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Hyperinvariant subspaces for sets of polynomially compact operators

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Abstract
We prove the existence of a non-trivial hyperinvariant subspace for several sets of polynomially compact operators. The main results of the paper are: (i) a non-trivial norm closed algebra $\mathcal{A} \subseteq \mathcal{B}(\mathscr{X})$ which consists of polynomially compact quasinilpotent operators has a non-trivial hyperinvariant subspace; (ii) if there exists a non-zero compact operator in the norm closure of the algebra generated by an operator band $\mathcal{S}$, then $\mathcal{S}$ has a non-trivial hyperinvariant subspace.

Keywords Polynomially compact operator · Hyperinvariant subspace

Mathematics Subject Classification 47A15 · 47B07 · 47L10 · 47B10

1 Introduction

Let $\mathscr{X}$ be a complex Banach space. Denote by $\mathcal{B}(\mathscr{X})$ the algebra of all bounded linear operators on $\mathscr{X}$. A closed subspace $\mathscr{M} \subseteq \mathscr{X}$ is said to be invariant for an operator $T \in \mathcal{B}(\mathscr{X})$ if $T\mathscr{M} \subseteq \mathscr{M}$. Let $\mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$ be a non-empty set of operators. Then, $\mathscr{M}$ is an invariant subspace of $\mathcal{S}$ if it is invariant for every operator in $\mathcal{S}$. If $\mathscr{M}$ is invariant for every operator in $\mathcal{S}$ and for every operator in the commutant

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$\mathcal{S}' = \{T \in \mathcal{B}(\mathscr{X}); TS = ST \text{ for every } S \in \mathcal{S}\}$, then it is a hyperinvariant subspace of $\mathcal{S}$. Of course, the trivial subspaces $\{0\}$ and $\mathscr{X}$ are (hyper)invariant for any set of operators. We are interested in the existence of non-trivial invariant and hyperinvariant subspaces. The problem of existence of invariant and hyperinvariant subspaces for a given operator or a non-empty set of operators is an extensively studied topic in operator theory. The problem is solved in the finite-dimensional setting by Burnside's theorem (see [14, Theorem 1.2.2]). In the context of infinite-dimensional Banach spaces, the problem is open for reflexive Banach spaces, in particular, for the infinite-dimensional separable Hilbert space. However, there are some Banach spaces for which we know either that every operator has a non-trivial invariant subspace or that there exist operators without it. For instance, Argyros and Haydon [1] have proved the existence of an infinite-dimensional Banach spaces $\mathscr{X}$ such that every operator in $\mathcal{B}(\mathscr{X})$ is of the form $\lambda I + K$, where I is the identity operator and K is compact. It follows, by the celebrated von Neumann-Aronszajn-Smith theorem [2] and Lomonosov's theorem [12], that any operator in $\mathcal{B}(\mathscr{X})$ has a non-trivial invariant subspace and any non-scalar operator in $\mathcal{B}(\mathscr{X})$ has a non-trivial hyperinvariant subspace. On the other hand, several examples of Banach spaces (including ℓ_1) with operators without a non-trivial invariant subspace are known (see [3, 15] for the first examples and [6] for a general approach to Read's type constructions of operators without non-trivial invariant closed subspaces).

With the von Neumann–Aronszajn–Smith theorem and Lomonosov's theorem in mind, it is not a surprise that suitable compactness conditions imply existence of non-trivial invariant and hyperinvariant subspaces for different classes of operators and sets of operators. For instance, Shulman [16, Theorem 2] proved that an algebra of operators whose radical contains a non-zero compact operator has a non-trivial hyperinvariant subspace. Turovskii [17, Corollary 5] extended this result to semigroups of quasinilpotent operators. Another type of results are those related to triangularizability of a set of operators. Recall that a non-empty set $\mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$ is triangularizable if there exists a chain $\mathscr{C}$ which is maximal as a chain of subspaces of $\mathscr{X}$ and every subspace in $\mathscr{C}$ is invariant for all operators in $\mathcal{S}$. Every commutative set of compact operators is triangularizable (see [14, Theorem 7.2.1]). Konvalinka [10, Corollary 2.6] has extended this result by showing that a commuting family of polynomially compact operators is triangularizable. Another result in this direction, obtained by the second author [9], says that for a norm closed subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathscr{X})$ of power compact operators the following assertions are equivalent: (a) $\mathcal{A}$ is triangularizable; (b) the Jacobson radical $\mathcal{R}(\mathcal{A})$ consists precisely of quasinilpotent operators in $\mathcal{A}$; (c) the quotient algebra $\mathcal{A}/\mathcal{R}(\mathcal{A})$ is commutative.

The aim of this paper is to consider the problem of existence of a non-trivial hyperinvariant subspace for sets of polynomially compact operators. For instance, we prove (Theorem 3.3) that a non-trivial norm closed algebra $\mathcal{A} \subseteq \mathcal{B}(\mathscr{X})$ which consists of polynomially compact quasinilpotent operators has a non-trivial hyperinvariant subspace. Another result (Theorem 4.3) which we mention here is related to operator bands, that is, to semigroups of idempotent operators. It says that an operator band $\mathcal{S}$ has a non-trivial hyperinvariant subspace if there exists a non-zero compact operator in the norm closure of the algebra generated by $\mathcal{S}$.

2 Preliminaries

2.1 Notation

Let $\mathscr{X}$ be a non-trivial complex Banach space. Since the results proved in this paper are either trivial or well known when $\mathscr{X}$ is finite-dimensional, we always assume that $\dim(\mathscr{X}) = \infty$. Let $\mathcal{B}(\mathscr{X})$ denote the Banach algebra of all bounded linear operators on $\mathscr{X}$ and let $\mathcal{K}(\mathscr{X}) \subseteq \mathcal{B}(\mathscr{X})$ be the ideal of compact operators. The identity operator is denoted by I and an operator is said to be scalar if it is a scalar multiple of I. The norm closure of a set $\mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$ is denoted by $\overline{\mathcal{S}}$.

For two operators $S_1, S_2 \in \mathcal{B}(\mathscr{X})$, we denote their commutator $S_1S_2 - S_2S_1$ by $[S_1, S_2]$. A non-empty set $\mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$ is commutative if any two operators from $\mathcal{S}$ commute, that is, $[S_1, S_2] = 0$ for all $S_1, S_2 \in \mathcal{S}$. Similarly, $\mathcal{S}$ is essentially commutative if $[S_1, S_2] \in \mathcal{K}(\mathscr{X})$. For an arbitrary non-empty subset $\mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$ the commutant of $\mathcal{S}$ is $\mathcal{S}' = \{T \in \mathcal{B}(\mathscr{X}); [T, S] = 0, \forall S \in \mathcal{S}\}$. It is clear that $\mathcal{S}'$ is a closed subalgebra of $\mathcal{B}(\mathscr{X})$. If $\mathcal{S}$ is commutative, then $\mathcal{S} \subseteq \mathcal{S}'$.

2.2 Invariant subspaces

A non-empty subset $\mathscr{M} \subseteq \mathscr{X}$ is a subspace if it is a closed linear manifold. It is said that a subspace $\mathscr{M}$ of $\mathscr{X}$ is invariant for the operator T if $T\mathscr{M} \subseteq \mathscr{M}$. An invariant subspace $\mathscr{M}$ is non-trivial if $\{0\} \neq \mathcal{M} \neq \mathscr{X}$. A non-empty set $\mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$ is reducible if there exists a non-trivial subspace $\mathscr{M} \subseteq \mathscr{X}$ which is invariant for every $T \in \mathcal{S}$. If there exists a chain $\mathscr{C}$ which is maximal as a chain of subspaces of $\mathscr{X}$ and every subspace in $\mathscr{C}$ is invariant for all operators in $\mathcal{S}$, then $\mathcal{S}$ is said to be triangularizable.

If a subspace $\mathscr{M}$ is invariant for every operator T in a set $\mathcal{S}$ and in its commutant $\mathcal{S}'$, then $\mathscr{M}$ is said to be a hyperinvariant subspace for $\mathcal{S}$.

Next theorem (see Assertion in the end of [12]) is one of the deepest results in the theory of invariant subspaces.

Theorem 2.1 (Lomonosov) *Every non-scalar operator which commutes with a non-zero compact operator has a non-trivial hyperinvariant subspace.*

The proof of Theorem 2.1 relies on the following useful lemma.

Lemma 2.2 (Lomonosov's Lemma) *Let $\mathcal{A} \subseteq \mathcal{B}(\mathscr{X})$ be an algebra. If $\mathcal{A}$ is not reducible, then for every non-zero compact operator $K \in \mathcal{B}(\mathscr{X})$ there exist an operator $A \in \mathcal{A}$ and a non-zero vector $x \in \mathscr{X}$ such that $AKx = x$.*

2.3 Spectral radius

The spectrum of an operator $T \in \mathcal{B}(\mathscr{X})$ is denoted by $\sigma(T)$ and the spectral radius of T is $\rho(T) = \max\{|z|;\, z \in \sigma(T)\}$. By the spectral radius formula (Gelfand's formula), $\rho(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$. An operator $T \in \mathcal{B}(\mathscr{X})$ is quasinilpotent if $\rho(T) = 0$. A compact quasinilpotent operator is called a Volterra operator.

Let $\mathcal{F}$ be a non-empty set of operators in $\mathcal{B}(\mathscr{X})$. For each $n \in \mathbb{N}$, let $\mathcal{F}^{(n)} = \{T_1 \cdots T_n;\, T_1, \ldots, T_n \in \mathcal{F}\}$. By $\|\mathcal{F}\| = \sup\{\|T\|;\, T \in \mathcal{F}\}$ we denote the joint norm of $\mathcal{F}$ and by $\rho(\mathcal{F})$, we denote the joint spectral radius of $\mathcal{F}$ defined as

$$\rho(\mathcal{F}) = \limsup_{n\to\infty} \|\mathcal{F}^{(n)}\|^{\frac{1}{n}}.$$

A subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathscr{X})$ is said to be finitely quasinilpotent if $\rho(\mathcal{F}) = 0$ for every finite subset $\mathcal{F}$ of $\mathcal{A}$. By [16, Theorem 1], every subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathscr{X})$ of Volterra operators is finitely quasinilpotent.

2.4 Polynomially compact operators

An operator $T \in \mathcal{B}(\mathscr{X})$ is polynomially compact if there exists a non-zero complex polynomial p such that $p(T)$ is a compact operator. In particular, algebraic operators (nilpotents, idempotents, etc.) are polynomially compact. Hence, T is polynomially compact if and only if $\pi(T)$, where $\pi : \mathcal{B}(\mathscr{X}) \to \mathcal{B}(\mathscr{X})/\mathcal{K}(\mathscr{X})$ is the quotient projection, is an algebraic element in the Calkin algebra $\mathcal{B}(\mathscr{X})/\mathcal{K}(\mathscr{X})$. If T^n is compact for some $n \in \mathbb{N}$, then T is said to be power compact. For a polynomially compact operator T, there exists a unique monic polynomial m_T of the smallest degree such that $m_T(T)$ is a compact operator. The polynomial m_T is called the minimal polynomial of T. The following is the structure theorem for polynomially compact operators proved by Gilfeather [4, Theorem 1].

Theorem 2.3 *Let $T \in \mathcal{B}(\mathscr{X})$ be a polynomially compact operator with minimal polynomial $m_T(z) = (z - \lambda_1)^{n_1} \cdots (z - \lambda_k)^{n_k}$. Then there exist invariant subspaces $\mathscr{X}_1, \ldots, \mathscr{X}_k$ for T such that $\mathscr{X} = \mathscr{X}_1 \oplus \cdots \oplus \mathscr{X}_k$ and $T = T_1 \oplus \cdots \oplus T_k$, where T_i is the restriction of T to $\mathscr{X}_i$. The operators $(T_j - \lambda_j I_j)^{n_j}$ are all compact.*

The spectrum of T consists of countably many points with $\{\lambda_1, \ldots, \lambda_k\}$ as the only possible limit points and such that all but possibly $\{\lambda_1, \ldots, \lambda_k\}$ are eigenvalues with finite-dimensional generalized eigenspaces. Each point λ_j $(j = 1, \ldots, k)$ is either the limit of eigenvalues of T or else $\mathscr{X}_j$ is infinite-dimensional and $T_j - \lambda_j I_j$ is a quasinilpotent operator on $\mathscr{X}_j$.

Corollary 2.4 *A quasinilpotent operator $T \in \mathcal{B}(\mathscr{X})$ is polynomially compact if and only if it is power compact.*

Proof It is clear that every power compact operator is polynomially compact. On the other hand, if T is a polynomially compact and quasinilpotent, then $\sigma(T) = \{0\}$ and therefore $m_T(z) = z^n$ for a positive integer n, by Theorem 2.3, that is, T is power compact. □

2.5 Algebras and ideals

For a non-empty set of operators $\mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$, let $\mathcal{A}(\mathcal{S})$ be the subalgebra of $\mathcal{B}(\mathscr{X})$ generated by $\mathcal{S}$ and let $\mathcal{A}_1(\mathcal{S})$ be the subalgebra of $\mathcal{B}(\mathscr{X})$ generated by $\mathcal{S}$ and I. By $\mathcal{H}(\mathcal{S})$ we denote the algebra which is generated by $\mathcal{S}$ and its commutant $\mathcal{S}'$. We will call it the *hyperalgebra* of $\mathcal{S}$. If $\mathcal{S}$ is a semigroup, then an operator $T \in \mathcal{B}(\mathscr{X})$ is in $\mathcal{H}(\mathcal{S})$ if and only if there exist $n \in \mathbb{N}$ and operators $S_1, \dots, S_n \in \mathcal{S}$ and $T_0, T_1, \dots, T_n \in \mathcal{S}'$ such that

$$T = T_0 + S_1T_1 + \cdots + S_nT_n = T_0 + T_1S_1 + \cdots + T_nS_n.$$

Here, we used the fact that $I \in \mathcal{S}'$. Since $\mathcal{S} \subseteq \mathcal{H}(\mathcal{S})$ and $\mathcal{A}(\mathcal{S})$ is the smallest algebra which contains $\mathcal{S}$, we have $\mathcal{A}(\mathcal{S}) \subseteq \mathcal{H}(\mathcal{S})$. On the other hand, it is obvious that $\mathcal{S}' = \mathcal{A}(\mathcal{S})'$ and therefore $\mathcal{A}(\mathcal{S})' \subseteq \mathcal{H}(\mathcal{S})$. We conclude that the hyperalgebra of $\mathcal{S}$ is generated by $\mathcal{A}(\mathcal{S})$ and $\mathcal{A}(\mathcal{S})'$, that is, $\mathcal{H}(\mathcal{S}) = \mathcal{H}\big(\mathcal{A}(\mathcal{S})\big)$.

If $\mathcal{M}$ and $\mathcal{N}$ are non-empty subsets of $\mathcal{B}(\mathscr{X})$, then let $\mathcal{M} + \mathcal{N} = \{M + N;\, M \in \mathcal{M}, N \in \mathcal{N}\}$ and let $\mathcal{M}\mathcal{N}$ be the set of all finite sums $M_1N_1 + \cdots + M_kN_k$, where $M_i \in \mathcal{M}$ and $N_i \in \mathcal{N}$ for each $i = 1, \dots, k$. Hence, if $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathscr{X})$, then its hyperalgebra is $\mathcal{H}(\mathcal{A}) = \mathcal{A}' + \mathcal{A}\mathcal{A}' = \mathcal{A}' + \mathcal{A}'\mathcal{A}$.

If $\mathcal{J}$ is an ideal in an algebra $\mathcal{A} \subseteq \mathcal{B}(\mathscr{X})$, then we write $\mathcal{J} \lhd \mathcal{A}$. We will denote by $\mathcal{J}_{\mathcal{H}}$ the ideal in $\mathcal{H}(\mathcal{A})$ generated by $\mathcal{J}$.

Lemma 2.5 *Let $\mathcal{A} \subseteq \mathcal{B}(\mathscr{X})$ be an algebra. If $\mathcal{J} \lhd \mathcal{A}$, then $\mathcal{J}_{\mathcal{H}} = \mathcal{A}'\mathcal{J} = \mathcal{J}\mathcal{A}'$.*

Proof From equalities $\mathcal{J}_{\mathcal{H}} = \mathcal{J} + \mathcal{H}(\mathcal{A})\mathcal{J} + \mathcal{J}\mathcal{H}(\mathcal{A}) + \mathcal{H}(\mathcal{A})\mathcal{J}\mathcal{H}(\mathcal{A})$ and $\mathcal{H}(\mathcal{A}) = \mathcal{A}' + \mathcal{A}'\mathcal{A}$, we conclude

$$\begin{aligned}\mathcal{J}_{\mathcal{H}} &= \mathcal{J} + \mathcal{H}(\mathcal{A})\mathcal{J} + \mathcal{J}\mathcal{H}(\mathcal{A}) + \mathcal{H}(\mathcal{A})\mathcal{J}\mathcal{H}(\mathcal{A}) \\ &= \mathcal{J} + \Leftarrow\mathcal{A}' + \mathcal{A}'\mathcal{A})\mathcal{J} + \mathcal{J}\Leftarrow\mathcal{A}' + \mathcal{A}'\mathcal{A}) + \Leftarrow\mathcal{A}' + \mathcal{A}'\mathcal{A})\mathcal{J}\Leftarrow\mathcal{A}' + \mathcal{A}'\mathcal{A}) \\ &\subseteq \mathcal{J} + \mathcal{A}'\mathcal{J} \subseteq \mathcal{A}'\mathcal{J}\end{aligned}$$

as $\mathcal{A}'$ contains the identity operator. On the other hand, since we also have $\mathcal{A}'\mathcal{J} \subseteq \mathcal{J}_{\mathcal{H}}$, we obtain the equality $\mathcal{J}_{\mathcal{H}} = \mathcal{A}'\mathcal{J}$. □

An algebra $\mathcal{A}$ over an arbitrary field is said to be a nil-algebra if every element of $\mathcal{A}$ is nilpotent. A nil-algebra $\mathcal{A}$ is of bounded nil-index if there exists a positive integer n such that $x^n = 0$ for each $x \in \mathcal{A}$. If there exists $n \in \mathbb{N}$ such that $a_1 \cdots a_n = 0$ for all $a_1, \dots, a_n \in \mathcal{A}$, then $\mathcal{A}$ is said to be a nilpotent algebra. By the celebrated Nagata-Higman theorem (see [8, 13]), every nil-algebra of bounded nil-index is nilpotent.

Lemma 2.6 *Let $\mathcal{A} \subseteq \mathcal{B}(\mathscr{X})$ be an algebra. An ideal $\mathcal{J} \lhd \mathcal{A}$ is nilpotent if and only if $\mathcal{J}_{\mathcal{H}} \lhd \mathcal{H}(\mathcal{A})$ is nilpotent. The nilpotency indices of $\mathcal{J}$ and $\mathcal{J}_{\mathcal{H}}$ are equal.*

Proof Since $\mathcal{A}'$ and $\mathcal{J}$ commute, an easy induction shows that for each $n \in \mathbb{N}$ we have $(\mathcal{A}'\mathcal{J})^n = \mathcal{A}'\mathcal{J}^n$. Hence, if $\mathcal{J}^n = \{0\}$, then $(\mathcal{A}'\mathcal{J})^n = \{0\}$, as well. If $(\mathcal{A}'\mathcal{J})^n = \{0\}$, then $\mathcal{J}^n \subseteq \mathcal{A}'\mathcal{J}^n = (\mathcal{A}'\mathcal{J})^n = \{0\}$ yields that $\mathcal{J}^n = \{0\}$. □

If $\mathcal{J}$ is a nil-ideal of bounded nil-index, then $\mathcal{J}$ is nilpotent by the Nagata-Higman theorem. This immediately implies that $\mathcal{J}_{\mathcal{H}}$ is nilpotent. In particular, if $\mathcal{A}$ is a nilpotent algebra, then $\mathcal{A}\mathcal{A}'$ is a nilpotent ideal in the hyperalgebra $\mathcal{H}(\mathcal{A})$.

3 Hyperinvariant subspaces of algebras of polynomially compact operators

The simplest polynomially compact operators which are not necessary compact are algebraic operators, in particular nilpotent operators. Hadwin et al. [7, Corollary 4.2] proved that a norm closed algebra of nilpotent operators on the separable infinite-dimensional complex Hilbert space is triangularizable. The following proposition shows that a norm closed algebra of nilpotent operators on an arbitrary complex Banach space has a non-trivial hyperinvariant subspace.

Proposition 3.1 *If a subalgebra $\{0\} \neq \mathcal{A} \subseteq \mathcal{B}(\mathscr{X})$ consists of nilpotent operators, then its hyperalgebra $\mathcal{H}(\mathcal{A})$ is reducible in either of the following cases.*

(a) *The algebra $\mathcal{A}$ is nilpotent.*
(b) *The algebra $\mathcal{A}$ is norm closed.*

Proof (a) There exists $n \in \mathbb{N}$ such that an arbitrary product of at least n operators from $\mathcal{A}$ is the zero operator. Let n_0 be the smallest positive integer with this property. Since $\mathcal{A} \neq \{0\}$ we have $n_0 > 1$. There exist operators $A_1, \ldots, A_{n_0-1} \in \mathcal{A}$ such that $A_0 := A_1, \ldots, A_{n_0-1} \neq 0$. Note that $A_0 T = TA_0 = 0$ for every operator $T \in \mathcal{A}$, that is, $\mathcal{A} \subseteq (A_0)'$, where $(A_0)'$ is the commutant of A_0. It is clear that $\mathcal{A}' \subseteq (A_0)'$. Hence, $\mathcal{H}(\mathcal{A}) \subseteq (A_0)'$. Since $A_0 \neq 0$ the kernel $\ker(A_0)$ is a non-trivial subspace of $\mathscr{X}$ and it is hyperinvariant for A_0. It follows that $\ker(A_0)$ is a non-trivial hyperinvariant subspace for $\mathcal{A}$, that is, $\mathcal{H}(\mathcal{A})$ is reducible.

(b) Since $\mathcal{A}$ is a closed subalgebra of $\mathcal{B}(\mathscr{X})$, it is a Banach algebra. Therefore, $\mathcal{A}$ is a Banach algebra which is also a nil-algebra, so that by Grabiner's theorem [5], $\mathcal{A}$ is a nilpotent algebra. Now, we apply (a). □

Hadwin et al. have constructed a semi-simple algebra of nilpotent operators on a separable Hilbert space such that for each pair (x, y) of vectors, where $x \neq 0$, there exists an operator $A \in \mathcal{A}$ such that $Ax = y$ (see [7, Section 4]). Hence, Proposition 3.1(b) (and consequently Theorem 3.3 below) does not hold, in general, for non-closed algebras. However, as the following example shows there are simple special cases when a not necessarily closed algebra generated by a set of nilpotents has a non-trivial hyperinvariant subspace.

Example The idea for this example is from [7, Theorem 4.3] where quadratic nilpotent operators are considered. Choose and fix $\lambda \in \mathbb{C}$. Let $\mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$ be a non-empty set of operators such that $A^2 = \lambda A$ for every $A \in \mathcal{S}$. If $\mathcal{S}$ is a linear manifold and contains a non-scalar operator, then $\mathcal{A}(\mathcal{S})$ has a non-trivial hyperinvariant subspace. To see this, choose a non-scalar operator $A \in \mathcal{S}$. Then its kernel $\ker(A)$ is a non-trivial hyperinvariant subspace for A. Since $\mathcal{A}(\mathcal{S})' \subseteq (A)'$, we see that $\ker(A)$ is invariant for every operator from $\mathcal{A}(\mathcal{S})'$. Let $B \in \mathcal{S}$ be arbitrary. It follows from $(A+B)^2 = \lambda(A+B)$ that $AB = -BA$. Hence, $ABx = -BAx = 0$ for every $x \in \ker(A)$, that is, $\ker(A)$ is invariant for every $B \in \mathcal{S}$. We conclude that $\ker(A)$ is invariant for every operator from the hyperalgebra $\mathcal{H}(\mathcal{S})$.

Proposition 3.2 *Let $\mathcal{A} \subseteq \mathcal{B}(\mathscr{X})$ be a non-trivial subalgebra. If there exists a non-trivial nilpotent ideal $\mathcal{J} \triangleleft \mathcal{A}$, then $\mathcal{A}$ has a non-trivial hyperinvariant subspace.*

Proof Let $\mathcal{J}$ be a non-trivial nilpotent ideal in $\mathcal{A}(\mathcal{S})$. Then, by Lemma 2.6, the ideal $\mathcal{J}_{\mathcal{H}}$ which is generated by $\mathcal{J}$ in the hyperalgebra $\mathcal{H}(\mathcal{A})$ is nilpotent, as well. By Proposition 3.1(a) the ideal $\mathcal{J}_{\mathcal{H}}$ has a non-trivial hyperinvariant subspace, in particular, it is reducible. By [14, Lemma 7.4.6], $\mathcal{H}(\mathcal{A})$ is reducible, as well. □

The following two theorems show that an algebra $\mathcal{A}$ of polynomially compact operators has a non-trivial hyperinvariant subspace if $\mathcal{A}$ satisfies some additional condition. For instance, as we already mentioned, the key assumption in Theorem 3.3 is that the involved algebra is norm closed.

Theorem 3.3 *Let $\{0\} \neq \mathcal{A} \subseteq \mathcal{B}(\mathscr{X})$ be a norm closed algebra. If every operator in $\mathcal{A}$ is quasinilpotent and polynomially compact, then $\mathcal{A}$ has a non-trivial hyperinvariant subspace.*

Proof Note first that each operator in $\mathcal{A}$ is power compact, by Corollary 2.4. If each operator in $\mathcal{A}$ is nilpotent, then $\mathcal{H}(\mathcal{A})$ is reducible, by Proposition 3.1(b). Assume therefore that there exists an operator $T \in \mathcal{A}$ which is not nilpotent. Since T is power compact there exists $m \in \mathbb{N}$ such that $K = T^m \neq 0$ is compact. Let $\mathcal{J}$ be the two-sided ideal in $\mathcal{A}_1$ generated by K. It is clear that $K \in \mathcal{J} \subseteq \mathcal{A}$. Hence, $\mathcal{J}$ is an algebra of Volterra operators. By [16, Theorem 1], $\mathcal{J}$ is finitely quasinilpotent. Let $A = B_0 + \sum_{i=1}^n A_i B_i$, where $A_i \in \mathcal{A}$ and $B_i \in \mathcal{A}'$, be an arbitrary operator in $\mathcal{H}(\mathcal{A})$ and let $\mathcal{M} = \{K, A_1 K, \ldots, A_n K\}$. Since $\mathcal{M}$ is a finite subset of $\mathcal{J}$ it is a quasinilpotent set, that is, $\rho(\mathcal{M}) = 0$. Let $\mathcal{N} = \{B_0, B_1, \ldots, B_n\} \subseteq \mathcal{A}'$. Since $B_0, B_1, \ldots, B_n$ commute with operators from $\mathcal{M}$ we have $\rho(AK) \leq (n+1)\rho(\mathcal{M})\rho(\mathcal{N}) = 0$, by [16, Lemma 1]. This shows that the operator AK is quasinilpotent for each $A \in \mathcal{H}(\mathcal{A})$. It follows, by Lemma 2.2, that the hyperalgebra $\mathcal{H}(\mathcal{A})$ is reducible.

Theorem 3.4 *Let $\mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$ be an essentially commutative set of polynomially compact operators which contains at least one non-scalar operator. If $\mathcal{S}$ is triangularizable, then $\mathcal{A}(\mathcal{S})$ has a non-trivial hyperinvariant subspace.*

Proof It is obvious that every subspace of $\mathscr{X}$ which is invariant for $\mathcal{S}$ is invariant for $\mathcal{A}(\mathcal{S})$, as well. Hence, $\mathcal{A}(\mathcal{S})$ is triangularizable. Since $\mathcal{S}$ consists of polynomially compact operators, the same holds for the algebra $\mathcal{A}(\mathcal{S})$, by [10, Theorem 1.5]. Denote by $\pi\big(\mathcal{A}(\mathcal{S})\big)$ and $\pi(\mathcal{S})$ the image of $\mathcal{A}(\mathcal{S})$ and $\mathcal{S}$, respectively, in the Calkin algebra $\mathcal{B}(\mathscr{X})/\mathcal{K}(\mathscr{X})$. Since $\pi\big(\mathcal{A}(\mathcal{S})\big)$ is the subalgebra of the Calkin algebra generated by the commutative set $\pi(\mathcal{S})$ we see that $\pi\big(\mathcal{A}(\mathcal{S})\big)$ is commutative, as well. It follows that $\mathcal{A}(\mathcal{S})$ is an essentially commutative subalgebra of $\mathcal{B}(\mathscr{X})$. By [9, Theorem 3.5], for an algebra of essentially commuting polynomially compact operators triangularizability of $\mathcal{A}(\mathcal{S})$ is equivalent to the fact that each commutator $[S, T]$, where $S, T \in \mathcal{A}(\mathcal{S})$, is a quasinilpotent operator.

Suppose that there exist operators $S, T \in \mathcal{A}(\mathcal{S})$ such that the commutator $K := [S, T]$ is non-zero. Let $\mathcal{J}$ be the ideal in $\mathcal{A}(\mathcal{S})$ generated by K. Of course, $\mathcal{J}$ consists of compact operators and, by Ringrose's Theorem (see [14, Theorem 7.2.3]), every operator in $\mathcal{J}$ is quasinilpotent. Hence, $\mathcal{J}$ is an algebra of Volterra operators. Let $\mathcal{J}_{\mathcal{H}}$ be the ideal in the hyperalgebra $\mathcal{H}(\mathcal{S})$ generated by $\mathcal{J}$. It is clear that $\mathcal{J}_{\mathcal{H}} \subseteq \mathcal{K}(\mathscr{X})$. We claim that $\mathcal{J}_{\mathcal{H}}$ consists of quasinilpotent operators, as well. Choose an arbitrary operator $A \in \mathcal{J}_{\mathcal{H}}$. By Lemma 2.5, there exist operators $J_1, \dots, J_n \in \mathcal{J}$ and operators $B_1, \dots, B_n \in \mathcal{A}(\mathcal{S})'$ such that $A = \sum_{i=1}^n J_i B_i$. By [16, Lemma 1], $\rho(A) \leq n\rho(\{J_1, \dots, J_n\})\rho(\{B_1, \dots, B_n\})$. Since each finite subset of an algebra of Volterra operators is quasinilpotent, by [16, Theorem 1], we have $\rho(\{J_1, \dots, J_n\}) = 0$ and consequently $\rho(A) = 0$. We have proved that $\mathcal{J}_{\mathcal{H}}$ is a non-trivial ideal of Volterra operators. By [16, Theorem 2], the hyperalgebra $\mathcal{H}(\mathcal{S})$ is reducible.

Suppose now that $\mathcal{A}(\mathcal{S})$ is commutative. Hence, $\mathcal{A}(\mathcal{S}) \subseteq \mathcal{A}(\mathcal{S})'$. Let T be any non-scalar polynomially compact operator in $\mathcal{A}(\mathcal{S})$ and let m_T be its minimal polynomial. Hence, $m_T(T)$ is a compact operator. If $m(T)$ is a non-zero compact operator, then it has a non-trivial hyperinvariant subspace $\mathscr{Y}$, by Lomonosov's Theorem. Since $\mathcal{A}(\mathcal{S}) \subseteq \mathcal{A}(\mathcal{S})' \subseteq (m_T(T))'$ subspace $\mathscr{Y}$ is invariant for every operator in $\mathcal{A}(\mathcal{S})$ and $\mathcal{A}(\mathcal{S})'$. Thus, $\mathcal{H}(\mathcal{S})$ is reducible. On the other hand, if $m_T(T) = 0$, then T is a non-scalar algebraic operator. Hence, for every $\lambda \in \sigma(T)$, the kernel $\ker(T - \lambda I)$ is a non-trivial hyperinvariant subspace for T, and consequently, for $\mathcal{A}(\mathcal{S})$ as $\mathcal{A}(\mathcal{S}) \subseteq \mathcal{A}(\mathcal{S})' \subseteq (T)'$. □

4 Hyperinvariant subspaces for operator bands

An operator band on a Banach space $\mathscr{X}$ is a (multiplicative) semigroup $\mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$ of idempotents, that is, $S^2 = S$ for each $S \in \mathcal{S}$. The linear span of an operator band $\mathcal{S}$ is the algebra $\mathcal{A}(\mathcal{S})$ called a band algebra. We will denote by $\mathcal{N}(\mathcal{S})$, the set of all nilpotent operators in the band algebra $\mathcal{A}(\mathcal{S})$. By [11, Theorem 5.2], $\mathcal{N}(\mathcal{S})$ is the linear span of $[\mathcal{S},\mathcal{S}]$, that is, $\mathcal{N}(\mathcal{S}) = \{\sum_{i=1}^{n}[A_i, B_i];\ n \in \mathbb{N},\ A_i, B_i \in \mathcal{A}(\mathcal{S})\}$. By [11, Corollary 5.5], the set $\mathcal{N}(\mathcal{S})$ of nilpotent operators in $\mathcal{A}(\mathcal{S})$ coincides with the Jacobson radical $\mathcal{R}\big(\mathcal{A}(\mathcal{S})\big)$ of $\mathcal{A}(\mathcal{S})$.

Proposition 4.1 *Let $\{0\} \neq \mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$ be an operator band. If $\mathcal{N} \subseteq \mathcal{N}(\mathcal{S})$ is a non-empty set, then the ideal $\mathcal{J}_{\mathcal{H}}$ in the hyperalgebra $\mathcal{H}(\mathcal{S})$ generated by $\mathcal{N}$ is a nil-ideal.*

Proof Denote by $\mathcal{J}$ the ideal in $\mathcal{A}(\mathcal{S})$ generated by $\mathcal{N}$. We have already mentioned that $\mathcal{R}\big(\mathcal{A}(\mathcal{S})\big) = \mathcal{N}(\mathcal{S})$. Since the radical is an ideal, we have that $\mathcal{J} \subseteq \mathcal{R}\big(\mathcal{A}(\mathcal{S})\big)$. Hence, $\mathcal{J}$ is a nil-ideal in $\mathcal{A}(\mathcal{S})$.

It is clear that $\mathcal{N}$ and ideal $\mathcal{J}$ generate the same ideal $\mathcal{J}_{\mathcal{H}}$ in the hyperalgebra $\mathcal{H}(\mathcal{S})$. Let $A \in \mathcal{J}_{\mathcal{H}}$ be arbitrary. By Lemma 2.5, there exist $n \in \mathbb{N}$, $J_i \in \mathcal{J}$ and $B_i \in \mathcal{A}(\mathcal{S})'$ such that $A = \sum_{i=1}^{n} J_i B_i$. Let $\mathcal{F}$ be a finite subset of $\mathcal{S}$ such that $J_1, \dots, J_n$ are contained in the algebra $\mathcal{A}(\mathcal{F})$ generated by $\mathcal{F}$. Note that such finite sets exist, because each J_i is of the form $p_i(S_1^{(i)}, \dots, S_{k_i}^{(i)})$, where p_i is a polynomial of k_i non-commuting variables and $S_1^{(i)}, \dots, S_{k_i}^{(i)}$ $(i = 1, \dots, n)$ are idempotents from $\mathcal{S}$. Denote by $\mathcal{S}_0$ the operator band generated by $\mathcal{F}$. Thus, $\mathcal{A}(\mathcal{F}) \subseteq \mathcal{A}(\mathcal{S}_0)$. Since $\mathcal{F}$ is finite, the operator band $\mathcal{S}_0$ is finite, as well, by the Green-Rees theorem (see [14, Theorem 9.3.11]). Now, we apply [14, Theorem 9.3.15] which says that there exists a finite chain $\{0\} = \mathscr{X}_0 \subseteq \mathscr{X}_1 \subseteq \cdots \subseteq \mathscr{X}_m = \mathscr{X}$ of invariant subspaces for $\mathcal{S}_0$ such that for each $E \in \mathcal{S}_0$ the operator induced by E on $\mathscr{X}_i/\mathscr{X}_{i-1}$ is either zero or the identity operator. This implies that operator induced by J_i on $\mathscr{X}_j/\mathscr{X}_{j-1}$ is a scalar multiple of the identity operator. Since each J_i is nilpotent, every operator J_i induces the zero operator on $\mathscr{X}_j/\mathscr{X}_{j-1}$. Hence, for each i and each j, we have $J_i(\mathscr{X}_j) \subseteq \mathscr{X}_{j-1}$. This implies that an arbitrary product of length at least m with letters from $\{J_1, \dots, J_n\}$ is zero. Now it is obvious that $A^m = \left(\sum_{i=1}^{n} J_i B_i\right)^m = 0$ as J_i and B_j commute. □

Proposition 4.2 *Let $\{0\} \neq \mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$ be an operator band. If $K \in \overline{\mathcal{A}(\mathcal{S})}$ is a compact operator, then for each operator $A \in \overline{\mathcal{H}(\mathcal{S})}$ the commutator $[K, A]$ is in the Jacobson radical of $\overline{\mathcal{H}(\mathcal{S})}$, that is, $[K, \overline{\mathcal{H}(\mathcal{S})}] \subseteq \mathcal{R}\big(\overline{\mathcal{H}(\mathcal{S})}\big)$.*

Proof Let $K \in \overline{\mathcal{A}(\mathcal{S})}$ be a compact operator and let $A \in \overline{\mathcal{H}(\mathcal{S})}$ be an arbitrary operator. To prove that the commutator $[K, A]$ is in $\mathcal{R}\big(\overline{\mathcal{H}(\mathcal{S})}\big)$, we need to show that $[K, A]C$ is quasinilpotent for each $C \in \overline{\mathcal{H}(\mathcal{S})}$. Since $[K, A]C$ is compact, the continuity of the spectral radius at compact operators yields that it suffices to prove that $[K, A]C$ is quasinilpotent for all $A, C \in \mathcal{H}(\mathcal{S})$.

Let $A, C \in \mathcal{H}(\mathcal{S})$ be arbitrary. Since K belongs to the closure of $\mathcal{A}(\mathcal{S})$, there is a sequence $(K_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}(\mathcal{S})$ which converges to K. We claim that for each $n \in \mathbb{N}$ the commutator $[K_n, A]$ is nilpotent and belongs to the ideal $\mathcal{N}(\mathcal{S})_{\mathcal{H}}$ in $\mathcal{H}(\mathcal{S})$ generated by the Jacobson radical $\mathcal{N}(\mathcal{S})$ of the band algebra $\mathcal{A}(\mathcal{S})$. Since $A \in \mathcal{H}(\mathcal{S})$, there exist operators $A_1, \dots, A_k \in \mathcal{A}(\mathcal{S})$ and $B_0, \dots, B_k \in \mathcal{A}(\mathcal{S})'$ such that $A = B_0 + \sum_{i=1}^{k} A_i B_i$. It follows that $[K_n, A] = \sum_{i=1}^{k}[K_n, A_i]B_i$. Since every commutator $[K_n, A_i]$ is in $\mathcal{N}(\mathcal{S})$

and $\mathcal{N}(\mathcal{S}) \subseteq \mathcal{N}(\mathcal{S})_{\mathcal{H}}$, it follows $[K_n, A_i] \in \mathcal{N}(\mathcal{S})_{\mathcal{H}}$. Since $\mathcal{N}(\mathcal{S})_{\mathcal{H}}$ is an ideal in the hyper-algebra $\mathcal{H}(\mathcal{S})$ we have $[K_n, A]C \in \mathcal{N}(\mathcal{S})_{\mathcal{H}}$. By Proposition 4.1, the operator $[K_n, A]C$ is nilpotent. To finish the proof, we apply the fact that the spectral radius is continuous at compact operators and that the compact operator $[K, A]C$ is the limit of the sequence $([K_n, A]C)_{n\in\mathbb{N}}$ of nilpotent operators. ☐

Theorem 4.3 *Let $\mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$ be an operator band. If there exists a non-zero compact operator in $\overline{\mathcal{A}(\mathcal{S})}$, then $\mathcal{S}$ has a non-trivial hyperinvariant subspace.*

Proof Let K be a non-zero compact operator in $\overline{\mathcal{A}(\mathcal{S})}$. Then, by Proposition 4.2, $[K, A]$ is a compact operator contained in the radical $\mathcal{R}\big(\overline{\mathcal{H}(\mathcal{S})}\big)$ for each $A \in \overline{\mathcal{H}(\mathcal{S})}$. If $[K, A] = 0$ for every $A \in \overline{\mathcal{H}(\mathcal{S})}$, then $\overline{\mathcal{H}(\mathcal{S})} \subseteq (K)'$ and, therefore, every non-trivial hyperinvariant subspace of K is a non-trivial hyperinvariant subspace for $\mathcal{A}(\mathcal{S})$. Hence, we may assume that for some $A \in \overline{\mathcal{H}(\mathcal{S})}$ the commutator $[K, A]$ is a non-zero operator. Let $\mathcal{J}$ be the ideal in $\overline{\mathcal{H}(\mathcal{S})}$ generated by $[K, A]$. Of course, each operator in $\mathcal{J}$ is compact. Since, by Proposition 4.2, $[K, A]$ is in the Jacobson ideal of $\mathcal{R}\big(\overline{\mathcal{H}(\mathcal{S})}\big)$ we have $\mathcal{J} = \overline{\mathcal{H}(\mathcal{S})}[K, A]\overline{\mathcal{H}(\mathcal{S})} \subseteq \overline{\mathcal{H}(\mathcal{S})}\mathcal{R}\big(\overline{\mathcal{H}(\mathcal{S})}\big)\overline{\mathcal{H}(\mathcal{S})} \subseteq \mathcal{R}\big(\overline{\mathcal{H}(\mathcal{S})}\big)$. It follows that operators in $\mathcal{J}$ are quasinilpotent. Thus, $\mathcal{J}$ is a Volterra ideal in $\overline{\mathcal{H}(\mathcal{S})}$. By [16, Theorem 2], $\mathcal{J}$ is reducible. Now we apply [14, Lemma 7.4.6] and conclude that $\overline{\mathcal{H}(\mathcal{S})}$ is reducible. ☐

Corollary 4.4 *Every essentially commuting non-scalar operator band $\mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$ has a non-trivial hyperinvariant subspace.*

Proof If $\mathcal{S}$ is commutative, then $\mathcal{S} \subseteq \mathcal{S}'$. In this case the kernel of any non-scalar operator from $\mathcal{S}$ is invariant for each $S \in \mathcal{S}'$. If $\mathcal{S}$ is not commutative, then there exist idempotents $E, F \in \mathcal{S}$ with a non-zero compact commutator $EF - FE \in \mathcal{A}(\mathcal{S})$. The assertion follows, by Theorem 4.3. ☐

Since every essentially commuting band of operators has an invariant subspace, an application of the Triangularization lemma (see [14, Lemma 7.1.11]) immediately implies the following result.

Corollary 4.5 *Every essentially commuting band of operators on a Banach space is triangularizable.*

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