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ORIGINAL PAPER

On weighted boundedness and compactness of operators generated by fractional heat semigroups related with Schrödinger operators

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Abstract

Let $L=-\Delta+V$ be a Schrödinger operator with the potential V belonging to the reverse Hölder class $B_q, q>n/2$. Denote by $\mathrm{CMO}_\theta(\rho)$ the vanishing mean oscillation type space associated with L. By the aid of the regularity estimates of the fractional heat kernel related with L, we investigate the weighted boundedness and compactness of the commutators of operators generated by fractional heat semigroups related to L and functions belonging to $\mathrm{CMO}_\theta(\rho)$.

Keywords Commutator · Compactness · Schrödinger operator · Heat semigroup · Weight function

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1 Introduction

As an important concept in the functional analysis, the compactness of operators is widely used in the research of many fields, such as partial differential equations, the potential theory and so on. A linear operator L from a Banach space X to another Banach space Y is called a compact operator if the image under L of any bounded subset of X is a relatively compact subset of Y. As a class of bounded linear operators, compact operators have many good properties. For examples, it is well known that if Y is a Hilbert space, the class of compact operators can be defined alternatively as the closure of the set of finite-rank operators in the norm topology. The compact operators on an infinite-dimensional separable Hilbert space form a maximal ideal and the related quotient algebra is simple. Since the embedding of Sobolev spaces is compact, by the aid of the Gärding inequality and the Lax–Milgram theorem, an elliptic boundary value problem can be converted into Fredholm integral equations, see [54]. We refer to Conway [21], Folland and Stein [25] and Kutateladze [34] and the references therein for further information on compact operators.

In the literature, the compactness of commutators of different operators has been widely investigated by many researchers. In 1978, Uchiyama [49] first studied the compactness of commutators of a class of singular integral operators T_Ω and proved that the commutator $[b, T_\Omega]$ is compact on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in \mathrm{CMO}(\mathbb{R}^n)$, where $\mathrm{CMO}(\mathbb{R}^n)$ denotes the closure of $C_c^\infty(\mathbb{R}^n)$ in the topology of $\mathrm{BMO}(\mathbb{R}^n)$. The compactness of commutators of linear Fourier multipliers and pseudodifferential operators was considered by Cordes [22]. Peng [39] gave the compactness of paracommutators $T_b^{s,t}$. Beatrous and Li [6] studied the boundedness and compactness of the commutators of Hankel type operators. Krantz and Li [32, 33] applied the compactness characterization of the commutator $[b, T_\Omega]$ to study Hankel type operators on Bergman spaces. Wang [50] showed that the commutators of fractional integral operators are compact from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. In 2009, Chen and Ding [18] proved that the commutators of singular integrals with variable kernels are compact on $L^p(\mathbb{R}^n)$ if and only if $b \in \mathrm{CMO}(\mathbb{R}^n)$. In [19], the authors also established the compactness of Littlewood–Paley square functions. After that, Chen et al. [20] obtained the compactness of commutators for Marcinkiewicz integrals on Morrey spaces. Liu and Tang [38] studied the compactness for higher order commutators of oscillatory singular integral operators. For more information about the compactness problems of commutators, see also [7–9, 14–17, 23, 24, 27–31, 47, 48, 52, 55–57] and the references therein.

In recent twenty years, the semigroup of compact operators are widely investigated and used by many researchers on numerous topics of PDEs. In the setting of a Banach space X, Bahuguna and Srvastavai [4] studied the local and global existence of mild solutions to a class of integro-differential equations in an arbitrary Banach space associated with operators generating compact semigroup. In 2009, the results in [4] were extended to the time-fractional case by Rashid and El-Qaderi, see [40]. In [3], Avicou, Chalendar and Partington investigated the analyticity and the compactness of semigroups of composition operators on

Hardy and Dirichlet spaces on the unit disc. Let $\{e^{-tL}\}_{t\geq 0}$ be the Schrödinger heat semigroup on $L^2(\mathbb{R}^n)$ generated by the self-adjoint linear operator $-L = \Delta - q(x)$. In [2], Alziary and Takáč focus on the equivalence of the intrinsic ultra-contractivity of $\{e^{-tL}\}_{t\geq 0}$ and the compactness of this semigroup. For further information on this topic, see [1, 5, 41] and the references therein.

Consider the Schrödinger operator: $L = -\Delta + V$ in $\mathbb{R}^n$, $n \geq 3$, where Δ is the Laplacian operator on $\mathbb{R}^n$ and V is a nonnegative potential belonging to certain reverse Hölder class RH_q with an exponent $q > n/2$, that is, there exists a constant $C > 0$ such that for every ball $B(x, r)$ in $\mathbb{R}^n$,

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^q(y)\mathrm{d}y\right)^{1/q} \leq \frac{C}{|B(x,r)|}\int_{B(x,r)} V(y)\mathrm{d}y.$$

Recently, Bongioanni et al. [10] proved $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ boundedness for commutators of Riesz transforms associated with Schrödinger operators with $\mathrm{BMO}_\theta(\rho)(\mathbb{R}^n)$ functions which include the class of BMO functions. In [11], Bongioanni et al. established the weighted boundedness for Riesz transforms, fractional integrals and Littlewood–Paley functions associated to Schrödinger operators with weight A_p^ρ class which includes the Muckenhoupt weight class. Tang and his collaborators [45, 46] established weighted norm inequalities for some Schrödinger type operators, which include commutators of Riesz transforms, fractional integrals, and Littlewood–Paley functions related to Schrödinger operators (cf. [12, 13]). Li and Peng [37] investigated compact commutators of Riesz transforms associated to Schrödinger operators. Li et al. [36] established a compactness criterion with applications to the commutators associated with Schrödinger operators.

In this paper, we investigate the weighted compactness of commutators of maximal operators and fractional operators generated by fractional Schrödinger heat semigroups. Let $\{\Phi^L_{t,\gamma_1,\gamma_2}\}_{t\geq 0}$ be a family of operators related L with the kernels satisfying the conditions (2.3) and (2.4) below. Define the maximal operator related with $\{\Phi^L_{t,\gamma_1,\gamma_2}\}_{t\geq 0}$ as

$$\Phi^{L,*}_{\gamma_1,\gamma_2}(f)(x) := \sup_{t>0}\left\{\left|\Phi^L_{t,\gamma_1,\gamma_2}(f)(x)\right|\right\}. \tag{1.1}$$

Let $b \in \mathrm{CMO}_\theta(\rho)(\mathbb{R}^n)$ in Theorem 3.3 and $w \in A_p^{\rho,\theta}$. We consider the compactness of commutators $[b, \Phi^{L,*}_{\gamma_1,\gamma_2}]$ on the weighted Lebesgue spaces $L^p(w)$. In the literature, the proof of the compactness on the classical Lebesgue spaces is based on the following Frechet–Kolomogorv theorem.

Theorem 1.1 [58, *page* 275] *A subset G of $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, is strongly precompact if and only if it satisfies*:

(i) $\sup_{f\in G}\|f\|_p < \infty$;

(ii) *For any* $\varepsilon > 0$, *there exists a closed region* K_{δ_ε} *and* $\delta_\varepsilon > 0$ *such that for any* $f \in G, \|f\|_{L^p(K^c_{\delta_\varepsilon})} < \varepsilon$;

(iii) *For any* $f \in G$, $\lim_{|z|\to 0} \|f(\cdot + z) - f(\cdot)\|_p = 0$, *uniformly*.

In the proofs of the main results of [37] and [36], when verifying the condition (iii), the authors used the translation invariance of classical Lebesgue spaces, i.e., $\|f(\cdot + z)\|_{L^p} = \|f(\cdot)\|_{L^p}$, see [37, page 731]. Obviously, this translation invariance is invalid for the weighted Lebesgue spaces $L^p(w)$ and hence, Theorem 1.1 mentioned above is invalid for the compactness on weighted Lebesgue spaces. In 2021, Xue et al. established a weighted version of Theorem 1.1, which holds for weights beyond A_∞. As applications, the authors obtained the weighted compactness theory for the commutators of multilinear vector-valued Calderón–Zygmund type operators, including the commutators of multilinear Littlewood–Paley type operators. In essence, the method used in [57] is a the smooth truncated method. Roughly speaking, let $b \in \mathrm{BMO}$ and T be a linear operator T which is bounded on weighted Lebesgue spaces $L^p(w)$. We utillize a sequence of commutators $\{[b_{\epsilon,T}]\}_{\epsilon>0}$ to approximate $[b, T]$ in the sense of $L^p(w)$ as $\epsilon \to 0$, where $b_\epsilon \in C_0^\infty$. Since the elements in C_0^∞ are sufficiently smooth, this method avoids the difficulty caused by the lack of translation invariance of $L^p(w)$, and can be applied to deal with the weighted compactness of operators on many function spaces, see [27, 51, 57].

Inspired by [51, 57], by conditions (2.3)–(2.4) and the weighted estimates for maximal functions, we first prove that the L^p weighted boundedness of the maximal function $\Phi^{L,*}_{\gamma_1,\gamma_2}$ and the related commutator $[b, \Phi^{L,*}_{\gamma_1,\gamma_2}]$, see Theorem 2.13 and Proposition 2.14. Via a function in $C^\infty([0,\infty))$, we introduce a family of truncated operators $\{\Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}\}$ and prove that $\{[b, \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}]\}$ is compact on the weighted L^p space. Hence the compactness results for $[b, \Phi^{L,*}_{\gamma_1,\gamma_2}]$ can be deduced from [58, p.278, Theorem (iii)], see Theorem 3.3.

We point out that the L^p-compactness of the commutator $[b, \Phi^{L,*}_{\gamma_1,\gamma_2}]$ obtained in Theorem 3.3 covers many maximal functions associated with Schrödinger operators. Let $\{e^{-tL^\alpha}\}_{t\ge 0}$, $\alpha \in (0,1)$, be the fractional heat semigroup associated with Schrödinger operators with the kernel $K^L_{\alpha,t}(\cdot,\cdot)$ and let $p^{L,\alpha}_{t,\sigma}(\cdot,\cdot)$ be the generalized Poisson operator related with L, respectively. Define the operators

$$T^L_{i,t}(f)(x) := \int_{\mathbb{R}^n} K^L_{i,t}(x,y)f(y)\mathrm{d}y,\ i = 1,2,\dots,6,$$

where

$$\begin{cases} K^L_{1,t}(x,y) := K^L_{\alpha,t}(x,y); \\ K^L_{2,t}(x,y) := t^{1/(2\alpha)}\nabla_x K^L_{\alpha,t}(x,y); \\ K^L_{3,t}(x,y) := t^m \partial^m_t K^L_{\alpha,t}(x,y), m > 0. \end{cases} \qquad \begin{cases} K^L_{4,t}(x,y) := p^{L,\alpha}_{t,\sigma}(x,y); \\ K^L_{5,t}(x,y) := t\nabla_x p^{L,\alpha}_{t,\sigma}(x,y); \\ K^L_{6,t}(x,y) := t^\upsilon \partial^\upsilon_t p^{L,\alpha}_{t,\sigma}(x,y), \upsilon > 0, \end{cases} \tag{1.2}$$

and

$$p_{t,\sigma}^{L,\alpha}(x,y) := \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-r} K_{\alpha,t^{2\alpha}/(4r)}^L(x,y) \frac{\mathrm{d}r}{r^{1-\sigma}},\ 0 < \sigma < 1. \tag{1.3}$$

The corresponding maximal operators are defined as

$$T_i^{L,*}(f)(x) := \sup_{t>0} |T_{i,t}^L(f)(x)|,\ i = 1, 2, \dots, 6.$$

By the regularity estimates of the fractional heat kernel $K_{\alpha,t}^L(\cdot,\cdot)$, we can verify that the kernels $K_{i,t}^L(\cdot,\cdot)$, $i = 1, 2, \dots, 6$, satisfy the conditions (2.3) and (2.4) for some γ_1, γ_2. Therefore the weighted L^p-compactness of $[b,\ T_i^{L,*}]$, $i = 1, 2, \dots, 6$, are immediate consequences of Theorem 3.3.

In Sect. 4, we focus on the weighted (L^p, L^q)-boundedness and compactness of the fractional operators and their commutators generated by Schrödinger operators. Let

$$\widetilde{\mathcal{I}}_i^L f(x) := \int_{\mathbb{R}^n} \mathcal{I}_i^L(x,y) f(y) \mathrm{d}y,\ i = 1, 2,$$

where

$$\begin{cases} \mathcal{I}_1^L(x,y) := \int_0^\infty K_{\alpha,t}^L(x,y) t^{\beta/2-1} \mathrm{d}t; \\ \mathcal{I}_2^L(x,y) := \int_0^\infty p_{t,\sigma}^L(x,y) t^{\beta/2-1} \mathrm{d}t, \end{cases}$$

and

$$p_{t,\sigma}^L(x,y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-r} K_{t^2/(4r)}^L(x,y) \frac{\mathrm{d}r}{r^{1-\sigma}}.$$

In Theorems 4.3 and 4.4, an application of weighted estimates of maximal functions indicates the weighted (L^p, L^q)-boundedness of $\widetilde{\mathcal{I}}_i^L$, $i = 1, 2$, and $[b,\ \widetilde{\mathcal{I}}_i^L]$, mutators $[b,\ \widetilde{\mathcal{I}}_i^L]$, $i = 1, 2$, are compact from $L^p(w^p)$ to $L^q(w^q)$, where $w \in A_{p,q}^\rho$, see Theorem 4.5.

We point out that our compactness results generalize those obtained in [51]. Let $\alpha = 1$. Then the weighted compactness for $[b, T_1^{L,*}]$ in Theorem 3.4 (i) comes back to [51, Theorem 1.1]. The compactness results for $[b, T_i^{L,*}]$, $i = 2, 3, \dots 6$, are new even for $\alpha = 1$ or the setting of the Lebesgue measure, i.e., $w(x) = 1$. Also, noting that for $\alpha = 1$, $\mathcal{I}_1^L = L^{-\beta}$, the fractional integrals related with L. Hence, Theorem 4.5 can be seen as a generalization of [51, Theorem 1.2].

Notations Throughout this article, we will use A and C to denote the positive constants, which are independent of main parameters and may be different at each occurrence. By $B_1 \simeq B_2$, we mean that there exists a constant $C > 1$ such that $C^{-1} \leq B_1/B_2 \leq C$.

2 Preliminaries

2.1 Fractional Schrödinger heat kernels and weight functions

As in [42], we will use the auxiliary function ρ defined on $\mathbb{R}^n$ as

$$\rho(x) = \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)\mathrm{d}y \leq 1 \right\}. \tag{2.1}$$

Definition 2.1 Let $L = -\Delta + V$, $V \in B_q$, be a Schrödinger operator. Assume that $\gamma_1 > 0$ and $\gamma_2 \in (0, 2)$. Let $\{\Phi^L_{t,\gamma_1,\gamma_2}\}_{t\geq 0}$ be a family of operators related to L and with the integral kernel $K^{L,\gamma_1}_{\gamma_2,t}(\cdot,\cdot)$, i.e.,

$$\Phi^L_{t,\gamma_1,\gamma_2}(f)(x) = \int_{\mathbb{R}^n} K^{L,\gamma_1}_{\gamma_2,t}(x,y)f(y)\mathrm{d}y, \tag{2.2}$$

where $K^{L,\gamma_1}_{\gamma_2,t}(\cdot,\cdot)$ satisfies the following conditions.

(i) For any $N > 0$, there exists a constant $C_N > 0$ such that

$$|K^{L,\gamma_1}_{\gamma_2,t}(x,y)| \leq \frac{C_N t^{\gamma_1}}{(t^{1/\gamma_2} + |x-y|)^{n+\gamma_1\gamma_2}} \left(1 + \frac{t^{1/\gamma_2}}{\rho(x)} + \frac{t^{1/\gamma_2}}{\rho(y)}\right)^{-N}. \tag{2.3}$$

(ii) For any $N > 0$, there exists a constant $C_N > 0$ such that for every $0 < \delta < \min\{\gamma_2, 1 - n/q\}$ and all $|h| \leq t^{1/\gamma_2}$,

$$|K^{L,\gamma_1}_{\gamma_2,t}(x+h,y) - K^{L,\gamma_1}_{\gamma_2,t}(x,y)| \leq \frac{C_N(|h|/t^{1/\gamma_2})^{\delta} t^{\gamma_1}}{(t^{1/\gamma_2} + |x-y|)^{n+\gamma_1\gamma_2}} \left(1 + \frac{t^{1/\gamma_2}}{\rho(x)} + \frac{t^{1/\gamma_2}}{\rho(y)}\right)^{-N}. \tag{2.4}$$

Below we state some preliminaries on the fractional heat semigroup associated with L. The fractional heat kernel associated with L is defined via the subordinative formula (cf. [26]). Precisely, for $0 < \alpha < 1$, there exists a continuous function $\eta^{\alpha}_t(\cdot)$ on $(0, \infty)$ such that

$$K^L_{\alpha,t}(x,y) = \int_0^{\infty} \eta^{\alpha}_t(s) K^L_s(x,y)\mathrm{d}s, \tag{2.5}$$

where $\eta_t^\alpha(\cdot)$ satisfies:

$$\begin{cases} \eta_t^\alpha(s) = \dfrac{1}{t^{1/\alpha}}\eta_1^\alpha(s/t^{1/\alpha}); \\ \eta_t^\alpha(s) \le C\dfrac{t}{s^{1+\alpha}} \quad \forall\, s,t > 0; \\ \displaystyle\int_0^\infty s^{-\gamma}\eta_1^\alpha(s)\mathrm{d}s < \infty,\ \gamma > 0; \\ \eta_t^\alpha(s) \simeq \dfrac{t}{s^{1+\alpha}} \quad \forall\, s \ge t^{1/\alpha} > 0. \end{cases}$$

The following regularity estimates of $K_{\alpha,t}^L(\cdot,\cdot)$ were obtained by Li et al. in [35].

Lemma 2.2 [35, *Propositions* 3.1 *and* 3.2] *Let* $\alpha \in (0,1)$.

(i) *For any* $N > 0$, *there exists a constant* $C_N > 0$ *such that*

$$\left|K_{\alpha,t}^L(x,y)\right| \le \frac{C_N t}{(t^{1/(2\alpha)} + |x-y|)^{n+2\alpha}}\left(1 + \frac{t^{1/(2\alpha)}}{\rho(x)} + \frac{t^{1/(2\alpha)}}{\rho(y)}\right)^{-N}.$$

(ii) *For any* $N > 0$, *there exists a constant* $C_N > 0$ *such that for every* $0 < \delta < \delta_0 = \min\{1, 2 - n/q\}$ *and all* $|h| \le t^{1/\alpha}$,

$$\left|K_{\alpha,t}^L(x+h,y) - K_{\alpha,t}^L(x,y)\right| \le \frac{C_N(|h|/t^{1/(2\alpha)})^\delta t}{(t^{1/(2\alpha)} + |x-y|)^{n+2\alpha}}\left(1 + \frac{t^{1/(2\alpha)}}{\rho(x)} + \frac{t^{1/(2\alpha)}}{\rho(y)}\right)^{-N}.$$

Lemma 2.3 [35, *Propositions* 3.6 *and* 3.9] *Suppose* $\alpha > 0$ *and* $V \in B_q$ *for some* $q > n$.

(i) *For every* $N > 0$, *there exists a constant* $C_N > 0$ *such that for all* $x, y \in \mathbb{R}^n$ *and* $t > 0$,

$$\left|t^{1/(2\alpha)}\nabla_x K_{\alpha,t}^L(x,y)\right| \le \frac{C_N t}{(t^{1/(2\alpha)} + |x-y|)^{n+2\alpha}}\left(1 + \frac{t^{1/(2\alpha)}}{\rho(x)} + \frac{t^{1/2(\alpha)}}{\rho(y)}\right)^{-N}.$$

(ii) *Let* $\delta = 1 - n/q$. *For every* $N > 0$, *there exists a constant* $C_N > 0$ *such that for all* $t > 0$ *and* $x, y \in \mathbb{R}^n$ *with* $|h| < |x-y|/4$,

$$\left|\nabla_x K^L_{\alpha,t}(x+h,y)-\nabla_x K^L_{\alpha,t}(x,y)\right|$$
$$\leq C_N\left(\frac{|h|}{t^{1/(2\alpha)}}\right)^{\delta}\frac{1}{t^{1/(2\alpha)}}\frac{t}{(t^{1/(2\alpha)}+|x-y|)^{n+2\alpha}}\left(1+\frac{t^{1/(2\alpha)}}{\rho(x)}+\frac{t^{1/(2\alpha)}}{\rho(y)}\right)^{-N}.$$

Lemma 2.4 [35, *Propositions* 3.11 *and* 3.12] *Let* $\alpha\in(0,1)$, $m>0$ *and* $V\in B_q, q>n/2$.

(i) *For every* $N>0$, *there exists a constant* $C_N>0$ *such that*

$$\left|t^m\partial_t^m K^L_{\alpha,t}(x,y)\right|\leq\frac{C_N t^m}{(t^{1/(2\alpha)}+|x-y|)^{n+2\alpha m}}\left(1+\frac{t^{1/(2\alpha)}}{\rho(x)}+\frac{t^{1/(2\alpha)}}{\rho(y)}\right)^{-N}.$$

(ii) *Let* $0<\delta\leq\min\{2\alpha,\delta_0\}$. *For every* $N>0$, *there exists a constant* $C_N>0$ *such that for all* $|h|\leq t^{1/(2\alpha)}$,

$$\left|t^m\partial_t^m K^L_{\alpha,t}(x+h,y)-t^m\partial_t^m K^L_{\alpha,t}(x,y)\right|$$
$$\leq C_N\left(\frac{|h|}{t^{1/(2\alpha)}}\right)^{\delta}\frac{t^m}{(t^{1/(2\alpha)}+|x-y|)^{n+2\alpha m}}\left(1+\frac{t^{1/(2\alpha)}}{\rho(x)}+\frac{t^{1/(2\alpha)}}{\rho(y)}\right)^{-N}.$$

Let B be a ball centered at x. For $\theta>0$, denote by $\Psi_\theta(B)$ the function $\Psi_\theta(B):=(1+r/\rho(x))^\theta$. Denote by $\mathcal{D}$ the set of dyadic cubes in $\mathbb{R}^n$. We also need the following dyadic maximal operator $M^{\triangle}_{V,\eta}$, $0<\eta<\infty$, which is defined as

$$M^{\triangle}_{V,\eta}(f)(x):=\sup_{x\in Q\in\mathcal{D}}\frac{1}{\Psi_\theta(Q)^\eta|Q|}\int_Q|f(y)|dy.$$

Let $0<\eta<\infty$ and $f_Q=\frac{1}{|Q|}\int_Q f(x)dx$. The dyadic sharp maximal operator $M^{\sharp}_{V,\eta}$ is defined by

$$\begin{aligned}M^{\sharp}_{V,\eta}(f)(x):=&\sup_{x\in Q\in\mathcal{D},r\geq\rho(x_0)}\frac{1}{|Q|}\int_{Q(x_0,r)}|f(y)-f_Q|dy\\&+\sup_{x\in Q\in\mathcal{D},r<\rho(x_0)}\frac{1}{\Psi_\theta(Q)^\eta|Q|}\int_{Q(x_0,r)}|f|dy\\ \simeq&\sup_{x\in Q\in\mathcal{D},r\geq\rho(x_0)}\inf_C\frac{1}{|Q|}\int_{Q(x_0,r)}|f(y)-C|dy\\&+\sup_{x\in Q\in\mathcal{D},r<\rho(x_0)}\frac{1}{\Psi_\theta(Q)^\eta|Q|}\int_{Q(x_0,r)}|f|dy,\end{aligned}$$

where the supremum is taken over all dyadic cubes $Q(x_0,r)$ centered at x_0 with the radius r.

A variant of the dyadic maximal operator and the dyadic sharp maximal operator are defined by

$$M^{\triangle}_{\delta,\eta}(f)(x) = M^{\triangle}_{V,\eta}(|f|^{\delta})^{1/\delta}(x)$$

and

$$M^{\sharp}_{\delta,\eta}(f)(x) = M^{\sharp}_{V,\eta}(|f|^{\delta})^{1/\delta}(x).$$

Definition 2.5 ($\mathrm{BMO}_\theta(\rho)(\mathbb{R}^n)$ space, [10]) For all $x \in \mathbb{R}^n$, $r > 0$ and $\theta > 0$, the class $\mathrm{BMO}_\theta(\rho)(\mathbb{R}^n)$ is defined as the set of all locally integrable functions f satisfying

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y) - f_B| \mathrm{d}y \le C\left(1 + \frac{r}{\rho(x)}\right)^{\theta}. \tag{2.6}$$

A norm for $f \in \mathrm{BMO}_\theta(\rho)(\mathbb{R}^n)$ is given by the infimum of the constants satisfying (2.6), after identifying functions that differ upon a constant. For convenience, the norm for $f \in \mathrm{BMO}_\theta(\rho)(\mathbb{R}^n)$ is denoted by $||f||_{\mathrm{BMO}_\theta(\rho)}$. It is clear that $\mathrm{BMO}(\mathbb{R}^n) \subset \mathrm{BMO}_\theta(\rho)(\mathbb{R}^n) \subset \mathrm{BMO}_{\theta'}(\rho)(\mathbb{R}^n)$ for $0 < \theta < \theta'$.

For $b \in \mathrm{BMO}_\theta(\rho)(\mathbb{R}^n)$, the commutators of $\Phi^{L,*}_{\gamma_1,\gamma_2}$ and $\widetilde{\mathcal{I}}^L_i$, $i = 1, 2$, with b are defined by

$$[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2}](f)(x) := \sup_{t>0}\left|\int_{\mathbb{R}^n} K^{L,\gamma_1}_{\gamma_2,t}(x,y)(b(x) - b(y))f(y)\mathrm{d}y\right| \tag{2.7}$$

and

$$[b,\ \widetilde{\mathcal{I}}^L_i](f)(x) = \int_{\mathbb{R}^n} \mathcal{I}^L_i(x,y)(b(x) - b(y))f(y)\mathrm{d}y. \tag{2.8}$$

Let $B = B(x, r)$ and $\lambda > 0$. Denote by λB the λ-dilate ball centered at x and with radius λr. For a cube Q with sides parallel to the axes, we use λQ, $\lambda > 0$, to denote the cube Q dilated by λ.

Lemma 2.6 [42, *Lemma* 1.4] *There exist* $k_0 \ge 1$ *and* $C > 0$ *such that for all* $x, y \in \mathbb{R}^n$,

$$C^{-1}\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-k_0} \le \rho(y) \le C\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{k_0/(k_0+1)}.$$

In particular, $\rho(x) \sim \rho(y)$ *if* $|x - y| < C\rho(x)$.

Definition 2.7 ($A^{\rho,\theta}_p$ weights class, [11]) Let w be a nonnegative locally integrable function on $\mathbb{R}^n$.

(i) For $1 < p < \infty$, we say that a weight w belongs to the class $A_p^{\rho,\theta}$ if there exists a positive constant C such that for all balls $B = B(x, r)$,

$$\left(\frac{1}{|B|}\int_B w(y)\mathrm{d}y\right)\left(\frac{1}{|B|}\int_B w(y)^{-1/(p-1)}\mathrm{d}y\right)^{p-1} \le C\big(\Psi_\theta(B)\big)^p. \tag{2.9}$$

(ii) For $p = 1$, w is said to satisfy the $A_1^{\rho,\theta}$ condition if there exists a constant C such that for $x \in \mathbb{R}^n$ a.e.,

$$M_V^\theta(w)(x) := \sup_{x\in B} \frac{1}{\Psi_\theta(B)|B|}\int_B |w(y)|\mathrm{d}y \le Cw(x), \tag{2.10}$$

where the supremum is taken over all balls B centered at x with radius r.

Definition 2.8 ($A_{p,q}^{\rho,\theta}$ weights class, [45]). We say that a weight w belongs to the class $A_{p,q}^{\rho,\theta}$ for $1 < p < \infty$ and $1 \le q < \infty$, if there is a constant $C > 0$ such that for any cube $Q = Q(x, r)$,

$$\left(\frac{1}{\Psi_\theta(Q)|Q|}\int_Q w(y)^q\mathrm{d}y\right)^{1/q}\left(\frac{1}{\Psi_\theta(Q)|Q|}\int_Q w(y)^{-p/(p-1)}\mathrm{d}y\right)^{(p-1)/p} \le C.$$

Remark 2.9 It is easy to see that $w^{1/p} \in A_{p,p}^{\rho,\theta}$ if and only if $w \in A_p^{\rho,\theta}$ for $1 \le p < \infty$ and $w \in A_{p,q}^{\rho,\theta}$ if and only if $w^q \in A_{1+q/p'}^{\rho,\theta}$.

Lemma 2.10 [11, *Proposition* 5] *and* [45, *Lemma* 2.2] *Let* $1 < p < \infty$ *and* $w \in A_p^{\rho,\infty} = \bigcup_{\theta\ge 0} A_p^{\rho,\theta}$. *Then*

(i) *If* $1 \le p_1 < p_2 < \infty$, *then* $A_{p_1}^{\rho,\theta} \subset A_{p_2}^{\rho,\theta}$.

(ii) $w \in A_p^{\rho,\theta}$ *if and only if* $w^{-1/(p-1)} \in A_{p'}^{\rho,\theta}$, *where* $1/p + /p' = 1$.

(iii) *If* $w \in A_p^{\rho,\infty}$, $1 < p < \infty$, *then there exists* $\varepsilon > 0$ *such that* $w \in A_{p-\varepsilon}^{\rho,\infty}$.

(iv) *If* $w \in A_p^{\rho,\theta}$, $1 \le p < \infty$, *then for any cube* Q, *we have*

$$\frac{1}{\Psi_\theta(Q)|Q|}\int_Q |f(y)|\mathrm{d}y \le C\left(\frac{1}{w(5Q)}\int_Q |f(y)|^p w(y)\mathrm{d}y\right)^{1/p},$$

where $w(E) = \int_E w(x)\mathrm{d}x$. In particular, for any set $E \subset Q$, let $f = \chi_E$. Then

$$\frac{|E|}{\Psi_\theta(Q)|Q|} \le C\left(\frac{w(E)}{w(5Q)}\right)^{1/p}.$$

2.2 Weighted boundedness of maximal functions and commutators related with *L*

To prove the boundedness of $[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2}]$, we consider the following variant of maximal operator $M_{V,\eta}$, $0<\eta<\infty$, defined as

$$M_{V,\eta}(f)(x) := \sup_{x\in B}\frac{1}{(\Psi_\theta(B))^\eta|B|}\int_B|f(y)|\mathrm{d}y$$

and its commutator

$$[b,\ M_{V,\eta}](f)(x) := \sup_{x\in B}\frac{1}{(\Psi_\theta(B))^\eta|B|}\int_B|b(x)-b(y)||f(y)|\mathrm{d}y,$$

where the supremum is taken over all balls B centered at x with radius r.

Tang in [45] introduced a fractional variant maximal operator $M_{\beta,V,\eta}$ defined by

$$M_{\beta,V,\eta}(f)(x) := \sup_{x\in B}\frac{1}{(\Psi_\theta(B))^\eta(\Psi_\theta(B)|B|)^{1-\beta/n}}\int_B|f(y)|\mathrm{d}y,\quad 0<\eta<\infty,$$

and obtained the following weighted strong type (p, q) inequality.

Lemma 2.11 [45, *Theorem* 2.10] *Let* $0\le\beta<n$, $1<p<n/\beta$, $p'=p/(p-1)$ *and* $1/q=1/p-\beta/n$. *If* $w\in A^{\rho}_{p,q}$ *and* $\eta\ge(1-\beta/n)p'/q$, *then there exists a constant* $C>0$ *such that*

$$\|M_{\beta,V,\eta}(f)\|_{L^q(w^q)}\le C\|f\|_{L^p(w^p)}.$$

As a corollary of Lemma 2.11, for $\beta=0$, we can get

Corollary 2.12 *Let* $1<p<\infty$, $p'=p/(p-1)$ *and suppose that* $w\in A^{\rho,\theta}_p$. *Then there exists a constant* $C>0$ *such that*

$$\|M_{V,p'}(f)\|_{L^p(w)}\le C\|f\|_{L^p(w)}.$$

Theorem 2.13 *Let* $1<p<\infty$, $\gamma_1\gamma_2>\theta\eta$ *and* $w\in A^{\rho,\theta}_p$. *The operator* $\Phi^{L,*}_{\gamma_1,\gamma_2}$ *is bounded on* $L^p(w)$ *when* $w\in A^{\rho,\theta}_p$.

Proof We first divide $\Phi^{L,*}_{\gamma_1,\gamma_2}(f)$ into two parts:

$$\Phi^{L,*}_{\gamma_1,\gamma_2}(f)(x)\le\sup_{t>0}\left\{\int_{\mathbb{R}^n}|K^{L,\gamma_1}_{\gamma_2,t}(x,y)||f(y)|\mathrm{d}y\right\} := I(x)+II(x),$$

where

$$\begin{cases} I(x) := \sup_{t>0} \left\{ \int_{|x-y|<\rho(x)} |K_{\gamma_2,t}^{L,\gamma_1}(x,y)||f(y)|\mathrm{d}y \right\}; \\ II(x) := \sup_{t>0} \left\{ \int_{|x-y|\geq\rho(x)} |K_{\gamma_2,t}^{L,\gamma_1}(x,y)||f(y)|\mathrm{d}y \right\}. \end{cases}$$

The term $I(x)$ is further divided into three parts, i.e., $I(x) \leq I_1(x) + I_2(x) + I_3(x)$, where

$$\begin{cases} I_1(x) := \sup_{0<t^{1/\gamma_2}<\rho(x)} \left\{ \int_{|x-y|<t^{1/\gamma_2}} |K_{\gamma_2,t}^{L,\gamma_1}(x,y)||f(y)|\mathrm{d}y \right\}; \\ I_2(x) := \sup_{0<t^{1/\gamma_2}<\rho(x)} \left\{ \int_{t^{1/\gamma_2}\leq|x-y|<\rho(x)} |K_{\gamma_2,t}^{L,\gamma_1}(x,y)||f(y)|\mathrm{d}y \right\}; \\ I_3(x) := \sup_{t^{1/\gamma_2}\geq\rho(x)} \left\{ \int_{|x-y|\leq\rho(x)} |K_{\gamma_2,t}^{L,\gamma_1}(x,y)||f(y)|\mathrm{d}y \right\}. \end{cases}$$

By (2.3), we have

$$\begin{aligned} I_1(x) &\leq C \sup_{0<t^{1/\gamma_2}<\rho(x)} \left\{ \int_{|x-y|<t^{1/\gamma_2}} \frac{t^{\gamma_1}}{(t^{1/\gamma_2} + |x-y|)^{n+\gamma_1\gamma_2}} |f(y)|\mathrm{d}y \right\} \\ &\leq C \sup_{0<t^{1/\gamma_2}<\rho(x)} \left\{ \int_{|x-y|<t^{1/\gamma_2}} (t^{1/\gamma_2})^{-n} |f(y)|\mathrm{d}y \right\} \\ &\leq C \sup_{0<t^{1/\gamma_2}<\rho(x)} \left\{ \frac{2^{\theta\eta}}{|B(x,t^{1/\gamma_2})|} \frac{1}{(1+t^{1/\gamma_2}/\rho(x))^{\theta\eta}} \int_{|x-y|<t^{1/\gamma_2}} |f(y)|\mathrm{d}y \right\} \\ &\leq CM_{V,\eta}(f)(x). \end{aligned}$$

For $I_2(x)$, noting that $\gamma_1\gamma_2 > \theta\eta$, we can get

$$
\begin{aligned}
I_2(x) &\le C \sup_{0<t^{1/\gamma_2}<\rho(x)} \left\{ \sum_{k=1}^{\infty} \int_{|x-y|\sim 2^k t^{1/\gamma_2}} \frac{t^{\gamma_1}}{|x-y|^{n+\gamma_1\gamma_2}} |f(y)| \mathrm{d}y \right\} \\
&\le C \sup_{0<t^{1/\gamma_2}<\rho(x)} \left\{ \sum_{k=1}^{\infty} \frac{1}{(2^k)^{\gamma_1\gamma_2}} \frac{1}{(2^k t^{1/\gamma_2})^n} \int_{|x-y|\sim 2^k t^{1/\gamma_2}} |f(y)| \mathrm{d}y \right\} \\
&\le C \sum_{k=1}^{\infty} \frac{1}{2^{k(\gamma_1\gamma_2-\theta\eta)}} \sup_{0<t^{1/\gamma_2}<\rho(x)} \left\{ \frac{1}{|B(x, 2^k t^{1/\gamma_2})|} \frac{1}{(1+2^k t^{1/\gamma_2}/\rho(x))^{\theta\eta}} \right. \\
&\quad \left. \times \int_{|x-y|\sim 2^k t^{1/\gamma_2}} |f(y)| \mathrm{d}y \right\} \le C M_{V,\eta}(f)(x)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3(x) &\le C \sup_{t^{1/\gamma_2}\ge\rho(x)} \left\{ \int_{|x-y|<\rho(x)} \frac{1}{(t^{1/\gamma_2})^n} |f(y)| \mathrm{d}y \right\} \\
&\le C \sup_{t^{1/\gamma_2}\ge\rho(x)} \left\{ \frac{1}{(\rho(x))^n} \int_{|x-y|<\rho(x)} |f(y)| \mathrm{d}y \right\} \\
&\le C \sup_{t^{1/\gamma_2}\ge\rho(x)} \left\{ \frac{(1+\rho(x)/\rho(x))^{\theta\eta}}{|B(x,\rho(x))|} \frac{1}{(1+\rho(x)/\rho(x))^{\theta\eta}} \right. \\
&\quad \left. \times \int_{|x-y|<\rho(x)} |f(y)| \mathrm{d}y \right\} \le C M_{V,\eta}(f)(x).
\end{aligned}
$$

For $II(x)$, by (2.3) again, taking N large enough, we obtain

$$
\begin{aligned}
II(x) &\le C \sup_{t>0} \left\{ \sum_{k=1}^{\infty} \int_{|x-y|\sim 2^k\rho(x)} \frac{t^{\gamma_1}}{|x-y|^{n+\gamma_1\gamma_2}} \left(1+\frac{t^{1/\gamma_2}}{\rho(x)}\right)^{-N} |f(y)| \mathrm{d}y \right\} \\
&\le C \sup_{t>0} \left\{ \sum_{k=1}^{\infty} \frac{t^{\gamma_1}}{(2^k\rho(x))^{n+\gamma_1\gamma_2}} \left(1+\frac{t^{1/\gamma_2}}{\rho(x)}\right)^{-N} \int_{|x-y|\sim 2^k\rho(x)} |f(y)| \mathrm{d}y \right\} \\
&\le C \sup_{t>0} \left\{ \sum_{k=1}^{\infty} \frac{1}{2^{k\gamma_1\gamma_2}} \left(1+\frac{t^{1/\gamma_2}}{\rho(x)}\right)^{-N+\gamma_1\gamma_2} \frac{1}{(2^k\rho(x))^n} \right. \\
&\quad \left. \times \int_{|x-y|\sim 2^k\rho(x)} |f(y)| \mathrm{d}y \right\} \le C M_{V,\eta}(f)(x).
\end{aligned}
$$

The estimates for $I(x)$ and $II(x)$, together with Corollary 2.12 and $p' \le \eta < \infty$, imply that

$$\|\Phi^{L,*}_{\gamma_1,\gamma_2}(f)\|_{L^p(w)} \le C\|M_{V,\eta}(f)(x)\|_{L^p(w)} \le C\|f\|_{L^p(w)}.$$

This completes the proof. □

Proposition 2.14 *Let* $1 < p < \infty$, $\gamma_1\gamma_2 > \theta\eta$ *and* $w \in A_p^{\rho,\theta}$. *For* $b \in \mathrm{BMO}_\theta(\rho)(\mathbb{R}^n)$, *there exists a constant C such that*

$$\|[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2}](f)\|_{L^p(w)} \le C\|b\|_{\mathrm{BMO}_\theta(\rho)}\|f\|_{L^p(w)}.$$

Proof Similar to the procedure of the Theorem 2.13, we can get

$$|[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2}](f)(x)| \le C[b,\ M_{V,\eta}](f)(x).$$

Then using [44, Lemmas 3.3–3.5, Theorem 2.2 and Proposition 2.2], we can see that Proposition 2.14 holds. □

3 Compactness of commutators of semi-group maximal function

Let X and Y be Banach spaces. Suppose that T and $\{T_n\}$ are operators from X to Y. The weighted compactness of commutators are based on the following lemmas.

Lemma 3.1 [58, *p.* 278, *Theorem*(*iii*)] *Let a sequence* $\{T_n\}$ *of compact operators which converge to an operator T in the uniform operator toplogy, i.e.,* $\lim_{n\to\infty}\|T - T_n\| = 0$. *Then T is also compact.*

Lemma 3.2 [57, *Theorem* 1.1] *Let w be a weight on* $\mathbb{R}^n$. *Assume that* $w^{-1/(p_0-1)}$ *is also a weight on* $\mathbb{R}^n$ *for some* $p_0 > 1$. *Let* $0 < p < \infty$ *and* $\mathcal{F}$ *is sequentially compact in* $L^p(w)$ *if the following three conditions are satisfied*:

(i) $\mathcal{F}$ *is bounded, i.e.,* $\sup_{f\in\mathcal{F}} ||f||_{L^p(w)} < \infty$;
(ii) $\mathcal{F}$ *uniformly vanishes at infinity, i.e.,*

$$\lim_{N\to\infty}\sup_{f\in\mathcal{F}}\int_{|x|>N}|f(x)|^p w(x)\mathrm{d}x = 0;$$

(iii) $\mathcal{F}$ *is uniformly equicontinuous, i.e.,*

$$\lim_{|h|\to 0}\sup_{f\in\mathcal{F}}\int_{\mathbb{R}^n}|f(\cdot+h)-f(\cdot)|^p w(x)\mathrm{d}x=0.$$

Let $\mathrm{CMO}_\theta(\rho)(\mathbb{R}^n)$ denote the closure of $C_c^\infty(\mathbb{R}^n)$ in the $\mathrm{BMO}_\theta(\rho)(\mathbb{R}^n)$ topology.

Theorem 3.3 *Let* $1<p<\infty$, $\gamma_1\gamma_2>1+k_0\theta\eta$, $w\in A_p^{\rho,\theta}$ *and* $b\in\mathrm{CMO}_\theta(\rho)(\mathbb{R}^n)$. *Then the commutator* $[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2}]$ *defined by* (2.7) *is a compact operator from* $L^p(w)$ *to itself.*

Proof Firstly, we will consider a smooth truncated function to prove this theorem. Let $\varphi\in C^\infty([0,\infty))$ satisfy

$$0\le\varphi\le 1\ \text{ and }\ \varphi(x)=\begin{cases}1, & x\in[0,1],\\ 0, & x\in[2,\infty).\end{cases}\tag{3.1}$$

For any $\gamma>0$, let

$$K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x,y)=K^{L,\gamma_1}_{\gamma_2,t}(x,y)\big(1-\varphi(\gamma^{-1}|x-y|)\big).\tag{3.2}$$

Define

$$\Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}(f)(x)=\sup_{t>0}\left\{\left|\int_{\mathbb{R}^n}K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x,y)f(y)\mathrm{d}y\right|\right\}\tag{3.3}$$

and

$$[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)(x)=\sup_{t>0}\left\{\left|\int_{\mathbb{R}^n}K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x,y)(b(x)-b(y))f(y)\mathrm{d}y\right|\right\}.\tag{3.4}$$

For any $b\in C_c^\infty(\mathbb{R}^n)$ and $\gamma,\theta,\eta>0$, by (3.2), (3.4) and (2.3) with $N=\theta\eta$, one has

$$
\begin{aligned}
&|[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2}](f)(x) - [b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)(x)| \\
&\quad \le \left| \sup_{t>0} \left\{ \left| \int_{\mathbb{R}^n} K^{L,\gamma_1}_{\gamma_2,t}(x,y)(b(x)-b(y))f(y)\mathrm{d}y \right| \right\} \right. \\
&\qquad \left. - \sup_{t>0} \left\{ \left| \int_{\mathbb{R}^n} K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x,y)(b(x)-b(y))f(y)\mathrm{d}y \right| \right\} \right| \\
&\quad \le \sup_{t>0} \left\{ \left| \int_{\mathbb{R}^n} K^{L,\gamma_1}_{\gamma_2,t}(x,y)(b(x)-b(y))f(y)\big(1-1+\varphi(\gamma^{-1}|x-y|)\big)\mathrm{d}y \right| \right\} \\
&\quad \le C \sup_{t>0} \left\{ \left| \int_{\mathbb{R}^n} \frac{t^{\gamma_1}|x-y||f(y)|}{(t^{1/\gamma_2}+|x-y|)^{n+\gamma_1\gamma_2}} \left(1+\frac{t^{1/\gamma_2}}{\rho(x)}\right)^{-N} |\varphi(\gamma^{-1}|x-y|)|\mathrm{d}y \right| \right\} \\
&\quad \le L_1(x) + L_2(x),
\end{aligned}
$$

where

$$
L_1(x) := \sup_{t>0} \left\{ \left| \int_{|x-y|<2\gamma} \frac{t^{\gamma_1}|x-y||f(y)||\varphi(\gamma^{-1}|x-y|)|}{(t^{1/\gamma_2}+|x-y|)^{n+\gamma_1\gamma_2}} \left(1+\frac{t^{1/\gamma_2}}{\rho(x)}\right)^{-N} \mathrm{d}y \right| \right\}
$$

and

$$
L_2(x) := \sup_{t>0} \left\{ \left| \int_{|x-y|\ge 2\gamma} \frac{t^{\gamma_1}|x-y||f(y)||\varphi(\gamma^{-1}|x-y|)|}{(t^{1/\gamma_2}+|x-y|)^{n+\gamma_1\gamma_2}} \left(1+\frac{t^{1/\gamma_2}}{\rho(x)}\right)^{-N} \mathrm{d}y \right| \right\}.
$$

For $L_2(x)$, since $|x-y| \ge 2\gamma$, we have $\varphi(\gamma^{-1}|x-y|) = 0$. Then we can get

$$
\begin{aligned}
&\left|[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2}](f)(x) - [b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)(x)\right| \\
&\quad \le C\gamma \sup_{t>0} \left\{ \left| \int_{|x-y|<2\gamma} \frac{t^{\gamma_1}}{(t^{1/\gamma_2}+|x-y|)^{n+\gamma_1\gamma_2}} \left(1+\frac{t^{1/\gamma_2}}{\rho(x)}\right)^{-N} |f(y)|\mathrm{d}y \right| \right\} \\
&\quad \le C\gamma \big\{ J_1(x) + J_2(x) + J_3(x) \big\},
\end{aligned}
$$

where

$$
\begin{cases}
J_1(x) := \sup\limits_{t^{1/\gamma_2}<\gamma} \left\{ \displaystyle\int_{|x-y|<t^{1/\gamma_2}} \frac{t^{\gamma_1}(1+t^{1/\gamma_2}/\rho(x))^{-N}}{(t^{1/\gamma_2}+|x-y|)^{n+\gamma_1\gamma_2}} |f(y)|\mathrm{d}y \right\}; \\
J_2(x) := \sup\limits_{t^{1/\gamma_2}<\gamma} \left\{ \displaystyle\int_{t^{1/\gamma_2}<|x-y|<2\gamma} \frac{t^{\gamma_1}(1+t^{1/\gamma_2}/\rho(x))^{-N}}{(t^{1/\gamma_2}+|x-y|)^{n+\gamma_1\gamma_2}} |f(y)|\mathrm{d}y \right\}; \\
J_3(x) := \sup\limits_{t^{1/\gamma_2}\ge\gamma} \left\{ \displaystyle\int_{|x-y|<2\gamma} \frac{t^{\gamma_1}(1+t^{1/\gamma_2}/\rho(x))^{-N}}{(t^{1/\gamma_2}+|x-y|)^{n+\gamma_1\gamma_2}} |f(y)|\mathrm{d}y \right\}.
\end{cases}
$$

For the terms $J_1(x)$ and $J_3(x)$, we have

$$J_1(x) \leq \sup_{t^{1/\gamma_2}<\gamma} \left\{ \left(1+\frac{t^{1/\gamma_2}}{\rho(x)}\right)^{-\theta\eta} \int_{|x-y|<t^{1/\gamma_2}} \frac{|f(y)|}{t^{n/\gamma_2}} dy \right\} \leq CM_{V,\eta}(f)(x). \tag{3.5}$$

and

$$\begin{aligned} J_3(x) &\leq \sup_{t^{1/\gamma_2}\geq\gamma} \left\{ \int_{|x-y|<2\gamma} \left(t^{1/\gamma_2}\right)^{-n} \left(1+\frac{t^{1/\gamma_2}}{\rho(x)}\right)^{-N} |f(y)| dy \right\} \\ &\leq 2^{n+\theta\eta} \sup_{t^{1/\gamma_2}\geq\gamma} \left\{ (2\gamma)^{-n} \left(1+\frac{2\gamma}{\rho(x)}\right)^{-\theta\eta} \int_{|x-y|<2\gamma} |f(y)| dy \right\} \\ &\leq CM_{V,\eta}(f)(x). \end{aligned} \tag{3.6}$$

It remains to estimate $J_2(x)$.

$$\begin{aligned} J_2(x) &\leq \sup_{t^{1/\gamma_2}<\gamma} \left\{ \sum_{k=1}^{\infty} \int_{|x-y|\sim 2^k t^{1/\gamma_2}} \frac{t^{\gamma_1}}{(2^k t^{1/\gamma_2})^{n+\gamma_1\gamma_2}} \left(1+\frac{t^{1/\gamma_2}}{\rho(x)}\right)^{-N} |f(y)| dy \right\} \\ &\leq \sup_{t^{1/\gamma_2}<\gamma} \left\{ \sum_{k=1}^{\infty} \frac{1}{(2^{k\gamma_1\gamma_2})(2^k t^{1/\gamma_2})^n} \left(1+\frac{t^{1/\gamma_2}}{\rho(x)}\right)^{-\theta\eta} \right. \\ &\quad \left. \times \int_{|x-y|\sim 2^k t^{1/\gamma_2}} |f(y)| dy \right\} \\ &\leq CM_{V,\eta}(f)(x). \end{aligned} \tag{3.7}$$

By using (3.7), (3.5) and (3.6), the above fact leads to

$$|[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2}](f)(x) - [b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)(x)| \leq C\gamma M_{V,\eta}(f)(x).$$

Then Corollary 2.12 with $p' \leq \eta < \infty$ gives

$$\|[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2}](f) - [b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)\|_{L^p(w)} \leq C\gamma \|f\|_{L^p(w)},$$

which implies that

$$\lim_{\gamma\to 0} \|[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2}](f) - [b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)\|_{L^p(w)} = 0. \tag{3.8}$$

On the other hand, if $b \in \mathrm{CMO}_\theta(\rho)(\mathbb{R}^n)$, then for any $\varepsilon > 0$, there exists $b_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ such that $\|b - b_\varepsilon\|_{\mathrm{BMO}_\theta(\rho)} < \varepsilon$. Then

$$\begin{aligned}\|[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2}](f)-[b_\varepsilon,\ \Phi^{L,*}_{\gamma_1,\gamma_2}](f)\|_{L^p(w)} &\le \|[b-b_\varepsilon,\ \Phi^{L,*}_{\gamma_1,\gamma_2}](f)\|_{L^p(w)}\\ &\le C\|b-b_\varepsilon\|_{\mathrm{BMO}_\theta(\rho)}\|f\|_{L^p(w)}\le C\varepsilon.\end{aligned}$$

Thus, to prove $[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2}]$ is compact on $L^p(w)$ for any $b\in \mathrm{CMO}_\theta(\rho)$, it suffices to verify that $[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2}]$ is compact on $L^p(w)$ for any $b\in C_c^\infty(\mathbb{R}^n)$. By (3.8) and Lemma 3.1, it suffices to show that $[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}]$ is compact for any $b\in C_c^\infty(\mathbb{R}^n)$ when $\gamma>0$ is sufficiently small. To this end, for arbitrary bounded set F in $L^p(w)$, let

$$\mathcal{F}=\left\{[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)\ :\ f\in F\right\}.$$

Then we need to verify that $\mathcal{F}$ satisfies the conditions (i)–(iii) of Lemma 3.2 for $b\in C_c^\infty(\mathbb{R}^n)$.

From the definition $K^{L,\gamma_1}_{\gamma_2,t,\gamma}$, we can see that $|K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x,y)|\le |K^{L,\gamma_1}_{\gamma_2,t}(x,y)|$. Then

$$\begin{aligned}\Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}(f)(x) &\le \sup_{t>0}\left\{\int_{\mathbb{R}^n}|K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x,y)||f(y)|\mathrm{d}y\right\}\\ &\le \sup_{t>0}\left\{\int_{\mathbb{R}^n}|K^{L,\gamma_1}_{\gamma_2,t}(x,y)||f(y)|\mathrm{d}y\right\}\\ &\le CM_{V,\eta}(f)(x),\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)(x) &\le \sup_{t>0}\left\{\int_{\mathbb{R}^n}|K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x,y)||(b(x)-b(y))f(y)|\mathrm{d}y\right\}\\ &\le \sup_{t>0}\left\{\int_{\mathbb{R}^n}|K^{L,\gamma_1}_{\gamma_2,t}(x,y)||(b(x)-b(y))f(y)|\mathrm{d}y\right\}\\ &\le C[b,\ M_{V,\eta}](f)(x).\end{aligned}$$

According to Theorem 2.13 and Proposition 2.14, $\Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}$ and $[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}]$ are bounded from $L^p(w)$ to $L^p(w)$. We have

$$\sup_{f\in\mathcal{F}}\|[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)\|_{L^p(w)}\le C\sup_{f\in\mathcal{F}}\|f\|_{L^p(w)}\le C_{\mathcal{F}},$$

which yields the fact that the set $\mathcal{F}$ is bounded.

To verify the condition (ii) of Lemma 3.2, we also need the following pointwise estimate for the kernel function, which can be found in [53, Theorem 3].

$$|K_{\gamma_2,t}^{L,\gamma_1}(x,y)| \le \frac{C_N}{|x-y|^n}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-N}. \tag{3.10}$$

Suppose that $b \in C_c^\infty(\mathbb{R}^n)$ and supp $b \subset B(0,R)$, where $B(0,R)$ is a ball of radius R and centered at origin in $\mathbb{R}^n$. For $\nu > 2$, set $B^c = \{x \in \mathbb{R}^d : |x| > \nu R\}$. Then we have

$$\begin{aligned}&\int_{|x|>\nu R} |[b,\ \Phi_{\gamma_1,\gamma_2,\gamma}^{L,*}](f)(x)|^p w(x)\mathrm{d}x\\ &\le C\int_{|x|>\nu R}\left(\sup_{t>0}\int_{|y|<R}|K_{\gamma_2,t}^{L,\gamma_1}(x,y)||b(y)||f(y)|\mathrm{d}y\right)^p w(x)\mathrm{d}x.\end{aligned}$$

It can be deduced from Lemma 2.6 and the scaling technique directly that for any $x, y \in \mathbb{R}^n$ and $c \in (0,1]$,

$$\frac{1}{C(1+|x-y|/\rho(y))^{k_0+1}} \le C\frac{1}{1+c|x-y|/\rho(x)} \le C\frac{C}{c(1+|x-y|\rho(y))^{1/(1+k_0)}}, \tag{3.11}$$

where the constants k_0 and C are as same as in Lemma 2.6.

Since $|x| > \nu R$ implies $|x-y| > (1-1/\nu)|x|$ with $\nu > 2$, applying (3.10) and Hölder's inequality, we have

$$\begin{aligned}&\sup_{t>0}\int_{|y|<R}|K_{\gamma_2,t}^{L,\gamma_1}(x,y)||b(y)||f(y)|\mathrm{d}y\\ &\le \int_{|y|<R}\frac{C_N}{(1+|x-y|/(\rho(x)+\rho(y)))^N}\frac{1}{|x-y|^n}|b(y)||f(y)|\mathrm{d}y\\ &\le \int_{|y|<R}\frac{C_N}{(1+(1-1/\nu)|x|\rho(x))^N}\frac{1}{((1-1/\nu)|x|)^n}|b(y)||f(y)|\mathrm{d}y\\ &\le \frac{C_N\|b\|_\infty}{(1-1/\nu)^n|x|^n}\frac{1}{(1+(1-1/\nu)|x|/\rho(x))^N}\left(\int_{|y|<R}|f(y)|\mathrm{d}y\right)\\ &\le \frac{C_N\|b\|_\infty}{(1-1/\nu)^n|x|^n}\frac{\|f\|_{L^p(w)}}{(1+(1-1/\nu)|x|/\rho(x))^N}\left(\int_{|y|<R}w^{-1/(p-1)}(y)\mathrm{d}y\right)^{1-1/p}.\end{aligned}$$

Thus, by using (3.11), it follows that

$$
\begin{aligned}
&\left(\int_{|x|>\nu R} |[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)(x)|^p w(x)\mathrm{d}x\right)^{1/p} \\
&\quad \le C\|b\|_\infty \|f\|_{L^p(w)} \sum_{j=0}^{\infty} \frac{(1-1/\nu)^{-n}(2^j\nu R)^{-n}}{(1+(1-1/\nu)(2^j\nu R)/\rho(0))^{N/(k_0+1)}} \\
&\qquad \times \left(\int_{2^j\nu R<|x|<2^{j+1}\nu R} w(x)\mathrm{d}x\right)^{1/p} \left(\int_{|y|<R} w^{-1/(p-1)}(y)\mathrm{d}y\right)^{1-1/p} \\
&\quad \le C\|b\|_\infty \|f\|_{L^p(w)} \sum_{j=0}^{\infty} \frac{(1-1/\nu)^{-n}(2^j\nu R)^{-n}}{(1+(1-1/\nu)(2^j\nu R)/\rho(0))^{N/(k_0+1)}} \\
&\qquad \times \left(\int_{B(0,2^{j+1}\nu R)} w(x)\mathrm{d}x\right)^{1/p} \left(\int_{B(0,R)} w^{-1/(p-1)}(x)\mathrm{d}x\right)^{1-1/p}.
\end{aligned}
$$

Taking $Q = B(0, 2^{j+1}\nu R)$ and $E = B(0, R)$, we use (iv) in Lemma 2.10 to obtain

$$
\begin{aligned}
w(5Q) &\le C\left(\frac{\Psi(Q)|Q|}{|E|}\right)^p w(E) \\
&\le Cw(B(0,R))\left(\frac{(1+2^{j+1}\nu R/\rho(0))^\theta (2^{j+1}\nu R)^n}{R^n}\right)^p \\
&\le Cw(B(0,R))(1+2^{j+1}\nu R/\rho(0))^{\theta p}(2^{j+1}\nu)^{np}.
\end{aligned}
$$

From the above estimation, we have

$$
\begin{aligned}
&\left(\int_{|x|>\nu R} |[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)(x)|^p w(x)\mathrm{d}x\right)^{1/p} \\
&\quad \le C\|b\|_{\infty)} \|f\|_{L^p(w)} \sum_{j=0}^{\infty} \frac{(2^{j+1}\nu)^n}{(1-1/\nu)^n(2^j\nu R)^n} \\
&\qquad \times \frac{(1+2^{j+1}\nu R/\rho(0))^\theta}{(1+(1-1/\nu)(2^j\nu R)/\rho(0))^{N/(k_0+1)}} \\
&\qquad \times \left(\int_{B(0,R)} w(x)\mathrm{d}x\right)^{1/p} \left(\int_{B(0,R)} w^{-1/(p-1)}(x)\mathrm{d}x\right)^{1-1/p} \\
&\quad \le C\|b\|_\infty \|f\|_{L^p(w)} \|w\|^{1/p}_{A^\rho_p} \frac{1}{(1-1/\nu)^{n+N/(k_0+1)}} \\
&\qquad \times \sum_{j=0}^{\infty} \frac{(1+2^j\nu R/\rho(0))^{2\theta}}{(1+(2^j\nu R)/\rho(0))^{N/(k_0+1)}}.
\end{aligned}
$$

We can see that the series

$$\sum_{j=0}^{\infty} \frac{(1+2^j \nu R/\rho(0))^{2\theta}}{(1+(2^j \nu R)/\rho(0))^{l/(k_0+1)}}$$

is convergent. If $R > \rho(0)$, since $R > \rho(0)$ implies $1/(1+(2^j\nu R)/\rho) < 1/(1+2^j\nu) \leq 1/2^j\nu$, it holds for $N > 2\theta(k_0+1)$,

$$\sum_{j=0}^{\infty} \frac{(1+2^j \nu R/\rho(0))^{2\theta}}{(1+(2^j \nu R)/\rho(0))^{N/(k_0+1)}} \leq \sum_{j=0}^{\infty} \frac{1}{(2^j\nu)^{N/(k_0+1)-2\theta}} \leq \frac{C}{\nu^{N/(k_0+1)-2\theta}}.$$

On the other hand, for the case $R \leq \rho(0)$, since R and ρ are finite, there exists a finite integer $N_0 \geq [\log_2(\rho(0)/R)] + 1$ such that $2^{N_0}R > \rho(0)$. Therefore, similar to the case $R > \rho(0)$, we can get

$$\sum_{j=0}^{\infty} \frac{(1+2^j \nu R/\rho(0))^{2\theta}}{(1+(2^j \nu R)/\rho(0))^{N/(k_0+1)}} \leq \sum_{j=0}^{\infty} \frac{1}{(2^j\nu)^{N/(k_0+1)-2\theta}} \leq \frac{C2^{N_0(N/(k_0+1)-2\theta)}}{\nu^{N/(k_0+1)-2\theta}}.$$

By the above arguments, we obtain

$$\begin{aligned}
&\left(\int_{|x|>\nu R} |[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)(x)|^p w(x)\mathrm{d}x\right)^{1/p} \\
&\quad \leq C\frac{\|b\|_\infty \|f\|_{L^p(w)} \|w\|^{1/p}_{A_p^\rho}}{(1-1/\nu)^{d+N/(k_0+1)}\nu^{N/(k_0+1)-2\theta}} \max\{2^{N_0(N/(k_0+1)-2\theta)}, 1\},
\end{aligned}$$

which implies that for any $p > 1$ and $N > 2\theta(k_0+1)$,

$$\lim_{\nu\to\infty} \int_{|x|>\nu R} |[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)(x)|^p w(x)\mathrm{d}x = 0$$

holds whenever $f \in \mathcal{F}$.

It remains to show that the set $\mathcal{F}$ is uniformly equicontinuous. It suffices to verify that for any $\epsilon > 0$, if $|h|$ is sufficiently small and only depends on ϵ, then

$$\lim_{|h|\to 0} \|[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)(h+\cdot) - [b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)(\cdot)\|_{L^p(w)} = C\epsilon \tag{3.12}$$

holds whenever $f \in \mathcal{F}$.

In what follows, fix $\gamma \in (0,1)$ and $|h| < \gamma/4$. Then

$$|[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)(x+h) - [b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)(x)| \leq I(x) + II(x), \tag{3.13}$$

where

$$\begin{cases} I(x) := \sup_{t>0}\left\{\int_{\mathbb{R}^n} |K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x+h,y) - K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x,y)||b(x+h)-b(y)||f(y)|\mathrm{d}y\right\}; \\ II(x) := \sup_{t>0}\left\{\int_{\mathbb{R}^n} |K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x,y)||b(x+h)-b(x)||f(y)|\mathrm{d}y\right\}. \end{cases}$$

For $II(x)$, it holds

$$II(x) = |b(x+h)-b(x)| \sup_{t>0}\left\{\int_{\mathbb{R}^n} |K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x,y)||f(y)|\mathrm{d}y\right\} \le C|h|M_{V,\eta}(f)(x).$$

Then Corollary 2.12, together with $p' \le \eta < \infty$, gives

$$\|II\|_{L^p(w)} \le C|h|\|M_{V,\eta}(f)\|_{L^p(w)} \le C|h|\|f\|_{L^p(w)}. \tag{3.14}$$

We write $I(x) \le I_1(x) + I_2(x)$, where

$$I_1(x) := \sup_{t^{1/\gamma_2}\ge|h|}\left\{\int_{\mathbb{R}^n} |K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x+h,y) - K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x,y)| \times |b(x+h)-b(y)||f(y)|\mathrm{d}y\right\}$$

and

$$I_2(x) := \sup_{t^{1/\gamma_2}<|h|}\left\{\int_{\mathbb{R}^n} |K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x+h,y) - K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x,y)| \times |b(x+h)-b(y)||f(y)|\mathrm{d}y\right\}.$$

For $I_1(x)$, if $|h| \le t^{1/\gamma_2}$, then by (2.3) and (2.4), we have

$$\begin{aligned} &\left|K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x+h,y) - K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x,y)\right| \\ &\le \left|K^{L,\gamma_1}_{\gamma_2,t}(x+h,y) - K^{L,\gamma_1}_{\gamma_2,t}(x,y)\right| \\ &\quad + \left|K^{L,\gamma_1}_{\gamma_2,t}(x+h,y) - K^{L,\gamma_1}_{\gamma_2,t}(x,y)\right|\varphi(\gamma^{-1}|x+h-y|) \\ &\quad + \left|K^{L,\gamma_1}_{\gamma_2,t}(x,y)\right|\left|\varphi(\gamma^{-1}|x+h-y|) - \varphi(\gamma^{-1}|x-y|)\right| \\ &\le C\left(\frac{|h|}{t^{1/\gamma_2}}\right)^{\delta} \frac{t^{\gamma_1}}{(t^{1/\gamma_2}+|x-y|)^{n+\gamma_1\gamma_2}}\left(1+\frac{t^{1/\gamma_2}}{\rho(x)}+\frac{t^{1/\gamma_2}}{\rho(y)}\right)^{-N} \\ &\quad + C\left(\frac{|h|}{\gamma}\right) \frac{t^{\gamma_1}}{(t^{1/\gamma_2}+|x-y|)^{n+\gamma_1\gamma_2}}\left(1+\frac{t^{1/\gamma_2}}{\rho(x)}+\frac{t^{1/\gamma_2}}{\rho(y)}\right)^{-N}. \end{aligned} \tag{3.15}$$

Therefore, we have

$$
\begin{aligned}
|I_1(x)| &\le C \sup_{t^{1/\gamma_2}\ge|h|} \left\{ \int_{\mathbb{R}^n} \left(\left(\frac{|h|}{t^{1/\gamma_2}} \right)^{\delta} + \frac{|h|}{\gamma} \right) \frac{t^{\gamma_1} |b(x+h)-b(y)||f(y)|}{(t^{1/\gamma_2}+|x-y|)^{n+\gamma_1\gamma_2}} \right. \\
&\quad \left. \times \left(1 + \frac{t^{1/\gamma_2}}{\rho(x)} + \frac{t^{1/\gamma_2}}{\rho(y)} \right)^{-N} \mathrm{d}y \right\} \\
&:= C\{I_{1,1}(x)+I_{1,2}(x)+I_{1,3}(x)+I_{1,4}(x)\},
\end{aligned}
\tag{3.16}
$$

where

$$
\begin{aligned}
I_{1,1}(x) &:= \sup_{t^{1/\gamma_2}\ge 1} \left\{ \int_{|x-y|<t^{1/\gamma_2}} \left(\left(\frac{|h|}{t^{1/\gamma_2}} \right)^{\delta} + \frac{|h|}{\gamma} \right) \frac{t^{\gamma_1} |b(x+h)-b(y)||f(y)|}{(t^{1/\gamma_2}+|x-y|)^{n+\gamma_1\gamma_2}} \right. \\
&\quad \left. \times \left(1 + \frac{t^{1/\gamma_2}}{\rho(x)} + \frac{t^{1/\gamma_2}}{\rho(y)} \right)^{-N} \mathrm{d}y \right\}; \\
I_{1,2}(x) &:= \sup_{t^{1/\gamma_2}\ge 1} \left\{ \int_{|x-y|\ge t^{1/\gamma_2}} \left(\left(\frac{|h|}{t^{1/\gamma_2}} \right)^{\delta} + \frac{|h|}{\gamma} \right) \frac{t^{\gamma_1} |b(x+h)-b(y)||f(y)|}{(t^{1/\gamma_2}+|x-y|)^{n+\gamma_1\gamma_2}} \right. \\
&\quad \left. \times \left(1 + \frac{t^{1/\gamma_2}}{\rho(x)} + \frac{t^{1/\gamma_2}}{\rho(y)} \right)^{-N} \mathrm{d}y \right\}; \\
I_{1,3}(x) &:= \sup_{|h|\le t^{1/\gamma_2}<1} \left\{ \int_{|x-y|<t^{1/\gamma_2}} \left(\left(\frac{|h|}{t^{1/\gamma_2}} \right)^{\delta} + \frac{|h|}{\gamma} \right) \frac{t^{\gamma_1} |b(x+h)-b(y)||f(y)|}{(t^{1/\gamma_2}+|x-y|)^{n+\gamma_1\gamma_2}} \right. \\
&\quad \left. \times \left(1 + \frac{t^{1/\gamma_2}}{\rho(x)} + \frac{t^{1/\gamma_2}}{\rho(y)} \right)^{-N} \mathrm{d}y \right\}; \\
I_{1,4}(x) &:= \sup_{|h|\le t^{1/\gamma_2}<1} \left\{ \int_{|x-y|\ge t^{1/\gamma_2}} \left(\left(\frac{|h|}{t^{1/\gamma_2}} \right)^{\delta} + \frac{|h|}{\gamma} \right) \frac{t^{\gamma_1} |b(x+h)-b(y)||f(y)|}{(t^{1/\gamma_2}+|x-y|)^{n+\gamma_1\gamma_2}} \right. \\
&\quad \left. \times \left(1 + \frac{t^{1/\gamma_2}}{\rho(x)} + \frac{t^{1/\gamma_2}}{\rho(y)} \right)^{-N} \mathrm{d}y \right\}.
\end{aligned}
$$

If $t^{1/\gamma_2} \ge 1$, then $(t^{1/\gamma_2})^{-\delta} \le 1$. For $I_{1,1}(x)$, choosing $N = \theta\eta$, we have

$$
\begin{aligned}
I_{1,1}(x) &\le \sup_{t^{1/\gamma_2}\ge 1}\left\{\frac{1}{(t^{1/\gamma_2})^n}\int_{|x-y|<t^{1/\gamma_2}}\left(\left(\frac{|h|}{t^{1/\gamma_2}}\right)^{\delta}+\frac{|h|}{\gamma}\right)\right.\\
&\quad\left.\times\left(1+\frac{t^{1/\gamma_2}}{\rho(x)}\right)^{-N}|f(y)|\mathrm{d}y\right\}\\
&\le C\gamma^{-1}\big(|h|^{\delta}+|h|\big)\sup_{t^{1/\gamma_2}\ge 1}\left\{\frac{1}{(t^{1/\gamma_2})^n}\left(1+\frac{t^{1/\gamma_2}}{\rho(x)}\right)^{-\theta\eta}\right.\\
&\quad\left.\times\int_{|x-y|<t^{1/\gamma_2}}|f(y)|\mathrm{d}y\right\}\\
&\le C\gamma^{-1}\big(|h|^{\delta}+|h|\big)M_{V,\eta}(f)(x).
\end{aligned}
\tag{3.17}
$$

For $I_{1,2}(x)$, we get

$$
\begin{aligned}
I_{1,2}(x) &\le C\gamma^{-1}\big(|h|^{\delta}+|h|\big)\sup_{t^{1/\gamma_2}\ge 1}\left\{\sum_{k=1}^{\infty}\int_{|x-y|\sim 2^k t^{1/\gamma_2}}\frac{t^{\gamma_1}|f(y)|}{|x-y|^{n+\gamma_1\gamma_2}}\right.\\
&\quad\left.\times\left(1+\frac{t^{1/\gamma_2}}{\rho(x)}\right)^{-N}\mathrm{d}y\right\}\\
&\le C\gamma^{-1}\big(|h|^{\delta}+|h|\big)\sup_{t^{1/\gamma_2}\ge 1}\left\{\sum_{k=1}^{\infty}\int_{|x-y|\sim 2^k t^{1/\gamma_2}}\frac{1}{(2^k t^{1/\gamma_2})^n}\frac{2^{k\theta\eta}t^{\gamma_1}}{(2^k t^{1/\gamma_2})^{\gamma_1\gamma_2}}\right.\\
&\quad\left.\times\left(1+\frac{2^k t^{1/\gamma_2}}{\rho(x)}\right)^{-\theta\eta}|f(y)|\mathrm{d}y\right\}\\
&\le C\gamma^{-1}\big(|h|^{\delta}+|h|\big)\sum_{k=1}^{\infty}\frac{1}{2^{k(\gamma_1\gamma_2-\theta\eta)}}\sup_{t^{1/\gamma_2}\ge 1}\left\{\frac{1}{|B(x,2^k t^{1/\gamma_2})|}\right.\\
&\quad\left.\times\left(1+\frac{2^k t^{1/\gamma_2}}{\rho(x)}\right)^{-\theta\eta}\int_{|x-y|\sim 2^k t^{1/\gamma_2}}|f(y)|\mathrm{d}y\right\}\\
&\le C\gamma^{-1}\big(|h|^{\delta}+|h|\big)M_{V,\eta}(f)(x).
\end{aligned}
\tag{3.18}
$$

If $t^{1/\gamma_2}<1$, then $(t^{1/\gamma_2})^{-\delta}<t^{-1/\gamma_2}$. For $I_{1,3}(x)$, choosing $N=\theta\eta$, if $|h|\le t^{1/\gamma_2}$, $|x-y|<t^{1/\gamma_2}$ and $b\in C_c^{\infty}(\mathbb{R}^n)$, then

$$
|b(x+h)-b(y)|\le C|x+h-y|\le C(|x-y|+|h|)\le Ct^{1/\gamma_2}.
$$

Then, it follows that

$$
\begin{aligned}
I_{1,3}(x) &\le C|h|^{\delta} \sup_{|h|\le t^{1/\gamma_2}<1} \left\{ \int_{|x-y|<t^{1/\gamma_2}} \frac{t^{\gamma_1}}{t^{1/\gamma_2}(t^{1/\gamma_2}+|x-y|)^{n+\gamma_1\gamma_2}} \right. \\
&\quad \left. \times \left(1+\frac{t^{1/\gamma_2}}{\rho(x)}+\frac{t^{1/\gamma_2}}{\rho(y)}\right)^{-N} |b(x+h)-b(y)||f(y)|\mathrm{d}y \right\} \\
&\quad + C\frac{|h|}{\gamma} \sup_{|h|\le t^{1/\gamma_2}<1} \left\{ \int_{|x-y|<t^{1/\gamma_2}} \frac{t^{\gamma_1}|f(y)|}{(t^{1/\gamma_2}+|x-y|)^{n+\gamma_1\gamma_2}} \right. \\
&\quad \left. \times \left(1+\frac{t^{1/\gamma_2}}{\rho(x)}+\frac{t^{1/\gamma_2}}{\rho(y)}\right)^{-N} \mathrm{d}y \right\} \\
&\le C|h|^{\delta} \sup_{|h|\le t^{1/\gamma_2}<1} \left\{ \left(\frac{1}{t^{1/\gamma_2}}\right)^{n} \left(1+\frac{t^{1/\gamma_2}}{\rho(x)}\right)^{-\theta\eta} \int_{|x-y|<t^{1/\gamma_2}} |f(y)|\mathrm{d}y \right\} \\
&\quad + C\frac{|h|}{\gamma} \sup_{|h|\le t^{1/\gamma_2}<1} \left\{ \left(\frac{1}{t^{1/\gamma_2}}\right)^{n} \left(1+\frac{t^{1/\gamma_2}}{\rho(x)}\right)^{-\theta\eta} \int_{|x-y|<t^{1/\gamma_2}} |f(y)|\mathrm{d}y \right\} \\
&\le C\gamma^{-1}(|h|^{\delta}+|h|)M_{V,\eta}(f)(x).
\end{aligned} \tag{3.19}
$$

For $I_{1,4}(x)$, since $b \in C_c^{\infty}(\mathbb{R}^n)$, if $|x-y| \sim 2^k t^{1/\gamma_2}$, $k=1,2,\dots$, and $|h| \le t^{1/\gamma_2}$, then

$$
|b(x+h)-b(y)| \le C|x+h-y| \le C(|x-y|+|h|) \le C2^k t^{1/\gamma_2}.
$$

It follows that

$$
\begin{aligned}
I_{1,4}(x) &\le C|h|^{\delta} \sup_{|h|\le t^{1/\gamma_2}<1} \left\{ \sum_{k=1}^{\infty} \int_{|x-y|\sim 2^k t^{1/\gamma_2}} \frac{2^k t^{\gamma_1}|f(y)|}{(|x-y|)^{n+\gamma_1\gamma_2}} \left(1+\frac{t^{1/\gamma_2}}{\rho(x)}\right)^{-N} \mathrm{d}y \right\} \\
&\quad + C\frac{|h|}{\gamma} \sup_{|h|\le t^{1/\gamma_2}<1} \left\{ \sum_{k=1}^{\infty} \int_{|x-y|\sim 2^k t^{1/\gamma_2}} \frac{t^{\gamma_1}|f(y)|}{|x-y|^{n+\gamma_1\gamma_2}} \left(1+\frac{t^{1/\gamma_2}}{\rho(x)}\right)^{-N} \mathrm{d}y \right\} \\
&\le C\frac{(|h|^{\delta}+|h|)}{\gamma} \sup_{|h|\le t^{1/\gamma_2}<1} \left\{ \sum_{k=1}^{\infty} \int_{|x-y|\sim 2^k t^{1/\gamma_2}} \frac{2^{k\theta\eta} t^{\gamma_1}|f(y)|}{(2^k t^{1/\gamma_2})^{n+\gamma_1\gamma_2}} \right. \\
&\quad \left. \times \left(1+\frac{2^k t^{1/\gamma_2}}{\rho(x)}\right)^{-\theta\eta} \mathrm{d}y \right\} \\
&\le C\gamma^{-1}\big(|h|^{\delta}+|h|\big)M_{V,\eta}(f)(x).
\end{aligned} \tag{3.20}
$$

The estimates for $I_{1,i}(x), i=1,2,3,4$, indicate that

$$
I_1(x) \le C\gamma^{-1}\big(|h|^{\delta}+|h|\big)M_{V,\eta}(f)(x). \tag{3.21}
$$

Next we estimate $I_2(x)$. If $|x-y|<\gamma/2$ and $|h|<\gamma/4$, then $|x+h-y|<3\gamma/4$. Hence

$$
\varphi(\gamma^{-1}|x+h-y|) = 1 = \varphi(\gamma^{-1}|x-y|).
$$

This implies $K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x+h,y)=0=K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x,y)$. Write $I_2(x)\le I_{2,1}(x)+I_{2,2}(x)$, where

$$I_{2,1}(x):=\sup_{t^{1/\gamma_2}<|h|<\rho(x)}\left\{\int_{|x-y|\ge\frac{\gamma}{2}}|K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x+h,y)-K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x,y)|\times|b(x+h)-b(y)||f(y)|\mathrm{d}y\right\}$$

and

$$I_{2,2}(x):=\sup_{\substack{t^{1/\gamma_2}<|h|\\ \rho(x)\le|h|}}\left\{\int_{|x-y|\ge\frac{\gamma}{2}}|K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x+h,y)-K^{L,\gamma_1}_{\gamma_2,t,\gamma}(x,y)|\times|b(x+h)-b(y)||f(y)|\mathrm{d}y\right\}.$$

For $I_{2,1}(x)$, since $|x-y|\ge\gamma/2>2|h|$, we have $|x+h-y|\sim|x-y|$. Since $t^{1/\gamma_2}<|h|<\rho(x)$, it implies that $\rho(x+h)\sim\rho(x)$, $|h|/\rho(x)<1$ and $(|h|/t^{1/\gamma_2})>1$. For $\gamma_1\gamma_2>1$, $|h|/t^{1/\gamma_2}<(|h|/t^{1/\gamma_2})^{\gamma_1\gamma_2}$. Then

$$\begin{aligned}I_{2,1}(x)&\le C\sup_{t^{1/\gamma_2}<|h|<\rho(x)}\left\{\int_{|x-y|\ge\frac{\gamma}{2}}\left(|K^{L,\gamma_1}_{\gamma_2,t}(x+h,y)|+|K^{L,\gamma_1}_{\gamma_2,t}(x,y)|\right)\times|b(x+h)-b(y)||f(y)|\mathrm{d}y\right\}\\&\le C\sup_{t^{1/\gamma_2}<|h|<\rho(x)}\left\{\int_{|x-y|\ge2|h|}\frac{t^{\gamma_1}}{(t^{1/\gamma_2}+|x-y|)^{n+\gamma_1\gamma_2}}\left(\frac{|h|}{t^{1/\gamma_2}}\right)\times\left(1+\frac{t^{1/\gamma_2}}{\rho(x)}+\frac{t^{1/\gamma_2}}{\rho(y)}\right)^{-N}|x-y||f(y)|\mathrm{d}y\right\}\\&\le C|h|\sum_{k=2}^{\infty}\frac{1}{2^{k(\gamma_1\gamma_2-1-\theta\eta)}}\sup_{t^{1/\gamma_2}<|h|<\rho(x)}\left\{\frac{1}{(2^k|h|)^n}\left(1+\frac{2^k|h|}{\rho(x)}\right)^{-\theta\eta}\times\int_{|x-y|\sim2^k|h|}|f(y)|\mathrm{d}y\right\}.\end{aligned}\tag{3.22}$$

By $\gamma_1\gamma_2-1-\theta\eta>0$, we can get $I_{2,1}(x)\le C|h|M_{V,\eta}(f)(x)$.

Finally, it remains to consider $I_{2,2}(x)$. Since $b\in C^\infty_c(\mathbb{R}^n)$, $|x-y|\ge2|h|$, $|h|/t^{1/\gamma_2}>1$ and $|b(x+h)-b(y)|\le C|x-y|$. In addition, if $|x-y|<2^l\rho(x)$, $l=1,2,\dots$, by Lemma 2.6, we have $\rho(y)\le C2^{k_0l/(k_0+1)}\rho(x)$. It follows that

$$\left(1+\frac{t^{1/\gamma_2}}{\rho(x)}+\frac{t^{1/\gamma_2}}{\rho(y)}\right)^{-N}+\left(1+\frac{t^{1/\gamma_2}}{\rho(x+h)}+\frac{t^{1/\gamma_2}}{\rho(y)}\right)^{-N}$$
$$\leq 2\left(1+\frac{t^{1/\gamma_2}}{\rho(y)}\right)^{-N}\leq C_N\left(1+\frac{2^{-k_0 l/(k_0+1)}t^{1/\gamma_2}}{\rho(x)}\right)^{-N}.$$

Choosing $N=\gamma_1\gamma_2-1$ and applying (2.3) again, we obtain

$$\begin{aligned}
I_{2,2}(x) &\leq C \sup_{\substack{t^{1/\gamma_2}<|h|\\ \rho(x)\leq|h|}}\left\{\int_{|x-y|\geq\frac{\gamma}{2}}\frac{t^{\gamma_1}}{(t^{1/\gamma_2}+|x-y|)^{n+\gamma_1\gamma_2}}\left(\frac{|h|}{t^{1/\gamma_2}}\right)\right.\\
&\quad\left.\times\left(1+\frac{t^{1/\gamma_2}}{\rho(y)}\right)^{-N}|b(x+h)-b(y)||f(y)|\mathrm{d}y\right\}\\
&\leq C \sup_{\substack{t^{1/\gamma_2}<|h|\\ \rho(x)\leq|h|}}\left\{\int_{|x-y|\geq 2\rho(x)}\frac{t^{\gamma_1}}{|x-y|^{n+\gamma_1\gamma_2}}\left(\frac{|h|}{t^{1/\gamma_2}}\right)\right.\\
&\quad\left.\times\left(1+\frac{2^{-k_0 l/(k_0+1)}t^{1/\gamma_2}}{\rho(x)}\right)^{-N}|x-y||f(y)|\mathrm{d}y\right\}\\
&\leq C|h| \sup_{\substack{t^{1/\gamma_2}<|h|\\ \rho(x)\leq|h|}}\left\{\sum_{l=2}^{\infty}\int_{|x-y|\geq 2^l\rho(x)}\frac{2^{(\gamma_1\gamma_2-1)k_0 l/(k_0+1)}}{(2^l\rho(x))^n 2^{l(\gamma_1\gamma_2-1)}}\right.\\
&\quad\left.\times\left(1+\frac{2^l\rho(x)}{\rho(x)}\right)^{\theta\eta}\left(1+\frac{2^l\rho(x)}{\rho(x)}\right)^{-\theta\eta}|f(y)|\mathrm{d}y\right\}\\
&\leq C|h|M_{V,\eta}(f)(x).
\end{aligned}\tag{3.23}$$

The estimates (3.23) and (3.22) indicate that

$$I_2(x)\leq C|h|M_{V,\eta}(f)(x).\tag{3.24}$$

Therefore, by (3.21) and (3.24), we have

$$I(x)\leq C(|h|+|h|^{\delta})M_{V,\eta}(f)(x).$$

By Corollary 2.12, for any $p'\leq\eta<\infty$, it holds

$$\|I\|_{L^p(w)}\leq C(|h|+|h|^{\delta})\|M_{V,\eta}(f)\|_{L^p(w)}\leq C(|h|+|h|^{\delta})\|f\|_{L^p(w)}.\tag{3.25}$$

Combining (3.13) with (3.14) and (3.25) gives that

$$\|[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)(h+\cdot)-[b,\ \Phi^{L,*}_{\gamma_1,\gamma_2,\gamma}](f)(\cdot)\|_{L^p(w)}\le C(|h|+|h|^{\delta})\|f\|_{L^p(w)}.$$

Hence, we obtain the desired result (3.12). This finishes the proof of Theorem 3.3. □

Let $T_i^{L,*}$, $i=1,2,\dots,6$, be the maximal functions defined by (1.2). As one of the main results, we have

Theorem 3.4 *Suppose that* $1<p<\infty$, $V\in B_q$, $q>n/2$, $w\in A_p^{\rho,\theta}$ *and* $b\in \mathrm{CMO}_\theta(\rho)(\mathbb{R}^n)$ *with* $\theta>0$.

(i) *For* $0<\alpha<1$, $[b,\ T_1^{L,*}]$ *is a compact operator from* $L^p(w)$ *to* $L^p(w)$.
(ii) *For* $m>0$ *and* $0<\alpha<1$, $[b,\ T_3^{L,*}]$ *is a compact operator from* $L^p(w)$ *to* $L^p(w)$.
(iii) *For* $0<\alpha<1$ *and* $0<\sigma<1$, $[b,\ T_4^{L,*}]$ *is a compact operator from* $L^p(w)$ *to* $L^p(w)$.
(iv) *For* $\upsilon>0$, $0<\alpha<1$ *and* $0<\sigma<1$, $[b,\ T_6^{L,*}]$ *is a compact operator from* $L^p(w)$ *to* $L^p(w)$.

Proof

(i) By Lemma 2.2, we can see that $K^L_{\alpha,t}(x,y)$ satisfies the conditions (2.3) and (2.4) for $\gamma_2=2\alpha$ and $\gamma_1=1$. Hence, by Theorem 3.3, $[b,\ T_1^{L,*}]$ is a compact operator from $L^p(w)$ to $L^p(w)$.
(ii) By Lemma 2.4, we can see that $t^m\partial_t^m K^L_{\alpha,t}(x,y)$ satisfies the conditions (2.3) and (2.4) for $\gamma_2=2\alpha$ and $\gamma_1=m$. Hence, by Theorem 3.3, $[b,\ T_3^{L,*}]$ is a compact operator from $L^p(w)$ to $L^p(w)$.
(iii) By Theorem 3.3, it suffices to show that $p^{L,\alpha}_{t,\sigma}(x,y)$ satisfying the conditions (2.3) and (2.4) for $\gamma_2=1$ and $\gamma_1=2\alpha$.

We first prove that $p^{L,\alpha}_{t,\sigma}(x,y)$ satisfies the condition (2.3). By (1.3) and (i) of Lemma 2.2 we can get

$$
\begin{aligned}
\left|p_{t,\sigma}^{L,\alpha}(x,y)\right| &\le C_N \int_0^\infty e^{-r} \frac{t^{2\alpha}/(4r)}{\left((t^{2\alpha}/(4r))^{1/(2\alpha)} + |x-y|\right)^{n+2\alpha}} \\
&\quad \times \left(1 + \frac{(t^{2\alpha}/(4r))^{1/(2\alpha)}}{\rho(x)}\right)^{-N} \left(1 + \frac{(t^{2\alpha}/(4r))^{1/(2\alpha)}}{\rho(y)}\right)^{-N} \frac{\mathrm{d}r}{r^{1-\sigma}} \\
&\le C_N \int_0^\infty e^{-r} \frac{(t^{2\alpha}/(4r))^{1-N/\alpha}}{(t^{2\alpha}/(4r))^{n/(2\alpha)+1}} (\rho(x))^N (\rho(y))^N \frac{\mathrm{d}r}{r^{1-\sigma}} \\
&\le C_N t^{-n} \left(\frac{t}{\rho(x)}\right)^{-N} \left(\frac{t}{\rho(y)}\right)^{-N}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\left|p_{t,\sigma}^{L,\alpha}(x,y)\right| &\le C_N \int_0^\infty e^{-r} \frac{t^{2\alpha}/(4r)}{|x-y|^{n+2\alpha}} (t^{2\alpha}/(4r))^{-N/\alpha} (\rho(x))^N (\rho(y))^N \frac{\mathrm{d}r}{r^{1-\sigma}} \\
&\le C_N \frac{(t^{2\alpha})^{1-N/\alpha}}{|x-y|^{n+2\alpha}} (\rho(x))^N (\rho(y))^N \int_0^\infty e^{-r} r^{-1+N/\alpha+\sigma-1} \mathrm{d}r \\
&\le C_N \frac{t^{2\alpha}}{|x-y|^{n+2\alpha}} \left(\frac{t}{\rho(x)}\right)^{-N} \left(\frac{t}{\rho(y)}\right)^{-N}.
\end{aligned}
$$

Case 1: $t \le |x-y|$. We can see that

$$
\left(\frac{t}{\rho(x)}\right)^N \left(\frac{t}{\rho(y)}\right)^N |p_{t,\sigma}^{L,\alpha}(x,y)| \le \frac{C_N t^{2\alpha}}{|x-y|^{n+2\alpha}} \le \frac{C_N t^{2\alpha}}{(t+|x-y|)^{n+2\alpha}}.
$$

Case 2: $t > |x-y|$. For this case, we have

$$
\left(\frac{t}{\rho(x)}\right)^N \left(\frac{t}{\rho(y)}\right)^N |p_{t,\sigma}^{L,\alpha}(x,y)| \le \frac{C_N}{t^n} \le \frac{C_N t^{2\alpha}}{(t+|x-y|)^{n+2\alpha}}.
$$

Hence,

$$
\left(\frac{t}{\rho(x)}\right)^N \left(\frac{t}{\rho(y)}\right)^N |p_{t,\sigma}^{L,\alpha}(x,y)| \le C_N \frac{t^{2\alpha}}{(t+|x-y|)^{n+2\alpha}}.
$$

By the arbitrariness of N, we get

$$
\left|p_{t,\sigma}^{L,\alpha}(x,y)\right| \le C_N \frac{t^{2\alpha}}{(t+|x-y|)^{n+2\alpha}} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}.
$$

Next we prove that $p_{t,\sigma}^{L,\alpha}(x,y)$ satisfies the condition (2.4). By (1.3) and (ii) of Lemma 2.2 we can get

$$\begin{aligned}
&\left|p_{t,\sigma}^{L,\alpha}(x+h,y)-p_{t,\sigma}^{L,\alpha}(x,y)\right| \\
&\quad\le C\int_0^\infty e^{-r}\left|K^L_{\alpha,t^{2\alpha}/(4r)}(x+h,y)-K^L_{\alpha,t^{2\alpha}/(4r)}(x,y)\right|\frac{\mathrm{d}r}{r^{1-\sigma}} \\
&\quad\le C_N\int_0^\infty e^{-r}\left(\frac{|h|}{(t^{2\alpha}/(4r))^{1/(2\alpha)}}\right)^{\delta}\frac{t^{2\alpha}/(4r)}{(t^{2\alpha}/(4r))^{n/(2\alpha)+1}} \\
&\qquad\times\left(1+\frac{(t^{2\alpha}/(4r))^{1/(2\alpha)}}{\rho(x)}\right)^{-N}\left(1+\frac{(t^{2\alpha}/(4r))^{1/(2\alpha)}}{\rho(y)}\right)^{-N}\frac{\mathrm{d}r}{r^{1-\sigma}} \\
&\quad\le C_N\left(\frac{|h|}{t}\right)^{\delta}t^{-n-2N}(\rho(x))^N(\rho(y))^N\int_0^\infty e^{-r}r^{\delta/(2\alpha)+n/(2\alpha)+N/\alpha+\sigma-1}\mathrm{d}r \\
&\quad\le C_N\left(\frac{|h|}{t}\right)^{\delta}t^{-n}\left(\frac{t}{\rho(x)}\right)^{-N}\left(\frac{t}{\rho(y)}\right)^{-N}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left|p_{t,\sigma}^{L,\alpha}(x+h,y)-p_{t,\sigma}^{L,\alpha}(x,y)\right| \\
&\quad\le C_N\int_0^\infty e^{-r}\left(\frac{|h|}{(t^{2\alpha}/(4r))^{1/(2\alpha)}}\right)^{\delta}\frac{t^{2\alpha}/(4r)}{|x-y|^{n+2\alpha}} \\
&\qquad\times\left(1+\frac{(t^{2\alpha}/(4r))^{1/(2\alpha)}}{\rho(x)}\right)^{-N}\left(1+\frac{(t^{2\alpha}/(4r))^{1/(2\alpha)}}{\rho(y)}\right)^{-N}\frac{\mathrm{d}r}{r^{1-\sigma}} \\
&\quad\le C_N\left(\frac{|h|}{t}\right)^{\delta}\frac{t^{2\alpha-2N}}{|x-y|^{n+2\alpha}}(\rho(x))^N(\rho(y))^N\int_0^\infty e^{-r}r^{\delta/(2\alpha)+N/\alpha+\sigma-1}\mathrm{d}r \\
&\quad\le C_N\left(\frac{|h|}{t}\right)^{\delta}\frac{t^{2\alpha}}{|x-y|^{n+2\alpha}}\left(\frac{t}{\rho(x)}\right)^{-N}\left(\frac{t}{\rho(y)}\right)^{-N}.
\end{aligned}$$

Similar to the proof that $p_{t,\sigma}^{L,\alpha}(x,y)$ satisfies the condition (2.3), by the arbitrariness of N, we get

$$\begin{aligned}
&\left|p_{t,\sigma}^{L,\alpha}(x+h,y)-p_{t,\sigma}^{L,\alpha}(x,y)\right| \\
&\quad\le C_N\left(\frac{|h|}{t}\right)^{\delta}\frac{t^{2\alpha}}{(t+|x-y|)^{n+2\alpha}}\left(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\right)^{-N}.
\end{aligned}$$

Hence, $[b,\ T_4^{L,*}]$ is compact operator from $L^p(w)$ to $L^p(w)$.

(iv) By Theorem 3.3, it suffices to show that $t^{\upsilon}\partial_t^{\upsilon}p_{t,\sigma}^{L,\alpha}(x,y)$ satisfying the conditions (2.3) and (2.4). We first prove that $t^{\upsilon}\partial_t^{\upsilon}p_{t,\sigma}^{L,\alpha}(x,y)$ satisfies the condition (2.3). Since

$$t^{\upsilon}\partial_t^{\upsilon}p_{t,\sigma}^{L,\alpha}(x,y)=C\int_0^\infty e^{-r}t^{\upsilon}\partial_t^{\upsilon}K^L_{\alpha,t^{2\alpha}/(4r)}(x,y)\frac{\mathrm{d}r}{r^{1-\sigma}},$$

by (i) of Lemma 2.4, we can get

$$\begin{aligned}\left|t^{\upsilon}\partial_t^{\upsilon}p_{t,\sigma}^{L,\alpha}(x,y)\right| &\le C_N\int_0^\infty \frac{e^{-r}(t^{2\alpha}/(4r))^{\upsilon}}{\left((t^{2\alpha}/(4r))^{1/(2\alpha)}+|x-y|\right)^{n+2\alpha\upsilon}}\\ &\quad\times\left(1+\frac{(t^{2\alpha}/(4r))^{1/(2\alpha)}}{\rho(x)}\right)^{-N}\left(1+\frac{(t^{2\alpha}/(4r))^{1/(2\alpha)}}{\rho(y)}\right)^{-N}\frac{\mathrm{d}r}{r^{1-\sigma}}\\ &\le C_N\frac{t^{2\alpha\upsilon-2N}}{t^{n+2\alpha\upsilon}}(\rho(x))^N(\rho(y))^N\int_0^\infty e^{-r}r^{n/(2\alpha)+N/\alpha+\sigma-1}\mathrm{d}r\\ &\le C_N t^{-n}\left(\frac{t}{\rho(x)}\right)^{-N}\left(\frac{t}{\rho(y)}\right)^{-N}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left|t^{\upsilon}\partial_t^{\upsilon}p_{t,\sigma}^{L,\alpha}(x,y)\right| &\le C_N\int_0^\infty e^{-r}\frac{(t^{2\alpha}/(4r))^{\upsilon}}{|x-y|^{n+2\alpha\upsilon}}\left(\frac{(t^{2\alpha}/(4r))^{1/(2\alpha)}}{\rho(x)}\right)^{-N}\\ &\quad\times\left(\frac{(t^{2\alpha}/(4r))^{1/(2\alpha)}}{\rho(y)}\right)^{-N}\frac{\mathrm{d}r}{r^{1-\sigma}}\\ &\le C_N\frac{t^{2\alpha\upsilon-2N}}{|x-y|^{n+2\alpha\upsilon}}(\rho(x))^N(\rho(y))^N\int_0^\infty e^{-r}r^{-\upsilon+N/\alpha+\sigma-1}\mathrm{d}r\\ &\le C_N\frac{t^{2\alpha\upsilon}}{|x-y|^{n+2\alpha\upsilon}}\left(\frac{t}{\rho(x)}\right)^{-N}\left(\frac{t}{\rho(y)}\right)^{-N}.\end{aligned}$$

Similar to the proofs of (2.3) and (2.4) for $p_{t,\sigma}^{L,\alpha}(\cdot,\cdot)$, we can deduce that

$$\left|t^{\upsilon}\partial_t^{\upsilon}p_{t,\sigma}^{L,\alpha}(x,y)\right|\le\frac{C_Nt^{2\alpha\upsilon}}{(t+|x-y|)^{n+2\alpha\upsilon}}\left(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\right)^{-N}$$

and

$$\begin{aligned}&\left|t^{\upsilon}\partial_t^{\upsilon}p_{t,\sigma}^{L,\alpha}(x+h,y)-t^{\upsilon}\partial_t^{\upsilon}p_{t,\sigma}^{L,\alpha}(x,y)\right|\\ &\quad\le\left(\frac{|h|}{t}\right)^{\delta}\frac{Ct^{2\alpha\upsilon}}{(t+|x-y|)^{n+2\alpha\upsilon}}\left(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\right)^{-N}.\end{aligned}$$

Hence, $[b,\ T_6^{L,*}]$ is compact from $L^p(w)$ to $L^p(w)$.

□

Theorem 3.5 *Suppose that* $1 < p < \infty$, $V \in B_q$, $q > n$, $w \in A_p^{\rho,\theta}$ *and* $b \in \mathrm{CMO}_\theta(\rho)(\mathbb{R}^n)$ *with* $\theta > 0$.

(i) *For* $0 < \alpha < 1$, $[b,\ T_2^{L,*}]$ *is compact from* $L^p(w)$ *to* $L^p(w)$.
(ii) *For* $0 < \alpha < 1$ *and* $0 < \sigma < 1$, $[b,\ T_5^{L,*}]$ *is compact from* $L^p(w)$ *to* $L^p(w)$.

Proof

(i) By Lemma 2.3, we can see that $t^{1/(2\alpha)}\nabla_x K_{\alpha,t}^L(x,y)$ satisfies the conditions (2.3) and (2.4) for $\gamma_2 = 2\alpha$ and $\gamma_1 = 1$. Hence, by Theorem 3.3, $[b,\ T_2^{L,*}]$ is compact from $L^p(w)$ to $L^p(w)$.
(ii) By Theorem 3.3, it suffices to show that $t\nabla_x p_{t,\sigma}^{L,\alpha}(x,y)$ satisfying the conditions (2.3) and (2.4) for $\gamma_2 = 1$ and $\gamma_1 = 2\alpha$.

We first prove that $p_{t,\sigma}^{L,\alpha}(x,y)$ satisfies the condition (2.3). Since

$$\nabla_x p_{t,\sigma}^{L,\alpha}(x,y) = C\int_0^\infty e^{-r}\nabla_x K_{\alpha,t^{2\alpha}/(4r)}^L(x,y)\frac{\mathrm{d}r}{r^{1-\sigma}},$$

by (i) of Lemma 2.3, we can get

$$\begin{aligned}\left|\nabla_x p_{t,\sigma}^{L,\alpha}(x,y)\right| &\le C_N\int_0^\infty e^{-r}\frac{(t^{2\alpha}/(4r))^{1-1/(2\alpha)}}{\left((t^{2\alpha}/(4r))^{1/(2\alpha)} + |x-y|\right)^{n+2\alpha}}\\ &\quad\times\left(1+\frac{(t^{2\alpha}/(4r))^{1/(2\alpha)}}{\rho(x)}\right)^{-N}\left(1+\frac{(t^{2\alpha}/(4r))^{1/(2\alpha)}}{\rho(y)}\right)^{-N}\frac{\mathrm{d}r}{r^{1-\sigma}}\\ &\le C_N\int_0^\infty e^{-r}\left(\frac{t^{2\alpha}}{4r}\right)^{-n/(2\alpha)-1/(2\alpha)-N/\alpha}(\rho(x))^N(\rho(y))^N\frac{\mathrm{d}r}{r^{1-\sigma}}\\ &\le C_N t^{-n-1-2N}(\rho(x))^N(\rho(y))^N\int_0^\infty e^{-r}r^{(n+1)/(2\alpha)+N/\alpha+\sigma-1}\mathrm{d}r\\ &\le C_N t^{-n-1}\left(\frac{t}{\rho(x)}\right)^{-N}\left(\frac{t}{\rho(y)}\right)^{-N}.\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\left|\nabla_x p_{t,\sigma}^{L,\alpha}(x,y)\right| \\
&\leq C_N \int_0^\infty e^{-r} \frac{(t^{2\alpha}/(4r))^{1-1/(2\alpha)}}{|x-y|^{n+2\alpha}} \left(\frac{(t^{2\alpha}/(4r))^{1/(2\alpha)}}{\rho(x)} \right)^{-N} \\
&\quad \times \left(\frac{(t^{2\alpha}/(4r))^{1/(2\alpha)}}{\rho(y)} \right)^{-N} \frac{\mathrm{d}r}{r^{1-\sigma}} \\
&\leq C_N \frac{t^{2\alpha-1-2N}}{|x-y|^{n+2\alpha}} (\rho(x))^{-N} (\rho(y))^N \int_0^\infty e^{-r} r^{-1+1/(2\alpha)+N/\alpha+\sigma-1} \mathrm{d}r \\
&\leq C_N \frac{t^{2\alpha-1}}{|x-y|^{n+2\alpha}} \left(\frac{t}{\rho(x)} \right)^{-N} \left(\frac{t}{\rho(y)} \right)^{-N}.
\end{aligned}
$$

Similar to the proof that $p_{t,\sigma}^{L,\alpha}(x,y)$ satisfies the condition (2.3), we obtain

$$
\left| t\nabla_x p_{t,\sigma}^{L,\alpha}(x,y) \right| \leq \frac{C_N t^{2\alpha}}{(t+|x-y|)^{n+2\alpha}} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)} \right)^{-N}.
$$

The proof that $t\nabla_x p_{t,\sigma}^{L,\alpha}(x,y)$ satisfies the condition (2.4) is similar to $p_{t,\sigma}^{L,\alpha}(x,y)$, we obtain

$$
\begin{aligned}
&\left| t\nabla_x p_{t,\sigma}^{L,\alpha}(x+h,y) - t\nabla_x p_{t,\sigma}^{L,\alpha}(x,y) \right| \\
&\quad \leq C_N \left(\frac{|h|}{t} \right)^{\delta} \frac{t^{2\alpha}}{(t+|x-y|)^{n+2\alpha}} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)} \right)^{-N}.
\end{aligned}
$$

Hence, $[b,\ T_5^{L,*}]$ is compact from $L^p(w)$ to $L^p(w)$. □

4 Compactness of commutators of fractional operators related to *L*

Let $B(t) = t\log(e+t)$. We define the B-average of a function f over a cube Q by means of the following Luxemburg norm:

$$
\|f\|_{L\log L,Q} = \inf\left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q B\left(\frac{|f(y)|}{\lambda} \right) \mathrm{d}y \leq 1 \right\}.
$$

We define the corresponding maximal functions:

$$M_{L\log L}(f)(x) = \sup_{Q:\ x\in Q} \|f\|_{L\log L,Q},$$

and for $0 < \eta < \infty$ and $0 \le \beta < n$,

$$M_{L\log L,\beta,V,\eta}(f)(x) = \sup_{Q:x\in Q} \Psi_\theta(Q)^{-\eta}(\Psi_\theta(Q)|Q|)^{\beta/n}\|f\|_{L\log L,Q}.$$

Proposition 4.1 *Let $0 < \beta < n$ and $1 < p < q < \infty$.*

(i) *Let $0 < \alpha < 1$, $0 < \alpha\beta < n$, $1 < p < n/(\alpha\beta)$ and $1/q = 1/p - \alpha\beta/n$. For any $N > 0$, there exist a constant C_N and a regularity exponent $\delta' > 0$ such that for $|x-y| > 2|h|$,*

$$\left|\mathcal{I}_1^L(x,y)\right| \le \frac{C_N}{|x-y|^{n-\alpha\beta}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}. \tag{4.1}$$

$$\left|\mathcal{I}_1^L(x+h,y) - \mathcal{I}_1^L(x,y)\right| \le \frac{C_N|h|^{\delta'}}{|x-y|^{n-\alpha\beta+\delta'}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}. \tag{4.2}$$

(ii) *Let $0 < \sigma < 1$, $0 < \beta/2 < n$, $1 < p < 2n/\beta$ and $1/q = 1/p - \beta/(2n)$. For any $N > 0$, there exist a constant C_N and a regularity exponent $\delta' > 0$ such that for $|x-y| > 2|h|$,*

$$\left|\mathcal{I}_2^L(x,y)\right| \le \frac{C_N}{|x-y|^{n-\beta/2}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}. \tag{4.3}$$

$$\left|\mathcal{I}_2^L(x+h,y) - \mathcal{I}_2^L(x,y)\right| \le \frac{C_N|h|^{\delta'}}{|x-y|^{n-\beta/2+\delta'}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}. \tag{4.4}$$

Proof

(i) By the subordinative formula (2.5), letting $s/t^{1/\alpha} = u$, we can get

$$\int_{\mathbb{R}} K^L_{\alpha,t}(x,y)t^{\beta/2}\frac{\mathrm{d}t}{t} = \int_0^\infty \left(\int_0^\infty \eta_t^\alpha(s)K_s^L(x,y)\mathrm{d}s\right)t^{\beta/2}\frac{\mathrm{d}t}{t}$$
$$= \int_0^\infty \left(\int_0^\infty \frac{1}{t^{1/\alpha}}\eta_1^\alpha(s/t^{1/\alpha})K_s^L(x,y)\mathrm{d}s\right)t^{\beta/2}\frac{\mathrm{d}t}{t}$$
$$= \int_0^\infty \left(\int_0^\infty \frac{1}{t^{1/\alpha}}\eta_1^\alpha(u)K^L_{t^{1/\alpha}u}(x,y)t^{1/\alpha}du\right)t^{\beta/2}\frac{\mathrm{d}t}{t}$$
$$= \int_0^\infty \eta_1^\alpha(u)\left\{\int_0^\infty K^L_{t^{1/\alpha}u}(x,y)t^{\beta/2}\frac{\mathrm{d}t}{t}\right\}du.$$

Let $t^{1/\alpha}u = w$. By [45, Lemma 3.7], we have

$$\left|\mathcal{I}_1^L(x,y)\right| = \alpha\left|\int_0^\infty \eta_1^\alpha(u)\left\{\int_0^\infty K_w^L(x,y)\left(\frac{w}{u}\right)^{\alpha\beta/2}\frac{dw}{w}\right\}du\right|$$
$$= \alpha\left|\int_0^\infty \eta_1^\alpha(u)u^{-\alpha\beta/2}\left\{\int_0^\infty K_w^L(x,y)w^{\alpha\beta/2}\frac{dw}{w}\right\}du\right|$$
$$\le \alpha\int_0^\infty \left|K_w^L(x,y)\right|w^{\alpha\beta/2}\frac{dw}{w}$$
$$\le \frac{C_N}{|x-y|^{n-\alpha\beta}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}.$$

By the subordinative formula (2.5) and [45, Lemma 3.7], we have

$$\left|\mathcal{I}_1^L(x+h,y)-\mathcal{I}_1^L(x,y)\right|$$
$$\le \int_0^\infty \eta_1^\alpha(u)u^{\alpha\beta/2}\left|\left\{\int_0^\infty \left(K_w^L(x+h,y)-K_w^L(x,y)\right)w^{\alpha\beta/2}\frac{dw}{w}\right\}\right|du$$
$$\le \frac{C|h|^{\delta'}}{|x-y|^{n-\alpha\beta+\delta'}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}.$$

(ii) We have

$$\mathcal{I}_2^L(x,y) = \frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^\infty e^{-r}K^L_{t^2/(4r)}(x,y)\frac{\mathrm{d}r}{r^{1-\sigma}}t^{\beta/2}\frac{\mathrm{d}t}{t}$$
$$= \frac{1}{\Gamma(\sigma)}\int_0^\infty \frac{e^{-r}}{r^{1-\sigma}}\left\{\int_0^\infty K^L_{t^2/(4r)}(x,y)t^{\beta/2}\frac{\mathrm{d}t}{t}\right\}\mathrm{d}r.$$

Let $t^2/(4r) = u$. By [45, Lemma 3.7], we have

$$
\begin{aligned}
\left|\mathcal{I}_2^L(x,y)\right| &\le C\int_0^\infty \frac{e^{-r}}{\Gamma(\sigma)r^{1-\sigma}}\left\{\int_0^\infty \left|K_u^L(x,y)\right|(4ur)^{\beta/4}\frac{r^{1/2}}{(4ur)^{1/2}u^{1/2}}du\right\}\mathrm{d}r\\
&\le C\int_0^\infty \frac{e^{-r}}{\Gamma(\sigma)r^{1-\sigma-\beta/4}}\left\{\int_0^\infty \left|K_u^L(x,y)\right|u^{\beta/4}\frac{du}{u}\right\}\mathrm{d}r\\
&\le \frac{C_N}{|x-y|^{n-\beta/2}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}\int_0^\infty \frac{e^{-r}}{\Gamma(\sigma)}r^{\beta/4+\sigma-1}\mathrm{d}r\\
&\le \frac{C_N}{|x-y|^{n-\beta/2}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}.
\end{aligned}
$$

Similar to (i), we have

$$
\begin{aligned}
&\left|\mathcal{I}_2^L(x+h,y)-\mathcal{I}_2^L(x,y)\right|\\
&\quad\le \left|\int_0^\infty \frac{e^{-r}}{\Gamma(\sigma)r^{1-\sigma-\beta/4}}\left\{\int_0^\infty \left(K_u^L(x+h,y)-K_u^L(x,y)\right)u^{\beta/4}\frac{du}{u}\right\}\mathrm{d}r\right|\\
&\quad\le \frac{C_N|h|^{\delta'}}{|x-y|^{n-\beta/2+\delta'}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}\int_0^\infty \frac{e^{-r}}{\Gamma(\sigma)}r^{\beta/4+\sigma-1}\mathrm{d}r\\
&\quad\le \frac{C_N|h|^{\delta'}}{|x-y|^{n-\beta/2+\delta'}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}.
\end{aligned}
$$

This completes the proof. □

Lemma 4.2 *For* $0<\beta<n$.

- (i) *Let* $0<\alpha<1, 0<\alpha\beta<n, 1<p<n/(\alpha\beta)$ *and* $1/q=1/p-\alpha\beta/n$. *The operator* $\widetilde{\mathcal{I}}_1^L$ *is bounded from* $L^p(\mathbb{R}^n)$ *into* $L^q(\mathbb{R}^n)$.
- (ii) *Let* $0<\sigma\le 1$ *and* $0<\beta/2<n, 1<p<2n/\beta$ *and* $1/q=1/p-\beta/(2n)$. *The operator* $\widetilde{\mathcal{I}}_2^L$ *is bounded from* $L^p(\mathbb{R}^n)$ *into* $L^q(\mathbb{R}^n)$.

Proof

(i) For $0<\alpha<1$ and $0<\alpha\beta<n$, by Proposition 4.1, it is easy to see that

$$
\left|\widetilde{\mathcal{I}}_1^L(f)(x)\right| \le C\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^{n-\alpha\beta}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}\mathrm{d}y \le C|I_{\alpha\beta}(|f|)(x)|,
$$

where $I_{\alpha\beta}$ denotes the classical fractional integral. For $1<p<n/(\alpha\beta)$ and $1/q=1/p-\alpha\beta/n$, by the (L^p,L^q)-boundedness of $I_{\alpha\beta}$, we have

$$\|\widetilde{\mathcal{I}}_1^L(f)\|_q \leq C\|I_{\alpha\beta}(|f|)\|_q \leq C\|f\|_p,$$

see [43, Chapter 5].

(ii) Similar to $\widetilde{\mathcal{I}}_1^L$, $\widetilde{\mathcal{I}}_2^L$ is also bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, we omit the details. □

Theorem 4.3 *Let* $0 < \beta < n$ *and* $w \in A_{p,q}^{\rho}$.

(i) *If* $0 < \alpha < 1, 0 < \alpha\beta < n, 1 < p < n/(\alpha\beta)$ *and* $1/q = 1/p - \alpha\beta/n$, *then*

$$\left(\int_{\mathbb{R}^n} |\widetilde{\mathcal{I}}_1^L(f)(x)|^q w(x)^q \mathrm{d}x\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p w(x)^p \mathrm{d}x\right)^{1/p}.$$

(ii) *If* $0 < \sigma < 1$ *and* $0 < \beta/2 < n, 1 < p < 2n/\beta$ *and* $1/q = 1/p - \beta/(2n)$, *then*

$$\left(\int_{\mathbb{R}^n} |\widetilde{\mathcal{I}}_2^L(f)(x)|^q w(x)^q \mathrm{d}x\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p w(x)^p \mathrm{d}x\right)^{1/p}.$$

Proof

(i) Following the procedure of [45, Theorem 3.8], since $\widetilde{\mathcal{I}}_1^L$ is of weak type $(1, n/(n - \alpha\beta))$, we can get

$$\int_{\mathbb{R}^n} |\widetilde{\mathcal{I}}_1^L(f)(x)|^q w(x) \mathrm{d}x \leq \int_{\mathbb{R}^n} |M_{\alpha\beta,V,\eta}(f)(x)|^q w(x) \mathrm{d}x.$$

By $\|M_{\alpha\beta,V,\eta}(f)\|_{L^q(w^q)} \leq \|f\|_{L^q(w^q)}$, we can see that (i) of Theorem 4.3 holds.

(ii) The proof of (ii) is similar to (i), we omit it. □

Theorem 4.4 *For* $0 < \beta < n$, $b \in \mathrm{BMO}_\theta(\rho)(\mathbb{R}^n)$ *and* $w \in A_{p,q}^{\rho}$.

(i) *Let* $0 < \alpha < 1, 0 < \alpha\beta < n, 1 < p < n/(\alpha\beta)$ *and* $1/q = 1/p - \alpha\beta/n$. *Then*

$$\left(\int_{\mathbb{R}^n} |[b, \widetilde{\mathcal{I}}_1^L](f)(x)|^q w(x)^q \mathrm{d}x\right)^{1/q} \leq C\|b\|_{\mathrm{BMO}_\theta(\rho)}\left(\int_{\mathbb{R}^n} |f(x)|^p w(x)^p \mathrm{d}x\right)^{1/p}.$$

(ii) *Let* $0 < \sigma < 1, 0 < \beta/2 < n, 1 < p < 2n/\beta$ *and* $1/q = 1/p - \beta/(2n)$. *Then*

$$\left(\int_{\mathbb{R}^n} |[b, \widetilde{\mathcal{I}}_2^L](f)(x)|^q w(x)^q \mathrm{d}x\right)^{1/q} \leq C\|b\|_{\mathrm{BMO}_\theta(\rho)} \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)^p \mathrm{d}x\right)^{1/p}.$$

Proof

(i) Following the procedure of [45, Theorem 4.4], since $\widetilde{\mathcal{I}}_1^L$ is of weak type $(1, n/(n-\alpha\beta))$, then we can get

$$M^{\sharp}_{\delta,\eta}([b, \widetilde{\mathcal{I}}_1^L])(f)(x) \leq C\|b\|_{\mathrm{BMO}_\theta(\rho)} \Big(M^{\Delta}_{\epsilon,\eta}(\widetilde{\mathcal{I}}_1^L(f))(x) + M_{L\log L,\alpha\beta,V,\eta}(f)(x)\Big).$$

It follows from the inequalities

$$C_1 M_{\alpha\beta,V,\eta} M_{V,\eta+1}(f)(x) \leq M_{L\log L,\alpha\beta,V,\eta}(f)(x) \leq C_2 M_{\alpha\beta,V,\eta/2}(f)(x)$$

that

$$\begin{aligned}
&\|[b, \widetilde{\mathcal{I}}_1^L](f)\|_{L^q(w^q)} \\
&\quad \leq C\|b\|_{\mathrm{BMO}_\theta(\rho)} \Big(\|M^{\Delta}_{\epsilon,\eta}(\widetilde{\mathcal{I}}_1^L(f))\|_{L^q(w^q)} + \|M_{L\log L,\alpha\beta,V,\eta}(f)\|_{L^q(w^q)}\Big) \\
&\quad \leq C\|b\|_{\mathrm{BMO}_\theta(\rho)} \Big(\|\widetilde{\mathcal{I}}_1^L(f)\|_{L^q(w^q)} + \|M_{\alpha\beta,V,\eta/2}(f)\|_{L^q(w^q)}\Big) \\
&\quad \leq C\|b\|_{\mathrm{BMO}_\theta(\rho)} \|f\|_{L^p(w^p)}.
\end{aligned}$$

(ii) The proof of (ii) is similar to (i), we omit it. □

Theorem 4.5 *Let* $0 < \beta < n$, $b \in \mathrm{CMO}_\theta(\rho)(\mathbb{R}^n)$ *and* $w \in A^{\rho}_{p,q}$.

(i) *For* $\alpha \in (0, 1)$, $0 < \alpha\beta < n$, $1 < p < n/(\alpha\beta)$ *and* $1/q = 1/p - \alpha\beta/n$, *the commutator* $[b, \widetilde{\mathcal{I}}_1^L]$ *is a compact operator from* $L^p(w^p)$ *to* $L^q(w^q)$.

(ii) *For* $\sigma \in (0, 1)$, $0 < \beta/2 < n$, $1 < p < 2n/\beta$ *and* $1/q = 1/p - \beta/(2n)$, *the commutator* $[b, \widetilde{\mathcal{I}}_2^L]$ *is a compact operator from* $L^p(w^p)$ *to* $L^q(w^q)$.

Proof We only prove (i). The statement (ii) can be dealt with similarly and we omit the details. Similar to the procedure of Theorem 3.3, we choose the smooth truncated function $\varphi \in C^\infty([0, \infty))$ satisfying (3.1). For any $\gamma > 0$, let

$$\mathcal{I}^L_{1,\gamma}(x, y) = \mathcal{I}^L_1(x, y)\big(1 - \varphi(\gamma^{-1}|x - y|)\big). \tag{4.5}$$

Define

$$\widetilde{\mathcal{I}}^{L}_{1,\gamma}(f)(x)=\int_{\mathbb{R}^n}\mathcal{I}^{L}_{1,\gamma}(x,y)f(y)\mathrm{d}y \tag{4.6}$$

and

$$[b,\,\widetilde{\mathcal{I}}^{L}_{1,\gamma}](f)(x)=\int_{\mathbb{R}^n}\mathcal{I}^{L}_{1,\gamma}(x,y)(b(x)-b(y))f(y)\mathrm{d}y. \tag{4.7}$$

Combining (4.5), (4.7) and (4.1) with $N=\theta(\eta+1-\alpha\beta/n)$, we show that for any $b\in C_c^\infty(\mathbb{R}^n)$ and $\gamma,\theta,\eta>0$,

$$\begin{aligned}
&\left|[b,\,\widetilde{\mathcal{I}}^{L}_{1}](f)(x)-[b,\,\widetilde{\mathcal{I}}^{L}_{1,\gamma}](f)(x)\right|\\
&\le C\int_{|x-y|<2\gamma}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-\theta(\eta+1-\alpha\beta/n)}\frac{|f(y)|}{|x-y|^{n-\alpha\beta-1}}\mathrm{d}y\\
&\le C\gamma\sum_{k=1}^{\infty}\left(1+\frac{2^{-k}\gamma}{\rho(x)}\right)^{-\theta(\eta+1-\alpha\beta/n)}\frac{2^{-k}}{(2^{-k}\gamma)^{n-\alpha\beta}}\int_{|x-y|<2^{-k}\gamma}|f(y)|\mathrm{d}y\\
&\le C\gamma M_{\alpha\beta,V,\eta}(f)(x).
\end{aligned} \tag{4.8}$$

Then Lemma 2.11 with $\eta\ge(1-\alpha\beta/n)p'/q$ gives

$$\|[b,\,\widetilde{\mathcal{I}}^{L}_{1}](f)-[b,\,\widetilde{\mathcal{I}}^{L}_{1,\gamma}](f)\|_{L^q(w^q)}\le C\gamma\|f\|_{L^p(w^p)},$$

which implies that

$$\lim_{\gamma\to0}\|[b,\,\widetilde{\mathcal{I}}^{L}_{1}](f)-[b,\,\widetilde{\mathcal{I}}^{L}_{1,\gamma}](f)\|_{L^q(w^q)}=0. \tag{4.9}$$

If $b\in \mathrm{CMO}_\theta(\rho)(\mathbb{R}^n)$ and $w\in A^{\rho}_{p,q}$, there exists $b_\epsilon\in C_c^\infty(\mathbb{R}^n)$ such that $\|b-b_\epsilon\|_{\mathrm{BMO}_\theta(\rho)}<\epsilon$ for any $\epsilon>0$. Hence, by Theorem 4.4, we have

$$\left\|[b,\,\widetilde{\mathcal{I}}^{L}_{1}](f)-[b_\epsilon,\,\widetilde{\mathcal{I}}^{L}_{1}](f)\right\|_{L^q(w^q)}\le C\|b-b_\epsilon\|_{\mathrm{BMO}_\theta(\rho)}\|f\|_{L^p(w^p)}\le C\epsilon.$$

Thus, to prove $[b,\,\widetilde{\mathcal{I}}^{L}_{1}]$ is compact from $L^p(w^p)$ to $L^q(w^q)$ for any $b\in\mathrm{CMO}_\theta(\rho)$, it suffices to show that $[b,\,\widetilde{\mathcal{I}}^{L}_{1,\gamma}]$ is compact from $L^p(w^p)$ to $L^q(w^q)$ for any $b\in C_c^\infty(\mathbb{R}^n)$ and sufficiently small $\gamma>0$. For arbitrary bounded set G in $L^q(w^q)$, let $\mathcal{G}=\{[b,\,\widetilde{\mathcal{I}}^{L}_{1,\gamma}](f):\,f\in G\}$. Then, we only need to verify that $\mathcal{G}$ satisfies the conditions (i)–(iii) of Lemma 3.2 for $b\in C_c^\infty(\mathbb{R}^n)$.

Firstly, we prove that $\mathcal{I}^{L}_{1,\gamma}(\cdot,\cdot)$ satisfies (4.1)–(4.2). For any $x,y\in\mathbb{R}^n$,

$$\left|\mathcal{I}^{L}_{1,\gamma}(x,y)\right|\le\left|\mathcal{I}^{L}_{1}(x,y)\right|\le\frac{C_N}{|x-y|^{n-\alpha\beta}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}. \tag{4.10}$$

For any $|h| < |x-y|/2$, it suffices to verify that

$$\left|\mathcal{I}^L_{1,\gamma}(x+h,y) - \mathcal{I}^L_{1,\gamma}(x,y)\right| \le C_N \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N} \frac{|h|^{\delta'}}{|x-y|^{n-\alpha\beta+\delta'}}. \tag{4.11}$$

We consider the following four cases:

$$\begin{cases} Case1: & |x+h-y| \ge 2\gamma \ \& \ |x-y| \ge 2\gamma; \\ Case2: & |x+h-y| < 2\gamma \ \& \ |x-y| \ge 2\gamma; \\ Case3: & |x+h-y| \ge 2\gamma \ \& \ |x-y| < 2\gamma; \\ Case4: & |x+h-y| < 2\gamma \ \& \ |x-y| < 2\gamma. \end{cases}$$

Case 1. In this case, $\mathcal{I}^L_{1,\gamma}(x+h,y) = \mathcal{I}^L_1(x+h,y)$ and $\mathcal{I}^L_{1,\gamma}(x,y) = \mathcal{I}^L_1(x,y)$. This together with (4.1) yields (4.10).

Case 2. Since $|h| < |x-y|/2$ imlpies $\mathcal{I}^L_{1,\gamma}(x,y) = \mathcal{I}^L_1(x,y)$, $|x+h-y| \ge |x-y| - |h| \ge |x-y|/2$, applying $\gamma \ge |x-y|/4$ and (4.2), we infer that

$$\begin{aligned}
&\left|\mathcal{I}^L_{1,\gamma}(x+h,y) - \mathcal{I}^L_{1,\gamma}(x,y)\right| \\
&\quad \le \left|\mathcal{I}^L_1(x+h,y) - \mathcal{I}^L_1(x,y)\right|(1 + \varphi(\gamma^{-1}|x+h-y|)) \\
&\qquad + \left|\mathcal{I}^L_1(x,y)\right|\left|\varphi(\gamma^{-1}|x+h-y|) - \varphi(\gamma^{-1}|x-y|)\right| \\
&\quad \le C_N \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N} \left(\frac{|h|^{\delta'}}{|x-y|^{n-\alpha\beta+\delta'}} + \frac{1}{|x-y|^{n-\alpha\beta}} \frac{|h|}{\gamma}\right) \\
&\quad \le C_N \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N} \left(\frac{|h|^{\delta'}}{|x-y|^{n-\alpha\beta+\delta'}} + \frac{4}{|x-y|^{n-\alpha\beta+\delta'}} \frac{|h|}{|x-y|^{1-\delta'}}\right) \\
&\quad \le C_N \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N} \left(\frac{|h|^{\delta'}}{|x-y|^{n-\alpha\beta+\delta'}} + \frac{4}{|x-y|^{n-\alpha\beta+\delta'}} \frac{|h|}{(2|h|)^{1-\delta'}}\right) \\
&\quad \le C_N \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N} \frac{|h|^{\delta'}}{|x-y|^{n-\alpha\beta+\delta'}},
\end{aligned}$$

which proves (4.11).

Case 3 and *Case 4* can be deal with similar to *Case 2*.

From the definition of $\mathcal{I}^L_{1,\gamma}$, we can see that $|\mathcal{I}^L_{1,\gamma}(x,y)| \le |\mathcal{I}^L_1(x,y)|$. Then

$$\left|\widetilde{\mathcal{I}}^L_{1,\gamma}(f)(x)\right| \le \int_0^\infty |\mathcal{I}^L_{1,\gamma}(x,y)||f(y)|\mathrm{d}y \le \int_0^\infty |\mathcal{I}^L_1(x,y)||f(y)|\mathrm{d}y$$

and

$$\left|[b, \widetilde{\mathcal{I}}^L_{1,\gamma}](f)(x)\right| \leq \int_0^\infty |\mathcal{I}^L_{1,\gamma}(x,y)||(b(x)-b(y))f(y)|\mathrm{d}y \\ \leq \int_0^\infty |\mathcal{I}^L_1(x,y)||f(y)|\mathrm{d}y.$$

Hence, the weighted L^p boundedness of $\widetilde{\mathcal{I}}^L_{1,\gamma}$ and $[b, \widetilde{\mathcal{I}}^L_{1,\gamma}]$ also holds. Thus, we have

$$\sup_{f\in G} \|[b, \widetilde{\mathcal{I}}^L_{1,\gamma}](f)\|_{L^q(w^q)} \leq C \sup_{f\in G} \|f\|_{L^p(w^p)} \leq C_G,$$

which yields the fact that the set $\mathcal{G}$ is bounded.

Assume $b \in C_c^\infty(\mathbb{R}^n)$ and $\mathrm{supp}(b) \subset B(0, R)$, where $B(0, R)$ is the ball of radius R and centered at origin in $\mathbb{R}^n$. Using (4.1), for $|x| > A > 2R$ we have

$$\begin{aligned} \left|[b, \widetilde{\mathcal{I}}^L_{1,\gamma}](f)(x)\right| &\leq \int_{|y|<R} |\mathcal{I}^L_1(x,y)||b(y)f(y)|\mathrm{d}y \\ &\leq C_N\|b\|_\infty \int_{|y|<R} \left(1+\frac{|x-y|}{\rho(x)}\right)^{-N} \frac{|f(y)|}{|x-y|^{n-\alpha\beta}}\mathrm{d}y \\ &\leq \frac{C_N\|b\|_\infty}{|x|^{n-\alpha\beta}} \frac{\|f\|_{L^p(w^p)}}{(1+|x|/(2\rho(x)))^N} \left(\int_{|y|<R} w^{-p'}(y)\mathrm{d}y\right)^{1/p'}. \end{aligned} \tag{4.12}$$

By Lemma 2.6, there exist $k_0 \leq 1$ and $C_0 > 0$ such that $\rho(x) \leq C_0\rho(0)(1+|x|/\rho(0))^{k_0/(k_0+1)}$. Hence we can obtain

$$\begin{aligned} \frac{1}{1+|x|/(2\rho(x))} \leq \frac{2}{1+|x|/\rho(x)} &\leq \frac{C}{1+|x|/\rho(0)(1+|x|/\rho(0))^{-k_0/(k_0+1)}} \\ &\leq \frac{C}{(1+|x|/\rho(0))(1+|x|/\rho(0))^{-k_0/(k_0+1)}} \\ &= \frac{C}{(1+|x|/\rho(0))^{1/(k_0+1)}}, \end{aligned}$$

which leads to

$$\left(\int_{|x|>A}|[b,\widetilde{\mathcal{I}}^{L}_{1,\gamma}](f)(x)|^q w^q(x)\mathrm{d}x\right)^{1/q}$$
$$\leq C\sum_{j=0}^{\infty}\frac{1}{(2^jA)^{n-\alpha\beta}}\frac{1}{(1+2^jA/\rho(0))^{N/(k_0+1)}}\left(\int_{2^jA<|x|<2^{j+1}A}w(x)^q\mathrm{d}x\right)^{1/q}$$
$$\times\left(\int_{|y|<R}w(y)^{-p'}\mathrm{d}y\right)^{1/p'}$$
$$\leq C\sum_{j=0}^{\infty}\frac{1}{(2^jA)^{n-\alpha\beta}}\frac{1}{(1+2^jA/\rho(0))^{N/(k_0+1)}}\left(\int_{B(0,2^{j+1}A)}w(x)^q\mathrm{d}x\right)^{1/q}$$
$$\times\left(\int_{B(0,R)}w(y)^{-p'}\mathrm{d}y\right)^{1/p'}.$$

Taking $Q=B(0,2^{j+1}A)$, $E=B(0,R)$, since $w\in A^{\rho}_{p,q}$, then $w^q\in A^{\rho,\theta}_{1+q/p'}$. Let $r=1+q/p'$. By Lemma 2.10, we can get

$$w^q(5Q)\leq C\left(\frac{\Psi_\theta(Q)|Q|}{|E|}\right)^r w^q(E)$$
$$\leq Cw^q(B(0,R))\left(\frac{(1+2^{j+1}A/\rho(0))^{\theta}(2^{j+1}A)^n}{R^n}\right)^r$$
$$\leq Cw^q(B(0,R))(1+2^{j+1}A/\rho(0))^{\theta r}\frac{(2^{j+1}A)^{nr}}{R^{nr}}.$$

Take $N>\max\{(k_0+1)2\theta r/q,(k_0+1)((n+2\theta)r/q+\alpha\beta-n)\}$. Then

$$\left(\int_{|x|>A}|[b,\widetilde{\mathcal{I}}^{L}_{1,\gamma}](f)(x)|^q w^q(x)\mathrm{d}x\right)^{1/q}$$
$$\leq C\sum_{j=0}^{\infty}\frac{(2^{j+1}A)^{nr/q}}{(2^jA)^{n-\alpha\beta}}\frac{(1+2^{j+1}A/\rho(0))^{\theta r/q}}{(1+2^jA/\rho(0))^{N/(k_0+1)}}\frac{1}{R^{nr/q}}\left(\int_{B(0,R)}w(x)^q\mathrm{d}x\right)^{1/q}$$
$$\times\left(\int_{B(0,R)}w(y)^{-p'}\mathrm{d}y\right)^{1/p'}$$
$$\leq C\sum_{j=0}^{\infty}(2^{j+1}A)^{-(N/(k_0+1)-(n+2\theta)r/q+n-\alpha\beta)}$$
$$\leq CA^{-(N/(k_0+1)-(n+2\theta)r/q+n-\alpha\beta)}.$$

Therefore, we obtain

$$\lim_{A\to\infty}\int_{|x|>A}|[b,\widetilde{\mathcal{I}}^{L}_{1,\gamma}](f)(x)|^q w^q(x)\mathrm{d}x=0$$

holds uniformly for $f \in \mathcal{G}$.

It remains to show that the set $\mathcal{G}$ is uniformly equicontinuous. It suffices to verify that for any $\epsilon > 0$, if $|h|$ is sufficiently small and only depends on ϵ, then

$$\lim_{|h|\to 0} \|[b, \widetilde{\mathcal{I}}^L_{1,\gamma}](f)(h+\cdot) - [b, \widetilde{\mathcal{I}}^L_{1,\gamma}](f)(\cdot)\|_{L^q(w^q)} = C\epsilon \tag{4.13}$$

holds uniformly for $f \in \mathcal{G}$.

In what follows, we fix $\gamma \in (0, 1/4)$ and $|h| < \gamma/4$. Then $[b, \widetilde{\mathcal{I}}^L_{1,\gamma}](f)(x+h) - [b, \widetilde{\mathcal{I}}^L_{1,\gamma}](f)(x) =: \widetilde{I}(x) + \widetilde{II}(x)$, where

$$\begin{cases} \widetilde{I}(x) := \int_{\mathbb{R}^n} \Big(\mathcal{I}^L_{1,\gamma}(x+h, y) - \mathcal{I}^L_{1,\gamma}(x, y)\Big)(b(x+h) - b(y))f(y)\mathrm{d}y; \\ \widetilde{II}(x) := \int_{\mathbb{R}^n} \mathcal{I}^L_{1,\gamma}(x, y)(b(x+h) - b(x))f(y)\mathrm{d}y. \end{cases}$$

For $\widetilde{II}(x)$, it holds

$$\left|\widetilde{II}(x)\right| \le |b(x+h) - b(x)| \left|\int_{\mathbb{R}^n} \mathcal{I}^L_{1,\gamma}(x, y)f(y)\mathrm{d}y\right| \le C|h| \int_{\mathbb{R}^n} |\mathcal{I}^L_1(x, y)||f(y)|\mathrm{d}y.$$

Then, by the $L^p(w^p)$-boundedness of $\widetilde{\mathcal{I}}^L_1$, we have

$$\|\widetilde{II}\|_{L^q(w^q)} \le C|h|\|f\|_{L^p(w^p)}. \tag{4.14}$$

When $|x-y| < \gamma/2$ and $|h| < \gamma/4$, then $|x+h-y| < 3\gamma/4$. Hence

$$\varphi(\gamma^{-1}|x+h-y|) = 1 = \varphi(\gamma^{-1}|x-y|).$$

This implies

$$\mathcal{I}^L_{1,\gamma}(x+h, y) = 0 = \mathcal{I}^L_{1,\gamma}(x, y).$$

Since $|x-y| \ge \gamma/2$ and $|h| < \gamma/4$ implies $|h| < |x-y|/2$. Thus, combining (4.11) together with $N = \theta(\eta + 1 - \alpha\beta/n)$, we have

$$\begin{aligned} \left|\widetilde{I}(x)\right| &\le C|h|^{\delta'} \int_{|x-y|\ge\gamma/2} \Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{-N} \frac{|f(y)|}{|x-y|^{n-\alpha\beta+\delta'}}\mathrm{d}y \\ &\le C\gamma^{-\delta'}|h|^{\delta'} \sum_{k=1}^{\infty} \Big(1 + \frac{2^k\gamma}{\rho(x)}\Big)^{-N} \frac{2^{-k\delta'}}{(2^k\gamma)^{n-\alpha\beta}} \int_{|x-y|\sim 2^k\gamma} |f(y)|\mathrm{d}y \\ &\le C\gamma^{-\delta'}|h|^{\delta'} M_{\alpha\beta,V,\eta}(f)(x). \end{aligned}$$

By Lemma 2.11 for any $\eta \ge (1 - \alpha\beta/n)p'/q$, it holds

$$\|\widetilde{I}\|_{L^q(w^q)} \le C|h|^{\delta'}\|M_{\alpha\beta,V,\eta}(f)\|_{L^q(w^q)} \le C|h|^{\delta'}\|f\|_{L^p(w^p)}. \tag{4.15}$$

Using (4.14) and (4.15), we get

$$\|[b, \widetilde{\mathcal{I}}^{L}_{1,\gamma}](f)(h+\cdot) - [b, \widetilde{\mathcal{I}}^{L}_{1,\gamma}](f)(\cdot)\|_{L^q(w^q)} \le C(|h| + |h|^{\delta'})\|f\|_{L^p(w^p)}.$$

Hence, we obtain the desired result (4.13). This completes the proof of (i). □

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