lakshmikantham [11]) *Let P be a cone in a real Banach space E. Assume $\Omega_1,\ \Omega_2$ are bounded open sets in E with $\theta \in \Omega_1,\ \bar{\Omega}_1 \subset \Omega_2$.*

If

$$A : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$$

is completely continuous such that either

(a) $\|Ax\| \le \|x\|, \ \ \forall x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \ge \|x\|, \ \forall x \in P \cap \partial\Omega_2$, *or*
(b) $\|Ax\| \ge \|x\|, \ \ \forall x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \le \|x\|, \ \forall x \in P \cap \partial\Omega_2$,
then A has at least one fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Remark 6.2 Comparing with Lemma 4.1 and Lemma 5.1, the fixed point of A in Lemmas 6.1 can reach the boundary of Ω_1 and Ω_2.

Proof of Theorem 1.11. Since $f^0 = 0$, then there is $0 < r < 1$ such that

$$f(x, u) \le \varepsilon_1 \|u\|_C^\alpha, \ \forall x \in \bar{\Omega}, \ 0 \le u \le r, \tag{6.1}$$

where $\varepsilon_1 > 0$ gratifies

$$\varepsilon_1 \|\phi_1\|_C \|\phi_1\|_C \le 1,$$

and ϕ_i are defined in (3.6) for $i \in \{1, 2\}$.

Thus, for $x \in \bar{\Omega}, \ u \in K \cap \partial D_{r_1}$, it hence follows from (3.2), (3.6), (6.1), $\alpha > 1$ and $0 < r = \|u\|_C < 1$ that

$$\begin{aligned}
\|T_1 u\|_C &= \max_{x\in\bar{\Omega}} \int_{\bar{\Omega}} G(x, y) f(y, u(y)) \mathrm{d}y \\
&= \max_{x\in\bar{\Omega}} \int_{\bar{\Omega}} \int_{\bar{\Omega}} G^*(x, z) G^*(z, y) f(y, u(y)) \mathrm{d}z \mathrm{d}y \\
&\le \varepsilon_1 \|u\|_C^\alpha \max_{x\in\bar{\Omega}} \int_{\bar{\Omega}} G^*(x, z) \mathrm{d}z \int_{\bar{\Omega}} G^*(z, y) \mathrm{d}y \\
&\le \varepsilon_1 \|u\|_C^\alpha \|\phi_1\|_C \|\phi_2\|_C \\
&\le \|u\|_C^\alpha \\
&< \|u\|_C.
\end{aligned}$$

This indicates that

$$\|T_1 u\|_C < \|u\|_C, \ \ \forall u \in K \cap \partial D_r. \tag{6.2}$$

Take $\eta > 1$. Next, we demonstrate that

$$\|T_1 u\|_C > \|u\|_C, \ \ \forall u \in K \cap \partial D_\eta. \tag{6.3}$$

If (1.5) holds, when a η is fixed, then there is a $l > 0$ so that

$$f(x, u) \ge l > \frac{\eta}{\tau_1 \tau_2 \text{mes } B_1 \text{mes } B_2 \text{mes } B_3 (\text{mes } \bar{\Omega})^{-1}}, \ \forall x \in \bar{\Omega}, \ u \ge \eta. \tag{6.4}$$

Therefore, for $u \in K \cap \partial D_\eta$, we get from (2.4), (2.5), (2.6), (3.2) and (6.4) that

$$
\begin{aligned}
(\text{mes } \bar{\Omega})\|T_1 u\|_C &\geq \int_{\bar{\Omega}} (T_1 u)(x) dx \\
&= \int_{\bar{\Omega}} dx \int_{\bar{\Omega}} G(x, y) f(y, u(y)) \mathrm{d}y \\
&= \int_{\bar{\Omega}} dx \int_{\bar{\Omega}} \int_{\bar{\Omega}} G^*(x, z) G^* z, y f(y, u(y)) \mathrm{d}z \mathrm{d}y \\
&\geq l \int_{B_1} dx \int_{B_2} \int_{B_3} G^*(x, z) G^* z, y \mathrm{d}z \mathrm{d}y \\
&\geq l \tau_1 \tau_2 \text{mes } B_1 \text{mes } B_2 \text{mes } B_3,
\end{aligned}
$$

which indicates that

$$
\|T_1 u\|_C \geq l \tau_1 \tau_2 \text{mes } B_1 \text{mes } B_2 \text{mes } B_3 \ (\text{mes} \bar{\Omega})^{-1} > \eta = \|u\|_C.
$$

This indicates that (6.3) is true.

Therefore, according to (a) of Lemma 6.1, (6.2) and (6.3) yields that operator T_1 possesses a fixed point u in $K \cap (\bar{D}_\eta \backslash D_r)$ with $r \leq \|u\|_C \leq \eta$. This derives that problem (1.3) admits at least one positive solution u with $r \leq \|x\|_C \leq \eta$. □

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