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ORIGINAL PAPER

Existence of capacity solution for a nonlocal thermistor problem in Musielak–Orlicz–Sobolev spaces

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Abstract
In this work, we study the existence of a capacity solution for a nonlocal thermistor problem in Musielak–Orlicz–Sobolev spaces. We get the existence of capacity solution using the approximate techniques and we prove the existence of a weak solution by introducing a sequence of approximate problems converging in a certain sense to a capacity solution. As a consequence, we obtain the existence of a capacity solution of the original problem in Musielak–Orlicz–Sobolev Lebesgue spaces.

Keywords Existence · Capacity solution · Nonlinear parabolic equation · Thermistor problem · Musielak–Orlicz–Sobolev spaces

Mathematics Subject Classification 35J60 · 32U20 · 35J60 · 35K61

1 Introduction

In recent decades, Sobolev spaces and Musielak–Orlicz spaces have become of great interest in the study of different problems [10]. In the context of Musielak–Orlicz spaces, the first work was done by Orlicz in 1930, followed by the work of Nakano in 1950 [24], in which the author presented a general study of these spaces. On the other hand, Czechoslovak and Polich investigated the modular function spaces. When the Leray Lions operator satisfies the nonpolynomial growth condition, the study of variational problems becomes more interesting in the applications to electro-rheology.

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Ruzicka and Rajagopal proposed a mathematical model of electro-rheological fluids (see [27, 28] for more details).

We consider the following problem modeling the temperature produced by a material crossed by an electric current flow:

$$\begin{cases} \dfrac{\partial u}{\partial s} - \Delta u = \lambda \dfrac{f(u)}{(\int_\Omega f(u)\mathrm{d}x)^2}, & \text{in } Q_S, \\ u(x,0) = u_0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega\times]0,S[, \end{cases} \tag{1.1}$$

where $f(u)$ is the electrical resistance of the conductor and $\dfrac{f(u)}{(\int_\Omega f(u)\mathrm{d}x)^2}$ represents the nonlocal term of (1.1). Whereas Q_S is defined as follows $Q_S := \Omega \times [0, S]$ where Ω is an open-bounded subset of $\mathbb{R}^N$, $N \geq 1$, S is a positive constant, and $]0, S[$ denotes the time horizon. The first equation in problem (1.1) describes the diffusion of temperature in the presence of a nonlocal term as a consequence of Joule effect in the second member. λ is a constant without dimension and can be identified by the square of the applied potential difference at the ends of the conductor. The function u represents the temperature generated by the electric current flowing through the material [9, 22]. There are various motivations behind the analysis of the heat and current flow in thermistors. One is the obvious question of design: how do the characteristics, such as the switch-off time in response to a current surge, depend on the physical parameters. Another is an issue of quality control: some thermistors can crack, because rapid thermal expansion caused by large temperature gradients stresses the material too much. In 1833, Micheal Faraday (1791–1867) discovered the thermistor and remarked that the augmentation of temperature implies a decrease of the Silver Sulfides resistance. The thermistor is defined as a temperature sensing device.

The thermistor problem has been excessively used by many authors (Antontsev and Chipot [4], [13]). They proved the existence of solution for thermistor problem in the context of vector-valued Sobolev spaces or in the standard Sobolev spaces. Hence, it is interesting to develop and analyze thermistor problem in the context of Musielak–Orlicz–Sobolev spaces.

Our aim is to prove the existence of a capacity solution in the sense of Definition 4.2 to system (1.1) which is the transformation of a coupled system consisting of an elliptic equation describing the quasistatic evolution of the electric potential and a nonlinear parabolic equation, which describes the temperature [22]. The literature on problems (1.1) and coupled system recall above is vast (see [1, 7, 14–16, 20, 25, 29]).

The rest of this paper is organized as follows: in Sect. 2, we state some basic concepts and a few known results that are useful for the results that will be established in this paper. In Sect. 3, we give the compactness results and the assumptions on data. In Sect. 4, we introduce the concept of capacity solutions. In Sect. 5, we develop the state of the main result of this paper. Finally, we give a conclusion and some perspectives.

2 Preliminaries

In this section, we introduce some definitions, proprieties, and basic notions of this work needed in the next sections.

Definition 2.1 (*See* [10]) Let $\psi : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, then ψ is Musielak–Orlicz functions if it satisfies two following conditions:

1. $\psi(., s)$ is a measurable function for all s in $\mathbb{R}$.
2. For each x in Ω, we have $\psi(x, .)$ is a N-function; also, it is convex in $\mathbb{R}$ and increasing in $\mathbb{R}^+$, such that

$$\lim_{s \longrightarrow 0} \frac{\psi(x,s)}{s} = 0, \quad \lim_{s \longrightarrow \infty} \frac{\psi(x,s)}{s} = \infty,$$
$$\psi(x,s) > 0, \text{ for all } s > 0,$$
$$\psi(x,s) = 0, \text{ for } s = 0.$$

Definition 2.2 (*See* [23]) Let ϕ and ψ be two Musielak–Orlicz functions defined in $\Omega \times \mathbb{R}$ with values in $\mathbb{R}$, then ψ dominates ϕ globally ($\phi << \psi$) if $\exists r > 0$ and $\exists s_0 \geq 0$, such that

$$\phi(x,s) \leq \psi(x,rs) \text{ for each } x \in \Omega \text{ and for all } s \geq s_0,$$

we also say ψ dominates ϕ globally if $s_0 = 0$ and beside infinity if $s_0 > 0$.

We define the space

$$F_\psi(\Omega) := \left\{u : \Omega \longrightarrow \mathbb{R} \text{ mesurable: } \varrho_{\psi,\Omega}(u) < \infty\right\},$$

where $\varrho_{\psi,\Omega}(u) = \int_\Omega \psi(x, u(x))\mathrm{d}x$.

Let $L_\psi(\Omega)$ the Musielak–Orlicz space generated by $F_\psi(\Omega)$, such that this last space is the Musielak–Orlicz class and it is the smallest vector space of the following space:

$$L_\psi(\Omega) := \{u : \Omega \longrightarrow \mathbb{R} \text{ mesurable: } \varrho_{\psi,\Omega}(u/\lambda) < \infty \text{ for each } \lambda > 0\}.$$

We define the complementary function of the Musielak function $\psi(x,r)$ in the sense of Young with respect to variable t as follows:

$$\overline{\psi}(x,t) = \sup_{r \geq 0}\{tr - \psi(x,r)\}.$$

Then, Young Fenchel inequality is defined by

$$| rt | \leq \overline{\psi}(x,t) + \psi(x,r) \text{ for all } r, t \in \mathbb{R} \text{ and } x \in \Omega.$$

We endow the space $L_\psi(\Omega)$ by Luxemburg norm

$$\| g \|_{\psi,\Omega} = \inf \left\{\lambda > 0 / \int_\Omega \psi\left(x, \frac{g(x)}{\lambda}\right) \mathrm{d}x < 1\right\},$$

or by Orlicz norm

$$\| g \|_{(\psi),\Omega} = \sup \left\{ \int_\Omega g(x)h(x)\mathrm{d}x : \; g \in E_{\overline{\psi}}(\Omega), \; \varrho_{\overline{\psi},\Omega}(h) < 1 \right\}.$$

Moreover, the following inequality holds:

$$\int_\Omega \psi(x, u(x))\mathrm{d}x \leq \| u \|_{(\psi),\Omega}, \text{ where } \| u \|_{(\psi),\Omega} \leq 1. \tag{2.1}$$

Using the above inequality, we get

$$\int_\Omega \psi(x, u(x)/ \| u \|_{(\psi),\Omega})\mathrm{d}x \leq 1, \text{ for all } u \in L_\psi(\Omega)\backslash\{0\}. \tag{2.2}$$

Also, we have the equivalent between Luxemburg norm and Orlicz norm

$$\| u \|_{\psi,\Omega} \leq \| u \|_{\psi,(\Omega)} \leq 2 \| u \|_{\psi,\Omega}. \tag{2.3}$$

For the proof, we refer to [23]. We also have the Hölder's inequality holds

$$\int_\Omega g(x)h(x)\mathrm{d}x \leq \| g \|_{\psi,\Omega} \| h \|_{\overline{\psi},(\Omega)} \text{ for all } g \in L_\psi(\Omega) \text{ and } h \in L_{\overline{\psi}}(\Omega); \tag{2.4}$$

if Ω has a finite measure, the inequality (2.4) implies the following continuous inclusion: $L_\psi(\Omega) \subset L^1(\Omega)$ which is strict in general.

We denote by $E_\psi(\Omega)$ the set of the closure of bounded measurable functions with compact support in the closure of Ω denoted by $\overline{\Omega}$ with respect to the norm of $L_\psi(\Omega)$.

Throughout this paper, we will use the standard reference for Musielak–Orlicz–Sobolev spaces [23]; see also [3]. Now, we introduce some definition and lemmas useful hereafter.

Definition 2.3 Let $(u_n)_{n\in\mathbb{N}} \subset L_\psi(\Omega)$, we say $(u_n)_{n\in\mathbb{N}}$ converges to $u \in L_\psi(\Omega)$ if there exists $l > 0$, such that

$$\lim_{n\longrightarrow\infty} \varrho_{\psi,\Omega}\left(\frac{u_n - u}{l}\right) = 0.$$

For all $p \in \mathbb{N}$, we defined a Musielak–Orlicz–Sobolev spaces as follows:

$$W^p L_\psi(\Omega) := \{u \in L_\psi(\Omega)/D^\alpha u \in L_\psi(\Omega) \text{ for all } \alpha, | \alpha | \leq p\},$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, ..., \alpha_{m-1}, \alpha_m) \in \mathbb{Z}^m$, $| \alpha | = \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_{m-1} + \alpha_m$ and

$$D^\alpha = \partial_1^{\alpha_1}.\partial_2^{\alpha_2}...\partial_n^{\alpha_n} \quad \text{with} \qquad \partial_j = \frac{\partial}{\partial x_j}.$$

D^α is the distributional derivative of multi-index α.

For each Musielak–Orlicz–Sobolev space $W^pL_\psi(\Omega)$, we define the modular as follows:

$$\varrho^{(p)}_{\psi,\Omega}(u) := \sum_{|\alpha|\leq p} \varrho_{\psi,\Omega}(D^\alpha u),$$

which is convex in $W^pL_\psi(\Omega)$. We can equipped Musielak–Orlicz–Sobolev space with

$$\| u \|^{(p)}_{\psi,\Omega} = \inf\{\lambda > 0 / \varrho^{(p)}_{\psi,\Omega}(u/\lambda) \leq 1\} \text{ or with } \| u \|_{p,\psi,\Omega} = \sum_{|\alpha|\leq p} \| D^\alpha u \|_{\psi,\Omega}.$$

The above two norms are equivalent on $W^pL_\psi(\Omega)$. The pair $(W^pL_\psi(\Omega), \| u \|^{(p)}_{\psi,\Omega})$ is a Banach space [23], if $\exists z_0 > 0$, such that

$$\operatorname*{ess\,inf}_{x\in\Omega} \psi(x, 1) > z_0. \tag{2.5}$$

Then, $(W^pL_\psi(\Omega), \| u \|_{p,\psi,\Omega})$ is a Banach space.

Hereafter, we suppose that the condition (2.5) is satisfied. The space $W^pL_\psi(\Omega)$ can be identified to a $\sigma(\Pi_{|\alpha|\leq p}L_\psi(\Omega), \Pi_{|\alpha|\leq p}E_{\overline{\psi}}(\Omega))$-closed subspace of $\Pi_{|\alpha|\leq p}L_\psi(\Omega)$. Let $W^p_0L_\psi(\Omega) = \overline{D(\Omega)}^{\sigma(\Pi_{|\alpha|\leq p}L_\psi(\Omega), \Pi_{|\alpha|\leq p}E_{\overline{\psi}}(\Omega))}$ and $W^pE_\psi(\Omega)$ the spaces of functions u, where u and its distribution derivatives up to order m lie in $E_\psi(\Omega)$. Moreover, $W^p_0E_\psi(\Omega)$ is the norm closure of $D(\Omega)$ in $W^pL_\psi(\Omega)$.

Lemma 2.4 (Poincaré's inequality see [2]) *Let Ω a subset of $\mathbb{R}^N$ a bounded Lipchitz-continuous set, then there exists a constant $C = C(\Omega) > 0$, such that*

$$\| u \|_{\psi,\Omega} \leq C \| \nabla u \|_{\psi,\Omega}, \textit{ for all } u \in W^p_0L_\psi(\Omega). \tag{2.6}$$

Remark 2.5 Let $u \in W^p_0L_\psi(\Omega)$ where ψ is a Musielak–Orlicz function, we suppose that there exists a positive constant C, such that

$$\int_\Omega \psi(x, \nabla u)\mathrm{d}x \leq C,$$

then

$$\int_\Omega \psi\left(x, \frac{\nabla u}{C}\right)\mathrm{d}x \leq 1.$$

Using the convexity of $\psi(x, .)$ and if $C \geq 1$, we get

$$C \in \left\{\lambda > 0 / \int_\Omega \psi\left(x, \frac{\nabla u}{\lambda}\right)\mathrm{d}x < 1\right\}, \textit{and} \text{ hence, } \| \nabla u \|_{\psi,\Omega} \leq C,$$

if not, i.e., $C < 1$, we obtain $\int_\Omega \psi(x, \nabla u)\mathrm{d}x \leq C < 1$, then $\| \nabla u \|_{\psi,\Omega} \leq 1$.

In view of the fact that $u \in W_0^p L_\psi(\Omega)$, we apply Lemma 2.4, we get that there exists a positive constant $C = C(\Omega)$, such that

$$\| u \|_{\psi,\Omega} \leq C \| \nabla u \|_{\psi,\Omega} \text{ for all } u \in W_0^p L_\psi(\Omega).$$

On the other hand, we have $\| u \|_{1,\psi,\Omega} = \| u \|_{\psi,\Omega} + \| \nabla u \|_{\psi,\Omega}$, and hence

$$\| u \|_{1,\psi,\Omega} \leq (C+1) \| \nabla u \|_{\psi,\Omega} \leq (C+1)\max(C, 1).$$

Then

$$\| u \|_{1,\psi,\Omega} \leq (C+1)\max(C, 1).$$

In the next of this paper, we suppose that ψ and ϕ are two generalized N-function, such that $\psi << \phi$. We also assume that the following conditions hold for complementary functions $\overline{\psi}$ and $\overline{\phi}$

$$\lim_{|s| \longrightarrow \infty} \operatorname*{ess\,inf}_{x \in \Omega} \frac{\overline{\psi}(x, s)}{| s |} = \infty, \tag{2.7}$$

$$\lim_{|s| \longrightarrow \infty} \operatorname*{ess\,inf}_{x \in \Omega} \frac{\overline{\phi}(x, s)}{| s |} = \infty. \tag{2.8}$$

Remark 2.6 (See [18], Remark 2.1) We suppose (2.7) and (2.8) hold, then

$$\sup_{s \in B(x,K)} \operatorname*{ess\,sup}_{x \in \Omega} \psi(x, s) < +\infty, \quad \text{for all } 0 < K < +\infty, \tag{2.9}$$

$$\sup_{s \in B(x,K)} \operatorname*{ess\,sup}_{x \in \Omega} \phi(x, s) < +\infty, \quad \text{for all } 0 < K < +\infty. \tag{2.10}$$

Definition 2.7 Let $(u_n)_{n\in\mathbb{N}} \subset W^p L_\psi(\Omega)$, we say that $(u_n)_{n\in\mathbb{N}}$ converges to $u \in W^p L_p si(\Omega)$ for the modular convergence in $W^p L_\psi(\Omega)$ if and only if

$$\lim_{n \longrightarrow \infty} \varrho^{(p)}_{\psi,\Omega}\left(\frac{u_n - u}{l}\right) = 0, \text{ for some } l > 0.$$

Also, we can define these spaces of distributions as follows:

$$W^{-p} L_{\bar{\psi}}(\Omega) := \left\{ g \in \mathcal{D}'(\Omega) : g = \sum_{|\alpha| \leq p} (-1)^{|\alpha|} D^\alpha g_\alpha \text{ for each } g_\alpha \in L_{\bar{\psi}}(\Omega) \right\},$$

$$W^{-p} E_{\bar{\psi}}(\Omega) := \left\{ g \in \mathcal{D}'(\Omega) : g = \sum_{|\alpha| \leq p} (-1)^{|\alpha|} D^\alpha g_\alpha \text{ for each } g_\alpha \in E_{\bar{\psi}}(\Omega) \right\}.$$

Lemma 2.8 *Let $(u_n)_{n\in\mathbb{N}} \subset L_{\psi}(\Omega)$, If $\phi << \psi$ and $(u_n)_{n\in\mathbb{N}}$ converges to $u \in L_{\psi}(\Omega)$, in the sense of modular convergent, then $(u_n)_{n\in\mathbb{N}}$ converges to u strongly in $E_{\phi}(\Omega)$. In particular, the following continuous injection hold: $L_{\psi}(\Omega) \subset E_{\phi}(\Omega)$ and $L_{\overline{\phi}}(\Omega) \subset E_{\overline{\psi}}(\Omega)$.*

Proof By hypothesis, for $\ell > 0$ and $\varepsilon > 0$, we have

$$\lim_{n\longrightarrow\infty} \int_{\Omega} \psi\left(x, \frac{u_n - u}{\ell}\right) \mathrm{d}x = 0.$$

Then, there exists $h_0 \in L^1(\Omega)$, such that

$$\psi\left(x, \frac{u_n - u}{\ell}\right) \le h_0 \quad \text{and} \quad u_n \longrightarrow u, \quad \text{a.e in } \Omega,$$

for a subsequence of $(u_n)_{n\in\mathbb{N}}$ which is still denoted $(u_n)_{n\in\mathbb{N}}$ for convenience. Knowing that $\phi << \psi$, then by applying Definition 2.2, there exists $k > 0$, such that $\lim_{s\longrightarrow\infty} \sup_{x\in\Omega} \dfrac{\phi(x, ks)}{\psi(x, s)} = 0$. As a consequence, there exists $s_0 \ge 0$, such that

$$\frac{\phi(x, ks)}{\psi(x, s)} \le 1, \text{ for each } x \in \Omega \text{ and for all } \ s \ge s_0.$$

Let set $k = \dfrac{\ell}{\varepsilon}$ and $s = \dfrac{t}{\ell}$ where $t = u_n - u$, and hence

$$\phi\left(x, \frac{u_n - u}{\varepsilon}\right) \le \psi\left(x, \frac{u_n - u}{\ell}\right) \text{ for each } x \in \Omega \text{ and for all } t \ge \ell s_0.$$

Using the characteristic function χ_{Ω}, we get

$$\phi\left(x, \frac{u_n - u}{\varepsilon}\right) \le \phi\left(x, \frac{u_n - u}{\varepsilon}\right) \chi_{[0,\ell s_0]} + \phi\left(x, \frac{u_n - u}{\varepsilon}\right) \chi_{]\ell s_0,\infty[}.$$

Then

$$\phi\left(x, \frac{u_n - u}{\varepsilon}\right) \le \sup_{s\in[0,\ell s_0]} \operatorname*{ess\,sup}_{x\in\Omega} \phi(x, s) + \phi\left(x, \frac{u_n - u}{\varepsilon}\right) \chi_{]\ell s_0,\infty[},$$
$$\phi\left(x, \frac{u_n - u}{\varepsilon}\right) \le \sup_{s\in[0,\ell s_0]} \operatorname*{ess\,sup}_{x\in\Omega} \phi(x, s) + \psi\left(x, \frac{u_n - u}{\ell}\right).$$

From Remark 2.1 in [18], we get $\sup_{s\in[0,\ell s_0]} \operatorname{ess\,sup}_{x\in\Omega} \phi(x, s) < +\infty$.

Then, there exists $h_1 > 0$, such that

$$h_0 + \sup_{s\in[0,\ell s_0]} \operatorname*{ess\,sup}_{x\in\Omega} \phi(x, s) \le h_1.$$

Thanks to Lebesgue's dominated convergence theorem, we have

$$\phi\left(x, \frac{u_n - u}{\varepsilon}\right) \longrightarrow 0 \text{ as } n \longrightarrow \infty \text{ in } L^1(\Omega);$$

for n near infinity, we obtain

$$\| u_n - u \|_{\phi,\Omega} \leq \varepsilon, \text{ then } u_n \longrightarrow u \text{ in } L_\phi(\Omega).$$

The continuous embedding $L_\psi(\Omega) \subset E_\phi(\Omega)$ is trivial, because the convergence in $L_\psi(\Omega)$ implies the modular convergence. On the other side, we have $\phi << \psi$ is equivalent to $\overline{\psi} << \overline{\phi}$; as a consequence, the following embedding $L_{\overline{\phi}}(\Omega) \subset E_{\overline{\psi}}(\Omega)$ is continuous. □

Lemma 2.9 (See [12]) *Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be two convergent sequences in $L_\psi(\Omega)$ and $L_{\overline{\psi}}(\Omega)$, respectively, and denote by $f \in L_\psi(\Omega)$ and $g \in L_{\overline{\psi}}(\Omega)$ their corresponding limits in the sense of modular convergence, then*

$$\lim_{n\longrightarrow\infty} \int_\Omega g_n f \,\mathrm{d}x = \int_\Omega g f \,\mathrm{d}x,$$
$$\lim_{n\longrightarrow\infty} \int_\Omega g_n f_n \mathrm{d}x = \int_\Omega g f \,\mathrm{d}x.$$

Lemma 2.10 (See [6]) *Let Ω be a bounded, Lipchitz-continuous subset of $\mathbb{R}^N$, ψ a Musielak–Orlicz function and $\overline{\psi}$ its complementary. Then*

- *$D(\Omega)$ is dense in $L_\psi(\Omega)$ with respect to the modular convergence.*
- *$D(\Omega)$ is dense in $W_0^1 L_\psi(\Omega)$ and $D(\overline{\Omega})$ is dense in $W^1 L_\psi(\Omega)$.*

The previous densities are with respect to the modular convergence. Moreover, all the previous densities hold true if the following conditions are satisfied:

(1) There exists a constant $\lambda > 0$, such that $\forall x, y \in \Omega$, $| x - y | \leq \frac{1}{2}$ implies

$$\frac{\phi(x,\ell)}{\phi(y,\ell)} \leq \ell^{-\frac{\lambda}{\log(| x - y |)}} \text{ for all } \ell \geq 1. \tag{2.11}$$

(2) There exists a constant $\beta > 0$, such that

$$\overline{\psi}(x, 1) \leq \beta, \text{ a.e in } \Omega. \tag{2.12}$$

Remark 2.11 Define the measurable function $q : \Omega \longrightarrow]1, \infty[$ and suppose that there exists a positive constant C, such that for all $x, y \in \Omega$ with $| x - y | < \frac{1}{2}$, we have

$$|q(y) - q(x)| \leq \frac{C}{| \log |y - x||}.$$

Then, the following Musielak–Orlicz functions:

(1) $\psi(x,\ell)=\ell^{q(x)}$,
(2) $\psi(x,\ell)=\ell^{q(x)}\log(1+\ell)$,
(3) $\psi(x,\ell)=\ell\log(1+\ell)(\log(e-1+\ell))^{q(x)}$,

satisfy the inequality (2.11).

Now, let us introduce inhomogeneous Musielak–Orlicz–Sobolev spaces. Let $\Omega\subset\mathbb{R}^N$ be an open-bounded set and ψ a Musielak–Orlicz function defined in $Q_S:=\Omega\times]0.S[$ with $S>0$. We denote by D_x^α the distributional derivative on Q_S of order $\alpha\in\mathbb{Z}^N$, where α is a multi-index with respect to the variable x. We define the inhomogeneous Musielak–Orlicz–Sobolev spaces as follows:

$$W^{p,x}L_\psi(Q_S):=\{g\in L_\psi(Q_S)/D_x^\alpha g\in L_\psi(Q_S)\ \text{ for all }\alpha, \mid\alpha\mid\leq p\},$$
$$W^{p,x}E_\psi(Q_S):=\{g\in E_\psi(Q_S)/D_x^\alpha g\in E_\psi(Q_S)\ \text{ for all }\alpha, \mid\alpha\mid\leq p\};$$

we equip the spaces $W^{p,x}L_\psi(Q_S)$ and $W^{p,x}E_\psi(Q_S)$ with the norm

$$\| g \|=\sum_{|\alpha|\leq p}\| D^\alpha g \|_{\psi,Q_S}.$$

For $p=1$, the pairs $(W^{p,x}L_\psi(Q_S),\|\ .\ \|)$ and $(W^{p,x}E_\psi(Q_S),\|\ .\ \|)$ are Banach spaces [17]. The two last spaces are considered as subspaces of the product space

$$\prod_{|\alpha|\leq m}L_\psi(Q_S)=\prod L_\psi.$$

We consider the weakly star topology $\sigma(\Pi_{|\alpha|\leq p}L_\psi(Q_S),\Pi_{|\alpha|\leq p}E_{\overline{\psi}}(Q_S))$ and $\sigma(\Pi_{|\alpha|\leq p}L_\psi(Q_S),\Pi_{|\alpha|\leq p}L_{\overline{\psi}}(Q_S))$. If $u\in W^{p,x}L_\psi(Q_S)$, then the following mapping:

$$u:]0.S[\longrightarrow W^1L_\psi(Q_S).$$
$$s\longrightarrow u(s)$$

is well defined. Moreover, if $u\in W^{1,x}E_\psi(Q_S)$, this function is a $W^1E_\psi(\Omega)$-valued function and is strongly measurable. We cannot assure the measurability of the function $u(s)$ on $]0.S[$. However, the function $s\longrightarrow\| u(s) \|_{\psi,\Omega}$ belongs to the space $L^1(]0.S[)$. We define the space $W_0^{1,x}E_\psi(Q_S)$ as follows:

$$W_0^{1,x}E_\psi(Q_S)=\overline{D(Q_S)}^{\|.\|_{W^{1,x}E_\psi(Q_S)}}.$$

If Ω is a Lipschitz-continuous domain, we can show as in [6] that each element u in the closure of $D(Q_S)$ with respect of weak-$*$ topology associated $(\sigma(\prod L_\psi,\prod E_{\overline{\psi}}))$ is a limit in $W^{1,x}L_\psi(Q_S)$, of subsequence $(u_n)_{n\in\mathbb{N}}\subset D(Q_S)$. We emphasize that the modular convergence, i.e., there exists a positive constant ℓ, such that for all $\mid \alpha \mid \leq 1$, we have

$$\lim_{n \longrightarrow \infty} \int_{Q_S} \psi\left(x, \frac{D_x^\alpha u_n - D_x^\alpha u}{\ell}\right) \mathrm{d}x \mathrm{d}s = 0,$$

implies that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to u in $W^{1,x}L_\psi(Q_S)$ for the weak-$*$ topology $\sigma(\prod L_\psi, \prod L_{\overline{\psi}})$. Consequently, we obtain

$$\overline{D(Q_S)}^{\sigma(\prod L_\psi, \prod L_{\overline{\psi}})} = \overline{D(Q_S)}^{\sigma(\prod L_\psi, \prod E_{\overline{\psi}})}.$$

This space is denoted by $W_0^{1,x}E_\psi(Q_S)$. Moreover, we have

$$W_0^{1,x}E_\psi(Q_S) = W_0^{1,x}L_\psi(Q_S) \cap \prod E_{\overline{\psi}}.$$

In $W_0^{1,x}L_\psi(Q_S)$, the following Poincaré's inequality holds:

$$\sum_{|\alpha|\leq 1} \left\| D_x^\alpha u \right\|_{\psi, Q_S} \leq C \sum_{|\alpha|=1} \left\| D_x^\alpha u \right\|_{\psi, Q_S}. \tag{2.13}$$

We denote by $W^{-1,x}L_\psi(Q_S)$ the topologic dual of $W_0^{1,x}E_\psi(Q_S)$ characterized by

$$W^{-1,x}L_{\overline{\psi}}(Q_S) = \left\{ g = \sum_{|\alpha|\leq 1} D_x^\alpha g_\alpha : g_\alpha \in L_{\overline{\psi}}(Q_S) \text{ for all } \alpha \right\},$$

which can be equipped by the usual quotient norm

$$\| g \| = \inf \sum_{|\alpha|\leq 1} \left\| D_x^\alpha g_\alpha \right\|_{\overline{\psi}, Q_S} \text{ for all } g_\alpha \in L_{\overline{\psi}}(Q_S) \text{ where } g = \sum_{|\alpha|\leq 1} D_x^\alpha g_\alpha.$$

Furthermore, we denote $W^{-1,x}E_{\overline{\psi}}(Q_S)$ the subspace of $W^{-1,x}L_{\overline{\psi}}(Q_S)$ consisting of linear forms which are $(\sigma(\prod L_\psi, \prod E_{\overline{\psi}}))$-continuous. It can be shown that

$$W^{-1,x}E_{\overline{\psi}}(Q_S) := \left\{ g = \sum_{|\alpha|\leq 1} D_x^\alpha g_\alpha : g_\alpha \in E_{\overline{\psi}}(Q_S) \text{ for all } \alpha \right\}.$$

In the sequel, we need the following lemma.

Lemma 2.12 *We assume that ϕ is a Musielak function verifying the condition (2.8) and we suppose that $s^2 \leq \psi(x, s)$ for all $x \in \Omega$ and $s \in \mathbb{R}$. Then, the following embedding:*

$$L_\phi(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L_{\overline{\phi}}(\Omega),$$

are continuous. In particular, $W_0^1L_\phi(\Omega) \hookrightarrow H_0^1(\Omega)$ *and* $H^{-1}(\Omega) \hookrightarrow W^{-1}L_{\overline{\phi}}(\Omega)$. *Moreover, if* ψ *is a Musielak function verifying (2.7) and* $\phi << \psi$, *then the following embeddings:*

$$L_\psi(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L_{\overline{\psi}}(\Omega),$$

are continuous. Consequently, the following embeddings $W_0^1L_\psi(\Omega) \hookrightarrow H_0^1(\Omega)$ *and* $H^{-1}(\Omega) \hookrightarrow W^{-1}L_{\overline{\psi}}(\Omega)$ *are continuous.*

Proof By hypothesis, we have $v^2 \leq \phi(x, v)\ \ for\ all\ x \in \Omega$, from whence follows that:

$$\int_\Omega v^2 \mathrm{d}x \leq \int_\Omega \psi(x, v)\mathrm{d}x \ \text{ for all } x \in \Omega \text{ and } \ v \in F_\phi(\Omega),$$

we set $v = \dfrac{u}{\| u \|_{(\phi),\Omega}}$ and $u \neq 0$

$$\int_\Omega u^2 \mathrm{d}x \leq \| u \|^2_{(\phi),\Omega} \int_\Omega \phi\left(x, \frac{u}{\| u \|_{(\phi),\Omega}}\right)\mathrm{d}x \ \text{ for all } \ x \in \Omega \text{ and } \ v \in F_\phi(\Omega).$$

It yields that

$$\| u \|_{L^2(\Omega)} \leq \| u \|_{(\phi),\Omega},$$

which proves the first embedding.

Now, let $\phi << \psi$, for $r \in]0.S[$

$$\phi(x, s) \leq \psi(x, rs) \qquad \text{for each} \quad x \in \Omega \quad \text{and for all } s \geq s_0. \tag{2.14}$$

Then, for $v \in F_\psi(\Omega)$ and using Remark 2.1 in [18], we deduce the existence of a positive constant C_1, such that

$$\int_\Omega v^2 \mathrm{d}x \leq \int_{\{|v| \geq s_0\}} \phi(x, v)\mathrm{d}x + \int_{\{|v| < s_0\}} \phi(x, v)\mathrm{d}x \ \text{ for all } x \in \Omega \text{ and } \ v \in F_\phi(\Omega),$$
$$\int_\Omega v^2 \mathrm{d}x \leq \int_\Omega \psi(x, rv)\mathrm{d}x + C_1 \ \text{ for all } x \in \Omega \text{ and } \ v \in F_\phi(\Omega),$$
$$\int_\Omega v^2 \mathrm{d}x \leq r \int_\Omega \psi(x, v)\mathrm{d}x + C_1 \ \text{ for all } x \in \Omega \text{ and } \ v \in F_\phi(\Omega).$$

We replace v in the above inequality by $\dfrac{u}{\| u \|_{(\psi),\Omega}}$ where $u \neq 0$, and we use (2.2) and we get

$$\| u \|_{L^2(\Omega)} \leq C_2 \| u \|_{(\psi),\Omega}, \quad u \in L_\psi(\Omega),$$

where $C_2 = (C_1 + r)^{1/2}$. □

Remark 2.13 We assume that the hypothesis of Lemma 2.12 is satisfied, then

$$L^2\left(]0.S[;\, H^{-1}(\Omega)\right) \hookrightarrow W^{-1,x}L_{\bar{\phi}}(Q_S) \hookrightarrow W^{-1,x}E_{\bar{\psi}}(Q_S).$$

For the proof, it suffices to assume that $g \in L^2(]0.S[;\, H^{-1}(\Omega))$, then for $g_\alpha \in L^2(Q_S)$, $g = \sum_{|\alpha|\le 1} D_x^\alpha g_\alpha$ and by Lemma 2.8, we have

$$L^2(Q_S) \subset L_{\bar{\phi}}(Q_S) \subset E_{\bar{\psi}}(Q_S),$$

and thus

$$g \in W^{-1,x}L_{\bar{\phi}}(Q_S) \hookrightarrow W^{-1,x}E_{\bar{\psi}}(Q_S).$$

We introduce the truncation operation $S_R : \mathbb{R} \longrightarrow \mathbb{R}$, appearing in [8]

$$S_R(r) = \begin{cases} r & \text{if } |r| \le R, \\ R\dfrac{r}{|r|} & \text{if } |r| > R. \end{cases} \tag{2.15}$$

Then, its primitive is defined as follows:

$$T_R(r) = \int_0^r S_R(s)\mathrm{d}s = \begin{cases} r^2/2 & \text{if } |r| \le R, \\ R|r| - R^2/2 & \text{if } |r| > R. \end{cases} \tag{2.16}$$

3 Compactness results

In this section, we state trace and mollification results.

Let Ω be an open-bounded subset of $\mathbb{R}^N$ with a Lipschitz-continuous boundary, and ψ be a Musielak function. We set $Q_S =]0.S[\times\Omega$. For $u \in L^1(Q_S)$, $\eta > 0$, $r \in [0, S]$ and $x \in \Omega$, we define u_η as follows:

$$u_\eta(x, r) = \eta \int_{-\infty}^{r} \widetilde{u}(x, t)\exp(\eta(t - r))\mathrm{d}t, \tag{3.1}$$

where $\widetilde{u}(x, t) = u(x, t)\chi_{]0,S[}$.

The following lemmas play a crucial role in the sequel of this paper.

Lemma 3.1 (See [11]) *The following assertions hold:*

(1) *Given any function $u \in L_\psi(Q_S)$, then $u_\eta \in C([0, S]; L_\psi(\Omega))$ and $\lim_{\eta\longrightarrow\infty} u_\eta = u$ in $L_\psi(Q_S)$ for the modular convergence.*
(2) *Let $u \in W^{1,x}L_\psi(Q_S)$, we have $u_\eta \in C([0, S]; W^1L_\psi(\Omega))$ and $\lim_{\eta\longrightarrow\infty} u_\eta = u$ in $W^{1,x}L_\psi(Q_S)$ for the modular convergence.*

(3) *Let $u \in E_\psi(Q_S)$ (resp, $u \in W^{1,x}E_\psi(Q_S)$). $\lim_{\eta \longrightarrow \infty} u_\eta = u$ strongly in $E_\psi(Q_S)$ (resp, strongly in $W^{1,x}E_\psi(Q_S)$).*

(4) *Let $u \in W^{1,x}L_\psi(Q_S)$, then $\dfrac{\partial u_\eta}{\partial s} = \eta(u - u_\eta) \in W^{1,x}L_\psi(Q_S)$.*

(5) *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W^{1,x}L_\psi(Q_S)$ and $u \in W^{1,x}L_\psi(Q_S)$, such that $u_n \longrightarrow u$ as $n \longrightarrow \infty$ strongly in $W^{1,x}L_\psi(Q_S)$ (resp, for the modular convergence). Then, for each $\eta > 0$, we obtain $(u_n)_\eta \longrightarrow u_\eta$ strongly in $W^{1,x}L_\psi(Q_S)$ (resp, for the modular convergence).*

Lemma 3.2 (See [11]) *The following embedding:*

$$E_\psi\left(Q_S\right) \hookrightarrow L^1\left(0, S; E_\psi(\Omega)\right), \tag{3.2}$$

is continuous.

Lemma 3.3 (See [11]) *The following embeddings:*

$$W^1E_\psi\left(Q_S\right) \hookrightarrow L^1\left(0, S; W^1E_\psi(\Omega)\right), \tag{3.3}$$

$$W^{-1,x}E_{\bar{\psi}}(Q_S) \hookrightarrow L^1\left(0, S; W^{-1}E_{\bar{\psi}}(\Omega)\right), \tag{3.4}$$

are continuous.

The Lemmas 3.4 and 3.6 play a key role in the proof of Theorem 5.2, whereas the Lemma 3.5 plays an important role in the Step III.

Lemma 3.4 (See [12]) *Given a Banach space Y, such that $L^1(\Omega) \hookrightarrow Y$ is a continuous embedding. If H is bounded in $W_0^{1,x}L_\psi(Q_S)$ and relatively compact in $L^1(0, S; Y)$, then H is relatively compact in $L^1(Q_S)$ and in $E_\phi(Q_S)$ for every $\phi << \psi$.*

Lemma 3.5 (See [12]) *Let Ω be an open-bounded subset of $\mathbb{R}^N$ with the segment property. Then, the following inclusion:*

$$F = \{u \in W_0^{1,x}L_\psi\left(Q_S\right) : \frac{\partial u}{\partial s} \in W^{-1,x}L_{\bar{\psi}}\left(Q_S\right) + L^1\left(Q_S\right)\} \subset C(]0, S[; L^1(\Omega))$$

holds with a continuous embedding.

Lemma 3.6 (See Theorem 2 in [12]) *Let ψ be a Musielak function. If H is a bounded subset of $W_0^{1,x}L_\psi\left(Q_S\right)$ and $\{\dfrac{\partial g}{\partial t}/g \in H\}$ is bounded in $W^{-1,x}L_{\bar{\psi}}\left(Q_S\right)$, then H is relatively compact in $L^1\left(Q_S\right)$.*

In our study, we obtain the existence of a weak solution by applying Theorem 3.7.

We define the partial differential operator as follows:
$B : D(B) \subset W^{1,x}L_\psi\left(Q_S\right) \longrightarrow W^{1,x}L_\psi\left(Q_S\right)$, such that $B(u) = -\text{div}\, a(x, s, \nabla u)$ where $a(., ., .) : \Omega\times]0.S[\times\mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function, and for all

$(x,s) \in Q_S$, two real numbers $\lambda > 0$, $k \geq 0$ and $z, y \in \mathbb{R}^N$ where $z \neq y$, the following conditions hold:

$$| a(x,s,z) | \leq \lambda(e_1(x,s) + \bar{\psi}^{-1}(x, \psi(x, k|z|))), \tag{3.5}$$

$$(a(x,s,z) - a(x,s,y))(z-y) > 0, \tag{3.6}$$

$$a(x,s,z)z \geq \alpha\psi(x,|z|). \tag{3.7}$$

For each function $g \in W^{-1,x}L_{\bar{\psi}}(Q_S)$ and for all $u_0 \in L^2(\Omega)$, we consider the following parabolic problem:

$$\begin{cases} \dfrac{\partial u}{\partial s} - \operatorname{div} a(x,s,\nabla u) = g, & \text{in } Q_S, \\ u(x,0) = u_0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \times]0,S[. \end{cases} \tag{3.8}$$

The following theorem plays a key role in the proof of Theorem 5.2.

Theorem 3.7 (See [26]) *We suppose that the conditions (3.5) to (3.7) are satisfied, then the problem (3.8) has a weak solution* $u \in D(B) \cap W_0^{1,x}L_\psi(Q_S) \cap C([0,S]; L^2(\Omega))$ *where* $a(x,s,\nabla u) \in W^{-1,x}L_{\bar{\psi}}(Q_S)$, *and for all* $\omega \in W_0^{1,x}L_\psi(Q_S)$ *with* $\dfrac{\partial \omega}{\partial s} \in W^{-1,x}L_{\bar{\psi}}(Q_S)$ *and for all* $r \in [0,S]$, *we have*

$$-\left\langle \frac{\partial \omega}{\partial s}, u \right\rangle_{Q_r} + \int_\Omega u(x,r)\omega(x,r)\mathrm{d}x + \int_0^r \int_\Omega a(x,s,\nabla u)\nabla\omega \mathrm{d}x\mathrm{d}s$$
$$= \langle g, \omega\rangle_{Q_r} + \int_\Omega u(x,0)\omega(x,0)\mathrm{d}x,$$

where $\langle .,.\rangle_{Q_r} = \langle .,.\rangle_{W^{-1,x}L_{\bar{\psi}}(Q_S), W_0^{1,x}L_\psi(Q_S)}$. *Moreover, for all* $r \in [0,S]$, *the following energy identity holds:*

$$\frac{1}{2}\int_\Omega |u(x,r)|^2\mathrm{d}x + \int_0^r \int_\Omega \nabla u \nabla u \mathrm{d}x\mathrm{d}t = \langle g, u\rangle_{Q_r} + \frac{1}{2}\int_\Omega |u(0,x)|^2\,\mathrm{d}x.$$

4 Concept of capacity solution

Here, we define the concept of capacity solution for problem (1.1) in the context of the Musielak–Orlicz–Sobolev spaces. Now, let $\Omega \subset \mathbb{R}^N$ be an open-bounded set and let ψ a Musielak function verifying the inequalities (2.11) and (2.12).

$$F = \left\{ \omega \in W_0^{1,x}L_\psi(Q_S) : \frac{\partial \omega}{\partial s} \in W^{-1,x}L_{\bar{\psi}}(Q_S) \right\}.$$

We equip the space F by the following norm:

$$\|\omega\|_F = \|\omega\|_{W_0^{1,x}L_\psi(Q_S)} + \left\|\frac{\partial\omega}{\partial s}\right\|_{W^{-1,x}L_{\bar{\psi}}(Q_S)}.$$

The pair $(F, \| \,.\, \|_F)$ is a Banach space.

In the sequel of this paper, we consider $\langle ., .\rangle_{Q_r} = \langle ., .\rangle_{W^{-1,x}L_{\bar{\psi}}(Q_S), W_0^{1,x}L_\psi(Q_S)}$, and we assume the following conditions:

$$\phi << \psi \text{ and } s^2 \leq \phi(x, s) \text{ for each } x \in \Omega \text{ and for all } s \in \mathbb{R}. \tag{4.1}$$

Let $\bar{\phi}$ and $\bar{\psi}$ be two complementary functions of the Musielak functions $\phi(x, r)$ and $\psi(x, r)$, respectively, satisfying the conditions (2.9) and (2.10), respectively. We consider also the operator

$$B : D(B) \subset W^{1,x}L_\psi\,(Q_S) \longrightarrow W^{1,x}L_\psi\,(Q_S),$$

where $Bu = -\operatorname{div} a(x, s, \nabla u)$, such that $a(., ., .)$ is a Leray–Lions operator where $a(x, s, \nabla u) =\mid \nabla u \mid^{p-2} \nabla u$. In our case, we take $p = 2$ where $\nabla : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ satisfies the following assumptions, for all $(x, s) \in Q_S$:

$$|\nabla u| \leq \zeta \left[c(x, s) + \bar{\psi}_x^{-1}(\psi(x, k|u|))\right], \tag{4.2}$$

$$\alpha\psi(x, |\nabla u - \nabla v|) \leq |\nabla u - \nabla v|^2, \tag{4.3}$$

where $c(x, s) \in E_{\bar{\psi}}(Q_S)$, $\alpha, k, \zeta > 0$ are given real numbers.

The initial condition is given by

$$u_0 \in L^2(\Omega). \tag{4.4}$$

We suppose that f is a locally L_1-Lipschitz function and there exists a positive constant σ, such that

$$\sigma \leq f(t), \ \text{ for all } t \in \mathbb{R}. \tag{4.5}$$

Remark 4.1 Under the condition (4.3) and for $\nabla v = 0$, we get

$$\alpha\psi(x, |\nabla u|) \leq |\nabla u|^2 \text{ for all } u \in \mathbb{R}^N. \tag{4.6}$$

We introduce the notion of capacity solution as follows:

Definition 4.2 The pair (u, f) is called a capacity solution for the problem (1.1), if the following conditions hold:

(1) $u \in F$ and $\nabla u \in L_{\bar{\psi}}(\Omega)^N$.

(2) (u, f) satisfies the following equation:

$$\frac{\partial u}{\partial s} - \Delta u = \frac{\lambda f(u)}{(\int_\Omega f(u)\mathrm{d}x)^2}, \text{ in } Q_S.$$

(3) $u(.,0) = u_0$, in Ω.

We obtain using the Lemma 3.5 and the regularity of u that $u \in C(]0, S[; L^1(\Omega))$. Then, u is well defined in $L^1(\Omega)$.

5 An existence result

In this section, we develop the proof of the main result.

Theorem 5.1 *We assume that hypotheses (2.7), (2.8), (2.11), (2.12) and (4.2)–(4.4) hold, then the problem (1.1) has a capacity solution in the sense of Definition 4.2.*

To prove this result, we need to apply the following theorem.

Theorem 5.2 *We suppose that the conditions (2.7), (2.8), (4.3) and (4.4) hold. Then, there exists a weak solution for the problem (1.1), that is*

$$\begin{aligned}
&u \in W_0^{1,x}L_\psi(Q_S) \cap C([0,S]; L^2(\Omega)), \quad \nabla u \in L_{\bar{\psi}}(Q_S)^N,\\
&u(.,0) = u_0, \ \textit{in } \Omega,\\
&\int_0^s \left\langle \frac{\partial u}{\partial s}, \varphi \right\rangle \mathrm{d}s + \int_0^s \int_\Omega \nabla u \nabla u \mathrm{d}x\mathrm{d}s = \left\langle \lambda \frac{f(u)}{\left(\int_\Omega f(u)\mathrm{d}x\right)^2}, u \right\rangle_{Q_s},
\end{aligned}$$

for each $\varphi \in W_0^{1,x}L_\psi(Q_S)$ *and* $s \in [0, S]$.

Proof To show the existence of a weak solution, Schauder's fixed point theorem will be applied. To this end, applying Theorem 3.7, we get the existence of a solution to the following problem, for all $v \in W_0^{1,x}L_\psi(Q_S)$:

$$\begin{aligned}
\left\langle \frac{\partial u}{\partial s}, v \right\rangle_{Q_S} + \int_0^S \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x\mathrm{d}s &= \left\langle \frac{\lambda f(w)}{(\int_\Omega f(w)\mathrm{d}x)^2}, v \right\rangle_{Q_S}\\
&= \int_0^S \frac{\lambda}{(\int_\Omega f(w)\mathrm{d}x)^2} \left(\int_\Omega f(w) \cdot v\mathrm{d}x \right) \mathrm{d}s,
\end{aligned} \tag{5.1}$$

$$u(.,0) = u_0, \ \text{in } \Omega.$$

From (4.5), we obtain

$$\left\| \frac{\lambda f(w)}{(\int_\Omega f(w)\mathrm{d}x)^2} \right\|_{L^2(0,S;H^{-1}(\Omega))} \leq \frac{\lambda}{(\sigma \cdot \mathrm{meas}(\Omega))^2} \| f(w) \|_{L^2(0,S;H^{-1}(\Omega))}.$$

We have to prove that

$$\|f(w)\|_{L^2(0,S;H^{-1}(\Omega))} \leq K,$$

where K is a positive constant.

In view of the fact that $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, we get

$$\|f(w)\|_{L^2(0,S;H^{-1}(\Omega))} \leq \|f(w)\|_{L^2(0,S;L^2(\Omega))}.$$

Since f is a Lipshitz function, then we get

$$\begin{aligned} \| f(w) \|_{L^2(0,S;L^2(\Omega))} \leq L_1 \| w \|_{L^2(0,S;L^2(\Omega))} + \Big((f(0))^2 \cdot (\text{meas}(Q))\Big)^{1/2} \\ + \Big(2f(0)\cdot \| w \|^2_{L^2(0,S;L^2(\Omega))}\Big)^{1/2}. \end{aligned} \tag{5.2}$$

All terms in the right-hand side are bounded due to $w \in E_\psi\,(Q_S) \hookrightarrow L^1(0, S; H^{-1}(\Omega))$. Then, there exists a positive constant C, such that

$$\left\| \frac{\lambda f(w)}{(\int_\Omega f(w)\mathrm{d}x)^2} \right\|_{L^2(0,S;H^{-1}(\Omega))} \leq C. \tag{5.3}$$

Hence

$$\frac{\lambda \cdot f(w)}{(\int_\Omega f(w)\mathrm{d}x)^2} \in L^2\Big(0, S; H^{-1}(\Omega)\Big).$$

Using the following continuous embedding $L^2\Big(0, S; H^{-1}(\Omega)\Big) \hookrightarrow W^{-1,x}E_{\bar{\psi}}\,(Q_S)$ obtained by applying Lemma 2.12 and Remark 2.13, we get that:

$$\frac{\lambda \cdot f(w)}{(\int_\Omega f(w)\mathrm{d}x)^2} \in W^{-1,x}E_{\bar{\psi}}\,(Q_S).$$

Now, we are in a position to employ Theorem 3.7, and we get the existence of a weak solution. Now, we prove that $|\nabla u| \in F_\psi(\Omega)$ and the following estimates:

$$\int_0^S \int_\Omega \psi(x, |\nabla u|)\mathrm{d}x\mathrm{d}s \leq C. \tag{5.4}$$

$$\|\nabla u\|_{\bar{\psi},Q_S} \leq C_2, \tag{5.5}$$

where C and C_2 are a positive constant. Let us prove (5.4). To this end, we use (4.3), to obtain

$$\alpha\psi(x, |\nabla u|) \leq |\nabla u|^2. \tag{5.6}$$

Hence

$$\alpha \int_0^S \int_\Omega \psi(x, |\nabla u|)\mathrm{d}x\mathrm{d}s \leq \int_0^S \int_\Omega |\nabla u|^2 \,\mathrm{d}x\mathrm{d}s. \tag{5.7}$$

From (5.1), we get

$$\int_0^r \int_\Omega \nabla u \nabla u \mathrm{d}x\mathrm{d}t = \left\langle \lambda \frac{f(w)}{(\int_\Omega f(w)\mathrm{d}x)^2}, u \right\rangle_{Q_r} + \frac{1}{2}\int_\Omega |u(x,0)|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega |u(x,r)|^2 \mathrm{d}x. \tag{5.8}$$

Using (4.5) and the hypothesis on f, we get that

$$\begin{aligned}
\int_0^r \int_\Omega \nabla u \nabla u \leq\ & \frac{1}{2} \parallel u(.,r) \parallel^2_{L^2(\Omega)} + \frac{\lambda}{(\sigma \cdot \mathrm{meas}(\Omega))^2} \int_0^r f(w) \cdot \mid u \mid \\
& \leq \frac{\lambda}{(\sigma \cdot \mathrm{meas}(\Omega))^2} \left(\int_0^r \int_\Omega |f(w) - f(0)| \cdot |u| + f(0) \cdot \int_0^r \int_\Omega |u| \right) \\
& \quad + \frac{1}{2} \parallel u(.,r) \parallel^2_{L^2(\Omega)} \\
& \leq \frac{\lambda L_1}{(\sigma \cdot \mathrm{meas}(\Omega))^2} \left(\int_0^r \int_\Omega |w| \cdot |u| + f(0) \cdot \int_0^r \int_\Omega |u| \right) \\
& \quad + \frac{1}{2} \|u(.,r)\|^2_{L^2(\Omega)} \\
& \leq \frac{\lambda L_1}{2(\sigma \cdot \mathrm{meas}(\Omega))^2} \left(\|u\|^2_{L^2(Q_r)} + \|w\|^2_{L^2(Q_r)} \right) + \frac{1}{2} \|u(.,r)\|^2_{L^2(\Omega)}.
\end{aligned}$$

Owing to $L_\psi(\Omega) \hookrightarrow L^2(\Omega)$ and $w \in E_\psi(Q_S) \hookrightarrow L^1\left(0, S; E_\psi(\Omega)\right) \hookrightarrow L^1(0, S; L^2(\Omega))$, there exists a positive constant C, such that

$$\int_0^r \int_\Omega \nabla u \cdot \nabla u \leq C. \tag{5.9}$$

From (5.7) and (5.9), we obtain

$$\alpha \int_0^S \int_\Omega \psi(x, |\nabla u|)\mathrm{d}x\mathrm{d}s \leq \int_0^S \int_\Omega \nabla u \cdot \nabla u \mathrm{d}x\mathrm{d}t \leq C; \tag{5.10}$$

it yields that $\mid \nabla u \mid \in F_\psi(\Omega)$.

Now, we state to prove the inequality (5.5). Knowing that

$$\int_0^S \int_\Omega (\nabla u - \nabla\varphi)(\nabla u - \nabla\varphi)\,\mathrm{d}x\mathrm{d}s \geq 0.$$

This implies that

$$\frac{1}{2}\left[\int_0^S \int_\Omega \mid \nabla u \mid^2 \mathrm{d}x\mathrm{d}s + \int_0^S \int_\Omega \mid \nabla\varphi \mid^2 \mathrm{d}x\mathrm{d}s \right] \geq \int_0^S \int_\Omega \mid \nabla u \mid\mid \nabla\varphi \mid \mathrm{d}x\mathrm{d}s.$$

Applying (5.9) and using (4.1), we get

$$\int_0^S \int_\Omega \mid \nabla\varphi \mid^2 \mathrm{d}x\mathrm{d}s \leq \int_0^S \int_\Omega \psi(x, |\nabla\varphi|)\mathrm{d}x\mathrm{d}s,$$

for each $\varphi \in W_0^{1,x} E_{\psi}(Q_S)$ where $\|\nabla\varphi\|_{\psi,Q_S} = \dfrac{1}{k+1}$. Consequently, we get

$$\int_0^S \int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x\mathrm{d}s \le C,$$

from whence follows, there exists a positive constant C_2, such that

$$\int_0^S \int_\Omega |\nabla u||\nabla\varphi| \,\mathrm{d}x\mathrm{d}s \le C_2.$$

It yields that $\|\nabla u\|_{\bar\psi,Q_S} \le C_2$, as a consequence $\nabla u \in E_{\bar\psi}(Q_S)$; hence, $\Delta u \in W_0^{-1,x} E_{\bar\psi}(Q_S)$. Keeping this in mind, using the following inclusion:

$$\frac{\lambda f(w)}{(\int_\Omega f(w)\mathrm{d}x)^2} \in L^2\left(0, S; H^{-1}(\Omega)\right) \hookrightarrow W^{-1,x} E_{\bar\psi}(Q_S),$$

and the first equation of the problem (5.1), it follows that:

$$\frac{\partial u}{\partial s} \in W^{-1,x} E_{\bar\psi}(Q_S) \text{ and } \left\|\frac{\partial u}{\partial s}\right\|_{W^{-1,x}L_{\bar\psi}(Q_S)} \le C_3. \tag{5.11}$$

We introduce the following operator:

$$\begin{aligned} G : E_\varphi(Q_S) &\longrightarrow F \\ v &\longmapsto G(v) = u, \end{aligned} \tag{5.12}$$

where u is the solution for the problem (5.1). G is compact operator. Indeed, $F \subset E_\varphi(Q_S)$ (i.e., F is included in $E_\varphi(Q_S)$ with the compact injection), we can show this embedding using Lemmas 3.6 and 3.4. From the inequality (5.11), we find that set $\left\{\dfrac{\partial u}{\partial s};\ u \in F\right\}$ is bounded in $W^{-1,x}L_{\bar\psi}(Q_S)$. From Lemmas 3.6 and 3.4, where $Y := L^1(\Omega)$, we get the following compact embedding $F \hookrightarrow E_\varphi(Q_S)$. This combined with (5.11) and (5.10) yields to the compactness of the mapping G.

We define

$$B_\nu := \left\{\omega \in E_\varphi(Q_S) / \|\omega\|_{\varphi,\Omega} \le \nu\right\}.$$

B_ν is bounded and closed. Keeping this and (5.10) in mind, we obtain $G(B_\nu) \subset B_\nu$.

To achieve the proof of the existence of a weak solution, it suffices to show that G is a continuous operator. To this end, we assume that $(v_n)_{n\in\mathbb{N}} \subset B_\nu$, such that $v_n \longrightarrow \omega$, also, let us consider $G(\omega) = u$ and $G(v_n) = u_n$. Hence

$$(v_n)_{n\in\mathbb{N}} \subset B_\nu \subset E_\varphi(Q_S) \subset L_\varphi(Q_S) \subset L^2(Q_S).$$

Since $L^2(Q_S)$ is a Banach space, then $v_n \longrightarrow \omega$ in $L^2(Q_S)$, so there exists a subsequence still denoted by $(v_n)_{n\in\mathbb{N}}$, such that $v_n \longrightarrow \omega$ a.e in Q_S. Knowing that

$$(v_n)_{n\in\mathbb{N}} \subset B_\nu \subset L_\varphi(Q_S).$$

Then, $(v_n)_{n\in\mathbb{N}}$ is bounded. Hence, there exists a subsequence, such that

$$u_n \longrightarrow V \text{ in } E_\varphi(Q_S), \tag{5.13}$$

and

$$\nabla u_n \longrightarrow \nabla V \text{ weakly in } L^2(Q_S)^N. \tag{5.14}$$

We choose $v = u_n - u$ in (5.1), and we obtain

$$\left\langle \frac{\partial u}{\partial s}, u_n - u \right\rangle_{Q_S} + \int_0^S \int_\Omega \nabla u \cdot \nabla (u_n - u) \mathrm{d}x \mathrm{d}s = \left\langle \lambda \frac{f(w)}{(\int_\Omega f(w) \mathrm{d}x)^2}, u_n - u \right\rangle_{Q_S},$$
$$\left\langle \frac{\partial u_n}{\partial s}, u_n - u \right\rangle_{Q_S} + \int_0^S \int_\Omega \nabla u_n \cdot \nabla (u_n - u) \mathrm{d}x \mathrm{d}s = \left\langle \lambda \frac{f(v_n)}{(\int_\Omega f(v_n) \mathrm{d}x)^2}, u_n - u \right\rangle_{Q_S}.$$

By subtracting the above two equations, we get

$$\begin{aligned} &\left\langle \frac{\partial (u_n - u)}{\partial s}, u_n - u \right\rangle_{Q_S} + \int_0^S \int_\Omega \mid \nabla (u_n - u) \mid^2 \mathrm{d}x \mathrm{d}s \\ &= \left\langle \lambda \frac{f(v_n)}{(\int_\Omega f(v_n) \mathrm{d}x)^2} - \lambda \frac{f(w)}{(\int_\Omega f(w) \mathrm{d}x)^2}, u_n - u \right\rangle_{Q_S}. \end{aligned} \tag{5.15}$$

On the other hand, we have the following identity:

$$\left\langle \frac{\partial (u_n - u)}{\partial s}, u_n - u \right\rangle_{Q_S} = \frac{1}{2} \int_0^S \int_\Omega \mid u_n(s) - u(s) \mid^2 \ \mathrm{d}x = \frac{1}{2} \parallel u_n - u \parallel^2_{L^2(Q_S)}. \tag{5.16}$$

Then, from (5.15) and (5.16), it yields that

$$\frac{1}{2} \parallel u_n - u \parallel^2_{L^2(Q_S)} \leq \left\langle \lambda \frac{f(v_n)}{(\int_\Omega f(v_n) \mathrm{d}x)^2} - \lambda \frac{f(w)}{(\int_\Omega f(w) \mathrm{d}x)^2}, u_n - u \right\rangle_{Q_S}. \tag{5.17}$$

Putting

$$\gamma_n(x, s) := \lambda \frac{f(v_n)}{(\int_\Omega f(v_n) \mathrm{d}x)^2} - \lambda \frac{f(w)}{(\int_\Omega f(w) \mathrm{d}x)^2}.$$

Then

$$\gamma_n(x, s) = \lambda \frac{f(v_n) - f(w)}{(\int_\Omega f(v_n) \mathrm{d}x)^2} + \lambda f(w) \frac{\int_\Omega [f(w) - f(v_n)] \, \mathrm{d}x \int_\Omega [f(w) + f(v_n)] \, \mathrm{d}x}{(\int_\Omega f(w) \mathrm{d}x)^2 (\int_\Omega f(v_n) \mathrm{d}x)^2}.$$

Hence

$$\gamma_n(x,s)\cdot(u_n-u)=\lambda\frac{f(v_n)-f(w)}{(\int_\Omega f(v_n)\mathrm{d}x)^2}(u_n-u)$$
$$+\lambda f(w)\frac{\int_\Omega[f(w)-f(v_n)]\,\mathrm{d}x\int_\Omega[f(w)+f(v_n)]\,\mathrm{d}x}{(\int_\Omega f(w)\mathrm{d}x)^2(\int_\Omega f(v_n)\mathrm{d}x)^2}(u_n-u). \tag{5.18}$$

Then

$$\int_0^S\int_\Omega \gamma_n(x,s)\cdot(u_n-u)\mathrm{d}x\mathrm{d}s$$
$$\leq \frac{\lambda L_1}{(\mathrm{meas}(\Omega)\sigma)^2}\int_0^S\int_\Omega |v_n-w|\cdot|u_n-u|\mathrm{d}x\mathrm{d}s$$
$$+\frac{\lambda L_1\cdot\mathrm{meas}(\Omega)}{(\mathrm{meas}(\Omega)\sigma)^4}\int_0^S\int_\Omega f(w)|v_n-w|\cdot|u_n-u|\left(\int_\Omega(f(v_n)+f(w))\mathrm{d}x\right)\mathrm{d}x\mathrm{d}s.$$

Knowing that $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, by applying the convergence dominate theorem, we get

$$\lim_{n\to\infty}\int_0^S\int_\Omega \gamma_n(x,s)\cdot(u_n-u)\mathrm{d}x\mathrm{d}s=0. \tag{5.19}$$

Combining (5.17) with (5.18), we get

$$\frac{1}{2}\|u_n-u\|^2_{L^2(Q)}\leq\int_0^S\int_\Omega \gamma_n(x,s)\cdot(u_n-u)\mathrm{d}x\mathrm{d}s. \tag{5.20}$$

From (5.17)–(5.20), we get $u_n\xrightarrow{\|\cdot\|_{L^2(\Omega)}}u$. Knowing that $u_n\longrightarrow V$ in $E_\varphi(Q_S)\subset L_\varphi(Q_S)\subset L^2(Q_S)$, we obtain that $u_n\xrightarrow{\|\cdot\|_{L^2(\Omega)}}V$. This implies that $V=u$, then $G(v_n)\longrightarrow G(\omega)=u$. Hence, G is continuous. This completes the proof of Theorem 5.2. □

We now proceed to prove Theorem 5.1.

Proof The proof consists of four steps. We begin by presenting a sequence of approximation problems, establishing a priori estimates for them, and demonstrating intermediate results, namely strong convergence in $L^1(\Omega)$ of $(\nabla u_n)_{n\in\mathbb{N}}$.

Step I

For every $n\in\mathbb{N}$, we consider the following approximate problem:

$$\begin{cases}\dfrac{\partial u_n}{\partial s}-\Delta u_n=\dfrac{\lambda f(u_n)}{(\int_\Omega f(u_n)\mathrm{d}x)^2}, & \text{in } Q_S,\\ u_n(.,.)=0, & \text{on } \partial\Omega\times]0,S[,\\ u_n(.,0)=u_0, & \text{in } \Omega.\end{cases} \tag{5.21}$$

Under the assumption (4.1), we obtain

$$| \nabla u - \nabla v |^2 \leq \psi(x, |\nabla u - \nabla v|) + \psi(x, s).$$

From assumptions (4.2)–(4.3), we get

$$| \nabla u | \leq \zeta \left[c(x, s) + \bar{\psi}_x^{-1}(\psi(x, k|u|)) \right].$$
$$\alpha \psi(x, |\nabla u - \nabla v|) \leq |\nabla u - \nabla v|^2 .$$

Applying Theorem 5.2, to get the existence of a weak solution to the approximate problem (5.21). We use u_n as a test function in (5.21). Then, we get

$$\left\langle \frac{\partial u_n}{\partial s}, u_n \right\rangle_{Q_S} + \int_0^S \int_\Omega \nabla u_n \cdot \nabla u_n \mathrm{d}x\mathrm{d}s = \left\langle \frac{\lambda f(u_n)}{(\int_\Omega f(u_n)\mathrm{d}x)^2}, u_n \right\rangle_{Q_S}.$$

Hence

$$\frac{1}{2} \int_\Omega |u_n(x, r)|^2 \mathrm{d}x + \int_0^r \int_\Omega |\nabla u_n|^2 \mathrm{d}x\mathrm{d}s = \int_0^r \int_\Omega \frac{\lambda f(u_n)}{(\int_\Omega f(u_n)\mathrm{d}x)^2} \cdot u_n + \frac{1}{2} \int_\Omega |u_n(x, 0)|^2 \, \mathrm{d}x.$$

Consequently

$$\int_0^r \int_\Omega | \nabla u_n |^2 \, \mathrm{d}x\mathrm{d}s \leq \int_0^r \int_\Omega \frac{\lambda f(u_n)}{(\int_\Omega f(u_n)\mathrm{d}x)^2} \cdot u_n + \|u_n(0, .)\|^2_{L^2(\Omega)}.$$

Keeping this and (5.6) in mind, we obtain

$$\alpha \int_0^r \int_\Omega \psi \left(x, | \nabla u_n |\right) \mathrm{d}x\mathrm{d}s \leq \int_0^r \int_\Omega \frac{\lambda f(u_n)}{(\int_\Omega f(u_n)\mathrm{d}x)^2} \cdot u_n + \| u_n(., 0) \|^2_{L^2(\Omega)}.$$

On the other hand, using the condition (4.5) and Hölder's inequality, we get

$$\begin{aligned}
\alpha \int_0^r \int_\Omega \psi \left(x, | \nabla u_n |\right) \mathrm{d}x\mathrm{d}s &\leq \frac{\lambda}{(\mathrm{meas}(\Omega)\sigma)^2} \int_0^r \int_\Omega |f(u_n)| \cdot |u_n| + \|u_n(., 0)\|^2_{L^2(\Omega)} \\
&\leq \frac{\lambda}{(\mathrm{meas}(\Omega)\sigma)^2} \int_0^r \int_\Omega (|f(u_n) - f(0)| + |f(0)|) \cdot |u_n| \\
&\quad + \| u_n(., 0) \|^2_{L^2(\Omega)} \\
&\leq \frac{\lambda}{(\mathrm{meas}(\Omega)\sigma)^2} \left(L_1 + f(0)(\mathrm{meas}(\Omega))^{1/2} \right) \int_0^r \int_\Omega |u_n|^2.
\end{aligned}$$

Since $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^2(Q_S)$. Then, there exists a positive constant C_5, such that

$$\alpha \int_0^r \int_\Omega \psi\left(x, \mid \nabla u_n \mid\right) \mathrm{d}x\mathrm{d}s \leq\ C_5.$$

$$\int_0^r \int_\Omega \psi\left(x, \mid \nabla u_n \mid\right) \mathrm{d}x\mathrm{d}s \leq\ C. \tag{5.22}$$

Recall from Remark 2.5 that

$$\| u_n \|_{1,\psi,\Omega} \leq\ (C_0 + 1)\max(C, 1);$$

this implies that $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $W_0^{1,x}L_\psi\left(Q_S\right)$. Then, there exists a subsequence still denoted $(u_n)_{n\in\mathbb{N}}$ weakly converges in $W_0^{1,x}L_\psi\left(Q_S\right)$ as $n \longrightarrow \infty$ to a limit u, such that

$$u_n \rightharpoonup u \text{ in } W_0^{1,x}L_\psi\left(Q_S\right) \text{ for } \sigma\left(\Pi L_\psi, \Pi E_{\bar{\psi}}\right). \tag{5.23}$$

On the other hand, for any function $\varphi \in W_0^{1,x}E_\psi\left(Q_S\right)^N$, such that $\| \nabla\varphi \|_{\psi,Q_S} = \dfrac{1}{m+1}$ where m is a positive real number, we have

$$\begin{aligned}
\int_0^S \int_\Omega \nabla u_n \cdot \nabla\varphi \mathrm{d}x\mathrm{d}s &\leq \frac{1}{2}\left[\int_0^S \int_\Omega \mid \nabla u_n \mid^2 \mathrm{d}x\mathrm{d}s + \int_0^S \int_\Omega \mid \nabla\varphi \mid^2 \mathrm{d}x\mathrm{d}s\right] \\
&\leq \frac{1}{2}\int_0^S \int_\Omega \psi\left(x, \mid \nabla u_n \mid\right) \mathrm{d}x\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^S \int_\Omega \psi\left(x, \mid \nabla\varphi \mid\right) \mathrm{d}x\mathrm{d}s \\
&\leq \frac{C}{2} + \frac{1}{2} \| \nabla\varphi \|_{\psi,Q_S} .
\end{aligned}$$

Using the equivalence between the Luxemburg norm and the Orlicz norm, and using (5.22), there exists a positive constant C_6, such that

$$\int_0^S \int_\Omega \nabla u_n \cdot \nabla\varphi \mathrm{d}x\mathrm{d}s \leq\ C_6,$$

from whence follows, $(\nabla u_n)_{n\in\mathbb{N}}$ is bounded in $L_{\bar{\psi}}\left(Q_S\right)^N$. This implies that there exists a subsequence still denoted $(\nabla u_n)_{n\in\mathbb{N}}$, such that

$$u_n \rightharpoonup u \text{ in } L_{\bar{\psi}}\left(Q_S\right)^N \text{ for } \sigma\left(\Pi L_{\bar{\psi}}, \Pi E_\psi\right). \tag{5.24}$$

Since the sequences $(\nabla u_n)_{n\in\mathbb{N}}$ and $\left(\lambda \dfrac{f(u_n)}{(\int_\Omega f(u_n)\mathrm{d}x)^2}\right)_{n\in\mathbb{N}}$ are bounded in $W^{-1,x}L_{\bar{\psi}}(Q_S)$. Hence, using the first equation of the problem (5.21), we get that the sequence $\left(\dfrac{\partial u_n}{\partial s}\right)_{n\in\mathbb{N}}$ is bounded in $W^{-1,x}L_{\bar{\psi}}(Q_S)$. Consequently, $(u_n)_{n\in\mathbb{N}}$ is bounded in F. In view of the fact that $F \hookrightarrow E_\phi(Q_S)$ is compact, then, for a subsequence still denoted in the same way, we get

$$u_n \longrightarrow u \text{ strongly in } E_\phi(Q_S) \text{ and a.e. in } Q_S, \tag{5.25}$$

where $u \in W_0^{1,x}L_\psi(Q_S)$ is also the same limit appearing in (5.23).

Step II

We introduce the following regularized sequences for $i, j \in \mathbb{N}$:

(1) $v_j \to u$ in $W_0^{1,x}L_\psi(Q_S)$ with the modular convergence;
(2) $v_j \to u$ and $\nabla v_j \to \nabla u$ a.e in Q_S;
(3) $\omega_i \to u_0$ in $L^2(\Omega)$ with the strong convergence;
(4) $\|\omega_i\|_{L^2(\Omega)} \leq 2\|u_0\|_{L^2(\Omega)}$, for all $i \geq 1$.

These fourth points are satisfied for all $\omega \in D(\Omega)$ and $v_j \in D(Q_S)$.

Let $R > 0$ a real number, and we define the truncation function as in (2.15). Then, for each $R, \eta > 0$ and for $i, j \in \mathbb{N}$, we consider the function $\omega^i_{\eta,j} \in W_0^{1,x}L_\psi(Q_S)$ defined as follows $\omega^i_{\eta,j} := S_R(v_j)_\eta + \exp(-\eta s)S_R(\omega_j)$, such that $S_R(v_j)_\eta$ is the mollification with respect to the time variable of $S_R(v_j)$ appearing in (3.1). From Lemma 3.1, it follows that:

$$\frac{\partial \omega^i_{\eta,j}}{\partial s} = \eta\left(S_R(v_j) - \omega^i_{\eta,j}\right), \quad \omega^i_{\eta,j}(.,0) = S_R(\omega_i), \ |\omega^i_{\eta,j}| \leq R \text{ a.e in } Q_S, \tag{5.26}$$

$$\omega^i_{\eta,j} \longrightarrow \omega^i_\eta := S_R(u)_\eta + \exp(-\eta s)S_R(\omega_i) \text{ as } j \longrightarrow \infty \text{ in } W_0^{1,x}L_\psi(Q_S), \tag{5.27}$$

$$S_R(u)_\eta + \exp(-\eta s)S_R(\omega_i) \longrightarrow S_R(u) \text{ as } \eta \longrightarrow \infty \text{ in } W_0^{1,x}L_\psi(Q_S), \tag{5.28}$$

with the modular convergence in the two last convergences. □

We consider subsequences in (5.26)–(5.28), without loss the generality that convergences in (5.26)–(5.28) hold a.e. in Q_S.

Proposition 5.3 *Let u_n be a solution to the problem (5.21). Then, for a subsequence, we have the following convergence:*

$$\nabla u_n \longrightarrow \nabla u \ \text{ as } n \longrightarrow \infty \text{ a.e. in } Q_S. \tag{5.29}$$

Proof Throughout this paper, we use χ^j_r and χ_r as the characteristic functions of the following sets:

$$Q^j_r = \left\{(x,s) \in Q_S / \mid \nabla S_R(v_j) \mid \leq r\right\}, \quad Q_r = \{(x,s) \in Q_S / \mid \nabla S_R(u) \mid \leq r\}.$$

For any real numbers $\eta, \vartheta > 0$ and for $i, j, n \in \mathbb{N}$, we use the admissible test function $\phi^{\eta,i}_{n,j,\vartheta} := S_\vartheta(u_n - \omega^i_{\eta,j})$ in the first equation of the approximate problem (5.21), and we get

$$\begin{aligned}&\left\langle \frac{\partial u_n}{\partial s}, \phi^{\eta,i}_{n,j,\vartheta} \right\rangle_{Q_S} + \int_0^S \int_\Omega \nabla S_\vartheta(u_n - \omega^i_{\eta,j}) \nabla u_n \mathrm{d}x\mathrm{d}s \\ &= \left\langle \lambda \frac{f(u_n)}{(\int_\Omega f(u_n)\mathrm{d}x)^2}, S_\vartheta(u_n - \omega^i_{\eta,j}) \right\rangle_{Q_S}.\end{aligned} \tag{5.30}$$

On the other hand, using the condition (4.5), and by the same reasoning done to get (5.2), we obtain

$$\int_\Omega \frac{\lambda f(u_n)}{(\int_\Omega f(u_n)\mathrm{d}x)^2}\mathrm{d}x \leq C_7, \text{ where } C_7 > 0. \tag{5.31}$$

From (2.15), we obtain

$$\int_0^S \int_\Omega \lambda \frac{f(u_n)}{(\int_\Omega f(u_n)\mathrm{d}x)^2} \cdot S_\vartheta(u_n - \omega^i_{\eta,j}) \leq \lambda\vartheta \int_0^S \int_\Omega \frac{f(u_n)}{(\int_\Omega f(u_n)\mathrm{d}x)^2}.$$

Using (5.31), we get

$$\left\langle \frac{\partial u_n}{\partial s}, \phi^{\eta,i}_{n,j,\vartheta} \right\rangle_{Q_S} + \int_0^S \int_\Omega \nabla S_\vartheta(u_n - \omega^i_{\eta,j}) \nabla u_n \mathrm{d}x\mathrm{d}s \leq C_7\vartheta. \tag{5.32}$$

Now, we split the first term on the left side of the above inequality into two parts and estimate each one separately

$$\left\langle \frac{\partial u_n}{\partial s}, \phi^{\eta,i}_{n,j,\vartheta} \right\rangle = \left\langle \frac{\partial u_n}{\partial s} - \frac{\partial \omega^i_{\vartheta,j}}{\partial s}, \phi^{\eta,i}_{n,j,\vartheta} \right\rangle + \left\langle \frac{\partial \omega^i_{\vartheta,j}}{\partial s}, \phi^{\eta,i}_{n,j,\vartheta} \right\rangle. \tag{5.33}$$

We start by estimating the first term on the right side of the above identity

$$\left\langle \frac{\partial u_n}{\partial s} - \frac{\partial \omega^i_{\vartheta,j}}{\partial s}, \phi^{\eta,i}_{n,j,\vartheta} \right\rangle = \int_\Omega T_\vartheta(u_n(S) - \omega^i_{\eta,j}(S))\mathrm{d}x - \int_\Omega T_\vartheta(u_0 - S_\vartheta(\omega_i))\mathrm{d}x. \tag{5.34}$$

From (2.16), it can be shown that

$$0 \leq T_R(r) \leq R|r|, \text{ for all } r \in \mathbb{R}.$$

It follows that:

$$\begin{aligned}0 \leq \int_\Omega T_\vartheta(u_0 - S_\vartheta(\omega_i))\mathrm{d}x &\leq \vartheta \int_\Omega |u_0 - S_\vartheta(\omega_i)|\mathrm{d}x \\ &\leq \vartheta(\mathrm{meas}(\Omega))^{1/2}\left(\int_\Omega |u_0 - S_\vartheta(\omega_i)|^2\mathrm{d}x\right)^{1/2} \\ &\leq 3\vartheta(\mathrm{meas}(\Omega))^{1/2}\,\|u_0\|_{L^2(\Omega)} := C_8\vartheta.\end{aligned}$$

Then, for all $\eta, \vartheta > 0$ and $i, j, n \geq 1$, and from (5.34), we get

$$-C_8\vartheta \leq \left\langle \frac{\partial u_n}{\partial s} - \frac{\partial \omega^i_{\vartheta,j}}{\partial s}, \phi^{\eta,i}_{n,j,\vartheta} \right\rangle. \tag{5.35}$$

Now, we derive an estimate for the second term on the right side of (5.33). Under assumption (5.26), we get

$$\left\langle \frac{\partial \omega^i_{\vartheta,j}}{\partial s}, \phi^{\eta,i}_{n,j,\vartheta} \right\rangle = \eta \int_0^S \int_\Omega (S_R(v_j) - \omega^i_{\eta,j}) S_\vartheta(u_n - \omega^i_{\eta,j}) \mathrm{d}x\mathrm{d}s. \tag{5.36}$$

Then

$$\lim_{n\to\infty} \lim_{j\to\infty} \left\langle \frac{\partial \omega^i_{\vartheta,j}}{\partial s}, \phi^{\eta,i}_{n,j,\vartheta} \right\rangle = \eta \int_0^S \int_\Omega (S_R(u) - \omega^i_\eta) S_\vartheta(u - \omega^i_\eta) \mathrm{d}x\mathrm{d}s.$$

Under hypotheses (5.26)–(5.27), we have $|\omega^i_\eta| \leq R$, and due to $rS_\vartheta(r) \geq 0,\ r \in \mathbb{R}$, we get that for all $\eta, \vartheta, R > 0$ and $i \geq 1$

$$\lim_{n\to\infty} \lim_{j\to\infty} \left\langle \frac{\partial \omega^i_{\vartheta,j}}{\partial s}, \phi^{\eta,i}_{n,j,\vartheta} \right\rangle \geq 0. \tag{5.37}$$

Keeping this and (5.35) in mind, we get

$$\lim_{n\to\infty} \inf \lim_{j\to\infty} \inf \left\langle \frac{\partial u_n}{\partial s}, \phi^{\eta,i}_{n,j,\vartheta} \right\rangle \geq -C_8\vartheta. \tag{5.38}$$

On the other hand, we have

$$\begin{aligned} I_{i,j,n,\vartheta} &:= \int_0^S \int_\Omega \nabla S_\vartheta(u_n - \omega^i_{\eta,j}) \nabla u_n \mathrm{d}x\mathrm{d}s \\ &= \int_{\{|u_n - \omega^i_{\eta,j}| \leq \vartheta\}} (\nabla u_n - \nabla \omega^i_{\eta,j}) \nabla u_n \mathrm{d}x\mathrm{d}s \\ &= \int_{\{|u_n - \omega^i_{\eta,j}| \leq \vartheta\} \cap \{|u_n| > R\}} (\nabla u_n - \nabla \omega^i_{\eta,j}) \nabla u_n \mathrm{d}x\mathrm{d}s \\ &\quad + \int_{\{|u_n - \omega^i_{\eta,j}| \leq \vartheta\} \cap \{|u_n| \leq R\}} (\nabla u_n - \nabla \omega^i_{\eta,j}) \nabla u_n \mathrm{d}x\mathrm{d}s. \end{aligned}$$

Then

$$I_{i,j,n,\vartheta} = \int_{\{|u_n-\omega^i_{\eta,j}|\le\vartheta\}\cap\{|u_n|>R\}} (\nabla u_n - \nabla\omega^i_{\eta,j})\nabla u_n \mathrm{d}x\mathrm{d}s + \int_{\{|u_n-\omega^i_{\eta,j}|\le\vartheta\}\cap\{|u_n|\le R\}} (\nabla u_n - \nabla\omega^i_{\eta,j})\nabla u_n \mathrm{d}x\mathrm{d}s$$
$$= \int_{\{|S_R(u_n)-\omega^i_{\eta,j}|\le\vartheta\}} (\nabla S_R(u_n) - \nabla\omega^i_{\eta,j})\nabla S_R(u_n)\mathrm{d}x\mathrm{d}s + \int_{\{|u_n-\omega^i_{\eta,j}|\le\vartheta\}\cap\{|u_n|>R\}} |\nabla u_n|^2 \mathrm{d}x\mathrm{d}s - \int_{\{|u_n-\omega^i_{\eta,j}|\le\vartheta\}\cap\{|u_n|>R\}} \nabla\omega^i_{\eta,j}\nabla u_n \mathrm{d}x\mathrm{d}s,$$

where

$$I_{0,i,j,n,\vartheta} = \int_{\{|S_R(u_n)-\omega^i_{\eta,j}|\le\vartheta\}} (\nabla S_R(u_n) - \nabla\omega^i_{\eta,j})\nabla S_R(u_n)\mathrm{d}x\mathrm{d}s,$$
$$I_{1,i,j,n,\vartheta} = \int_{\{|u_n-\omega^i_{\eta,j}|\le\vartheta\}\cap\{|u_n|>R\}} |\nabla u_n|^2 \mathrm{d}x\mathrm{d}s,$$
$$I_{2,i,j,n,\vartheta} = \int_{\{|u_n-\omega^i_{\eta,j}|\le\vartheta\}\cap\{|u_n|>R\}} \nabla\omega^i_{\eta,j}\nabla u_n \mathrm{d}x\mathrm{d}s.$$

Under assumption (4.3), we have

$$I_{1,i,j,n,\vartheta} \ge \alpha \int_{\{|u_n-\omega^i_{\eta,j}|\le\vartheta\}\cap\{|u_n|>R\}} \psi(x, |\nabla u_n|)\mathrm{d}x\mathrm{d}s \ge 0.$$

Then, $I_{i,j,n,\vartheta} \ge I_{0,i,j,n,\vartheta} - I_{2,i,j,n,\vartheta}$; from (5.26), we have $|\omega^i_{\eta,j}| \le R$, a.e. in Q_S, and this implies that

$$|u_n| \le |u_n - \omega^i_{\eta,j}| + |\omega^i_{\eta,j}| \le R + \vartheta. \tag{5.39}$$

It follows that for $n > R + \vartheta$:

$$I_{2,i,j,n,\vartheta} = \int_{\{|u_n-\omega^i_{\eta,j}|\le\vartheta\}\cap\{|u_n|>R\}} \nabla\omega^i_{\eta,j}\nabla S_{\vartheta+R}(u_n)\mathrm{d}x\mathrm{d}s,$$

which gives

$$I_{i,j,n,\vartheta} \ge I_{0,i,j,n,\vartheta} - I_{2,i,j,n,\vartheta} \ge \int_{\{|S_R(u_n)-\omega^i_{\eta,j}|\le\vartheta\}} (\nabla S_R(u_n) - \nabla\omega^i_{\eta,j})\nabla S_R(u_n)\mathrm{d}x\mathrm{d}s - I_{2,i,j,n,\vartheta}. \tag{5.40}$$

Using again inequality (5.39), we get by definition $S_{\vartheta+R}(u_n) = u_n$, and hence, $\nabla S_{\vartheta+R}(u_n) = \nabla u_n$. In view of the fact that $(\nabla u_n)_{n\in\mathbb{N}}$ is bounded in $L_\psi(Q_S)^N$, we get that $(\nabla S_{\vartheta+R}(u_n))_{n\in\mathbb{N}}$ is also bounded in $L_\psi(Q_S)^N$. Then, there exists a_1, such that $\nabla S_{\vartheta+R}(u_n) \rightharpoonup a_1$ as $n \longrightarrow \infty$ in $L_\psi(Q_S)^N$ for the weak topology $\sigma(\prod L_\psi, \prod E_{\bar{\psi}})$. In view of the fact that

$$\nabla\omega^i_{\eta,j}\chi_{\{|u_n-\omega^i_{\eta,j}|\leq\vartheta\}\cap\{|u_n|>R\}} \longrightarrow \nabla\omega^i_{\eta,j}\chi_{\{|u-\omega^i_{\eta,j}|\leq\vartheta\}\cap\{|u|>R\}},$$

strongly in $E_\psi(Q_S)^N$ as $n \longrightarrow \infty$, we get

$$\lim_{n\to\infty}\int_{\{|u_n-\omega^i_{\eta,j}|\leq\vartheta\}\cap\{|u_n|>R\}} \nabla\omega^i_{\eta,j}\nabla S_{\vartheta+R}(u_n) = \int_{\{|u-\omega^i_{\eta,j}|\leq\vartheta\}\cap\{|u|>R\}} \nabla\omega^i_{\eta,j}a_1.$$

Under assumptions (5.27) and (5.28), we obtain

$$\nabla S_{\vartheta+R}(u_n)\nabla\omega^i_{\eta,j}\chi_{\{|u_n-\omega^i_{\eta,j}|\leq\vartheta\}\cap\{|u_n|>R\}} \longrightarrow \nabla S_{\vartheta+R}(u)\nabla\omega^i_{\eta,j}\chi_{\{|u-\omega^i_{\eta,j}|\leq\vartheta\}\cap\{|u|>R\}},$$

as $n,\ j \longrightarrow \infty$. We apply Lemma 2.8, and letting $\vartheta,\ j \longrightarrow \infty$, we obtain

$$\int_Q \nabla S_{\vartheta+R}(u)\nabla\omega^i_{\eta,j}\chi_{\{|u-\omega^i_{\eta,j}|\leq\vartheta\}\cap\{|u|>R\}} \longrightarrow I_3 := \int_{\{|u-\omega^i_{\eta,j}|\leq\vartheta\}\cap\{|u|>R\}} \nabla S_R(u)a_1.$$

For $|u| > R$, we get $S_R(u) = 0$, which yields $I_3 = 0$. Thus

$$\begin{array}{rl} I_{2,i,j,n,\vartheta} & \longrightarrow\ 0. \\ i,\ j,\ \vartheta & \longrightarrow\ \infty. \end{array} \tag{5.41}$$

By using (5.40), we obtain

$$I_{i,j,n,\vartheta} \geq \int_{\{|S_R(u_n)-\omega^i_{\eta,j}|\leq\vartheta\}} (\nabla S_R(u_n) - \nabla\omega^i_{\eta,j})\nabla S_R(u_n)\mathrm{d}x\mathrm{d}s - I_2.$$

$$\int_{\{|S_R(u_n)-\omega^i_{\eta,j}|\leq\vartheta\}} (\nabla S_R(u_n) - \nabla\omega^i_{\eta,j})\nabla S_R(u_n)\mathrm{d}x\mathrm{d}s \leq I_{i,j,n,\vartheta} + I_{2,i,j,n,\vartheta}.$$

Using (5.32), we get

$$I_{i,j,n,\vartheta} \leq C_7\vartheta - \left\langle \frac{\partial u_n}{\partial s}, \phi^{\eta,i}_{n,j,\vartheta} \right\rangle_{Q_S}.$$

Owing to (5.38), it follows that:

$$-\left\langle \frac{\partial u_n}{\partial s}, \phi^{\eta,i}_{n,j,\vartheta} \right\rangle_{Q_S} \leq C_8\vartheta.$$

Then

$$I_{i,j,n,\vartheta} \le (C_7 + C_8)\vartheta := C\vartheta.$$

We have

$$\begin{aligned}
&\int_{\{|S_R(u_n)-\omega^i_{\eta,j}|\le\vartheta\}} (\nabla S_R(u_n) - \nabla\omega^i_{\eta,j})\nabla S_R(u_n)\mathrm{d}x\mathrm{d}s \\
&= \int_A (\nabla S_R(u_n) - \nabla S_R(v_j)\chi^r_j)\nabla S_R(u_n)\mathrm{d}x\mathrm{d}s \\
&\quad + \int_A (\nabla S_R(v_j)\chi^r_j - \nabla\omega^i_{\eta,j})\nabla S_R(u_n)\mathrm{d}x\mathrm{d}s \\
&= I_{4,i,j,n,\eta} + I_{5,i,j,n,\eta},
\end{aligned} \tag{5.42}$$

where $A := \{|S_R(u_n) - \omega^i_{\eta,j}| \le \vartheta\}$.

Now, we show that $I_{5,i,j,n,\eta} \longrightarrow 0$. Knowing that $(S_R(u_n))_{n\in\mathbb{N}}$ is bounded. Hence, there exits a_0, such that

$$S_R(u_n) \rightharpoonup a_0 \ \ as \ \ n \longrightarrow \infty.$$

Since

$$\begin{aligned}
&\Gamma^n(v_j) := (\nabla S_R(v_j)\chi^r_j - \nabla\omega^i_{\eta,j})\chi_A \\
&\longrightarrow (\nabla S_R(v_j)\chi^r_j - \nabla\omega^i_{\eta,j})\chi_{\{(S_R(u)-\omega^i_{\eta,j})\le\vartheta\}} \text{ in } E_{\bar{\psi}}(Q_S).
\end{aligned}$$

It follows that $(\Gamma^n(v_j))_{n\in\mathbb{N}}$ is bounded in $E_{\bar{\psi}}(Q_S)$. Hence, $(\nabla S_R(u_n)\Gamma^n(v_j))_{n\in\mathbb{N}}$ is bounded as well. Applying the dominated convergence theorem, we get

$$\lim_{i,j,n,\eta\longrightarrow\infty} I_{5,i,j,n,\eta} = \int_A \lim_{i,j,n,\eta\longrightarrow\infty} (\nabla S_R(v_j)\chi^r_j - \nabla\omega^i_{\eta,j})\nabla S_R(u_n)\mathrm{d}x\mathrm{d}s = 0.$$

$$I_{5,i,j,n,\eta} \longrightarrow 0 \text{ as } i, j, n, \eta \longrightarrow \infty. \tag{5.43}$$

Recall (5.40) and (5.42), we get

$$\begin{aligned}
&I_{4,i,j,n,\eta} + I_{5,i,j,n,\eta} \le I_{i,j,n,\vartheta} + I_{2,i,j,n,\vartheta}, \\
&\qquad I_{4,i,j,n,\eta} \le I_{i,j,n,\vartheta} + I_{2,i,j,n,\vartheta} - I_{5,i,j,n,\eta} \le C\vartheta + I_{2,i,j,n,\vartheta} - I_{5,i,j,n,\eta}.
\end{aligned}$$

From (5.41) and (5.43), we can take $\epsilon(n, i, \eta, j) := I_{2,i,j,n,\vartheta} - I_{5,i,j,n,\eta}$ where $\epsilon(n, i, \eta, j) \longrightarrow 0$ as $i, j, n, \eta \longrightarrow \infty$. This implies that $I_{4,i,j,n,\eta} \le C\vartheta + \epsilon(n, i, \eta, j)$.

Putting

$$N_n := (\nabla S_R(u_n) - \nabla S_R(u))(\nabla S_R(u_n) - \nabla S_R(u)),$$

which is a nonnegative quantity. Since $(\nabla S_R(u_n))$ is bounded in $L_\psi(Q_S)^N$, then the same holds for N_n. Let us $J_n^r := \int_{Q_r} N_n^\theta \mathrm{d}x\mathrm{d}s$ for each θ in $]0, 1[$, we get

$$\int_{Q_r} N_n^\theta \chi_{A^c}\mathrm{d}x\mathrm{d}s \leq \left(\int_{Q_r} N_n \mathrm{d}x\mathrm{d}s\right)^\theta \left(\int_{Q_r} \chi_{A^c}\mathrm{d}x\mathrm{d}s\right)^{(1-\theta)} \leq C_9 \mathrm{meas}(A^c).$$

Using Hölder's inequality, we obtain

$$\int_{Q_r} N_n^\theta \chi_A \mathrm{d}x\mathrm{d}s \leq \left(\int_{Q_r} N_n \mathrm{d}x\mathrm{d}s\right)^\theta \left(\int_{Q_r} \chi_A \mathrm{d}x\mathrm{d}s\right)^{(1-\theta)} \leq C_{10}\left(\int_{Q_r\cap A} N_n \mathrm{d}x\mathrm{d}s\right)^\theta.$$

It follows that:

$$J_n^r = \int_{Q_r} N_n^\theta \mathrm{d}x\mathrm{d}s \leq \left(C_9 \mathrm{meas}(A^c) + C_{10}\left(\int_{Q_r\cap A} N_n \mathrm{d}x\mathrm{d}s\right)^\theta\right).$$

On the other hand, for $s \geq r$ and $r > 0$, we have

$$\begin{aligned}
\int_{Q_r\cap A} N_n \mathrm{d}x\mathrm{d}s &\leq \int_{Q_r\cap A} (\nabla S_R(u_n) - \nabla S_R(u))(\nabla S_R(u_n) - \nabla S_R(u))\mathrm{d}x\mathrm{d}s \\
&\leq \int_{Q_s\cap A} (\nabla S_R(u_n) - \nabla S_R(u)\chi_s)(\nabla S_R(u_n) - \nabla S_R(u)\chi_s)\mathrm{d}x\mathrm{d}s \\
&\leq \int_{Q_s\cap A} (\nabla S_R(u_n) - \nabla S_R(v_j)\chi_s)(\nabla S_R(u_n) - \nabla S_R(v_j)\chi_s)\mathrm{d}x\mathrm{d}s \\
&\leq J_{1,n,j} + J_{2,n,j} + J_{3,n,j} + J_{4,n,j},
\end{aligned}$$

where

$$\begin{aligned}
J_{1,n,j} &:= \int_A (\nabla S_R(u_n) - \nabla S_R(v_j)\chi_j^s)(\nabla S_R(u_n) - \nabla S_R(v_j)\chi_j^s)\mathrm{d}x\mathrm{d}s, \\
J_{2,n,j} &:= \int_A \left(\nabla S_R(v_j)\chi_j^s - \nabla S_R(u)\chi_s\right)\left(\nabla S_R(u_n) - \nabla S_R(v_j)\chi_j^s\right)\mathrm{d}x\mathrm{d}s, \\
J_{3,n,j} &:= \int_A \left(\nabla S_R(u_n) - \nabla S_R(v_j)\chi_j^s\right)\left(\nabla S_R(v_j)\chi_j^s - \nabla S_R(v_j)\chi^s\right)\mathrm{d}x\mathrm{d}s, \\
J_{4,n,j} &:= \int_A \left(\nabla S_R(v_j)\chi_j^s - \nabla S_R(u)\chi_s\right)\left(\nabla S_R(v_j)\chi_j^s - \nabla S_R(v_j)\chi^s\right)\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Then

$$\begin{aligned} J_{1,n,j} &= \int_A \left(\nabla S_R(u_n) - \nabla S_R(v_j)\chi_j^s\right) \nabla S_R(u_n) \mathrm{d}x\mathrm{d}s \\ &\quad - \int_A \nabla S_R(v_j)\chi_j^s \left(\nabla S_R(u_n) - \nabla S_R(v_j)\chi_j^s\right) \mathrm{d}x\mathrm{d}s \\ &\le I_{4,i,j,n,\eta} - \int_A \nabla S_R(v_j)\chi_j^s \left(\nabla S_R(u_n) - \nabla S_R(v_j)\chi_j^s\right) \mathrm{d}x\mathrm{d}s \\ &\le C\vartheta + \epsilon(n,i,\eta,j) - \int_A \nabla S_R(v_j)\chi_j^s \left(\nabla S_R(u_n) - \nabla S_R(v_j)\chi_j^s\right) \mathrm{d}x\mathrm{d}s. \end{aligned}$$

Putting

$$\begin{aligned} M_{j,n} &:= \nabla S_R(v_j)\chi_j^s \left(\nabla S_R(u_n) - \nabla S_R(v_j)\chi_j^s\right) \\ &= \nabla S_R(v_j)\chi_j^s \nabla S_R(u_n) - \nabla S_R(v_j)\chi_j^s \nabla S_R(v_j)\chi_j^s. \end{aligned}$$

By virtue of $\nabla S_R(v_j)\chi_j^s \longrightarrow \nabla S_R(u)\chi^s$ as $j \longrightarrow \infty$ and $\nabla S_R(u_n) \rightharpoonup \nabla S_R(u)$ weakly in $E_\psi(Q_S)^N$, it follows that $M_{j,n} \longrightarrow 0$ as $n, j \longrightarrow \infty$. Since $(\nabla S_R(v_j)\chi_j^s)_j$ converges to $\nabla S_R(u)\chi_s$ strongly in $E_\psi(Q_S)^N$. Applying the dominated convergence theorem, we get $J_{4,n,j} \longrightarrow 0$ as $n, j \longrightarrow \infty$. For $J_{3,n,j}$, knowing that sequence $(\nabla S_R(u_n))_{n\in\mathbb{N}}$ is bounded and $(\nabla S_R(v_j)\chi_j^s)_j$ converge strongly to $\nabla S_R(u)\chi^s$ in $E_\psi(Q_S)^N$, then $\left(\nabla S_R(u_n) - \nabla S_R(v_j)\chi_j^s\right)$ is bounded. Using again the convergence of $(\nabla S_R(v_j)\chi_j^s)_j$ to $\nabla S_R(u)\chi^s$, then $J_{3,n,j} \longrightarrow 0$ as $n, j \longrightarrow \infty$. For $J_{2,n,j}$, we have $\left(\nabla S_R(v_j)\chi_j^s - \nabla S_R(u)\chi_s\right) \longrightarrow 0$ as $j \longrightarrow \infty$, $(\nabla S_R(u_n))_{n\in\mathbb{N}}$ and $(\nabla S_R(v_j)\chi_j^s)_j$ are convergent. Consequently, $\left(\nabla S_R(u_n) - \nabla S_R(v_j)\chi_j^s\right)$, is bounded. Applying again the convergence dominate theorem, we get $J_{2,n,j} \longrightarrow 0$ as $n, j \longrightarrow \infty$, and

$$\lim_{n,j\to\infty} \int_A M_{j,n} \mathrm{d}x\mathrm{d}t = 0.$$

Letting n, j, then η, i, s, ϑ to infinity, we get

$$\lim_{n\to\infty} \sup J_n^r = \lim_{n\to\infty} \int_{Q_r} N_n^\theta \mathrm{d}x\mathrm{d}s = 0.$$

On the other hand, from (4.3), we obtain

$$0 \le \int_{Q_r} [\psi(x, |\nabla S_R(u_n) - \nabla S_R(u)|)]^\theta \, \mathrm{d}x\mathrm{d}s \le \int_{Q_r} N_n^\theta \mathrm{d}x\mathrm{d}s.$$

We recall that

$$\nabla S_R(u_n) \longrightarrow \nabla S_R(u) \text{ as } n \longrightarrow \infty,$$

almost everywhere in Q_r. Since $r > 0$ is arbitrary, we recall that for another subsequence $\nabla S_R(u_n) \longrightarrow \nabla S_R(u)$ almost everywhere in Q_S. Finally, for $R > 0$ arbitrary, we get

$$\nabla u_n \longrightarrow \nabla u \text{ a.e. in } Q_S. \tag{5.44}$$

This concludes the proof. □

Step III

We obtain the first condition of Definition 4.2, by applying (5.31), (5.24), and (5.26). For the second condition of the same definition obtained using the convergence (5.44) and the smoothness of the function f. For the regularity of the solution u, we use (5.11) and we apply directly Lemma 3.5, we get that $u \in C\left([0, S]; L^1(\Omega)\right)$. This concludes the proof of Theorem 5.1 □ .

6 Conclusion and perspectives

In this paper, we showed the existence result for a capacity solution to a nonlocal thermistor problem in Musielak–Orlicz–Sobolev spaces. In the future, we plan on studying the regularity of a global attractor. Other intriguing problems about this capacity solution surround the development of specific qualitative properties [5], such as the calculation of an energy estimate, the study of long-term behavior, or even the possibility of a blow-up event.

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