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ORIGINAL PAPER

Bergman projections on weighted mixed norm spaces and duality

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Abstract

We prove that the weighted Bergman projection P_γ is a bounded operator on the weighted Lebesgue space $L^p(\Omega, r(x)^\lambda \mathrm{d}m(x))$ for a certain range of parameters p, γ and λ. Here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. This result is used to prove boundedness of P_γ acting on weighted mixed norm space $L^{p,q}_\alpha(\Omega)$, again assuming certain conditions on the parameters. We describe the dual of harmonic mixed norm space $B^{p,q}_\alpha(\Omega)$ for a certain range of parameters.

Keywords Bergman projections · Mixed norm spaces · Duality

Mathematics Subject Classification 42B35 · 31B05

1 Introduction

Boundedness of the Bergman projection on harmonic mixed norm space on a smoothly bounded domain in $\mathbb{R}^n$ was proved by Hu and Lv [6]. The proof relies on an equivalent norm on the mixed norm space. As a consequence, they obtained the dual of such space. In this paper, we generalize these results to the case of weighted mixed norm spaces, see Theorems 3.2 and 4.5. The same general scheme of proof presented in [6] for the unweighted case works in the weighted case, however, we need delicate estimates for the weighted Bergman kernel obtained in [3]. These

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estimates are used to prove boundedness of the weighted Bergman projection P_γ acting on weighted Lebesgue spaces $L^p(\Omega, r(x)^\lambda \mathrm{d}m(x))$, see Theorem 3.1.

Note that the harmonic weighted case in the unit ball of $\mathbb{R}^n$ has been considered in [8]. It is worth mentioning that the problem of the boundedness of Bergman projection on certain holomorphic mixed norm spaces has been considered by several authors. About three decades ago, sufficient conditions for boundedness of the Bergman projection on certain weighted mixed norm spaces of functions on the unit ball in $\mathbb{C}^n$ are given in [4, 7] and the dual of such space is identified. In [5], it was proven that the Bergman operator is bounded on holomorphic weighted mixed norm spaces on the unit ball with radial weights satisfying Békollé's conditions and characterization of the corresponding dual space was obtained. Sufficient conditions for boundedness of the Bergman type operators on certain weighted mixed norm spaces of functions on the unit ball in $\mathbb{C}^n$, for a certain range of parameters, are given in [10].

The boundedness of the weighted Bergman projection on weighted mixed norm space on symmetric tube domains was discussed in [2], and on homogeneous Siegel domains of type II in [9]. The boundedness of the multifunctional Bergman type operators in symmetric tube domains was considered in [1].

2 Notation and preliminary results

Throughout the paper Ω denotes a bounded domain in $\mathbb{R}^n$ (i.e. open and connected) with C^∞ boundary and $h(\Omega)$ denotes the vector space of all real-valued harmonic functions in Ω. Let $\rho(x)$ be a defining function for Ω. This means ρ is a real valued function on $\mathbb{R}^n$ which is C^∞ in a neighborhood of the boundary $\partial\Omega$ of Ω such that $\Omega = \{x \in \mathbb{R}^n : \rho(x) > 0\}$ is bounded and $|\nabla\rho(x)| \neq 0$ on $\partial\Omega$. Throughout this paper such a domain Ω is fixed. It is convenient to work with a particular defining function, namely the distance function $r(x)$ defined by $r(x) = d(x, \partial\Omega)$ for $x \in \overline{\Omega}$ and $r(x) = -d(x, \partial\Omega)$ for $x \notin \overline{\Omega}$. Indeed, there is an $\epsilon > 0$ such that for all $0 < r \leq \epsilon$ the set $\Omega_r = \{x \in \mathbb{R}^n : r(x) > r\}$ is a smoothly bounded subdomain of Ω with defining function $r(x) - r$. We fix such $\epsilon > 0$. We denote by Γ_r the boundary $\partial\Omega_r = \{x \in \mathbb{R}^n : r(x) = r\}$.

We denote by $\mathrm{d}\sigma_r$ the induced surface measure on $\partial\Omega_r$. $\mathrm{d}m$ denotes the Lebesgue volume measure on $\mathbb{R}^n$. We also work with weighted measures $\mathrm{d}m_\gamma(x) = r(x)^\gamma \mathrm{d}m(x)$ on Ω, where $\gamma \in \mathbb{R}$ and set $L^p_\gamma(\Omega) = L^p(\Omega, \mathrm{d}m_\gamma)$. The exponent conjugate to $1 \leq p \leq +\infty$ is denoted by p'.

The weighted Bergman spaces are $b^p_\gamma(\Omega) = L^p(\Omega, \mathrm{d}m_\gamma) \cap h(\Omega)$ where $0 < p \leq +\infty$ and $\gamma > -1$.

For $0 < p < \infty$ and $0 < r \leq \epsilon$, we set

$$M_p(f, r) = \left\{ \int_{\Gamma_r} |f(\zeta)|^p \mathrm{d}\sigma_r(\zeta) \right\}^{\frac{1}{p}}, \tag{2.1}$$

with obvious modification for the case $p = +\infty$. Now let $0 < p \le +\infty$, $0 < q < +\infty$ and $\alpha > 0$. We define a mixed norm space $B_\alpha^{p,q}(\Omega)$ as the space of all $f \in h(\Omega)$ such that the (quasi) norm

$$\|f\|_{B_\alpha^{p,q}} = \left\{ \int_0^\epsilon r^{\alpha q - 1} M_p^q(f, r) \mathrm{d}r \right\}^{1/q} \tag{2.2}$$

is finite, again with obvious modification to include the case $q = +\infty$. The space $B_\alpha^{p,q}(\Omega)$ is a Banach space for $1 \le p \le +\infty$ and $1 \le q \le +\infty$. This scale of spaces includes weighted Bergman spaces: $b_\gamma^p(\Omega) = B_{(\gamma+1)/p}^{p,p}(\Omega)$, $\gamma > -1$, $0 < p < \infty$.

Throughout this paper we will use the convention of using C to denote any positive constant which may change from one occurrence to the next. Given two positive quantities A and B, we write $A \asymp B$ if there are constants $0 < c \le C < +\infty$ such that $cA \le B \le CA$.

Note that $r_0 = \max\{r(x) : x \in \Omega\}$ is a finite strictly positive number and set $r_j = \frac{r_0}{2^j}$ for all $j \in \mathbb{N}$. For $j \in \mathbb{N}$, we define

$$S_j = \{x \in \Omega : r_j < r(x) \le r_{j-1}\}.$$

It is obvious that $\Omega = \cup_{j=1}^\infty S_j$.

For $0 < p, q, \alpha < \infty$, we define weighted mixed norm space $L_\alpha^{p,q}(\Omega)$ as the set of all Lebesgue measurable functions f on Ω such that

$$\|f\|_{L_\alpha^{p,q}} = \left\{ \sum_{j=1}^\infty \left[\int_{S_j} |f(x)|^p \mathrm{d}m(x) \right]^{\frac{q}{p}} 2^{j(\frac{q}{p} - \alpha q)} \right\}^{\frac{1}{q}} < \infty.$$

Since

$$\|f\|_{L_\alpha^{p,q}} \asymp \|a_j(f)\|_{l^q}, \quad \text{where} \quad a_j(f) = \left(\int_{S_j} |f(x)|^p r(x)^{\alpha p - 1} \mathrm{d}m(x) \right)^{\frac{1}{p}} \tag{2.3}$$

we easily deduce the following lemma.

Lemma 2.1 *If* $0 < p < +\infty$ *and* $\alpha > 0$, *then* $L_\alpha^{p,p}(\Omega) = L_{\alpha p - 1}^p(\Omega)$ *and the two* (*quasi*) *norms on the space are equivalent.*

For $0 < p < \infty$ and $q = \infty$, we define a space $L_\alpha^{p,\infty}(\Omega)$ as the set of all Lebesgue measurable functions f on Ω such that

$$\|f\|_{L_\alpha^{p,\infty}} = \sup_{j \ge 1} \left[\int_{S_j} |f(x)|^p \mathrm{d}m(x) \right]^{\frac{1}{p}} 2^{j(\frac{1}{p} - \alpha)} < \infty.$$

It is clear that these definitions extend to the case $p = +\infty$ in a standard manner.

For $\gamma > -1$, let $R_\gamma(x, y)$ be the reproducing kernel of the harmonic Bergman space $b_\gamma^2(\Omega)$. For every function $f \in b_\gamma^2(\Omega)$, we have a reproducing formula

$$f(x) = \int_\Omega R_\gamma(x,y) f(y) \mathrm{d}m_\gamma(y), \qquad x \in \Omega.$$

The kernel $R_\gamma(x,y)$ is symmetric and real-valued. The (weighted) Bergman projection P_γ is the orthogonal projection from $L^2_\gamma(\Omega)$ onto its subspace $b^2_\gamma(\Omega)$; it is given by the following integral formula

$$P_\gamma f(x) = \int_\Omega R_\gamma(x,y) f(y) \mathrm{d}m_\gamma(y), \qquad x \in \Omega. \tag{2.4}$$

We prove below, see Corollary to Theorem 3.1, that the weighted Bergman projection defined by the above formula is a bounded operator from $L^p_\gamma(\Omega)$ onto $b^p_\gamma(\Omega) = h(\Omega) \cap L^p(\Omega, \mathrm{d}m_\gamma)$, for $1 < p < \infty$.

For $x, y \in \Omega$, we introduce a quasi distance $D(x,y) = r(x) + r(y) + |x - y|$ on Ω. In the next proposition, we state delicate estimates of the weighted Bergman kernel in terms of $D(x, y)$. These estimates are very special cases of results obtained by Engliš in [3].

Proposition 2.2 [3] *Let $\gamma > -1$. There is a positive constant $C = C_{\gamma,\Omega}$ such that $|R_\gamma(x,y)| \le C\frac{1}{D(x,y)^{n+\gamma}}$ and $|\frac{\partial R_\gamma(y,x)}{\partial y}| \le C\frac{1}{D(x,y)^{n+\gamma+1}}$. Moreover, for some constant $c > 0$ we have*

$$|R_\gamma(x,x)| \ge c\frac{1}{r(x)^{n+\gamma}}.$$

We will need next lemma from [6] to prove a theorem about boundedness of Bergman projections.

Lemma 2.3 [6] *For $\gamma > -1$, $t < 1$ and $\gamma + t > 0$, there exists a constant C such that*

$$\int_\Omega \frac{\mathrm{d}m(y)}{D(x,y)^{n+\gamma} r(y)^t} \le \frac{C}{r(x)^{\gamma+t}}.$$

The next lemma was also proven in [6]. We use it to obtain certain estimates for integral means $M_p(P_\gamma f, r)$ of the Bergman projections.

Lemma 2.4 [6] *For any $s > n-1$ there is a constant $C = C_{s,\Omega}$ such that for all $x \in \Omega$ and $0 < r \le \epsilon$*

$$\int_{\Gamma_r} \frac{\mathrm{d}\sigma_r(y)}{D(x,y)^s} \le \frac{C}{(r(x)+r)^{s-(n-1)}}.$$

Also, we use the next auxiliary result about the equivalence of norms on the space $B^{p,q}_\alpha$, obtained in [11].

Theorem 2.5 [11] *Let $1 \le p, q < \infty$ and $\alpha > 0$. Then, we have*

$$\|f\|^q_{B^{p,q}_\alpha} \asymp \sum_{j=1}^{\infty} \left[\int_{S_j} |f(x)|^p \mathrm{d}m(x) \right]^{\frac{q}{p}} 2^{j(\frac{q}{p}-\alpha q)}, \qquad f \in h(\Omega).$$

Notice that

$$\sum_{j=1}^{\infty} \left[\int_{S_j} |f(x)|^p \mathrm{d}m(x) \right]^{\frac{q}{p}} 2^{j(\frac{q}{p}-\alpha q)} \asymp \sum_{j=1}^{\infty} \left[\int_{S_j} |f(x)|^p \mathrm{d}m_\gamma(x) \right]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)},$$

where $\gamma = p(\alpha - \frac{1}{q})$.

3 Boundedness of weighted Bergman projections

Our first main result is Theorem 3.1 below about the boundedness of Bergman projections P_γ acting on weighted Lebesgue spaces $L^p_\lambda(\Omega)$.

Theorem 3.1 *Assume $\gamma > -1, 1 \le p < +\infty$ and $\lambda > -1$ satisfy inequality*

$$\mu_0 = \frac{\lambda - \gamma}{p} < \min\left(\frac{1+\lambda}{p}, \frac{1+\gamma}{p'} \right) = \mu_1. \tag{3.1}$$

Then P_γ is a bounded linear operator on the space $L^p_\lambda(\Omega)$.

Proof Since

$$(P_\gamma f)(x) = \int_\Omega R_\gamma(x,y) r(y)^{\gamma-\lambda} f(y) \mathrm{d}m_\lambda(y), \qquad x \in \Omega, \tag{3.2}$$

the weighted Bergman projection P_γ is an integral operator with kernel $K_\lambda(x,y) = R_\gamma(x,y) r(y)^{\gamma-\lambda}$, when considered as an operator acting on $L^p_\lambda(\Omega)$.

Case $1 < p < +\infty$.

In this case $\mu_1 > 0$ and we can fix $s > 0$ in the interval (μ_0, μ_1). We we are going to use Schur's test, with auxiliary function $h(x) = r(x)^{-s}$. Since the conditions $sp' - \gamma < 1$ and $sp' > 0$ are satisfied by the assumption (3.1) we can use Proposition 2.2 and Lemma 2.3 to obtain

$$\begin{aligned} \int_\Omega |K_\lambda(x,y)| h(y)^{p'} \mathrm{d}m_\lambda(y) &= \int_\Omega |R_\gamma(x,y)| r(y)^{\gamma-\lambda} r(y)^{-sp'+\lambda} \mathrm{d}m(y) \\ &\le C \int_\Omega \frac{\mathrm{d}m(y)}{D(x,y)^{n+\gamma} r(y)^{sp'-\gamma}} \le C r(x)^{-sp'} \\ &= C h(x)^{p'}. \end{aligned}$$

Also, by (3.1) conditions $sp - \lambda < 1$ and $\gamma + sp - \lambda > 0$ are satisfied so we can use Proposition 2.2 and Lemma 2.3 to obtain

$$\begin{aligned}\int_\Omega |K_\lambda(x,y)|h(x)^p \mathrm{d}m_\lambda(x) &= \int_\Omega |R_\gamma(x,y)| r(y)^{\gamma-\lambda} r(x)^{-sp+\lambda} \mathrm{d}m(x)\\ &\le Cr(y)^{\gamma-\lambda}\int_\Omega \frac{\mathrm{d}m(x)}{D(x,y)^{n+\gamma} r(x)^{sp-\lambda}}\\ &\le Cr(y)^{\gamma-\lambda} r(y)^{\lambda-sp+\gamma}\\ &= Ch(y)^p.\end{aligned}$$

The above two estimates show that conditions of the Schur's test are satisfied. Therefore P_γ is bounded on $L^p_\lambda(\Omega)$.

Case $p = 1$.

Now $\mu_1 = 0$ and $\gamma > \lambda$. Again using Proposition 2.2 and Lemma 2.3, we obtain

$$\begin{aligned}\|P_\gamma f\|_{L^1_\lambda} &= \int_\Omega \left|\int_\Omega K_\lambda(x,y) f(y)\mathrm{d}m_\lambda(y)\right| \mathrm{d}m_\lambda(x)\\ &\le \int_\Omega \left(\int_\Omega |R_\gamma(x,y)| r(y)^{\gamma-\lambda} |f(y)| \mathrm{d}m_\lambda(y)\right)\mathrm{d}m_\lambda(x)\\ &\le C\int_\Omega r(y)^{\gamma-\lambda}|f(y)|\left(\int_\Omega \frac{\mathrm{d}m(x)}{D(x,y)^{n+\gamma} r(x)^{-\lambda}}\right)\mathrm{d}m_\lambda(y)\\ &\le C\int_\Omega r(y)^{\gamma-\lambda}|f(y)| r(y)^{\lambda-\gamma}\mathrm{d}m_\lambda(y)\\ &= C\|f\|_{L^1_\lambda}.\end{aligned}$$

□

Two special cases, $p = 1$ and $\lambda = \gamma$, of the above theorem are worth mentioning separately. This is the content of the following corollary.

Corollary *Let $\gamma > -1$. Then P_γ is bounded linear operator from $L^p_\gamma(\Omega)$ to $b^p_\gamma(\Omega)$ for all $1 < p < +\infty$. Also, for $-1 < \lambda < \gamma$, P_γ is a bounded linear operator from $L^p_\lambda(\Omega)$ to $b^p_\lambda(\Omega)$ for all $1 \le p < +\infty$.*

Theorem 3.2 *Let $1 \le p, q < \infty$, $\alpha > 0$ and set $\gamma = p(\alpha - \frac{1}{q})$. Assume $\gamma > \alpha - 1$. If $p \le q$, then P_γ is a bounded projection from $L^{p,q}_\alpha(\Omega)$ onto $B^{p,q}_\alpha(\Omega)$. Also, if $p > q$ and $\gamma \ge 1$ i.e. $\alpha \ge \frac{1}{p} + \frac{1}{q}$, then P_γ is a bounded projection from $L^{p,q}_\alpha(\Omega)$ onto $B^{p,q}_\alpha(\Omega)$.*

Proof From the assumption $\gamma > \alpha - 1$, we see that $\lambda = \alpha p - 1$ satisfies condition (3.1) of the above theorem. Since μ_0 and μ_1 from (3.1) depend continuously on λ, it follows that there is an $\eta > 0$ such that $\delta = \lambda + \eta$ and $\zeta = \lambda - \eta$ also satisfy this condition. Thus, by the above theorem, we have

$$\|P_\gamma f\|_{L^p_\delta} \le C\|f\|_{L^p_\delta}, \quad f \in L^p_\delta(\Omega)$$

and

$$\|P_\gamma f\|_{L^p_\zeta} \le C\|f\|_{L^p_\zeta}, \quad f \in L^p_\zeta(\Omega).$$

In particular, if $f \in L^p_\delta(\Omega)$ and $\mathrm{supp}(f) \subset S_j$, then

$$\int_{S_k} |P_\gamma f(x)|^p r(x)^{\alpha p-1} 2^{-\eta k} \mathrm{d}m(x) \le C \int_{S_j} |f(x)|^p r(x)^{\alpha p-1} 2^{-\eta j} \mathrm{d}m(x)$$

for all $k \in \mathbb{Z}$. If $f \in L^p_\zeta(\Omega)$ and $\mathrm{supp}(f) \subset S_j$, then for all $k \in \mathbb{Z}$ we have

$$\int_{S_k} |P_\gamma f(x)|^p r(x)^{\alpha p-1} 2^{\eta k} \mathrm{d}m(x) \le C \int_{S_j} |f(x)|^p r(x)^{\alpha p-1} 2^{\eta j} \mathrm{d}m(x).$$

Let $f \in L^{p,q}_\alpha(\Omega)$. Since Ω is disjoint union of shells S_j, $f = \sum_j f\chi_{S_j}$. First, we assume that the sum is finite i.e. $f\chi_{S_j} \ne 0$ only for finitely many $j \ge 1$. By Minkowski's inequality and the above two inequalities, we obtain

$$\begin{aligned} \|(P_\gamma f)\chi_{S_k}\|_{L^p_{\alpha p-1}} &= \left\| \left(\sum_j P_\gamma(f\chi_{S_j}) \right) \chi_{S_k} \right\|_{L^p_{\alpha p-1}} \\ &\le \sum_j \|(P_\gamma(f\chi_{S_j}))\chi_{S_k}\|_{L^p_{\alpha p-1}} \\ &\le C_1 \sum_{j\le k} 2^{\frac{-\eta(k-j)}{p}} \|f\chi_{S_j}\|_{L^p_{\alpha p-1}} + C_2 \sum_{j>k} 2^{\frac{\eta(k-j)}{p}} \|f\chi_{S_j}\|_{L^p_{\alpha p-1}}. \end{aligned} \tag{3.3}$$

For $j \in \mathbb{Z}$, let $x_j = 2^{-\frac{\eta|j|}{p}}$ and

$$y_j = \begin{cases} \|f\chi_{S_j}\|_{L^p_{\alpha p-1}}, & j \ge 1 \\ 0, & j < 1. \end{cases}$$

Now for two sequences $X = \{x_j\}_{j=-\infty}^{+\infty}$ and $Y = \{y_j\}_{j=-\infty}^{+\infty}$, we define convolution $X * Y$ as sequence $Z = \{z_j\}_{j=-\infty}^{+\infty}$, where

$$z_j = \sum_{k=-\infty}^{\infty} x(k-j)y(k).$$

This is the classical convolution of Y with the sequence $W = \{x_{-j}\}_{j=-\infty}^{+\infty}$ and therefore we can apply the standard norm estimates like Young's inequality. Note that

$$\|(P_\gamma f)\chi_{S_k}\|_{L^p_{\alpha p-1}} \asymp 2^{-k(\alpha-\frac{1}{p})} \|(P_\gamma f)\chi_{S_k}\|_{L^p}$$

and

$$\|f\chi_{S_j}\|_{L^p_{\alpha p-1}} \asymp 2^{-j(\alpha-\frac{1}{p})} \|f\chi_{S_j}\|_{L^p}.$$

Now, using the notion of convolution, inequality (3.3) means that

$$\|(P_\gamma f)\chi_{S_k}\|_{L^p_{\alpha p-1}} \le C(X * Y)(k), \qquad k \in \mathbb{Z}. \tag{3.4}$$

Young's inequality for convolutions, (3.4) and (2.3) give

$$\begin{aligned}\|P_\gamma f\|_{L^{p,q}_\alpha} &\le C\|a_k(P_\gamma f)\|_{l^q} = C\| \|(P_\gamma f)\chi_{S_k}\|_{L^p_{\alpha p-1}}\|_{l^q}\\ &\le C\|X\|_{l^1}\| \|f\chi_{S_j}\|_{L^p_{\alpha p-1}}\|_{l^q} \le C\|a_j(f)\|_{l^q} \le C\|f\|_{L^{p,q}_\alpha}.\end{aligned}$$

The vector subspace of compactly supported functions in $L^{p,q}_\alpha(\Omega)$ is dense in $L^{p,q}_\alpha(\Omega)$, and it easily follows that P_γ is bounded from $L^{p,q}_\alpha(\Omega)$ to $B^{p,q}_\alpha(\Omega)$.

Next we show that for a constant C we have $\|f\|_{b^t_\gamma} \le C\|f\|_{B^{p,q}_\alpha}$, where $t = \min(p, q)$. We are going to use Theorem 2.5 on the equivalence of norms. Let us consider the cases $p \le q$ and $p > q$ separately.

Case $p \le q$. In this case, $t = p$. Since $\frac{q}{p} - 1 \ge 0$ and $2^{j(\frac{q}{p}-1)} \ge 1$ for $j \in \mathbb{N}$, we have

$$\begin{aligned}\|f\chi_{S_j}\|^q_{L^t_\gamma} &= \left(\int_{S_j} |f(x)|^p \mathrm{d}m_\gamma(x)\right)^{\frac{q}{p}}\\ &\le \left(\int_{S_j} |f(x)|^p \mathrm{d}m_\gamma(x)\right)^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} = \|f\chi_{S_j}\|^q_{L^{p,q}_\alpha}.\end{aligned}$$

Hence, summation over j gives $\|f\|^q_{b^t_\gamma} \le \|f\|^q_{B^{p,q}_\alpha}$, for harmonic function f.

Case $p > q$. In this case $t = q$ and $\frac{q}{p} - 1 \le 0$. We use Hölder's inequality with conjugate exponents $\frac{p}{t}$ and $\frac{p}{p-t}$ and the fact that $r(x) \asymp 2^{-j}$ on S_j, to obtain

$$\begin{aligned}\|f\chi_{S_j}\|^t_{L^t_\gamma} &= \int_{S_j} |f(x)|^t \mathrm{d}m_\gamma(x)\\ &\le \left(\int_{S_j} (|f(x)|^t)^{\frac{p}{t}} \mathrm{d}m_\gamma(x)\right)^{\frac{t}{p}} \left(\int_{S_j} \mathrm{d}m_\gamma(x)\right)^{1-\frac{q}{p}}\\ &\le C\left(\int_{S_j} |f(x)|^p \mathrm{d}m_\gamma(x)\right)^{\frac{q}{p}} 2^{-j\gamma(1-\frac{q}{p})}\\ &\le C\left(\int_{S_j} |f(x)|^p \mathrm{d}m_\gamma(x)\right)^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} = \|f\chi_{S_j}\|^q_{L^{p,q}_\alpha},\end{aligned}$$

where the last inequality holds because of the assumption $\gamma \ge 1$ in this case. Summation over j gives $\|f\|^q_{b^t_\gamma} \le C\|f\|^q_{B^{p,q}_\alpha}$, for harmonic function f.

This means that $B^{p,q}_\alpha(\Omega) \subset b^t_\gamma(\Omega)$, so every function $f \in B^{p,q}_\alpha(\Omega)$ belongs to the Bergman space $b^t_\gamma(\Omega)$, with given assumptions on parameters p, q and α. Now we conclude that $P_\gamma f = f$, which means $P_\gamma(L^{p,q}_\alpha(\Omega)) = B^{p,q}_\alpha(\Omega)$. □

Corollary *Let $\gamma > -1$, $\alpha > 0$, $1 < q < +\infty$ and set $\gamma = \alpha - 1/q$. Then P_γ is a bounded projection from $L^{1,q}_\alpha(\Omega)$ onto $B^{1,q}_\alpha(\Omega)$.*

We note that the corollary is false for $q = 1$: then $\gamma = \alpha - 1$ and, by Lemma 2.1, $L^{1,1}_\alpha(\Omega) = L^1_\gamma(\Omega)$ and $B^{1,1}_\alpha(\Omega) = b^1_\gamma(\Omega)$. However, P_γ is not bounded on $L^1_\gamma(\Omega)$. Of course, for $p = 1$ and $q = 1$ the condition $\gamma > \alpha - 1$ is not satisfied since then $\gamma = \alpha - 1$.

On the other hand, if $q = 1$ condition $\gamma > \alpha - 1$ from Theorem 3.2 is equivalent to $\alpha > 1$ and $p > 1$.

Corollary *Under the assumptions of the previous theorem, $h(\Omega) \cap C^\infty(\overline{\Omega})$ is dense in $B^{p,q}_\alpha(\Omega)$.*

Proof Let us choose $f \in B^{p,q}_\alpha$. For $\epsilon > 0$, let χ_ϵ be the characteristic function of Ω_ϵ and set $F_\epsilon = P_\gamma(f\chi_\epsilon)$. Then $\lim_{\epsilon\to 0} f\chi_\epsilon = f$ in $L^{p,q}_\alpha(\Omega)$ and hence, by Theorem 3.2,

$$f = P_\gamma(f) = P_\gamma(\lim_{\epsilon\to 0} f\chi_\epsilon) = \lim_{\epsilon\to 0} P_\gamma(f\chi_\epsilon) = \lim_{\epsilon\to 0} F_\epsilon.$$

The last two limits are taken, by Theorem 3.2, in $B^{p,q}_\alpha(\Omega)$ (quasi)-norm.

Since harmonic Bergman kernel is a smooth function on $\overline{\Omega} \times \overline{\Omega} \setminus \{(\xi, \xi) : \xi \in \partial\Omega\}$ (see [3]), we have $P_\gamma(\varphi) \in C^\infty(\overline{\Omega})$ for compactly supported φ in $L^1(\Omega)$. Hence $F_\epsilon \in C^\infty(\overline{\Omega}) \cap h(\Omega)$ and the proof is completed. □

4 Duality

Now we consider duality for mixed norm spaces. Since the argument presented in [6] works in abstract situations, it seems a general treatment is natural here. We start with a σ- finite measure μ on a σ- algebra $\mathcal{M}$ on a set X and consider a partition $X = \cup_{j=1}^\infty X_j$ of X into measurable pairwise disjoint subsets X_j. For $j \geq 1$ we can restrict μ to X_j and obtain a measure space $(X_j, \mathcal{M}_j, \mu_j)$, where $\mathcal{M}_j = \{E \in \mathcal{M} : E \subset X_j\}$, $\mu_j(E) = \mu(E)$ for $E \in \mathcal{M}_j$. Hence we have spaces $L^p(X_j, \mu_j)$, $1 \leq p \leq \infty$.

Next we fix a sequence $\omega = (\omega_j)_{j=1}^\infty$ of strictly positive integers and define a sequence space l^q_ω, $0 < q \leq \infty$ with

$$\|(\zeta_j)_{j=1}^\infty\|_{l^q_\omega} = \Big(\sum_{j=1}^\infty |\zeta_j|^q \omega_j\Big)^{\frac{1}{q}}, \; 0 < q < \infty$$

$$\|(\zeta_j)_{j=1}^\infty\|_{l^\infty_\omega} = \sup_{j\geq 1} |\zeta_j|$$

Note that if we set $\zeta = (\zeta_j)_{j=1}^\infty$, we can write $\|\zeta\|_{l^q_\omega} = \|\omega^{\frac{1}{q}}\zeta\|_{l^q}$. Also, we define a sequence space $l^{\infty,q}_\omega$, $0 < q \leq 1$ with

$$\|(\zeta_j)_{j=1}^{\infty}\|_{l_{\omega}^{\infty,q}} = \sup_{j\geq 1} |\zeta_j|\omega_j^{1-\frac{1}{q}}.$$

Note that $l_{\omega}^{\infty,1} = l^{\infty}$. For $0 < q \leq 1$, we use $l_{\omega,q}^{\infty}$ to denote a sequence space with norm

$$\|(\zeta_j)_{j=1}^{\infty}\|_{l_{\omega,q}^{\infty}} = \sup_{j\geq 1} |\zeta_j|\omega_j^{-\frac{1}{q}}.$$

These sequence spaces are just Lebesgue spaces formed with respect to weighted counting measure $\nu_{\omega} : \nu_{\omega}(E) = \sum_{j\in E} \omega_j$ on the set $\mathbb{N}$. With the above data, we form a mixed norm space $L_{\omega}^{p,q}(X,\mu)$ consisting of all measurable functions $f : X \to \mathbb{C}$ such that $f|_{X_j} \in L^p(X_j, \mu_j)$ for all $j \geq 1$ and such that $(\|f|_{X_j}\|_{L^p(X_j,\mu_j)})_{j=1}^{\infty}$ belongs to l_{ω}^{q} (i.e. $(\omega_j^{\frac{1}{q}} \|f|_{X_j}\|_{L^p(X_j,\mu_j)})_{j=1}^{\infty}$ belongs to l^q). We use $\|f\|_{p,j}$ to denote $\|f|_{X_j}\|_{L^p(X_j,\mu_j)}$. Explicitly, for finite p and q, the condition that $f \in L_{\omega}^{p,q}(X,\mu)$ means

$$\|f\|_{p,q}^{\omega,\mu} = \left\{ \sum_{j=1}^{\infty} \left(\int_{X_j} |f|^p \mathrm{d}\mu \right)^{\frac{q}{p}} \omega_j \right\}^{\frac{1}{q}} < +\infty.$$

The spaces $L_{\omega}^{p,q}(X,\mu)$ are Banach spaces for $1 \leq p, q \leq \infty$. The vector subspace $L_{\omega,0}^{p,q}(X,\mu)$ consisting of all $f \in L_{\omega}^{p,q}(X,\mu)$ such that $f = 0$ on X_j for all but finitely many integers j is dense in $L_{\omega}^{p,q}(X,\mu)$ if $q < +\infty$. For $0 < q < 1$, we define a space $L_{q,\omega}^{p,\infty}(X,\mu)$ consisting of all measurable functions $f : X \to \mathbb{C}$ such that $\sup_{j\geq 1} \omega_j^{\frac{-1}{q}} \|f\|_{p,j} < +\infty$. In the following theorem we work with the above abstract framework.

Theorem 4.1 *Assume* $1 \leq p < \infty$, $1 \leq q < \infty$ *and set* $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. *Set* $\omega_j' = \omega_j^{-\frac{q'}{q}}$. *For every* ϕ *in* $L_{\omega'}^{p',q'}(X,\mu)$ *formula*

$$\Lambda_{\phi}(f) = \int_X f\phi \mathrm{d}\mu = \sum_{j=1}^{\infty} \int_{X_j} f\phi \mathrm{d}\mu_j$$

defines a continuous linear functional Λ_{ϕ} *on* $L_{\omega}^{p,q}(X,\mu)$. *Conversely, for each* $\Lambda \in (L_{\omega}^{p,q}(X,\mu))^*$ *there exist a unique* $\phi \in L_{\omega'}^{p',q'}(X,\mu)$ *such that* $\Lambda = \Lambda_{\phi}$. *Moreover,* $\|\phi\| = \|\Lambda\|$.

Proof Fix ϕ in $L_{\omega'}^{p',q'}(X,\mu)$. Then, for f in $L_{\omega}^{p,q}(X,\mu)$, using Hölder's inequality for integrals, we have

$$\left|\int_{X_j} f\phi \mathrm{d}\mu_j\right| \leq \|f\|_{p,j}\|\phi\|_{p',j}$$

and therefore

$$\begin{aligned}|\Lambda_\phi f| &\leq \sum_{j=1}^{\infty}\|f\|_{p,j}\|\phi\|_{p',j} = \sum_{j=1}^{\infty}\|f\|_{p,j}\omega_j^{\frac{1}{q}}\|\phi\|_{p',j}\omega_j^{-\frac{1}{q}}\\ &\leq \left(\sum_{j=1}^{\infty}\|f\|_{p,j}^q\omega_j\right)^{\frac{1}{q}}\left(\sum_{j=1}^{\infty}\|\phi\|_{p',j}^{q'}\omega_j^{-\frac{q'}{q}}\right)^{\frac{1}{q'}} = \|\phi\|_{p',q'}^{\omega',\mu}\|f\|_{p,q}^{\omega,\mu}.\end{aligned}$$

Hence, $\Lambda_\phi \in (L^{p,q}_\omega(X,\mu))^*$, moreover $\|\Lambda_\phi\| \leq \|\phi\|^{\omega',\mu}_{p',q'}$.

Conversely, let $\Lambda \in (L^{p,q}_\omega(X,\mu))^*$. For any $j \geq 1$ we have a continuous linear functional Λ_j on $L^p(X_j, \mathrm{d}\mu_j)$ given by $L^p(X_j, \mathrm{d}\mu_j) \hookrightarrow L^{p,q}_\omega(X,\mu) \xrightarrow{\Lambda} \mathbb{C}$ and therefore we have $\phi_j \in L^{p'}(X_j, \mathrm{d}\mu_j)$ such that $\Lambda_j g = \int_{X_j} g\phi_j \mathrm{d}\mu_j$ for $g \in L^p(X_j, \mathrm{d}\mu_j)$. Hence

$$\Lambda f_j = \Lambda_j f_j = \int_{X_j} f_j\phi_j \mathrm{d}\mu_j = \int_{X_j} f_j\phi_j \mathrm{d}\mu$$

if $f_j = 0$ outside X_j and $f_j \in L^p(X_j, \mathrm{d}\mu_j)$. Moreover, $\|\Lambda_j\| = \|\phi_j\|_{p',j}$ for $j \geq 1$. In fact, the norm of Λ_j is attained: there exist a function g_j in $L^p(X_j, \mathrm{d}\mu_j)$ such that $\|g_j\|_{p,j} = 1$ and $\Lambda_j g_j = \|\Lambda_j\|$. We define $\phi : X \to \mathbb{C}$ by setting $\phi(x) = \phi_j(x)$ for $x \in X_j$. Next we prove that ϕ belongs to the space $L^{p',q'}_{\omega'}(X,\mu)$. Let us fix a positive integer N and set

$$f = \sum_{j=1}^{N}\lambda_j g_j, \quad \lambda_j \geq 0.$$

Then $\Lambda f = \sum_{j=1}^N \lambda_j\|\phi_j\|_{p',j}$ and

$$\|f\|^{\omega,\mu}_{p,q} = \left(\sum_{j=1}^{N}\lambda_j^q\omega_j\right)^{1/q}.$$

Now we introduce $\xi_j = \lambda_j\omega_j^{1/q}$, $1 \leq j \leq N$. Since $\Lambda f \leq \|\Lambda\|\|f\|^{\omega,\mu}_{p,q}$, we obtain

$$\sum_{j=1}^{N}\xi_j\left\|\frac{\phi_j}{\omega_j^{1/q}}\right\|_{p',j} \leq \|\Lambda\|\left(\sum_{j=1}^{N}\xi_j^q\right)^{1/q}$$

for all scalars $\xi_1, \ldots, \xi_N \geq 0$. By duality, the vector $(\omega_j^{-q'1/q}\|\phi_j\|_{p',j})_{j=1}^N$ in $\mathbb{R}^N$ has q'-norm bounded by $\|\Lambda\|$, and that implies

$$\sum_{j=1}^{N} \|\phi_j\|_{p',j}^{q'} \omega_j^{-q'/q} \leq \|\Lambda\|^{q'}.$$

Letting $N \to \infty$, we obtain $\|\phi\|_{p',q'}^{\omega',\mu} \leq \|\Lambda\|$. Since $\phi \in L_{\omega'}^{p',q'}(X,\mu)$, functional Λ_ϕ is continuous. By construction of ϕ, continuous linear functionals Λ and Λ_ϕ coincide on a dense subspace $L_{\omega,0}^{p,q}(X,\mu)$ of $L_\omega^{p,q}(X,\mu)$, therefore $\Lambda = \Lambda_\phi$. □

Now we consider duality for $0 < q < 1$. Let $\Lambda : l_\omega^q \to \mathbb{C}$ be a continuous linear functional on l_ω^1. We consider multiplication operators $M_{\omega^{\frac{1}{q}}} : l_\omega^q \to l^q$ and $M_{\omega^{-\frac{1}{q}}} : l_q \to l_\omega^q$ which are in fact isometries. Then the composition $\tilde{\Lambda} = \Lambda \circ M_{\omega^{-\frac{1}{q}}} : l^q \to \mathbb{C}$ is continuous. We have $\Lambda = \tilde{\Lambda} \circ M_{\omega^{\frac{1}{q}}}$ and $\Lambda\zeta = \sum_{j=1}^\infty \zeta_j \omega_j^{\frac{1}{q}} \theta_j$ for some $\theta \in l^\infty$. Here we used a well known fact that the dual of l^q is l^∞ for all $0 < q \leq 1$. We can represent Λ as $\Lambda\zeta = \sum_{j=1}^\infty \zeta_j \phi_j \omega_j$, where $\phi = \omega^{\frac{1}{q}-1}\theta$. Sequence ϕ is obviously in space $l_\omega^{\infty,q}$ since $\|\phi\|_{l_\omega^{\infty,q}} = \sup_{j\geq 1} \omega_j^{1-\frac{1}{q}} |\phi_j| < \infty$. Hence, $(l_\omega^q)^* \cong l_\omega^{\infty,q}$ and the pairing is given by

$$\langle \zeta, \phi \rangle = \sum_{j=1}^{\infty} \zeta_j \phi_j \omega_j.$$

Equivalently, $(l_\omega^q)^* \cong l_{\omega,q}^\infty$ and the pairing is given by

$$\langle \zeta, \psi \rangle = \sum_{j=1}^{\infty} \zeta_j \psi_j.$$

Now we can formulate duality theorem for $1 \leq p < \infty$ and $0 < q < 1$.

Theorem 4.2 *Assume $1 \leq p < \infty$, $0 < q < 1$ and set $\frac{1}{p} + \frac{1}{p'} = 1$. For every $\phi \in L_{q,\omega}^{p',\infty}(X,\mu)$ formula*

$$\Lambda_\phi(f) = \int_X f\phi \mathrm{d}\mu = \sum_{j=1}^{\infty} \int_{X_j} f\phi \mathrm{d}\mu_j$$

defines a continuous linear functional Λ_ϕ on $L_\omega^{p,q}(X,\mu)$. Conversely, for each $\Lambda \in (L_\omega^{p,q}(X,\mu))^*$ there exist a unique $\phi \in L_{q,\omega}^{p',\infty}(X,\mu)$ such that $\Lambda = \Lambda_\phi$. Moreover, $\|\phi\| = \|\Lambda\|$.

Combining Theorems 4.1 and 4.2, we obtain the dual space of weighted mixed norm space for $0 < q < \infty$.

Theorem 4.3 *Assume $1 \leq p < \infty$, $0 < q < \infty$ and set $\frac{1}{p} + \frac{1}{p'} = 1$. Set $\omega_j' = w_j^{-\frac{q'}{q}}$, $j \in \mathbb{N}$. Then, we have*

$$(L^{p,q}_{\omega}(X,\mu))^* = \begin{cases} L^{p',q'}_{\omega'}(X,\mu), \ 1 \le q < \infty, \ q' = \frac{q}{q-1} \\ L^{p',\infty}_{q,\omega}(X,\mu), \ 0 < q < 1. \end{cases}$$

Now we obtain some consequences of the above theorems. For $X = \Omega$, $X_j = S_j$, $\mu = m$ and $\omega_j = 2^{jq(\frac{1}{p}-\alpha)}$, the space $L^{p,q}_{\omega}(X,\mu)$ becomes the weighted mixed norm space $L^{p,q}_{\alpha}(\Omega)$, so we obtain the next consequence of Theorem 4.1.

Proposition 4.4 *For* $1 \le p, q < \infty$ *and* $0 < \alpha < 1$, *we have* $(L^{p,q}_{\alpha}(\Omega))^* \cong L^{p',q'}_{1-\alpha}(\Omega)$ *and the pairing is given by*

$$\langle f, \phi \rangle = \int_{\Omega} f \phi \mathrm{d}m.$$

Proof Since $\omega_j = 2^{jq(\frac{1}{p}-\alpha)}$, we have

$$\omega'_j = \omega_j^{-\frac{q'}{q}} = 2^{-jq'(\frac{1}{p}-\alpha)} = 2^{jq'(\alpha-(1-\frac{1}{p'}))} = 2^{jq'(\frac{1}{p'}-(1-\alpha))}.$$

Hence, the space $L^{p',q'}_{\omega'}(X)$ becomes the space $L^{p',q'}_{1-\alpha}(\Omega)$ in this setting. □

Finally, in the following theorem, we obtain the dual space of $B^{p,q}_{\alpha}(\Omega)$, for a certain range of parameters p, q and α.

Theorem 4.5 *Let* $1 \le p < \infty, 1 < q < \infty, 0 < \alpha < 1$ *and* $(p-1)\alpha > \frac{p-q}{q}$. *If* $p \le q$, *we have* $B^{p,q}_{\alpha}(\Omega)^* \cong B^{p',q'}_{1-\alpha}(\Omega)$ *and duality is given by*

$$\langle f, \phi \rangle = \int_{\Omega} f \phi \mathrm{d}m.$$

Also, if $p > q$ *and* $\alpha \ge \frac{1}{p} + \frac{1}{q}$, *we have the same duality result.*

Proof Theorem 3.2 asserts that assuming a certain conditions on parameters p, q and α, the operator P_{γ} is a surjective continuous map from $L^{p,q}_{\alpha}(\Omega)$ to $B^{p,q}_{\alpha}(\Omega)$. Hence, its adjoint P^*_{γ} is an embedding of $(B^{p,q}_{\alpha}(\Omega))^*$ into $(L^{p,q}_{\alpha}(\Omega))^* \cong L^{p',q'}_{1-\alpha}(\Omega)$. One easily checks that the range of P^*_{γ} consists of harmonic functions and the result follows from $L^{p',q'}_{1-\alpha}(\Omega) \cap h(\Omega) = B^{p',q'}_{1-\alpha}(\Omega)$. □

For $0 < q < 1$, we obtain the next duality result for mixed norm Lebesgue spaces $L^{p,q}_{\alpha}(\Omega)$.

Proposition 4.6 *Let* $1 \le p < \infty$, $0 < q < 1$ *and* $0 < \alpha < 1$. *Then* $(L^{p,q}_{\alpha}(\Omega))^* \cong L^{p',\infty}_{1-\alpha}(\Omega)$ *and the pairing is given by*

$$\langle f, \phi \rangle = \int_{\Omega} f \phi \mathrm{d}m.$$

Proof In the definition of the space $L^{p',\infty}_{q,\omega}(X, \mu)$, set $X = \Omega$, $X_j = S_j$, $\mu = m$ and $\omega_j = 2^{jq(\frac{1}{p'} - \alpha)}$. We obtain $\omega_j^{-\frac{1}{q}} = 2^{j(\alpha - \frac{1}{p'})} = 2^{j(\alpha - (1 - \frac{1}{p}))} = 2^{j(\frac{1}{p} - (1-\alpha))}$. Hence, in this setting we get the space $L^{p',\infty}_{1-\alpha}(\Omega)$ and the result is direct consequence of Theorem 4.2. □

Combining Propositions 4.4 and 4.6, we obtain the next result.

Proposition 4.7 *Assume* $1 \le p < \infty$, $0 < q < \infty$, $0 < \alpha < 1$ *and set* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then, we have*

$$(L^{p,q}_{\alpha}(\Omega))^* = \begin{cases} L^{p',q'}_{1-\alpha}(\Omega), \ 1 \le q < \infty, \ q' = \frac{q}{q-1} \\ L^{p',\infty}_{1-\alpha}(\Omega), \ 0 < q < 1. \end{cases}$$

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