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ORIGINAL PAPER

Singular value inequalities for convex functions of positive semidefinite matrices

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Abstract

In this paper, we give new singular value inequalities for matrices. It is shown that if A, B, X are $n \times n$ matrices such that X is positive semidefinite, and if $f : [0, \infty) \to \mathbb{R}$ is an increasing nonnegative convex function, then

$$s_j\left(f\left(\frac{|AXB^*|}{\|X\|}\right)\right) \leq \frac{\left\|f\left(\frac{A^*A+B^*B}{2}\right)\right\|}{\|X\|} s_j(X)$$

and

$$s_j\left(AXB^*\right) \leq \frac{1}{2}\left\|\frac{A^*A}{\|A\|^2} + \frac{B^*B}{\|B\|^2}\right\| \|A\| \|B\| s_j(X)$$

for $j = 1, 2, ..., n$. Some of our inequalities present refinements of some known singular value inequalities.

Keywords Singular value · Spectral norm · Unitarily invariant norm · Positive semidefinite matrix · Convex function · Inequality

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1 Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. For $A \in \mathbb{M}_n(\mathbb{C})$, the singular values of A, denoted by $s_1(A), ..., s_n(A)$, are the eigenvalues of $|A| = (A^*A)^{1/2}$ arranged in decreasing order and repeated according to multiplicity.

A norm $|||\cdot|||$ on $\mathbb{M}_n(\mathbb{C})$ is called unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathbb{M}_n(\mathbb{C})$ and all unitary matrices $U, V \in \mathbb{M}_n(\mathbb{C})$. Some of the typical examples of unitarily invariant norms are the spectral norm $\|A\| = s_1(A)$ and the Ky Fan k-norm $\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A)$ for $k = 1, 2, ..., n$.

One of the most famous inequalities for the singular values of matrices (see, e.g., [4, p. 75] or [7, p. 27]) asserts that if $A, B, X \in \mathbb{M}_n(\mathbb{C})$, then

$$s_j(AXB) \leq \|A\| \, \|B\| \, s_j(X) \tag{1.1}$$

for $j = 1, 2, ..., n$. Another important result is the celebrated matrix Young inequality (see, e.g., [2]), which says that if $A, B \in \mathbb{M}_n(\mathbb{C})$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$s_j(AB^*) \leq s_j\left(\frac{1}{p}|A|^p + \frac{1}{q}|B|^q\right) \tag{1.2}$$

for $j = 1, 2, ..., n$. Since unitarily invariant norms are increasing functions of singular values, we have

$$|||AB^*||| \leq \left|\left|\left|\frac{1}{p}|A|^p + \frac{1}{q}|B|^q\right|\right|\right|$$

for every unitarily invariant norm. The case $p = q = 2$ of the inequality (1.2), which is a matrix arithmetic–geometric mean inequality, has been obtained earlier in [5].

It has been shown in [1] that if $A, B, X \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite, then

$$s_j(A^{1/2}XB^{1/2}) \leq \frac{\|X\|}{2} s_j(A + B)$$

for $j = 1, 2, , ..., n$.

For $A, B \in \mathbb{M}_n(\mathbb{C})$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, let $C_{p,A,B} = \frac{1}{p}A^*A + \frac{1}{q}B^*B$. It has been shown in [3] that

$$s_j(AB^*) \leq s_j(C_{p,A,B}^{1/2} C_{q,A,B}^{1/2}) \tag{1.3}$$

for $j = 1, 2, , ..., n$.

It has been shown in [11] that

$$2s_j(K) \leq s_j\left(\begin{bmatrix} M & K \\ K^* & N \end{bmatrix}\right), \; j = 1, 2, ..., r \tag{1.4}$$

for any positive semidefinite matrix $\begin{bmatrix} M & K \\ K^* & N \end{bmatrix}$, where $M \in \mathbb{M}_m(\mathbb{C})$, $N \in \mathbb{M}_n(\mathbb{C})$, and $r = \min(m, n)$.

It has been shown in [9] that if $X, Y \in \mathbb{M}_n(\mathbb{C})$, then

$$s_j(X + Y) \leq 2s_j\,(X \oplus Y) \tag{1.5}$$

for $j = 1, 2, ..., n$.

In this paper, we give singular value inequalities for matrices. In Sect. 2, we give new inequalities involving singular values and unitarily invariant norms of products of matrices. In Sect. 3, we give applications of some of our results given in Sect. 2. In fact, we introduce a refinement of the inequality (1.1) when X is positive semidefinite. This refinement enables us to investigate the equality conditions of the inequality (1.1) in this special case. In addition, we establish the inequality (1.4) using a new approach of analysis. Moreover, we introduce a general version of the inequality (1.5) when the matrices X and Y are positive semidefinite.

2 Main results

We start with the following lemmas. For the first lemma, see, e.g., [4, p. 291]. The second lemma is a consequence of Theorem 3.2 in [6]. For the third lemma, see, e.g., [4, p. 72] or (III.19) in [4].

Lemma 2.1 *Let $A \in \mathbb{M}_n(\mathbb{C})$ and let $f : [0, \infty) \to \mathbb{R}$ be a nonnegative increasing function. Then,*

$$s_j\,(f\,(|A|)) = f\left(s_j\,(A)\right)$$

for $j = 1, 2, ..., n$.

Lemma 2.2 *Let $A, X \in \mathbb{M}_n(\mathbb{C})$ be such that A is positive semidefinite and X is contraction. If $f : [0, \infty) \to \mathbb{R}$ is convex, then*

$$s_j(f(X^*AX)) \leq s_j(X^*f(A)X)$$

for $j = 1, 2, ..., n$.

Lemma 2.3 *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then,*

$$\prod_{j=1}^{k} s_j(AB) \leq \prod_{j=1}^{k} s_j(A)s_j\,(B)$$

for $k = 1, 2, ..., n$.

The notions of weak majorization and weak-log-majorization are defined for sequences of real numbers as follows: Let $a = \left(a_j\right)_{j=1}^{n}$ and $b = \left(b_j\right)_{j=1}^{n}$ be two sequences of real numbers arranged in such a way that $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$. We say that a is weakly majorized by b, written as $a \prec_w b$, if $\sum_{j=1}^k a_j \leq \sum_{j=1}^k b_j$ for $k = 1, 2, ..., n$. In addition, when $a_n \geq 0$ and $b_n \geq 0$, we say that a is weakly log-majorized by b, written as $a \prec_{w\log} b$, if $\prod_{j=1}^k a_j \leq \prod_{j=1}^k b_j$ for $k = 1, 2, ..., n$. It is known that weak log-majorization implies weak majorization (see, e.g., [12, p. 19]). Moreover, a function $f : [0, \infty) \to \mathbb{R}$ is said to preserve weak-log-majorization if $f(a) \prec_{w\log} f(b)$ whenever $a \prec_{w\log} b$, where $f(a) = \left(f\left(a_j\right)\right)_{j=1}^n$ and $f(b) = \left(f\left(b_j\right)\right)_{j=1}^n$. For more details on such functions, we refer the reader to [8]. In addition, f is called submultiplicative if $f(xy) \leq f(x)f(y)$ whenever $x, y \in [0, \infty)$.

Theorem 2.4 *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$, let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and let $f : [0, \infty) \to \mathbb{R}$ be a nonnegative increasing submultiplicative convex function that preserves weak-log majorization.*

(a) *If X is a positive semidefinite contraction, then*

$$\prod_{j=1}^k s_j \left(f\left(\left|AXB^*\right|^2\right)\right) \leq \left\|f\left(C_{p,A,B}\right)\right\|^k \left\|f\left(C_{q,A,B}\right)\right\|^k \prod_{j=1}^k s_j^2(X)$$

for $k = 1, 2, ..., n$.

(b) *If X is a nonzero positive semidefinite matrix, then*

$$\prod_{j=1}^k s_j \left(f\left(\frac{|AXB^*|^2}{\|X\|^2}\right)\right) \leq \frac{\left\|f\left(C_{p,A,B}\right)\right\|^k \left\|f\left(C_{q,A,B}\right)\right\|^k}{\|X\|^{2k}} \prod_{j=1}^k s_j^2(X)$$

for $k = 1, 2, ..., n$. In particular, letting $f(t) = t$, we have

$$\prod_{j=1}^k s_j^2\left(AXB^*\right) \leq \left\|C_{p,A,B}\right\|^k \left\|C_{q,A,B}\right\|^k \prod_{j=1}^k s_j^2(X)$$

for $k = 1, 2, ..., n$.

Proof Suppose that X is a positive semidefinite contraction and let $\mathcal{A} = AX^{1/2}$, $\mathcal{B} = BX^{1/2}$. Then,

$$\begin{aligned} s_j\left(f\left(C_{p,\mathcal{A},\mathcal{B}}\right)\right) &= s_j\left(f\left(X^{1/2}C_{p,A,B}X^{1/2}\right)\right) \\ &\leq s_j\left(X^{1/2}f\left(C_{p,A,B}\right)X^{1/2}\right) \text{ (by Lemma 2.2)} \\ &= s_j\left(f^{1/2}\left(C_{p,A,B}\right)Xf^{1/2}\left(C_{p,A,B}\right)\right) \\ &\leq \left\|f\left(C_{p,A,B}\right)\right\| s_j(X) \text{ (by the inequality (1.1)).} \end{aligned} \tag{2.1}$$

Similarly, we have

$$s_j\left(f\left(C_{q,\mathcal{A},\mathcal{B}}\right)\right) \leq \left\| f\left(C_{q,A,B}\right)\right\| s_j\left(X\right). \tag{2.2}$$

Now,

$$\begin{aligned}
\prod_{j=1}^{k} s_j(\left|AXB^*\right|^2)) &= \prod_{j=1}^{k} s_j^2(AXB^*) \\
&= \prod_{j=1}^{k} s_j^2(\mathcal{A}\mathcal{B}^*) \\
&\leq \prod_{j=1}^{k} s_j^2\left(C_{p,\mathcal{A},\mathcal{B}}^{1/2} C_{q,\mathcal{A},\mathcal{B}}^{1/2}\right) \text{ (by the inequality (1.3))} \\
&\leq \prod_{j=1}^{k} s_j^2\left(C_{p,\mathcal{A},\mathcal{B}}^{1/2}\right) s_j^2\left(C_{q,\mathcal{A},\mathcal{B}}^{1/2}\right) \text{ (by Lemma 2.3)} \\
&= \prod_{j=1}^{k} s_j\left(C_{p,\mathcal{A},\mathcal{B}}\right) s_j\left(C_{q,\mathcal{A},\mathcal{B}}\right) \qquad (2.3)
\end{aligned}$$

for $k = 1, 2, ..., n$, which means that

$$\left(s_j(\left|AXB^*\right|^2)\right)_{j=1}^{n} \prec_{w \log} \left(s_j\left(C_{p,\mathcal{A},\mathcal{B}}\right) s_j\left(C_{q,\mathcal{A},\mathcal{B}}\right)\right)_{j=1}^{n}.$$

Since f preserves weak-log-majorization, we have

$$\left(f\left(s_j(\left|AXB^*\right|^2)\right)\right)_{j=1}^{n} \prec_{w \log} \left(f\left(s_j\left(C_{p,\mathcal{A},\mathcal{B}}\right) s_j\left(C_{q,\mathcal{A},\mathcal{B}}\right)\right)\right)_{j=1}^{n} \tag{2.4}$$

and so,

$$\begin{aligned}
\prod_{j=1}^{k} s_j\left(f\left(\left|AXB^*\right|^2\right)\right) &= \prod_{j=1}^{k} f\left(s_j\left(\left|AXB^*\right|^2\right)\right) \\
&\leq \prod_{j=1}^{k} f\left(s_j\left(C_{p,\mathcal{A},\mathcal{B}}\right) s_j\left(C_{q,\mathcal{A},\mathcal{B}}\right)\right) \text{ (by the inequality (2.4))} \\
&\leq \prod_{j=1}^{k} f\left(s_j\left(C_{p,\mathcal{A},\mathcal{B}}\right)\right) f\left(s_j\left(C_{q,\mathcal{A},\mathcal{B}}\right)\right) \\
&\quad \text{(since } f \text{ is submultiplicative)} \\
&= \prod_{j=1}^{k} s_j\left(f\left(C_{p,\mathcal{A},\mathcal{B}}\right)\right) s_j\left(f\left(C_{q,\mathcal{A},\mathcal{B}}\right)\right)
\end{aligned}$$

$$\leq \left\| f\left(C_{p,A,B}\right)\right\|^{k} \left\| f\left(C_{q,A,B}\right)\right\|^{k} \prod_{j=1}^{k} s_j^2\left(X\right)$$
(by the inequalities (2.1) and (2.2))

for $k = 1, 2, ..., n$. This proves part (a). Part (b) follows by applying part (a) to the positive semidefinite contraction matrix $\frac{X}{\|X\|}$. □

For $A, B \in \mathbb{M}_n(\mathbb{C})$, the Ky Fan dominance principle asserts that $|||A||| \leq |||B|||$ for all unitarily invariant norms if and only if $\|A\|_{(k)} \leq \|B\|_{(k)}$ for $k = 1, 2, ..., n$ (see, e.g., [4, p. 93]or[7, p.72]). Since weak log-majorization implies weak majorization, the Ky Fan dominance principle, enables us to give the following application of Theorem 2.4.

Corollary 2.5 *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and let $f : [0, \infty) \to \mathbb{R}$ be a nonnegative increasing submultiplicative convex function that preserves weak-log-majorization.*

(a) *If X is a positive semidefinite contraction, then*

$$\left|\left|\left| f\left(\left|AXB^*\right|^2\right)\right|\right|\right| \leq \left\| f\left(C_{p,A,B}\right)\right\| \left\| f\left(C_{q,A,B}\right)\right\| \left|\left|\left| \, |X|^2 \right|\right|\right|$$

for every unitarily invariant norm.

(b) *If X is a nonzero positive semidefinite, then*

$$\left|\left|\left| f\left(\frac{|AXB^*|^2}{\|X\|^2}\right)\right|\right|\right| \leq \frac{\left\| f\left(C_{p,A,B}\right)\right\| \left\| f\left(C_{q,A,B}\right)\right\|}{\|X\|^2} \left|\left|\left| \, |X|^2 \right|\right|\right|$$

for every unitarily invariant norm. In particular, letting $f(t) = t$, we have

$$\left|\left|\left| \left|AXB^*\right|^2 \right|\right|\right| \leq \left\| f\left(C_{p,A,B}\right)\right\| \left\| f\left(C_{q,A,B}\right)\right\| \left|\left|\left| \, |X|^2 \right|\right|\right|$$

for every unitarily invariant norm.

For $p = q = 2$, a stronger version of our result given in Theorem 2.4 can be seen as follows.

Theorem 2.6 *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$, and let $f : [0, \infty) \to \mathbb{R}$ be a nonnegative increasing convex function.*

(a) *If X is a positive semidefinite contraction, then*

$$s_j\left(f\left(\left|AXB^*\right|\right)\right) \leq \left\| f\left(\frac{A^*A + B^*B}{2}\right)\right\| s_j\left(X\right)$$

for $j = 1, 2, ..., n$.

(b) *If X is a nonzero positive semidefinite matrix, then*

$$s_j\left(f\left(\frac{|AXB^*|}{\|X\|}\right)\right) \le \frac{\left\|f\left(\frac{A^*A+B^*B}{2}\right)\right\|}{\|X\|} s_j(X)$$

for $j = 1, 2, ..., n$. In particular, letting $f(t) = t$, we have

$$s_j\left(AXB^*\right) \le \frac{\|A^*A + B^*B\|}{2} s_j(X)$$

for $j = 1, 2, ..., n$.

Proof Suppose that X is a positive semidefinite contraction, and let $\mathcal{A} = AX^{1/2}$, $\mathcal{B} = BX^{1/2}$. Then,

$$\begin{aligned}
s_j(f(\left|AXB^*\right|)) &= f(s_j(AXB^*)) \\
&= f(s_j(\mathcal{A}\mathcal{B}^*)) \\
&\le f\left(s_j\left(C_{2,\mathcal{A},\mathcal{B}}\right)\right) \text{ (by the inequality (1.3))} \\
&= s_j\left(f\left(X^{1/2}C_{2,A,B}X^{1/2}\right)\right) \qquad (2.5) \\
&\le s_j\left(X^{1/2}f\left(C_{p,A,B}\right)X^{1/2}\right) \text{ (by Lemma 2.2)} \\
&\le \left\|f\left(C_{2,A,B}\right)\right\| s_j(X) \text{ (by the inequality (1.1))} \\
&= \left\|f\left(\frac{A^*A + B^*B}{2}\right)\right\| s_j(X). \qquad (2.6)
\end{aligned}$$

This proves part (a). Part (b) follows by applying part (a) to the positive semidefinite contraction matrix $\frac{X}{\|X\|}$. □

We need the following special case of Theorem V.2.3 in [4].

Lemma 2.7 *Let $A \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite, and let $f : [0, \infty) \to \mathbb{R}$ be a convex function with $f(0) = 0$. Then, $f(\alpha A) \le \alpha f(A)$ for all $\alpha \in [0, 1]$.*

It can be seen that if $f : [0, \infty) \to \mathbb{R}$ is *a nonnegative* convex function with $f(0) = 0$, then f is increasing.

Theorem 2.8 *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ be such that $A^*A + B^*B \le 2I_n$, X is positive semidefinite, and let $f : [0, \infty) \to \mathbb{R}$ be a nonnegative convex function with $f(0) = 0$. Then,*

$$s_j\left(f\left(\left|AXB^*\right|\right)\right) \le \frac{\|A^*A + B^*B\|}{2} s_j(f(X))$$

for $j = 1, 2, ..., n$.

Proof It follows from Theorem 2.6 that

$$s_j\left(AXB^*\right) \leq \frac{\|A^*A + B^*B\|}{2} s_j\left(X\right)$$

The function f is nonnegative and convex with $f\left(0\right) = 0$, so it is increasing. Consequently,

$$\begin{aligned} s_j(f(\left|AXB^*\right|)) &\leq f\left(\frac{\|A^*A + B^*B\|}{2} s_j\left(X\right)\right) \\ &\leq \frac{\|A^*A + B^*B\|}{2} f\left(s_j\left(X\right)\right) \text{ (by Lemma 2.7)} \\ &= \frac{\|A^*A + B^*B\|}{2} s_j\left(f\left(X\right)\right), \end{aligned}$$

as required. □

3 Applications

In this section, we give applications of some of our results given in Sect. 2. First, we start with an application of the particular case given in Theorem 2.6 (b). Our result can be regarded as a refinement of the inequality (1.1) when X is positive semidefinite.

Corollary 3.1 *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ be such that X is positive semidefinite. Then,*

$$s_j\left(AXB^*\right) \leq \frac{1}{2} \left\| \frac{A^*A}{\|A\|^2} + \frac{B^*B}{\|B\|^2} \right\| \|A\| \|B\| s_j\left(X\right) \tag{3.1}$$

for $j = 1, 2, ..., n$.

Proof The result follows from Theorem 2.6 (b) by replacing A and B by $\frac{A}{\|A\|}$ and $\frac{B}{\|B\|}$, respectively. □

Remark 3.2 It can be seen that for the case when $X \in \mathbb{M}_n(\mathbb{C})$ is positive semidefinite, the inequality (3.1) is sharper than the inequality (1.1). Indeed, the triangle inequality implies that

$$\begin{aligned} s_j\left(AXB^*\right) &\leq \frac{1}{2} \left\| \frac{A^*A}{\|A\|^2} + \frac{B^*B}{\|B\|^2} \right\| \|A\| \|B\| s_j\left(X\right) \\ &\leq \frac{1}{2} \left(\left\| \frac{A^*A}{\|A\|^2} \right\| + \left\| \frac{B^*B}{\|B\|^2} \right\| \right) \|A\| \|B\| s_j\left(X\right) \\ &= \|A\| \|B\| s_j\left(X\right) \end{aligned}$$

for $j = 1, 2, ..., n$.

The following lemma can be found in [10].

Lemma 3.3 *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be nonzero positive semidefinite matrices. Then,*

$$\|A+B\| = \|A\| + \|B\| \text{ if and only if } \left\|A^{1/2}B^{1/2}\right\| = \|A\|^{1/2}\|B\|^{1/2}.$$

As an application of Corollary 3.1, we investigate the equality conditions of the inequality (1.1) when X is positive semidefinite.

Corollary 3.4 *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ be nonzero matrices such that X is positive semidefinite. If $s_j(AXB^*) = \|A\|\|B\| s_j(X)$ and $s_j(X) \neq 0$ for some $j \in \{1, 2, ..., n\}$, then $\|AB^*\| = \|A\|\|B\|$ and $\|A^*A + B^*B\| = \|A\|^2 + \|B\|^2$.*

Proof Suppose that $s_j(AXB^*) = \|A\|\|B\| s_j(X)$ and $s_j(X) \neq 0$ for some $j \in \{1, 2, ..., n\}$. It follows from Corollary 3.1 that

$$\begin{aligned} s_j\left(AXB^*\right) &\leq \frac{1}{2}\left\|\frac{A^*A}{\|A\|^2} + \frac{B^*B}{\|B\|^2}\right\| \|A\|\|B\| s_j(X) \\ &\leq \|A\|\|B\| s_j(X) \\ &= s_j\left(AXB^*\right). \end{aligned}$$

Therefore, $\left\|\frac{A^*A}{\|A\|^2} + \frac{B^*B}{\|B\|^2}\right\| = 2 = \left\|\frac{A^*A}{\|A\|^2}\right\| + \left\|\frac{B^*B}{\|B\|^2}\right\|$. This, together with Lemma 3.3, implies that

$$\|\,|A|\ |B|\,\| = \|A\|\|B\|,$$

which is equivalent to

$$\left\|AB^*\right\| = \|A\|\|B\|.$$

On the other hand, since $\|\,|A|\ |B|\,\| = \|A\|\|B\|$, we have

$$\left\|\left(A^*A\right)^{1/2}\left(B^*B\right)^{1/2}\right\| = \left\|A^*A\right\|^{1/2}\left\|B^*B\right\|^{1/2}. \tag{3.2}$$

Consequently, the relation (3.2) together with Lemma 3.3, implies that

$$\left\|A^*A + B^*B\right\| = \|A\|^2 + \|B\|^2,$$

as required. □

Remark 3.5 It should be mentioned here that the converse of Corollary 3.4 is not always true as the following example reveals: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then, X is positive semidefinite and $s_1(AXB^*) = 1 < 2 = \|A\|\|B\| s_1(X)$, while $\|AB^*\| = \|A\|\|B\| = 1$ and $\|A^*A + B^*B\| = \|A\|^2 + \|B\|^2 = 2$.

An application of Theorem 2.4 can be seen as follows.

Corollary 3.6 *Let $M \in \mathbb{M}_m(\mathbb{C})$ and $N \in \mathbb{M}_n(\mathbb{C})$ be such that $X = \begin{bmatrix} M & K \\ K^* & N \end{bmatrix}$ is positive semidefinite. Let $r = \min(m, n)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and let $f : [0, \infty) \to \mathbb{R}$ be a nonnegative submultiplicative convex function that preserves weak-log majorization.*

(a) *If X is contraction and f* is increasing *, then*

$$\prod_{j=1}^{k} s_j \left(f \left(|K|^2 \right) \right) \leq \max \left(f^{2k} \left(\frac{1}{p} \right), f^{2k} \left(\frac{1}{q} \right) \right) \prod_{j=1}^{k} s_j^2 (X)$$

for $k = 1, 2, ..., r$.

(b) *If X is nonzero and f* is *increasing, then*

$$\prod_{j=1}^{k} s_j \left(f \left(\frac{|K|^2}{\|X\|^2} \right) \right) \leq \frac{\max \left(f^{2k} \left(\frac{1}{p} \right), f^{2k} \left(\frac{1}{q} \right) \right)}{\|X\|^{2k}} \prod_{j=1}^{k} s_j^2 (X)$$

for $k = 1, 2, ..., r$. In particular, letting $f(t) = t$, we have

$$\prod_{j=1}^{k} s_j (K) \leq \max \left(\frac{1}{p^k}, \frac{1}{q^k} \right) \prod_{j=1}^{k} s_j (X)$$

for $k = 1, 2, ..., r$.

(c) *If $f(0) = 0$, then*

$$\prod_{j=1}^{k} s_j \left(f \left(|K|^2 \right) \right) \leq \max \left(\frac{1}{p^{2k}}, \frac{1}{q^{2k}} \right) \prod_{j=1}^{k} s_j \left(f^2 (X) \right)$$

for $k = 1, 2, ..., r$.

Proof Let $A = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix}$. Then, $A^*A + B^*B = I_{m+n}$, and so

$$\begin{aligned} \prod_{j=1}^{k} s_j \left(f \left(|K|^2 \right) \right) &= \prod_{j=1}^{k} s_j \left(f \left(\left| AXB^* \right|^2 \right) \right) \\ &\leq \left\| f \left(C_{p,A,B} \right) \right\|^k \left\| f \left(C_{q,A,B} \right) \right\|^k \prod_{j=1}^{k} s_j^2 (X) \ \text{(by Theorem 2.4)} \\ &= \max \left(f^{2k} \left(\frac{1}{p} \right), f^{2k} \left(\frac{1}{q} \right) \right) \prod_{j=1}^{k} s_j^2 (X). \end{aligned}$$

This proves part (a).

Part (b) follows by applying part (a) to the contraction matrix $\frac{X}{\|X\|}$.

For part (c), the particular case of part (b) is equivalent to $a \prec_{w\log} b$, where $a = \left(s_j^2(K)\right)_{j=1}^r$ and $b = \left(\max\left(\frac{1}{p^2}, \frac{1}{q^2}\right) s_j^2(X)\right)_{j=1}^r$. Since f preserves weak-log-majorization, we have $f(a) \prec_{w\log} f(b)$, and so

$$\prod_{j=1}^{k} f\left(s_j^2(K)\right) \leq \prod_{j=1}^{k} f\left(\max\left(\frac{1}{p^2}, \frac{1}{q^2}\right) s_j^2(X)\right). \tag{3.3}$$

Since $0 < \max\left(\frac{1}{p^2}, \frac{1}{q^2}\right) < 1$ and f is convex with $f(0) = 0$, the inequality (3.3), together with Lemma 2.7, implies that

$$\prod_{j=1}^{k} s_j\left(f\left(|K|^2\right)\right) \leq \max\left(\frac{1}{p^{2k}}, \frac{1}{q^{2k}}\right) \prod_{j=1}^{k} f\left(s_j^2(X)\right). \tag{3.4}$$

Now, part (c) follows from the inequality (3.4) and the submultiplicativity of f. □

The following result is an application of Corollary 3.1. In this result, we establish the inequality (1.4) using a different approach of analysis other than that used in [11].

Corollary 3.7 *Let $M \in \mathbb{M}_m(\mathbb{C})$ and $N \in \mathbb{M}_n(\mathbb{C})$ be such that $X = \begin{bmatrix} M & K \\ K^* & N \end{bmatrix}$ is positive semidefinite. Let $r = \min(m, n)$ and $f : [0, \infty) \to \mathbb{R}$ be a nonnegative convex function with $f(0) = 0$. Then,*

(a)

$$2s_j(K) \leq s_j(X) \tag{3.5}$$

for $j = 1, 2, ..., r$.

(b)

$$2s_j(f(|K|)) \leq s_j(f(X)) \tag{3.6}$$

and

$$s_j\left(f\left(\frac{|K|}{\|X\|}\right)\right) \leq \frac{f\left(\frac{1}{2}\right)}{\|X\|} s_j(X) \tag{3.7}$$

for $j = 1, 2, ..., r$..

Proof Let $A = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix}$. Then, $A^*A + B^*B = I_{m+n}$. Applying Corollary 2.8 to the function $f(t) = t$, we have

$$s_j\left(AXB^*\right) \leq \frac{\|A^*A + B^*B\|}{2} s_j(X)$$

$$= \frac{1}{2} s_j (X) . \tag{3.8}$$

Now, the inequality (3.5) follows from the inequality (3.8) and the fact that

$$s_j \left(AXB^* \right) = s_j (K) .$$

For part (b), applying both sides of the inequality in part (a) to the increasing function f, we have

$$s_j (f (|K|)) \leq s_j \left(f \left(\frac{X}{2} \right) \right) . \tag{3.9}$$

Since f is convex with $f(0) = 0$, the inequality (3.6) follows from the inequality (3.9) and Lemma 2.7. Part (c) follows by an argument similar to that used in the proof of the inequality (3.6). □

In the rest of this section, we give inequalities related to the inequality (1.5), from which we get a general version of this inequality for the case when X and Y are positive semidefinite matrices.

We start with the following application of Theorem 2.4.

Corollary 3.8 *Let $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and let $f : [0, \infty) \to \mathbb{R}$ be a nonnegative increasing submultiplicative convex function that preserves weak-log majorization.*

(a) *If X and Y are positive semidefinite contractions, then*

$$\begin{aligned} &\prod_{j=1}^{k} s_j \left(f \left(|AX + YB^*|^2 \right) \right) \\ &\quad \leq 2 \max \left(\left\| f \left(C_{p,A,I_n} \right) \right\|^k , \left\| f \left(C_{q,B,I_n} \right) \right\|^k \right) \\ &\quad \times \max \left(\left\| f \left(C_{q,A,I_n} \right) \right\|^k , \left\| f \left(C_{p,B,I_n} \right) \right\|^k \right) \prod_{j=1}^{k} s_j^2 (X \oplus Y) \end{aligned}$$

for $k = 1, 2, ..., n$.

(b) *If X and Y* are *nonzero positive semidefinite matrices, then*

$$\begin{aligned} &\prod_{j=1}^{k} s_j \left(f \left(\frac{|AX + YB^*|^2}{\max \left(\|X\|^2 , \|Y\|^2 \right)} \right) \right) \\ &\quad \leq \frac{2 \max \left(\left\| f \left(C_{p,A,I_n} \right) \right\|^k , \left\| f \left(C_{q,B,I_n} \right) \right\|^k \right)}{\max \left(\|X\|^{2k} , \|Y\|^{2k} \right)} \\ &\quad \times \max \left(\left\| f \left(C_{q,A,I_n} \right) \right\|^k , \left\| f \left(C_{p,B,I_n} \right) \right\|^k \right) \prod_{j=1}^{k} s_j^2 (X \oplus Y) \end{aligned}$$

for $k = 1, 2, ..., n$. *In particular, letting* $f(t) = t$, *we have*

$$\prod_{j=1}^{k} s_j^2 \left(AX + YB^*\right) \leq 2 \max \left(\left\| C_{p,A,I_n} \right\|^k, \left\| C_{q,B,I_n} \right\|^k \right) \times \max \left(\left\| C_{q,A,I_n} \right\|^k, \left\| C_{p,B,I_n} \right\|^k \right) \prod_{j=1}^{k} s_j^2 (X \oplus Y)$$

for $k = 1, 2, ..., n$.

Proof Let $\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & I_n \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} I_n & 0 \\ 0 & B \end{bmatrix}$, and $\mathcal{X} = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$. Then, $\mathcal{X}$ is positive semidefinite. Now,

$$\begin{aligned} s_j \left(f \left(\left| AX + YB^* \right| \right) \right) &\leq 2 s_j \left(f \left(\left| AX \oplus YB^* \right| \right) \right) \text{ (by the inequality (1.5))} \\ &= 2 s_j \left(f \left(\left| \mathcal{A} \mathcal{X} \mathcal{B}^* \right| \right) \right). \end{aligned} \tag{3.10}$$

The results now follow from the inequality (3.10) by applying Theorem 2.4 to the matrices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{X}$. □

When $p = q = 2$ a stronger version of Corollary 3.8 can be obtained using Theorem 2.6. The proof is similar to that given for Corollary 3.8. We leave the details for the interested reader.

Corollary 3.9 *Let* $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$, *and let* $f : [0, \infty) \to \mathbb{R}$ *be a nonnegative increasing convex function.*

(a) *If* X and Y *are positive semidefinite contractions, then*

$$s_j \left(f \left(\left| AX + YB^* \right| \right) \right) \leq 2 \max \left(f \left(\frac{\|A\|^2 + 1}{2} \right), f \left(\frac{\|B\|^2 + 1}{2} \right) \right) s_j (X \oplus Y)$$

for $j = 1, 2, ..., n$.

(b) *If* X and Y *are nonzero positive semidefinite matrices, then*

$$s_j \left(f \left(\frac{|AX + YB^*|}{\max(\|X\|, \|Y\|)} \right) \right) \leq \frac{2 \max \left(f \left(\frac{\|A\|^2 + 1}{2} \right), f \left(\frac{\|B\|^2 + 1}{2} \right) \right)}{\max(\|X\|, \|Y\|)} s_j (X \oplus Y)$$

for $j = 1, 2, ..., n$. *In particular, letting* $f(t) = t$, *we have*

$$s_j \left(AX + YB^* \right) \leq \left(\max \left(\|A\|^2, \|B\|^2 \right) + 1 \right) s_j (X \oplus Y)$$

for $j = 1, 2, ..., n$.

As a consequence of the particular case of Corollary 3.9, we have the following result.

Corollary 3.10 *Let $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ be such that X and Y are positive semidefinite. Then,*

$$s_j\left(AX + YB^*\right) \leq 2\max\left(\|A\|, \|B\|\right) s_j\left(X \oplus Y\right)$$

for $j = 1, 2, ..., n$.

Proof In the particular case of Corollary 3.9, replacing A and B by tA and tB, respectively, where $t > 0$, we get

$$s_j\left(AX + YB^*\right) \leq \left(\frac{t^2 \max\left(\|A\|^2, \|B\|^2\right) + 1}{t}\right) s_j\left(X \oplus Y\right) \tag{3.11}$$

for all $t > 0$. Now, the result follows by taking the infimum to both sides of the inequality (3.11) over all positive real numbers t and observing that

$$\inf_{t>0} \frac{1 + t^2 \max\left(\|A\|^2, \|B\|^2\right)}{t} = 2\max\left(\|A\|, \|B\|\right).$$

□

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