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ORIGINAL PAPER

Spectrality and non-spectrality of some Moran measures in $\mathbb{R}^3$

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Abstract

Let $\mu_{\{M_n\},\{D_n\}}$ be the Moran measures generated by expanding diagonal matrix M_n with entries $p_n, q_n, r_n \in \mathbb{Z}\backslash\{0,2\}$ and the digit sets

$$D_n = \{(0,0,0)^{\mathrm{T}}, (a_n,0,0)^{\mathrm{T}}, (0,b_n,0)^{\mathrm{T}}, (0,0,c_n)^{\mathrm{T}}\},$$

where a_n, b_n, c_n are bounded integers. In this paper, we show $\mu_{\{M_n\},\{D_n\}}$ is a spectral measure provided that $2a_n \mid p_n$, $2b_n \mid q_n$, $2c_n \mid r_n$. In particular, if $M_n = M = \mathrm{diag}\{p,q,r\}$ and p,q,r,a_n,b_n,c_n are odd integers, we obtain that there exists at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,\{D_n\}})$, and the number 4 is the best.

Keywords Moran spectral measure · Non-spectral · Hadamard triple · Spectrum

Mathematics Subject Classification 28A80 · 42C05 · 46C05

1 Introduction

As it is well known, J. B. Fourier discovered that the exponential functions $\{e^{2\pi i\langle n,x\rangle} : n \in \mathbb{Z}^d\}$ form an orthonormal basis for $L^2([0,1]^d)$ and his discovery is now one of the fundamental pillars in modern mathematics. It is natural to ask what other measures have this property, that there is a family of exponential functions which form an orthonormal basis for their L^2-space?

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Definition 1.1 A Borel probability measure μ on $\mathbb{R}^n$ is said to be a spectral measure if there exists a countable set $\Lambda \subset \mathbb{R}^n$ such that $E_\Lambda := \{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ is an orthogonal basis for $L^2(\mu)$. In this case, we call Λ a spectrum of μ and (μ, Λ) a spectral pair.

The existence of a spectrum of μ is a basic problem in harmonic analysis on $L^2(\mu)$, it was initiated by well-known Fuglede conjecture [15].

Conjecture 1.2 *A measurable set Ω is a spectral set in $\mathbb{R}^n$ if and only if Ω is a translational tile.*

Tao [37] proved that this conjecture is wrong above five dimensions, and Tao's work was subsequently improved by Kolountzakis and Matolcsi [32–34] who proved that the conjecture was false in $\mathbb{R}^d (d = 3, 4)$. Nevertheless, many researchers still devoted themselves to the research on the spectrality and non-spectrality of fractal measures. In recent years, the middle-fourth Cantor measure was the first example of such spectral measures, which was discovered by Jorgensen and Pedersen [31]. From then on, many other self-similar/self-affine/Moran spectral measures have been discovered, see [2, 3, 5, 7–10, 13, 28, 30] and so on. For a non-spectral measure, it belongs to one of the following two cases:

(1) there exists an infinite set of orthogonal exponential functions but no such set forms a basis of $L^2(\mu)$;
(2) there are only a finite number of orthogonal exponential functions.

Let D_k be a finite subset of $\mathbb{R}^n$, $\#D$ means the cardinality of D and let R_k be an $n \times n$ expanding real matrix (all the eigenvalues of R_k have moduli larger than 1). We call the function system $\{f_{k,d} = R_k^{-1}(x + d) : d \in D_k\}_{k=1}^{\infty}$ a *Moran iterated function system* (Moran IFS), which is the generalization of an IFS [14, 21]. Let δ_d be the Dirac measure, and denote the measure

$$\delta_D = \frac{1}{\#D} \sum_{d \in D} \delta_d$$

for a finite set D. Denote the norm of an $n \times n$ real matrix A by $||A|| := \sup\{||Ax|| : ||x|| = 1\}$, where $||x|| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$ is the Euclidean norm of the n-tuple vector $x = (x_1, \ldots, x_n)^{\mathrm{T}}$. Then we introduce the following known theorem (see, e.g. [36, 41]).

Theorem A *Assume* $\sum_{k=1}^{\infty} ||(R_k R_{k-1} \cdots R_1)^{-1}||$ *and* $\sup\{||x|| : x \in R_k^{-1} D_k, k \geq 1\}$ *are finite. Then the sequence of measures*

$$\mu_k = \delta_{R_1^{-1} D_1} * \delta_{(R_2 R_1)^{-1} D_2} * \cdots * \delta_{(R_k R_{k-1} \cdots R_1)^{-1} D_k}$$

converges to a Radon measure $\mu := \mu_{\{R_k\},\{D_k\}}$ *with compact support* T *and* $\mu(\mathbb{R}^n) = 1$ *in a weak sense, where* $*$ *denotes the convolution sign and*

$$T=\sum_{k=1}^{\infty}(R_kR_{k-1}\cdots R_1)^{-1}D_k=\left\{\sum_{k=1}^{\infty}(R_kR_{k-1}\cdots R_1)^{-1}d_k : d_k\in D_k, k\geq 1\right\}.$$

The measure μ is called a *Moran measure*, and T is called a *Moran set*. In 2000, Strichartz [36] first studied the spectrality of the Moran measure when the two sets $\{R_k : k\in\mathbb{N}\}$ and $\{D_k : k\in\mathbb{N}\}$ have only finite elements. Unlike the self-similar measure, existence of Hadamard triples is not a sufficient condition for the Moran measure being a spectral measure, even for universal Hadamard triples (see [5, Example 5.2]). Strichartz [36] proved that the Moran measure is a spectral measure for an infinite compatible tower under some other assumptions. Later on, He et al. [3, 5, 11, 12, 17, 18, 20, 26, 35, 38, 40, 42, 44] investigated the spectral properties of the Moran measures with two, three, four and some special consecutive elements in digit sets in $\mathbb{R}$. In 2019, An, Fu and Lai [2] proved that a large class of Hadamard triples in $\mathbb{R}$ generated Moran spectral measures. Dong et al. [6, 25, 27, 39, 41, 43, 45] considered some special cases in $\mathbb{R}^2$. However, there are a few Moran spectral measures in $\mathbb{R}^n(n\geq 3)$. Recently, Fu and Zhu [19] considered a class of homogeneous Moran spectral measures with eight-element digit sets in $\mathbb{R}^4$. One of the authors and Peng [1] obtained some special Moran spectral measures in $\mathbb{R}^n$ for any $n\geq 1$.

The main purpose of this paper is to study the spectral and non-spectral properties of Moran measures $\mu_{\{M_n\},\{D_n\}}$ in $\mathbb{R}^3$, its expanding matrix and digit sets are as follows:

$$M_n=\begin{pmatrix}p_n&0&0\\0&q_n&0\\0&0&r_n\end{pmatrix},\quad D_n=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}a_n\\0\\0\end{pmatrix},\begin{pmatrix}0\\b_n\\0\end{pmatrix},\begin{pmatrix}0\\0\\c_n\end{pmatrix}\right\},\tag{1}$$

where $p_n,q_n,r_n\in\mathbb{Z}\backslash\{0,2\}$ and $\sup\{a_n,b_n,c_n\}<\infty$. In this paper, our main results are as follows.

Theorem 1.3 *For the Moran measures $\mu_{\{M_n\},\{D_n\}}$ corresponding to (1), if $2a_n\mid p_n$, $2b_n\mid q_n$, $2c_n\mid r_n$ and $|a_n|,|b_n|,|c_n|\in\mathbb{Z}^+\backslash\{1\}$, then $\mu_{\{M_n\},\{D_n\}}$ is a spectral measure.*

In particular, if $M_n=M=\begin{pmatrix}p&0&0\\0&q&0\\0&0&r\end{pmatrix}$ for all $n\geq 1$, we have the following two theorems.

Theorem 1.4 *If $p,q,r\in 2\mathbb{Z}\backslash\{0,2\}$ and $a_n,b_n,c_n\in\{-1,1\}$, then $\mu_{M,\{D_n\}}$ is a spectral measure.*

Theorem 1.5 *For the non-spectral properties of Moran measures $\mu_{M,\{D_n\}}$, we have following results.*

(i) *If* p *is even*, q, r, a_n, b_n, c_n *are odd and* $\frac{b_n}{c_n} = \frac{b_m}{c_m}$, *then there does not exist an infinite orthogonal exponential functions set in* $L^2(\mu_{M,\{D_n\}})$.

(ii) *If* p, q, r, a_n, b_n, c_n *are odd, then there exists at most* 4 *mutually orthogonal exponential functions in* $L^2(\mu_{M,\{D_n\}})$, *and the number* 4 *is the best.*

Remark 1.6 In the same manner we can replace the assumptions of Theorem 1.5 (i) with any of the following conditions.

(1) q is even, p, r, a_n, b_n, c_n are odd and $\frac{a_n}{c_n} = \frac{a_m}{c_m}$.

(2) r is even, p, q, a_n, b_n, c_n are odd and $\frac{a_n}{b_n} = \frac{a_m}{b_m}$.

Remark 1.7 If $a_n = b_n = c_n = 1$, the corresponding measure $\mu_{M,D}$ is a Sierpinski gasket type measure. Li [22–24] proved that if $p, q, r \in 2\mathbb{Z} + 1$, then $\mu_{M,D}$ is a non-spectral measure, and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. Subsequently, Zheng et al. [46] showed that if two of the three numbers p, q, r are odd and the other is even, then there does not exist infinite families of orthogonal exponential functions in $L^2(\mu_{M,D})$. More generally, Lu et al. [29] generalized the non-spectralities of the self-similar measures $\mu_{M,D}$ to real matrix M.

An outline of this paper is as follows. In Sect. 2, we give some preliminary knowledge. We focus on proving Theorems 1.3 and 1.4 in Sect. 3. In Sect. 4, we firstly prove Theorem 1.5 (i) by Ramsey's theorem and some established lemmas. Finally we solve the problem of the number of orthogonal exponential functions under conditions of Theorem 1.5 (ii).

2 Preliminaries

This section is mainly used to introduce definitions and basic results that will be used. We assume that $\mu_{\{M_n\},\{D_n\}}$ is a probability measure with compact support. The Fourier transform of $\mu_{\{M_n\},\{D_n\}}$ is defined as

$$\hat{\mu}_{\{M_n\},\{D_n\}}(\xi) = \int e^{2\pi i\langle x,\xi\rangle} \mathrm{d}\mu_{\{M_n\},\{D_n\}}(x).$$

Recall that its Fourier transform can be written as follows

$$\hat{\mu}_{\{M_n\},\{D_n\}}(\xi) = \prod_{n=1}^{\infty} m_{D_n}((M_1 \cdots M_n)^{-1}\xi), \tag{2}$$

where $m_{D_n}(x)$ is the Mask polynomial of D_n, which is defined by

$$m_{D_n}(x) = \hat{\delta}_{D_n}(x) = \frac{1}{\# D_n} \sum_{d\in D_n} e^{2\pi i\langle x,d\rangle}.$$

Let $\mathcal{Z}(f) := \{\xi : f(\xi) = 0\}$ be the zero set of f. It is easy to see that

$$\mathcal{Z}(\hat{\mu}_{\{M_n\},\{D_n\}}) = \bigcup_{n=1}^{\infty} M_1 \cdots M_n(\mathcal{Z}(m_{D_n})). \tag{3}$$

Obviously, for a discrete set $\Lambda \in \mathbb{R}^n$, $E_\Lambda = \{e^{2\pi i\langle \lambda, x\rangle} : \lambda \in \Lambda\}$ is an orthogonal set of $L^2(\mu)$ if and only if $(\Lambda - \Lambda)\backslash\{0\} \subset \mathcal{Z}(\hat{\mu})$. We also say that Λ is a bi-zero set of μ. Since the bi-zero set is invariant under translation, it will be convenient to assume that $0 \in \Lambda$.

Definition 2.1 (Hadamard triple). Let $M \in M_n(\mathbb{Z})$ be an expanding matrix with integer entries. Let $D, C \subset \mathbb{Z}^n$ be two finite subsets of integer vectors with $\#D = \#C$. We say that the system (M, D, C) forms a Hadamard triple (or $(M^{-1}D, C)$ is a compatible pair) if the matrix

$$H = \frac{1}{\sqrt{\#D}}\left[e^{2\pi i\langle M^{-1}d, c\rangle}\right]_{d\in D, c\in C}$$

is unitary, i.e., $H^*H = I$, where H^* denotes the conjugated transposed matrix of H.

The following theorem is a basic criterion for the spectrality of a measure μ.

Theorem 2.2 ([18]; see also [31]) *Let μ be a compactly supported Borel probability measure on $\mathbb{R}^n$ and let Λ be an orthogonal set for μ. Then the following statements are equivalent.*

(i) *The orthogonal set Λ is a spectrum for μ;*
(ii) $Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} |\hat{\mu}(\xi + \lambda)|^2 = 1, \forall\, \xi \in \mathbb{R}^n$;
(iii) $Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} |\hat{\mu}(\xi + \lambda)|^2 = 1, \forall\, \xi \in (-r, r)^n$, *where* $r > 0$.

The following theorem is the famous infinite Ramsey's Theorem ([46, Theorem 2.3]; see also [3, Theorem 4.1]), which is a foundational result in combinatorics.

Theorem 2.3 (Ramsey's Theorem). *Let $\mathcal{A}$ be a countable infinite set and let $\mathcal{A}^k$ be the set of all k elements subsets of $\mathcal{A}$. For any splitting of $\mathcal{A}^k$ into r classes, there exists an infinite subset $\mathcal{T} \subset \mathcal{A}$ such that $\mathcal{T}^k$ is contained in the same class.*

Let $M_n = M$ and $a_n, b_n, c_n \in \{-1, 1\}$. After we set a_n, b_n, c_n as 1 or -1 respectively, there are eight kinds of D_n denoted by $D_{(i)}$, $i \in \{1, 2, \ldots, 8\}$. In such a case, we have the following lemmas.

Lemma 2.4 *Let S be a finite subset of $\mathbb{Z}^n$ with $0 \in S$. Then the following two statements are equivalent.*

(i) $\mathcal{Z}(m_{M^{-1}D_{(i)}}) \cap T_{M,S} = \emptyset$ *for all* $i \in \{1, 2, \ldots, 8\}$.

(ii) $\mathcal{Z}(\hat{\mu}_{M,D_{(i)}}) \cap T_{M,S} = \emptyset$ *for all* $i \in \{1, 2, \ldots, 8\}$.

Proof (i) ⇒ (ii) Suppose $\mathcal{Z}(m_{M^{-1}D_{(i)}}) \cap T_{M,S} = \emptyset$ for all $i \in \{1, 2, \ldots, 8\}$. Since $0 \in S$, we have

$$M^{-j} T_{M,S} = \left\{ \sum_{n=1}^{\infty} M^{-(n+j)} s_n \,:\, s_n \in S \right\} \subseteq T_{M,S} \quad \text{for all } j \in \mathbb{N}.$$

This leads to $\mathcal{Z}(m_{M^{-1}D_{(i)}}) \cap M^{-j} T_{M,S} = \emptyset$, so we have $M^j \mathcal{Z}(m_{M^{-1}D_{(i)}}) \cap T_{M,S} = \emptyset$ for all $i \in \{1, 2, \ldots, 8\}$ and $j \in \mathbb{N}$. Notice that $\mathcal{Z}(m_{M^{-1}D_{(i)}}) = M\mathcal{Z}(m_{D_{(i)}})$. Then our conclusion holds.

(ii) ⇒ (i) Due to that $\mathcal{Z}(\hat{\mu}_{M,D_{(i)}}) = \cup_{n=1}^{\infty} M^n \mathcal{Z}(m_{D_{(i)}})$. The proof is complete. □

Lemma 2.5 [36, *Theorem* 2.8] *Suppose* (M, D_n, S) *forms a Hadamard tower with* $0 \in D_n \cap S$ *and* $T_{M,S} = \{\sum_{n=1}^{\infty} M^{-n} s_n \,:\, s_n \in S\}$. *If* $\mathcal{Z}(\hat{\mu}_{M,D_{(i)}}) \cap T_{M,S} = \emptyset$ *for all* $i \in \{1, 2, \ldots, 8\}$, *then*

$$\Lambda_{M,S} = \left\{ \sum_{n=0}^{\infty} M^n s_n \,:\, s_n \in S \right\}$$

is a spectrum of $\mu_{M,\{D_n\}}$.

3 Spectrality

In this section, we will give the proofs of Theorem 1.3 and Theorem 1.4.

Let $2a_n \mid p_n$, $2b_n \mid q_n$, $2c_n \mid r_n$ and

$$L_n = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2a_n} \\ \frac{1}{2b_n} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2b_n} \\ \frac{1}{2c_n} \end{pmatrix}, \begin{pmatrix} \frac{1}{2a_n} \\ 0 \\ \frac{1}{2c_n} \end{pmatrix} \right\}. \tag{4}$$

It is easy to check that $(M_n^{-1} D_n, M_n L_n)$ is a compatible pair. Denote

$$\Lambda_n = \sum_{k=1}^{n} M_1 M_2 \cdots M_k L_k$$

for any $n \geq 1$ and $\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$. Define

$$\mu_n = \delta_{M_1^{-1} D_1} * \cdots * \delta_{(M_n \cdots M_2 M_1)^{-1} D_n},$$
$$\mu_{>n} = \delta_{(M_{n+1} \cdots M_2 M_1)^{-1} D_{n+1}} * \delta_{(M_{n+2} \cdots M_2 M_1)^{-1} D_{n+2}} * \cdots.$$

Then $\mu_{\{M_n\},\{D_n\}} = \mu_n * \mu_{>n}$. As a matter of fact, Λ_n is a spectrum of μ_n and Λ is an orthogonal set of $\mu_{\{M_n\},\{D_n\}}$ [36].

To prove Theorem 1.3, we need a lemma proved by An et al. [4].

Lemma 3.1 [4, *Theorem* 2.3] *Let μ be a Borel probability measure with compact support in $\mathbb{R}^n$ and $\xi \in \mathbb{R}^n$ with $\| \xi \| = \sqrt{\xi_1^2 + \cdots + \xi_n^2} \le \frac{1}{3}$. Suppose that $\Lambda = \bigcup_{n=1}^{\infty} \Lambda_{\alpha_n}$ is an orthogonal set of μ and $\{\alpha_n\}_{n=1}^{\infty}$ is an increasing sequence of integers. If Λ_{α_n} is a spectrum of μ_{α_n} and*

$$\inf_{\lambda \in \Lambda_{\alpha_{n+s}} \setminus \Lambda_{\alpha_n}} |\hat{\mu}_{>\alpha_{n+s}}(\xi + \lambda)|^2 \ge c > 0$$

for all $n, s \ge 1$, then Λ is a spectrum of μ.

Theorem 1.3 will be proved if we can show that there exists a constant $c > 0$ such that for any $\xi \in \mathbb{R}^3$ with $\| \xi \| \le \frac{1}{3}$, we have $|\hat{\mu}_{>n}(\xi + \lambda)|^2 \ge c$ for any $\lambda \in \Lambda_n$ and $n \ge 1$ by Lemma 3.1.

Proof of Theorem 1.3 Denote $p_n := 2a_n p_n'$, $q_n := 2b_n q_n'$, $r_n := 2c_n r_n'$, where $p_n', q_n', r_n' \in \mathbb{Z} \setminus \{0\}$. Note that $|\hat{\mu}_{>n}(\xi + \lambda)|^2 = \prod_{k=n+1}^{\infty} |\hat{\delta}_{(M_1 M_2 \cdots M_k)^{-1} D_k}(\xi + \lambda)|^2$. Firstly we need to estimate $|\hat{\delta}_{(M_1 M_2 \cdots M_k)^{-1} D_k}(\xi + \lambda)|^2$ for all $k \ge n + 1$. Take any $\lambda \in \Lambda_n$ and write

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} p_1 p_2 \cdots p_i l_{i1} \\ \sum_{i=1}^{n} q_1 q_2 \cdots q_i l_{i2} \\ \sum_{i=1}^{n} r_1 r_2 \cdots r_i l_{i3} \end{pmatrix}, \begin{pmatrix} l_{i1} \\ l_{i2} \\ l_{i3} \end{pmatrix} \in L_i.$$

For any $k \ge n + 1$ and $j = 1, 2, 3$, since $|a_n|, |b_n|, |c_n| \ge 2$, we have

$$\begin{aligned} \left| \frac{a_k(\xi_1 + \lambda_1)}{p_1 p_2 \cdots p_k} \right| &\le \frac{1}{6} \left| \frac{1}{p_1 \cdots p_{k-1} p_k'} + \frac{3}{2} \sum_{i=1}^{n} \frac{1}{a_i p_{i+1} \cdots p_{k-1} p_k'} \right| \le \frac{1}{6} \frac{1}{4^{k-n-1}}, \\ \left| \frac{b_k(\xi_2 + \lambda_2)}{q_1 q_2 \cdots q_k} \right| &\le \frac{1}{6} \left| \frac{1}{q_1 \cdots q_{k-1} q_k'} + \frac{3}{2} \sum_{i=1}^{n} \frac{1}{b_i q_{i+1} \cdots q_{k-1} q_k'} \right| \le \frac{1}{6} \frac{1}{4^{k-n-1}}, \\ \left| \frac{c_k(\xi_3 + \lambda_3)}{r_1 r_2 \cdots r_k} \right| &\le \frac{1}{6} \left| \frac{1}{r_1 \cdots r_{k-1} r_k'} + \frac{3}{2} \sum_{i=1}^{n} \frac{1}{c_i r_{i+1} \cdots r_{k-1} r_k'} \right| \le \frac{1}{6} \frac{1}{4^{k-n-1}}. \end{aligned} \tag{5}$$

A direct calculation shows that

$$\begin{aligned} &\left| \hat{\delta}_{(M_1 \cdots M_k)^{-1} D_k}(\xi + \lambda) \right|^2 \\ &= \left| \frac{1}{4}(1 + e^{-2\pi i x_1} + e^{-2\pi i x_2} + e^{-2\pi i x_3}) \right|^2 \\ &= \frac{1}{16} \big| 4 + 2(\cos 2\pi x_1 + \cos 2\pi x_2 + \cos 2\pi x_3) \\ &\quad + 2(\cos 2\pi(x_1 - x_2) + \cos 2\pi(x_2 - x_3) + \cos 2\pi(x_3 - x_1)) \big|, \end{aligned} \tag{6}$$

where

$$x_1=\frac{a_k(\xi_1+\lambda_1)}{p_1\dots p_k},\quad x_2=\frac{b_k(\xi_2+\lambda_2)}{q_1\dots q_k},\quad x_3=\frac{c_k(\xi_3+\lambda_3)}{r_1\dots r_k}.$$

For $k=n+1$, (5) shows that $\cos x_i\geq\cos\frac{\pi}{3}$ for $i=1,2,3$. By (6),

$$\begin{aligned}&\left|\hat{\delta}_{(M_1\dots M_k)^{-1}D_k}(\xi+\lambda)\right|^2\\ &\geq\frac{1}{16}\left|7+2(\cos 2\pi(x_1-x_2)+\cos 2\pi(x_2-x_3)+\cos 2\pi(x_3-x_1))\right|\\ &\geq\frac{1}{16}.\end{aligned}$$

For $k\geq n+2$, by $\cos x\geq 1-x^2$, we have

$$\begin{aligned}&\cos 2\pi x_i\geq 1-(2\pi x_i)^2\geq 1-\frac{\pi^2}{9}\frac{1}{16^{k-n-1}},\quad i=1,2,3,\\ &\cos 2\pi(x_i-x_j)\geq 1-\frac{4\pi^2}{9}\frac{1}{16^{k-n-1}},\quad i<j, j\in\{2,3\}.\end{aligned}$$

According to (6),

$$|\hat{\delta}_{(M_1\dots M_k)^{-1}D_k}(\xi+\lambda)|^2\geq 1-\frac{15\pi^2}{98}\frac{1}{64^{k-n-1}}.$$

Therefore,

$$|\hat{\mu}_{>n}(\xi+\lambda)|^2\geq\frac{1}{16}\prod_{k=1}^{\infty}(1-\frac{5\pi^2}{24}\frac{1}{64^k}):=c>0,$$

which completes the proof of Theorem 1.3. □

In the rest of this section, we will prove Theorem 1.4 by Lemmas 2.4 and 2.5.

Proof of Theorem 1.4 We construct the set

$$S=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}\frac{p}{2}\\\frac{q}{2}\\0\end{pmatrix},\begin{pmatrix}0\\\frac{q}{2}\\\frac{r}{2}\end{pmatrix},\begin{pmatrix}\frac{p}{2}\\0\\\frac{r}{2}\end{pmatrix}\right\}\subset\mathbb{Z}^3$$

such that $(M^{-1}D_n,S)$ is a compatible pair for any n. Then the invariant set $T_{M,S}$ is given by

$$T_{M,S}=\left\{\sum_{n=1}^{\infty}M^{-n}s_n\ :\ s_n\in S\right\}=\left\{\sum_{n=1}^{\infty}\begin{pmatrix}\frac{s_{1,n}}{p^n}\\\frac{s_{2,n}}{q^n}\\\frac{s_{3,n}}{r^n}\end{pmatrix}:\begin{pmatrix}s_{1,n}\\s_{2,n}\\s_{3,n}\end{pmatrix}\in S\right\}.\qquad(7)$$

For any $x=(x_1,x_2,x_3)^{\mathrm{T}}\in T_{M,S}$, we have

$$|x_1| \le \left|\frac{p}{2(p-1)}\right|, \qquad |x_2| \le \left|\frac{q}{2(q-1)}\right|, \qquad |x_3| \le \left|\frac{r}{2(r-1)}\right|.$$

If $p,q,r=-2$, then $x_j \in [-\frac{1}{3},\frac{2}{3}]$ for $j\in\{1,2,3\}$. Furthermore, for $p,q,r \in 2\mathbb{Z}\backslash\{0,\pm 2\}$, we also have $T_{M,S} \subseteq [-\frac{2}{3},\frac{2}{3}]^3$.

Secondly, for the given digit set D_n, we have

$$\mathcal{Z}(m_{D_n}(x)) = \begin{cases} E_1 \cup E_2 \cup E_3, & (a_n,b_n,c_n)=(1,1,1) \text{ or } (-1,-1,-1); \\ F_1 \cup F_2 \cup E_3, & (a_n,b_n,c_n)=(1,1,-1) \text{ or } (-1,-1,1); \\ F_1 \cup E_2 \cup F_3, & (a_n,b_n,c_n)=(1,-1,1) \text{ or } (-1,1,-1); \\ E_1 \cup F_2 \cup F_3, & (a_n,b_n,c_n)=(-1,1,1) \text{ or } (1,-1,-1), \end{cases} \tag{8}$$

where E_1,E_2,E_3,F_1,F_2,F_3 are given by

$$\begin{aligned} E_1 &= \left\{\begin{pmatrix} \frac{1}{2}+k_1 \\ a+k_2 \\ \frac{1}{2}+a+k_3 \end{pmatrix}\right\}, E_2 = \left\{\begin{pmatrix} \frac{1}{2}+a+k_1 \\ \frac{1}{2}+k_2 \\ a+k_3 \end{pmatrix}\right\}, E_3 = \left\{\begin{pmatrix} a+k_1 \\ \frac{1}{2}+a+k_2 \\ \frac{1}{2}+k_3 \end{pmatrix}\right\}, \\ F_1 &= \left\{\begin{pmatrix} \frac{1}{2}+k_1 \\ a+k_2 \\ \frac{1}{2}-a+k_3 \end{pmatrix}\right\}, F_2 = \left\{\begin{pmatrix} \frac{1}{2}-a+k_1 \\ \frac{1}{2}+k_2 \\ a+k_3 \end{pmatrix}\right\}, F_3 = \left\{\begin{pmatrix} a+k_1 \\ \frac{1}{2}-a+k_2 \\ \frac{1}{2}+k_3 \end{pmatrix}\right\} \end{aligned} \tag{9}$$

for any $a\in\mathbb{R}, k_1,k_2,k_3\in\mathbb{Z}$.

Now, for any $x=(x_1,x_2,x_3)^{\mathrm{T}} \in \mathcal{Z}(m_{M^{-1}D_{(i)}}(x)) = M\mathcal{Z}(m_{D_{(i)}}(x))$, it follows from (8) and (9) that

$$\begin{aligned} |x_1| &= \left|\left(\frac{1}{2}+k_1\right)\right| p \ge 1 \quad \text{if} \quad x\in ME_1\cup MF_1, \\ |x_2| &= \left|\left(\frac{1}{2}+k_2\right)\right| q \ge 1 \quad \text{if} \quad x\in ME_2\cup MF_2, \\ |x_3| &= \left|\left(\frac{1}{2}+k_3\right)\right| r \ge 1 \quad \text{if} \quad x\in ME_3\cup MF_3. \end{aligned}$$

This shows that

$$\mathcal{Z}(m_{M^{-1}D_{(i)}}(x)) \cap \left[-\frac{2}{3},\frac{2}{3}\right]^3 = \emptyset.$$

By Lemmas 2.4 and 2.5, we obtain that $\Lambda_{M,S}$ is a spectrum for $\mu_{M,\{D_n\}}$. This completes the proof of Theorem 1.4. □

4 Non-spectrality

In this section, we will prove Theorem 1.5. By simple calculation, we obtain that

$$\mathcal{Z}(\hat{\mu}_{M,\{D_n\}}) = \bigcup_{n=1}^{\infty} M^n \mathcal{Z}(m_{D_n}) = B_1 \cup B_2 \cup B_3, \tag{10}$$

where $B_i = \bigcup_{n=1}^{\infty} M^n B_i^{(n)}$ and

$$B_1^{(n)} = \left\{ \begin{pmatrix} \frac{1}{2a_n} + k_1 \\ \frac{1}{b_n} + d_n + k_2 \\ \frac{1}{2c_n} + \frac{d_n b_n}{c_n} + \frac{k_3}{c_n} \end{pmatrix}, \ d_n \in \mathbb{R}, k_1, k_2, k_3 \in \mathbb{Z} \right\},$$

$$B_2^{(n)} = \left\{ \begin{pmatrix} \frac{1}{2a_n} + \frac{d_n c_n}{a_n} + \frac{k_1}{a_n} \\ \frac{1}{2b_n} + k_2 \\ \frac{1}{c_n} + d_n + k_3 \end{pmatrix}, \ d_n \in \mathbb{R}, k_1, k_2, k_3 \in \mathbb{Z} \right\},$$

$$B_3^{(n)} = \left\{ \begin{pmatrix} \frac{1}{a_n} + d_n + k_1 \\ \frac{1}{2b_n} + \frac{d_n a_n}{b_n} + \frac{k_2}{b_n} \\ \frac{1}{2c_n} + k_3 \end{pmatrix}, \ d_n \in \mathbb{R}, k_1, k_2, k_3 \in \mathbb{Z} \right\}.$$

Lemma 4.1 [46, *Lemma* 2.4]. *Suppose* $\{j_n\}_{n=1}^{\infty}$ *is a strictly increasing positive integer sequence. Let* $t_{n,n'} := \max\{s : 2^s | (j_n - j_{n'})\}$, *then there must exist three positive integer* N *and* $m, m' < N$ *such that* $t_{N,m} \neq t_{N,m'}$.

Lemma 4.2 *If* p *is even,* q, r, a_n, b_n, c_n *are odd and* $B_1^j = M^j B_1^{(j)}$, *then the following statements hold*:

(i) *for each* $j \in N$ *and an element* $\xi \in (B_1^j - B_1^j)$, *if* $\xi \in B_1^i$ *for some integer* $i \in N$, *then* $i > j$;
(ii) *let* $j, l \in N$ *and* $j \neq l$, *for any element* $\xi \in (B_1^j - B_1^l)$, *if* $\xi \in B_1^i$ *for some integer* $i \in N$, *then* $i = \min\{j, l\}$.

Proof

(i) Since $\xi \in (B_1^j - B_1^j) \cap B_1^i$, we have

$$p^j\left(\frac{1}{2a_j} + k_1\right) - p^j\left(\frac{1}{2a_j} + k_2\right) = p^i\left(\frac{1}{2a_i} + k_3\right),$$

then

$$p^{i-j}\left(\frac{1}{2a_i} + k_3\right) = k_1 - k_2 \in \mathbb{Z}.$$

Hence $i > j$.

(ii) Suppose $\xi_j \in B_1^j, \xi_l \in B_1^l, \xi_i \in B_1^i$, where $j \neq l$. If $i \neq \min\{j,l\}$, then we have

$$p^j\left(\frac{1}{2a_j}+k_{j,1}\right)-p^l\left(\frac{1}{2a_l}+k_{l,1}\right)=p^i\left(\frac{1}{2a_i}+k_{i,1}\right).$$

Reorganize this equation, we obtain

$$a_la_ip^j-a_ja_ip^l-a_ja_lp^i=2a_ja_la_i(p^ik_{i,1}+p^lk_{l,1}-p^jk_{j,1}).$$

Without loss of generality, suppose $l=\min\{i,j,l\}$, then

$$a_la_ip^{j-l}-a_ja_i-a_ja_lp^{i-l}=2a_ja_la_i(p^{i-l}k_{j,1}+k_{l,1}-p^{j-l}k_{i,1}).$$

The parity is different on the left and right sides of this equation according to $p\in 2\mathbb{Z}, a_n\in 2\mathbb{Z}+1$, this is a contradiction. □

Lemma 4.3 ([46, *Lemma* 2.2]; *see also* [22]). *For two different odd numbers u and v. If α,β have different parity, then for any $k,\tilde{k}\in\mathbb{Z}$,*

$$(2k+1)(u^\alpha-v^\alpha)\neq(2\tilde{k}+1)(u^\beta-v^\beta).$$

To get the following two lemmas, we decompose the real numbers by

$$H_i=\left\{\frac{a}{2^ib}\ :\ a,b\in 2\mathbb{Z}+1, i\in\mathbb{Z}\right\},\quad G=\left\{\frac{a}{b}\ :\ a\in\mathbb{Z}, b\in 2\mathbb{Z}+1\right\}\tag{11}$$

and

$$K=(\mathbb{R}\backslash\mathbb{Q})\cup(\bigcup_{i=2}^{\infty}H_i).$$

It is easy to see that $\mathbb{R}=K\cup G\cup H_1=\{0\}\cup(\mathbb{R}\backslash\mathbb{Q})\cup(\cup_{i=-\infty}^{\infty}H_i)$ are pairwise disjoint union. We also have the following lemmas.

Lemma 4.4 [29, *Lemma* 2.1]

(i) *If $f,g\in H_1$, then $f\pm g\in G$.*
(ii) *If $f,g\in G$, then $f\pm g\in G$.*
(iii) *If $f\in H_{m_1}$ and $g\in H_{m_2}$ for two integers $m_1\neq m_2$, then*

$$f\pm g\in H_{\max\{m_1,m_2\}}.$$

In particular, if $f\in G\cup H_1$ and $g\in K$, then $f\pm g\in K$.
(iv) *If $f\in H_{m_1}, h\in H_{m_2}$ for two integers m_1,m_2, then $g\in H_{m_2-m_1}$.*

Lemma 4.5 *Let $B_i (i=1,2,3)$ be defined in* (10) *and H_i, G be defined in* (11). *Suppose that p,q,r,a_n,b_n,c_n are odd. Then the following statements hold.*

(i) *For all* $i = 1, 2, 3$, *if* $\xi = (\xi_1, \xi_2, \xi_3)^{\mathrm{T}} \in B_i$, *then* $\xi_i \in H_1$.
(ii) *For all* $i = 1, 2, 3$, *if* $\xi = (\xi_1, \xi_2, \xi_3)^{\mathrm{T}} \in B_i \pm B_i$, *then* $\xi \notin B_i$ *and* $\xi_i \in G$.
(iii) *Let* $\xi = (\xi_1, \xi_2, \xi_3)^{\mathrm{T}} \in B_1 \cup B_2 \cup B_3$. *If* $\xi_i \in G$, *then* $\xi \notin B_i$, *and the other two components of* ξ *belong to* H_1.

Proof The statement (i) is obvious. It is also easy to prove (ii) by looking at the structure of $B_i \pm B_i$ for all $i = 1, 2, 3$. For (iii), if $\xi_1 \in G$, we obtain $\xi \notin B_1$ from (i), then $\xi \in B_2 \cup B_3$. Suppose that $\xi \in B_2$, then $\xi_2 \in H_1$ by (i). We just need to prove $\xi_3 \in H_1$. Let

$$\xi_1 = \left(\frac{1}{2a_n} + \frac{d_n c_n}{a_n} + \frac{k_1}{a_n}\right)p^n := \frac{p'}{q'} \in G,$$

where $d_n \in \mathbb{R}$, $k_1, p' \in \mathbb{Z}$ and $q' \in 2\mathbb{Z} + 1$. Then

$$d_n = \frac{2p'a_n - (2k_1 + 1)q'p^n}{2q'p^n c_n} := \frac{p_n}{2q_n} \in H_1,$$

and

$$\xi_3 = \left(\frac{1}{c_n} + d_n + k_3\right)r^n = \frac{(2q_n + p_n c_n + 2q_n c_n k_3)r^n}{2c_n q_n} \in H_1.$$

The same proof works for $\xi_2 \in G$ or $\xi_3 \in G$. □

Now we are devoted to the proof of Theorem 1.5 (i).

Proof Suppose on contrary that there exists an infinite orthogonal exponential functions set $\Lambda := \{\lambda_n\}$ in $L^2(\mu_{M,\{D_n\}})$, then $(\Lambda - \Lambda)\backslash\{0\} \subset \mathcal{Z}(\hat{\mu}_{M,\{D_n\}})$. Let

$$\Lambda^{(2)} = \{(\lambda, \lambda') : \lambda - \lambda' \in B_1\} \cup \{(\lambda, \lambda') : \lambda - \lambda' \in B_2\} \cup \{(\lambda, \lambda') : \lambda - \lambda' \in B_3\}$$

be the set of all 2-elements subsets of Λ. By Ramsey's Theorem(Theorem 2.3), there exists an infinite subset $\Lambda' \subset \Lambda$ such that $(\Lambda' - \Lambda')\backslash\{0\} \subset B_i$ for $i = 1, 2$ or 3. We claim that $i = 1$, i.e., $(\Lambda' - \Lambda')\backslash\{0\} \subset B_1$. In fact, if $(\Lambda' - \Lambda')\backslash\{0\} \subset B_2$, let λ_1', $\lambda_2' \in \Lambda'$, then there exist j_1, j_2 and j such that

$$\lambda_1' - \lambda_2' = \begin{pmatrix} \left(\frac{1}{2a_{j_1}} + \frac{d_{j_1}c_{j_1}}{a_{j_1}} + \frac{k_{j_1,1}}{a_{j_1}}\right)p^{n_{j_1}} \\ \left(\frac{1}{2b_{j_1}} + k_{j_1,2}\right)q^{n_{j_1}} \\ \left(\frac{1}{c_{j_1}} + d_{j_1} + k_{j_1,3}\right)r^{n_{j_1}} \end{pmatrix} - \begin{pmatrix} \left(\frac{1}{2a_{j_2}} + \frac{d_{j_2}c_{j_2}}{a_{j_2}} + \frac{k_{j_2,1}}{a_{j_2}}\right)p^{n_{j_2}} \\ \left(\frac{1}{2b_{j_2}} + k_{j_2,2}\right)q^{n_{j_2}} \\ \left(\frac{1}{c_{j_2}} + d_{j_2} + k_{j_2,3}\right)r^{n_{j_2}} \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{2a_j} + \frac{d_jc_j}{a_j} + \frac{k_{j,1}}{a_j}\right)p^{n_j} \\ \left(\frac{1}{2b_j} + k_{j,2}\right)q^{n_j} \\ \left(\frac{1}{c_j} + d_j + k_{j,3}\right)r^{n_j} \end{pmatrix}. \tag{12}$$

By $(\frac{1}{2b_{j_1}} + k_{j_1,2})q^{n_{j_1}} - (\frac{1}{2b_{j_2}} + k_{j_2,2})q^{n_{j_2}} = (\frac{1}{2b_j} + k_{j,2})q^{n_j}$, we have

$$b_jb_{j_2}(1 + 2b_{j_1}k_{j_1,2})q^{n_{j_1}} - b_jb_{j_1}(1 + 2b_{j_2}k_{j_2,2})q^{n_{j_2}} = b_{j_1}b_{j_2}(1 + 2b_jk_{j,2})q^{n_j}.$$

The parity on the left and right sides of this equation is different since $q, b_n \in 2\mathbb{Z} + 1$, this is a contradiction. Hence $i \neq 2$. Similarly, we have $i \neq 3$.

According to the properties of Λ', we will divide the proof into two cases.

Case 1. There exists an infinite increasing sequence $\{j_n\}_{n=1}^{\infty}$ such that $\Lambda' \cap B_1^{j_n} \neq \emptyset$. Choosing $\hat{\lambda}_n \in \Lambda' \cap B_1^{j_n}$, let $\hat{\Lambda} := \{\hat{\lambda}_n\}_{n=1}^{\infty}$, then $\hat{\Lambda}, (\hat{\Lambda} - \hat{\Lambda})\backslash\{0\} \subset B_1$. For the sequence $\{j_n\}_{n=1}^{\infty}$, by Lemma 4.1, there exist $N, m_1 \neq m_2 < N$ and $0 \le t_1 < t_2$ such that

$$j_N - j_{m_1} = 2^{t_1}d_{N,m_1} \text{ and } j_N - j_{m_2} = 2^{t_2}d_{N,m_2}, \tag{13}$$

where $d_{N,m_1}, d_{N,m_2} \in 2\mathbb{Z} + 1$. By Lemma 4.2 and $j_{m_1}, j_{m_2} < j_N$, we have

$$\begin{aligned} &\hat{\lambda}_N - \hat{\lambda}_{m_i} \\ &= \begin{pmatrix} p^{j_N}\left(\frac{1}{2a_{j_N}} + k_{N,1}\right) \\ q^{j_N}\left(\frac{1}{b_{j_N}} + d_N + k_{N,2}\right) \\ r^{j_N}\left(\frac{1}{2c_{j_N}} + \frac{d_Nb_{j_N}}{c_{j_N}} + \frac{k_{N,3}}{c_{j_N}}\right) \end{pmatrix} - \begin{pmatrix} p^{j_{m_i}}\left(\frac{1}{2a_{j_{m_i}}} + k_{m_i,1}\right) \\ q^{j_{m_i}}\left(\frac{1}{b_{j_{m_i}}} + d_{m_i} + k_{m_i,2}\right) \\ r^{j_{m_i}}\left(\frac{1}{2c_{j_{m_i}}} + \frac{d_{m_i}b_{j_{m_i}}}{c_{j_{m_i}}} + \frac{k_{m_i,3}}{c_{j_{m_i}}}\right) \end{pmatrix} \\ &= M^{j_{m_i}} \begin{pmatrix} p^{j_N - j_{m_i}}\left(\frac{1}{2a_{j_N}} + k_{N,1}\right) - \left(\frac{1}{2a_{j_{m_i}}} + k_{m_i,1}\right) \\ q^{j_N - j_{m_i}}\left(\frac{1}{b_{j_N}} + d_N + k_{N,2}\right) - \left(\frac{1}{b_{j_{m_i}}} + d_{m_i} + k_{m_i,2}\right) \\ r^{j_N - j_{m_i}}\left(\frac{1}{2c_{j_N}} + \frac{d_Nb_{j_N}}{c_{j_N}} + \frac{k_{N,3}}{c_{j_N}}\right) - \left(\frac{1}{2c_{j_{m_i}}} + \frac{d_{m_i}b_{j_{m_i}}}{c_{j_{m_i}}} + \frac{k_{m_i,3}}{c_{j_{m_i}}}\right) \end{pmatrix}. \end{aligned}$$

Multiplying the second component by $\frac{b_{j_{m_i}}}{c_{j_{m_i}}}$, subtracting the third component to the second component, we obtain

$$d_N\left(\frac{b_{j_{m_i}}}{c_{j_{m_i}}}q^{j_N-j_{m_i}}-\frac{b_{j_N}}{c_{j_N}}r^{j_N-j_{m_i}}\right)\in\frac{2\mathbb{Z}+1}{2c_{j_{m_i}}c_{j_N}b_{j_N}}.$$

We use the condition $\frac{b_n}{c_n}=\frac{b_m}{c_m}$ to get

$$d_N b_{j_N}(q^{j_N-j_{m_1}}-r^{j_N-j_{m_1}})\in\frac{2\mathbb{Z}+1}{2f_1}, \tag{14}$$

$$d_N b_{j_N}(q^{j_N-j_{m_2}}-r^{j_N-j_{m_2}})\in\frac{2\mathbb{Z}+1}{2f_2}, \tag{15}$$

where $f_1=b_{j_{m_1}}c_{j_N}=b_{j_N}c_{j_{m_1}}$, $f_2=b_{j_{m_2}}c_{j_N}=b_{j_N}c_{j_{m_2}}$ and $f_1,f_2\in 2\mathbb{Z}+1$.

Subcase 1.1. If $|q|\neq|r|$, let $Q=q^{2^{t_1}}$, $R=r^{2^{t_1}}$ and $\alpha=d_{N,m_1}$, $\beta=2^{t_2-t_1}d_{N,m_2}$. Substituting (13), then (14) and (15) implies that

$$(Q^\alpha-R^\alpha)(2k_1+1)=(Q^\beta-R^\beta)(2k_2+1)$$

for some $k_1,k_2\in\mathbb{Z}$. Note that $Q,R\in 2\mathbb{Z}+1$ and α,β have different parity since $t_2>t_1$. By Lemma 4.3, the above equation is impossible.

Subcase 1.2. If $|q|=|r|$, then $0\le t_1<t_2$ implies $q^{2^{t_2}d_{N,m_2}}-r^{2^{t_2}d_{N,m_2}}=0$, which contradicts with (14) and (15).

Case 2. There exists a positive integer $\mathcal{N}<\infty$ such that $\Lambda'\subset\cup_{j=1}^{\mathcal{N}}B_1^j$.

Subcase 2.1. If there exists $\lambda_0\in\Lambda'$ such that $(\Lambda'-\{\lambda_0\})\cap B_1^i\neq\emptyset$ for infinite many i. We can replace Λ' by $\Lambda'-\{\lambda_0\}$ in **Case 1** and get a contradiction similarly.

Subcase 2.2. If there are only finite many j such that $(\Lambda'-\{\lambda'\})\cap B_1^j\neq\emptyset$ for any $\lambda'\in\Lambda'$, then for any n, by the pigeonhole principle, there exists j_n such that $\#(\Lambda-\lambda_n)\cap B_1^{j_n}=\infty$. Therefore, there exist j_1 and an infinite subsequence $\{\lambda_n^{(1)}\}_{n=1}^\infty$ of $\{\lambda_n'\}_{n=2}^\infty$ such that $\{\lambda_n^{(1)}-\lambda_1'\}_{n=1}^\infty\subset B_1^{j_1}$. Similarly, there exist j_2 and an infinite subsequence $\{\lambda_n^{(2)}\}_{n=1}^\infty$ of $\{\lambda_n^{(1)}\}_{n=2}^\infty$ such that $\{\lambda_n^{(2)}-\lambda_1^{(1)}\}_{n=1}^\infty\subset B_1^{j_2}$. We have $j_2>j_1$ since $\lambda_n^{(2)}-\lambda_1^{(1)}=(\lambda_n^{(2)}-\lambda_1')-(\lambda_1'-\lambda_1^{(1)})$ and $\lambda_n^{(2)}-\lambda_1',\lambda_1'-\lambda_1^{(1)}\in B_1^{j_1}$ by the Lemma 4.2. Continuing this process, we get an increasing sequence $\{j_s\}_{s=1}^\infty$ and a sequence set $\{\{\lambda_n^{(s)}\}_{n=1}^\infty\}_{s=1}^\infty$ which satisfies $\{\lambda_n^{(s)}\}_{n=1}^\infty\subset\{\lambda_n^{(s-1)}\}_{n=2}^\infty$ and $\{\lambda_n^{(s)}-\lambda_1^{(s-1)}\}_{n=1}^\infty\subset B_1^{j_s}$ for all $s\ge 1$, where $\{\lambda_n^{(0)}\}_{n=1}^\infty=\{\lambda_n'\}_{n=1}^\infty$, that is,

$$\lambda_1^{(s)}-\lambda_1^{(s-1)},\lambda_2^{(s)}-\lambda_1^{(s-1)},\lambda_3^{(s)}-\lambda_1^{(s-1)},\ldots,\lambda_n^{(s)}-\lambda_1^{(s-1)},\ldots\in B_1^{j_s}, s\ge 1.$$

For the sequence $\{j_n\}_{n=1}^\infty$, similar to **Case 1**, there exist N, $m_1\neq m_2<N$ and $0\le t_1<t_2$ such that (13) hold, i,e., $j_N-j_{m_1}=2^{t_1}d_{N,m_1}$, $j_N-j_{m_2}=2^{t_2}d_{N,m_2}$. Let $\bar\lambda_N:=\lambda_1^{(N)}-\lambda_1^{(N-1)}\in B_1^{j_N}$ and $\bar\lambda_{m_i}:=\lambda_1^{(N)}-\lambda_1^{(m_i-1)}$, $i=1,2$. Note that

$$\lambda_1^{(N)}-\lambda_1^{(m_1-1)}=(\lambda_1^{(N)}-\lambda_1^{(N-1)})+(\lambda_1^{(N-1)}-\lambda_1^{(N-2)})+\cdots+(\lambda_1^{(m_1)}-\lambda_1^{(m_1-1)}),$$
$$\lambda_1^{(N)}-\lambda_1^{(m_2-1)}=(\lambda_1^{(N)}-\lambda_1^{(N-1)})+(\lambda_1^{(N-1)}-\lambda_1^{(N-2)})+\cdots+(\lambda_1^{(m_2)}-\lambda_1^{(m_2-1)}).$$

According to the above two formulas and Lemma 4.2, we have $\bar\lambda_{m_i}\in B_1^{j_{m_i}}$ for $i=1,2$. Now we replace $\hat\lambda_N$, $\hat\lambda_{m_1}$ and $\hat\lambda_{m_2}$ of **Case 1** by $\bar\lambda_N$, $\bar\lambda_{m_1}$ and $\bar\lambda_{m_2}$ respectively, then get a contradiction similarly.

Hence, Λ is a finite set. This completes the proof of Theorem 1.5 (i). □

When p, q, r, a_n, b_n, c_n are restricted to be odd numbers, we can get the exact number of orthogonal exponential functions in $L^2(\mu_{M,\{D_n\}})$. Using Lemma 4.4 and Lemma 4.5, we can now prove Theorem 1.5 (ii).

Proof of Theorem 1.5 (ii) Let Λ be a bi-zero set of $\mu_{M,\{D_n\}}$. Suppose that $\#\Lambda = 5$ and $\Lambda = \{0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, we have $\Lambda \backslash \{0\} \subset B_1 \cup B_2 \cup B_3$. By the pigeonhole principle, there exist at least two distinct elements that belong to the same B_{i_0}, $i_0 \in \{1, 2, 3\}$. Without loss of generality, we assume that $\lambda_1, \lambda_2 \in B_1$. Let $\lambda_n = (x_n, y_n, z_n)^{\mathrm{T}}$ for $n = 1, 2, 3, 4$. Hence, $x_1, x_2 \in H_1$ by Lemma 4.5 (i). For two distinct numbers $n_1, n_2 \in \{1, 2, 3, 4\}$, we set $\lambda_{n_2} - \lambda_{n_1} := (x_{n_2,n_1}, y_{n_2,n_1}, z_{n_2,n_1})^{\mathrm{T}}$. From $x_2, x_1 \in H_1$ and Lemma 4.4 (i), we obtain $x_{2,1} = x_2 - x_1 \in G$. According to the property of bi-zero set Λ, we have $\lambda_2 - \lambda_1 \in B_1 \cup B_2 \cup B_3$. By Lemma 4.5 (iii) we have $\lambda_2 - \lambda_1 \notin B_1$. So $\lambda_2 - \lambda_1 \in B_2 \cup B_3$ and $y_{2,1}, z_{2,1} \in H_1$. Assume that $\lambda_2 - \lambda_1 \in B_2$, we consider the following three cases, since $x_3 \in \mathbb{R} = H_1 \cup K \cup G$ is a disjoint union.

Case 1. $x_3 \in H_1$. Let

$$\lambda_3 - \lambda_1 = (x_{3,1}, y_{3,1}, z_{3,1}),\ \lambda_3 - \lambda_2 = (x_{3,2}, y_{3,2}, z_{3,2}) \in B_1 \cup B_2 \cup B_3.$$

Then $x_{3,1}, x_{3,2} \in G$ by Lemma 4.4 (i). Combining with Lemma 4.5 (iii), we have $y_{3,1}, y_{3,2}, z_{3,1}, z_{3,2} \in H_1$. Then $y_{2,1} = y_{3,1} - y_{3,2} \in G$, $z_{2,1} = z_{3,1} - z_{3,2} \in G$ by Lemma 4.4 (i). This is a contradiction to $H_1 \cap G = \emptyset$.

Case 2. $x_3 \in K$. From Lemma 4.4 (iii), we know $x_{3,2}, x_{3,1} \in K$, and because of the Lemma 4.5 (i) we obtain $\lambda_3 - \lambda_2, \lambda_3 - \lambda_1 \in B_2 \cup B_3$.

- If $\lambda_3 - \lambda_2, \lambda_3 - \lambda_1 \in B_2$, on the basic of Lemma 4.5 (i), we have $y_{3,2}, y_{3,1} \in H_1$. Hence $y_{2,1} \in G$, a contradiction to $H_1 \cap G = \emptyset$.
- If $\lambda_3 - \lambda_2, \lambda_3 - \lambda_1 \in B_3$, on the basic of Lemma 4.5 (i), we have $z_{3,2}, z_{3,1} \in H_1$. Hence $z_{2,1} \in G$, a contradiction to $H_1 \cap G = \emptyset$.
- If $\lambda_3 - \lambda_2 \in B_2$, $\lambda_3 - \lambda_1 \in B_3$ and $\lambda_2 - \lambda_1 \in B_2$, by Lemma 4.5 (i), we obtain $y_{3,2}, y_{2,1} \in H_1$, then we have $y_{3,1} \in G$. Applying Lemma 4.5 (iii), we know $x_{3,1}, z_{3,1} \in H_1$. This contradicts to $H_1 \cap K = \emptyset$.

Case 3. $x_3, x_4 \in G$. By $\lambda_3, \lambda_4 \in B_1 \cup B_2 \cup B_3$ and Lemma 4.5 (iii), we have $y_3, y_4, z_3, z_4 \in H_1$, therefore $y_{4,3}, z_{4,3} \in G$. Since $\lambda_4 - \lambda_3 \in B_1 \cup B_2 \cup B_3$, we have $x_{4,3}, y_{4,3} \in H_1$ by Lemma 4.5 (iii), this is a contradiction.

Now we construct a bi-zero set of $\mu_{M,\{D_n\}}$ to show that the number 4 is the best. Let

$$\Lambda = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{p}{2a_1} \\ -\frac{q}{2b_1} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{p}{2a_1} \\ 0 \\ \frac{r}{2c_1} \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{q}{2b_1} \\ \frac{r}{2c_1} \end{pmatrix} \right\}.$$

It is easy to see $(\Lambda - \Lambda) \backslash \{0\} \subset \mathcal{Z}(\hat{\mu}_{M,\{D_n\}})$. The proof is complete. □

At the end of this paper, we propose some nature questions.

Question 4.6 If $p, q, r \in 2\mathbb{Z}+1$ and $a_n, b_n, c_n \in 2\mathbb{Z}$, what is the exact number of mutually orthogonal exponential functions in $L^2(\mu_{M,\{D_n\}})$?

Question 4.7 Does there exist an infinite orthogonal exponential functions in $L^2(\mu_{M,\{D_n\}})$ if and only if two of the three numbers p, q, r are even?

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