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ORIGINAL PAPER

Singular value inequalities for operators and matrices

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Abstract
Let $A_1, A_2, B_1, B_2, X_1, X_2, Y_1$ and Y_2 be compact operators on a complex separable Hilbert space. Then

$$2s_j(A_1^*X_1^*X_2A_2 + B_1^*Y_1^*Y_2B_2) \leq s_j((L + |N|) \oplus (M + |N^*|))$$

for $j = 1, 2, \ldots$ where

$$L = X_1A_1A_1^*X_1^* + X_2A_2A_2^*X_2^*,$$
$$M = Y_1B_1B_1^*Y_1^* + Y_2B_2B_2^*Y_2^*$$

and

$$N = Y_1B_1A_1^*X_1^* + Y_2B_2A_2^*X_2^*.$$

Several singular value inequalities for compact operators and matrices are also given.

Keywords Singular value · Compact operator · Positive operator · Matrix

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1 Introduction

Let $\mathbb{B}(\mathbb{H})$ denote the space of all bounded linear operators on a complex separable Hilbert space $\mathbb{H}$ and let $\mathbb{K}(\mathbb{H})$ denote the two-sided ideal of compact operators in $\mathbb{B}(\mathbb{H})$. For $A \in \mathbb{K}(\mathbb{H})$, the singular values of A denoted by $s_1(A), s_2(A), \ldots$ are the eigenvalues of the positive operator $|A| = (A^*A)^{1/2}$, which is denoted by $|A| \geq 0$, enumerated as

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$s_1(A) \geq s_2(A) \geq \cdots$ and repeated according to multiplicity. Properties of singular values where $A, B \in \mathbb{K}(\mathbb{H})$ are listed below:

(a)
$$s_j(A) = s_j(A^*) = s_j(|A|) = s_j(|A^*|) \tag{1}$$
for $j = 1, 2, \ldots$

(b) If $A, B \geq 0$ and $A \leq B$, then
$$s_j(A) \leq s_j(B) \tag{2}$$
for $j = 1, 2, \ldots$ This fact follows by applying Weyl's monotonicity principle (see, e.g., [7, p. 63] or [10, p. 26]). Moreover, $s_j(A) \leq s_j(B)$ if and only if $s_j(A \oplus A) \leq s_j(B \oplus B)$ for $j = 1, 2, \ldots$. Here, we use the direct sum notation $A \oplus B$ for the block-diagonal operator $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ defined on $\mathbb{H} \oplus \mathbb{H}$.

(c)
$$s_j \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = s_j \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix} \tag{3}$$
for $j = 1, 2, \ldots$, and they consist of those of A together with those of B.

Some related inequalities with our study are summarized below where $A, B, X, Y \in \mathbb{K}(\mathbb{H})$:

Bhatia and Kittaneh proved in [8] that if A is self-adjoint, $B \geq 0$ and $\pm A \leq B$, then
$$s_j(A) \leq s_j(B \oplus B) \tag{4}$$
for $j = 1, 2, \ldots$

Audeh and Kittaneh obtained in [6] an equivalent inequality of (4):

If $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then
$$s_j(B) \leq s_j(A \oplus C) \tag{5}$$
for $j = 1, 2, \ldots$

Bhatia and Kittaneh in [9] obtained the arithmetic-geometric mean inequality of singular values,
$$2s_j(AB^*) \leq s_j(A^*A + B^*B) \tag{6}$$
for $j = 1, 2, \ldots$ Zhan proved in [12] that if $A, B \geq 0$, then
$$s_j(A - B) \leq s_j(A \oplus B) \tag{7}$$
for $j = 1, 2, \ldots$ Hirzallah in [11] generalized inequality (6):
$$\sqrt{2} s_j\left(|A_1 A_2^* + A_3 A_4^*|^{1/2} \right) \leq s_j\left(\begin{bmatrix} A_1 & A_3 \\ A_2 & A_4 \end{bmatrix} \right) \tag{8}$$
for $j = 1, 2, \ldots$ Audeh in [4] gave another generalization of inequality (6):

$$s_j(AXY^*B^*) \leq \frac{1}{2}s_j\left(X^*|A|^2X + Y^*|B|^2Y\right) \tag{9}$$

for $j = 1, 2, \ldots$ Moreover, it has been shown in the same paper that if $X_i, Y_i \geq 0$, $i = 1, 2, \ldots, n$. Then

$$2s_j\left(\sum_{i=1}^{n} A_i X_i^{1/2} Y_i^{1/2} B_i^*\right) \leq s_j^2(W) \tag{10}$$

for $j = 1, 2, \ldots$, where $W = \begin{bmatrix} A_1X_1^{1/2} & A_2X_2^{1/2} & \ldots & A_nX_n^{1/2} \\ B_1Y_1^{1/2} & B_2Y_2^{1/2} & \ldots & B_nY_n^{1/2} \end{bmatrix}$. Several results are demonstrated as special cases for this inequality, some of these results are summarized below:

(i) Let $X, Y \geq 0$. Then

$$2s_j\left(AX^{1/2}Y^{1/2}B^* + BX^{1/2}Y^{1/2}A^*\right) \leq s_j^2\left(\begin{bmatrix} AX^{1/2} & BX^{1/2} \\ BY^{1/2} & AY^{1/2} \end{bmatrix}\right) \tag{11}$$

for $j = 1, 2, \ldots$ In particular, replacing Y by X in inequality (11), leads to the following inequality:

$$2s_j(AXB^* + BXA^*) \leq s_j^2\left(\begin{bmatrix} AX^{1/2} & BX^{1/2} \\ BX^{1/2} & AX^{1/2} \end{bmatrix}\right) \tag{12}$$

for $j = 1, 2, \ldots$

(ii) Let $A, B, X \geq 0$. Then

$$s_j\left(A^{1/2}XA^{1/2} + B^{1/2}XB^{1/2}\right) \leq s_j((P + |Q^*|) \oplus (R + |Q|)) \tag{13}$$

for $j = 1, 2, \ldots$, where $P = X^{1/2}AX^{1/2}$, $Q = X^{1/2}A^{1/2}B^{1/2}X^{1/2}$, and $R = X^{1/2}BX^{1/2}$. Let $X = I$, we have

$$s_j(A + B) \leq s_j\left(\left(A + \left|B^{1/2}A^{1/2}\right|\right) \oplus \left(B + \left|A^{1/2}B^{1/2}\right|\right)\right) \tag{14}$$

for $j = 1, 2, \ldots$

(iii) Let $X_1, X_2, Y_1, Y_2 \geq 0$. Then

$$2s_j(E - F) \leq s_j((H + |L^*|) \oplus (K + |L|)) \tag{15}$$

for $j = 1, 2, \ldots$, where $E = AX_1^{1/2}Y_1^{1/2}A^*$, $F = BX_2^{1/2}Y_2^{1/2}B^*$, $H = X_1^{1/2}A^*AX_1^{1/2} + Y_1^{1/2}A^*AY_1^{1/2}$, $L = X_1^{1/2}A^*BX_2^{1/2} - Y_1^{1/2}A^*BY_2^{1/2}$, and $K = X_2^{1/2}B^*BX_2^{1/2} + Y_2^{1/2}B^*BY_2^{1/2}$. For recent studies about generalizations and applications for singular value inequalities, we refer the reader to [1–6].

In Sect. 2, we provide generalizations of the inequalities (6)–(15).

2 Singular value inequalities for compact operators

The following lemma is well-known.

Lemma 2.1 *Let A be self-adjoint. Then*

$$\pm A \leq |A|. \tag{16}$$

We are ready to state the first main result in this section.

Theorem 2.2 *Let* $A_1, A_2, B_1, B_2, X_1, X_2, Y_1, Y_2 \in \mathbb{K}(\mathbb{H})$. *Then*

$$2s_j(A_1^*X_1^*X_2A_2 + B_1^*Y_1^*Y_2B_2) \leq s_j((L + |N|) \oplus (M + |N^*|)) \tag{17}$$

for $j = 1, 2, \ldots$ *where*

$$L = X_1A_1A_1^*X_1^* + X_2A_2A_2^*X_2^*,$$
$$M = Y_1B_1B_1^*Y_1^* + Y_2B_2B_2^*Y_2^*$$

and

$$N = Y_1B_1A_1^*X_1^* + Y_2B_2A_2^*X_2^*.$$

Proof In what follows in this proof, let

$$E = \begin{bmatrix} X_1A_1A_1^*X_1^* + X_2A_2A_2^*X_2^* & 0 \\ 0 & Y_1B_1B_1^*Y_1^* + Y_2B_2B_2^*Y_2^* \end{bmatrix}$$

and

$$F = \begin{bmatrix} 0 & X_1A_1B_1^*Y_1^* + X_2A_2B_2^*Y_2^* \\ Y_1B_1A_1^*X_1^* + Y_2B_2A_2^*X_2^* & 0 \end{bmatrix}.$$

Let $S = \begin{bmatrix} X_1A_1 & 0 \\ Y_1B_1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} X_2B_2 & 0 \\ Y_2B_2 & 0 \end{bmatrix}$. Note that

$$S^*T = A_1^*X_1^*X_2A_2 + B_1^*Y_1^*Y_2B_2$$

and

$$SS^* + TT^* = C + D$$

where

$$C = \begin{bmatrix} X_1A_1A_1^*X_1^* & X_1A_1B_1^*Y_1^* \\ Y_1B_1A_1^*X_1^* & Y_1B_1B_1^*Y_1^* \end{bmatrix}$$

and

$$D = \begin{bmatrix} X_2A_2A_2^*X_2^* & X_2A_2B_2^*Y_2^* \\ Y_2B_2A_2^*X_2^* & Y_2B_2B_2^*Y_2^* \end{bmatrix}.$$

Apply inequality (6) for the operator matrices S and T, we get

$$\begin{aligned} 2s_j(A_1^*X_1^*X_2A_2 + B_1^*Y_1^*Y_2B_2) &\le s_j(C+D) \\ &= s_j(E+F) \\ &= s_j\begin{bmatrix} L & N^* \\ N & M \end{bmatrix} \\ &= s_j\left(\begin{bmatrix} L & 0 \\ 0 & M \end{bmatrix} + \begin{bmatrix} 0 & N^* \\ N & 0 \end{bmatrix}\right) \\ &\le s_j\left(\begin{bmatrix} L & 0 \\ 0 & M \end{bmatrix} + \begin{bmatrix} |N| & 0 \\ 0 & |N^*| \end{bmatrix}\right), \\ &\text{(By applying inequalities (1.2) and (2.1)).} \\ &= s_j((L+|N|) \oplus (M+|N^*|)). \end{aligned}$$

Thus inequality (17) has thus been substantiated. □

In the following, we will see some special cases of inequality (17).

Remark 2.3 Letting $X_1 = X_2 = I$, $B_1 = B_2 = Y_1 = Y_2 = 0$ in inequality (17), we give inequality (6).

Remark 2.4 Letting $A_1 = A_2 = A^{1/2}$, $B_1 = -B_2 = B^{1/2}$, $X_1 = X_2 = Y_1 = Y_2 = I$ in inequality (17), implies inequality (7).

Remark 2.5 Letting $B_1 = B_2 = Y_1 = Y_2 = 0$ in inequality (17), leads to inequality (9).

Remark 2.6 Letting $A_1 = A_2 = A^{1/2}$, $B_1 = B_2 = B^{1/2}$, and $X_1 = X_2 = Y_1 = Y_2 = I$ in inequality (17), one can get inequality (14).

In the following, we will present special case of inequality (17) which in turns a generalization of several known results.

Corollary 2.7 *Let* $A_1, A_2, B_1, B_2, X \in \mathbb{K}(\mathbb{H})$ *such that* $X \ge 0$. *Then*

$$2s_j(A_1^*XA_2 + B_1^*XB_2) \le s_j((L+|N|) \oplus (M+|N^*|)) \tag{18}$$

for $j = 1, 2, \ldots$ *where*

$$\begin{aligned} L &= X^{1/2}\left(A_1A_1^* + A_2A_2^*\right)X^{1/2}, \\ M &= X^{1/2}\left(B_1B_1^* + B_2B_2^*\right)X^{1/2} \end{aligned}$$

and

$$N = X^{1/2}B_1A_1^*X^{1/2} + X^{1/2}B_2A_2^*X^{1/2}.$$

Proof Letting $X_1 = X_2 = Y_1 = Y_2 = X^{1/2}$ in inequality (17), we give inequality (18). □

Example Let $A_1 = B_1 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, $A_2 = B_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $X = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$. Then $2s_j(A_1^*XA_2 + B_1^*XB_2) = 16, 4, 0, 0$ for $j = 1, 2, 3, 4$, and $s_j((L + |N|) \oplus (M + |N^*|)) = 16, 16, 4, 4$ for $j = 1, 2, 3, 4$.

For $A, B, X \in \mathbb{B}(\mathbb{H})$, an operator of the form $AX - XA$ is called a commutator and an operator of the form $AX + XA$ is called anticommutator. Now we are ready to state the following generalization of singular value inequality for anticommutators.

Corollary 2.8 *Let* $A, B, X \in \mathbb{K}(\mathbb{H})$ *such that* $X \geq 0$. *Then*

$$2s_j(AXB + BXA) \leq s_j((L + |N|) \oplus (M + |N^*|)) \tag{19}$$

for $j = 1, 2, \ldots$ *where*

$$L = X^{1/2}(A^*A + BB^*)X^{1/2},$$
$$M = X^{1/2}(AA^* + B^*B)X^{1/2}$$

and

$$N = X^{1/2}B^*AX^{1/2} + X^{1/2}AB^*X^{1/2}.$$

Proof Let $A_1^* = B_2 = A$ and $A_2 = B_1^* = B$ in inequality (18), we give inequality (19). □

Example Let $A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2i \\ i & 0 \end{bmatrix}$, and $X = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$. Then $2s_j(AXB + BXA) = 24, 6, 0, 0$ for $j = 1, 2, 3, 4$, and $s_j((L + |N|) \oplus (M + |N^*|)) = 32, 20, 8, 5$ for $j = 1, 2, 3, 4$.

A remarkable inequality for singular value inequalities of anticommutators is now ready to present.

Corollary 2.9 *Let* $A, B \in \mathbb{K}(\mathbb{H})$. *Then*

$$2s_j(AB + BA) \leq s_j((L + |N|) \oplus (M + |N^*|)) \tag{20}$$

for $j = 1, 2, \ldots$ *where*

$$L = (A^*A + BB^*),$$
$$M = (AA^* + B^*B)$$

and

$$N = B^*A + AB^*.$$

Proof Letting $X = I$ in inequality (19), we give inequality (20). □

Corollary 2.10 *Let* $A_1, A_2, B_1, B_2, X_1, X_2, Y_1, Y_2 \in \mathbb{K}(\mathbb{H})$. *Then*

$$2s_j(A_1^*X_1^*X_2A_2 - B_1^*Y_1^*Y_2B_2) \le s_j((L + |N|) \oplus (M + |N^*|)) \tag{21}$$

for $j = 1, 2, \ldots$ *where*

$$L = X_1A_1A_1^*X_1^* + X_2A_2A_2^*X_2^*,$$
$$M = Y_1B_1B_1^*Y_1^* + Y_2B_2B_2^*Y_2^*$$

and

$$N = Y_1B_1A_1^*X_1^* - Y_2B_2A_2^*X_2^*.$$

Proof Substituting B_2 by $-B_2$ in inequality (17), we give inequality (21). □

We will present the following inequality which extends singular value inequality of commutators.

Corollary 2.11 *Let* $A_1, A_2, B_1, B_2 \in \mathbb{K}(\mathbb{H})$. *Then*

$$2s_j(A_1^*A_2 - B_1^*B_2) \le s_j((L + |N|) \oplus (M + |N^*|)) \tag{22}$$

for $j = 1, 2, \ldots$ *where*

$$L = A_1A_1^* + A_2A_2^*,$$
$$M = B_1B_1^* + B_2B_2^*$$

and

$$N = B_1A_1^* - B_2A_2^*.$$

Proof Letting $X_1 = X_2 = Y_1 = Y_2 = I$ in inequality (21), we give inequality (22). □

Now we state the singular value inequality of commutators.

Corollary 2.12 *Let* $A, B \in \mathbb{K}(\mathbb{H})$. *Then*

$$2s_j(AB - BA) \le s_j((L + |N|) \oplus (M + |N^*|)) \tag{23}$$

for $j = 1, 2, \ldots$ *where*

$$L = A^*A + BB^*,$$
$$M = AA^* + B^*B$$

and

$$N = B^*A - AB^*.$$

Proof Letting $A_1^* = B_2 = A$ and $A_2 = B_1^* = B$ in inequality (22), we give inequality (23). □

We are ready to state the second general result of this section.

Theorem 2.13 *Let* $A_i, B_i, X_i, Y_i \in \mathbb{K}(\mathbb{H})$, $i = 1, 2, 3, 4$. *Then*

$$2s_j(K + L) \le s_j((O + |T|) \oplus (V + |T^*|)) \tag{24}$$

where

$$K = A_1^*X_1^*X_3A_3 + A_2^*X_2^*X_4A_4,$$
$$L = B_1^*Y_1^*Y_3B_3 + B_2^*Y_2^*Y_4B_4,$$
$$O = \begin{bmatrix} X_1A_1A_1^*X_1^* + X_3A_3A_3^*X_3^* & X_1A_1A_2^*X_2^* + X_3A_3A_4^*X_4^* \\ X_2A_2A_1^*X_1^* + X_4A_4A_3^*X_3^* & X_2A_2A_2^*X_2^* + X_4A_4A_4^*X_4^* \end{bmatrix},$$
$$V = \begin{bmatrix} Y_1B_1B_1^*Y_1^* + Y_3B_3B_3^*Y_3 & Y_1B_1B_2^*Y_2^* + Y_3B_3B_4^*Y_4^* \\ Y_2B_2B_1^*Y_1^* + Y_4B_4B_3^*Y_3^* & Y_2B_2B_2^*Y_2^* + Y_4B_4B_4^*Y_4^* \end{bmatrix}$$

and

$$T = \begin{bmatrix} Y_1B_1A_1^*X_1^* + Y_3B_3A_3^*X_3 & Y_1B_1A_2^*X_2^* + Y_3B_3A_4^*X_4^* \\ Y_2B_2A_1^*X_1^* + Y_4B_4A_3^*X_3^* & Y_2B_2A_2^*X_2^* + Y_4B_4A_4^*X_4^* \end{bmatrix}$$

for $j = 1, 2, ...$

Proof On $\oplus_{j=1}^2 H$, let $C_1 = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix}$, $C_2 = \begin{bmatrix} A_3 & 0 \\ A_4 & 0 \end{bmatrix}$, $D_1 = \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix}$, $D_2 = \begin{bmatrix} B_3 & 0 \\ B_4 & 0 \end{bmatrix}$, $S_1 = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$, $S_2 = \begin{bmatrix} X_3 & 0 \\ 0 & X_4 \end{bmatrix}$, $T_1 = \begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix}$, and $T_2 = \begin{bmatrix} Y_3 & 0 \\ 0 & Y_4 \end{bmatrix}$. It follows that $C_1^*S_1^*S_2C_2 + D_1^*T_1^*T_2D_2 = K + L$, $S_1C_1C_1^*S_1^* + S_2C_2C_2^*S_2^* = O$, $T_1D_1D_1^*T_1^* + T_2D_2D_2^*T_2^* = V$ and $T_1D_1C_1^*S_1^* + T_2D_2C_2^*S_2^* = T$. Substitute the operators $A_1, A_2, B_1, B_2, X_1, X_2, Y_1$ and Y_2 by $C_1, C_2, D_1, D_2, S_1, S_2, T_1$ and T_2, respective'ly, in inequality (17), we give inequality (24). □

Inequality (24) is an extension of several known results, some of them are listed below.

Remark 2.14 Letting $B_i = 0$ for $i = 1, 2, 3, 4$, $X_1 = X_1^{1/2}$, $X_2 = X_2^{1/2}$, $X_3 = Y_1^{1/2}$, $X_4 = Y_2^{1/2}$ in inequality (24), we give inequality (10) for $n = 2$.

Remark 2.15 Letting $B_i = Y_i = 0$ for $i = 1, 2, 3, 4$, $A_1 = A_4 = A$, $A_2 = A_3 = B$, $X_1 = X_2 = X^{1/2}$ and $X_3 = X_4 = Y^{1/2}$in inequality (24), one can get inequality (11).

Remark 2.16 Letting $B_i = Y_i = 0$ for $i = 1, 2, 3, 4$, $A_1 = A_4 = A$, $A_2 = A_3 = B$, $X_1 = X_2 = X_3 = X_4 = X^{1/2}$ in inequality (24), leads to inequality (12).

Remark 2.17 Letting $B_i = Y_i = 0$ and $X_i = I$ for $i = 1, 2, 3, 4$ in inequality (24), implies inequality (8).

Remark 2.18 Letting $B_i = Y_i = 0$ for $i = 1, 2, 3, 4$ and $A_i = 0$ for $i = 2, 4$ in inequality (24), we have inequality (9).

Remark 2.19 Letting $B_i = Y_i = 0$ for $i = 1, 2, 3, 4$, $A_i = 0$ for $i = 2, 4$, and $X_1 = X_3 = I$ in inequality (24), we get inequality (6).

We will give a special case of inequality (24) which is a generalization of inequality (13).

Corollary 2.20 *Let* $A, B, X, Y \in \mathbb{K}(\mathbb{H}) \geq 0$. *Then*

$$s_j\left(A^{1/2}XA^{1/2} + B^{1/2}YB^{1/2}\right) \leq s_j((P + |Q^*|) \oplus (R + |Q|) \tag{25}$$

for $j = 1, 2, ...$ *where* $P = X^{1/2}AX^{1/2}$, $Q = X^{1/2}A^{1/2}B^{1/2}Y^{1/2}$ *and* $R = Y^{1/2}BY^{1/2}$. *In particular, letting* $Y = X$ *in inequality* (25), *we give*

$$s_j\left(A^{1/2}XA^{1/2} + B^{1/2}XB^{1/2}\right) \leq s_j((P + |T^*|) \oplus (S + |T|)$$

for $j = 1, 2, ...,$ *where* $P = X^{1/2}AX^{1/2}$, $T = X^{1/2}A^{1/2}B^{1/2}X^{1/2}$ *and* $S = X^{1/2}BX^{1/2}$. *Moreover, letting* $X = I$ *in inequality* (13), *we give inequality* (14).

Proof Letting $B_i = Y_i = 0$ for $i = 1, 2, 3, 4$, $A_1 = A_3 = A^{1/2}$, $A_2 = A_4 = B^{1/2}$, $X_1 = X_3 = X^{1/2}$ and $X_2 = X_4 = Y^{1/2}$ in inequality (24), leads to

$$\begin{aligned} &2s_j(A^{1/2}XA^{1/2} + B^{1/2}YB^{1/2}) \\ &\leq 2s_j\left(\begin{bmatrix} X^{1/2}AX^{1/2} & X^{1/2}A^{1/2}B^{1/2}Y^{1/2} \\ Y^{1/2}B^{1/2}A^{1/2}X^{1/2} & Y^{1/2}BY^{1/2} \end{bmatrix}\right) \\ &= 2s_j\left(\begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix}\right) \\ &= 2s_j\left(\begin{bmatrix} P & 0 \\ 0 & R \end{bmatrix} + \begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix}\right). \end{aligned} \tag{26}$$

But

$$\begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} \leq \left|\begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix}\right| = \begin{bmatrix} |Q^*| & 0 \\ 0 & |Q| \end{bmatrix}$$

combining this with inequality (26), one can get

$$\begin{aligned} & s_j(A^{1/2}XA^{1/2} + B^{1/2}YB^{1/2}) \\ & \le s_j\left(\begin{bmatrix} P & 0 \\ 0 & R \end{bmatrix} + \begin{bmatrix} |Q^*| & 0 \\ 0 & |Q| \end{bmatrix}\right) \\ & = s_j((P + |Q^*|) \oplus (R + |Q|)). \end{aligned} \tag{27}$$

□

In the following, we will give another special case of inequality (24) which has been proved in [13].

Corollary 2.21 *Let* $A, B, X_1, X_2, Y_1, Y_2 \in \mathbb{K}(\mathbb{H})$ *such that* $X_1, X_2, Y_1, Y_2 \ge 0$. *Then*

$$2s_j(E - F) \le s_j((H + |L^*|) \oplus (K + |L|))$$

for $j = 1, 2, ...,$ *where* $E = AX_1^{1/2}Y_1^{1/2}A^*$, $F = BX_2^{1/2}Y_2^{1/2}B^*$, $H = X_1^{1/2}A^*AX_1^{1/2} + Y_1^{1/2}A^*AY_1^{1/2}$, $L = X_1^{1/2}A^*BX_2^{1/2} - Y_1^{1/2}A^*BY_2^{1/2}$ *and* $K = X_2^{1/2}B^*BX_2^{1/2} + Y_2^{1/2}B^*BY_2^{1/2}$.

Proof Letting $B_i = Y_i = 0$ for $i = 1, 2, 3, 4$, $A_1 = A_3 = A^*$, $A_2 = -A_4 = B^*$, $X_1 = X_1^{1/2}$, $X_2 = X_2^{1/2}$, $X_3 = Y_1^{1/2}$ and $X_4 = Y_2^{1/2}$ in inequality (24), we give

$$\begin{aligned} & 2s_j\left(AX_1^{1/2}Y_1^{1/2}A^* - BX_2^{1/2}Y_2^{1/2}B^*\right) \\ & \le s_j\left(\begin{bmatrix} H & L \\ L^* & K \end{bmatrix}\right) \\ & = s_j\left(\begin{bmatrix} H & 0 \\ 0 & K \end{bmatrix} + \begin{bmatrix} 0 & L \\ L^* & 0 \end{bmatrix}\right) \\ & \le s_j\left(\begin{bmatrix} H & 0 \\ 0 & K \end{bmatrix} + \begin{bmatrix} |L^*| & 0 \\ 0 & |L| \end{bmatrix}\right) \\ & = s_j((H + |L^*|) \oplus (K + |L|)), \end{aligned}$$

which is exactly inequality (15). □

3 Singular value inequalities for matrices

Let $\mathbb{M}_n$ be the space of all $n \times n$ complex matrices In this section, attractive generalizations of inequalities (6) and (7) for matrices are proved.

Bhatia and Kittaneh in [8] proved that if $A, B \in \mathbb{M}_n$ and $Q = AA^* + BB^*$, then

$$s_j(AB^* + BA^*) \le s_j(Q \oplus Q) \tag{28}$$

for$j = 1, 2, ..., n$. Among our results in this section, we obtained an inequality that is sharper than inequality (28).

Recently in [13] a new generalization of inequality (6) has been given: If $A, B, X \in \mathbb{M}_n$ such that $X \geq 0$, then

$$2s_j(AXB^*) \leq s_j\left[\left(|A|^2 + |B|^2\right)^{1/2} X \left(|A|^2 + |B|^2\right)^{1/2}\right] \tag{29}$$

for $j = 1, 2, ..., n$. In this section, we have established singular value inequality that is equivalent to inequality (29). Several relevant singular value inequalities are also given.

We start this section with the following lemmas.

Lemma 3.1 *Let A be self-adjoint matrix. Then*

$$\pm A \leq |A| \tag{30}$$

Lemma 3.2 *Let* $A, B, X \in \mathbb{M}_n$ *such that* $X \geq 0$. *Then*

$$s_j(AXB^*) \leq \frac{1}{2} s_j\left(X^{1/2}|A|^2 X^{1/2} + X^{1/2}|B|^2 X^{1/2}\right) \tag{31}$$

for $j = 1, 2, ..., n$.

Proof Inequality (31) is a direct consequence of inequality (6) by substituting $A = AX^{1/2}$ and $B = BX^{1/2}$. □

Corollary 3.3 *Let*$A, B, X \geq 0$. *Then*

$$s_j(A^{1/2}XB^{1/2}) \leq \frac{1}{2} s_j\left(X^{1/2}AX^{1/2} + X^{1/2}BX^{1/2}\right) \tag{32}$$

for $j = 1, 2, ..., n$.

Proof Inequality (32) is followed from Lemma 3.2 by substituting $A = A^{1/2}$ and $B = B^{1/2}$. □

Now, we can present the first result of this section, which is an impressive generalization of arithmetic–geometric mean inequality.

Theorem 3.4 *Let* $A_i, B_i, X_i \in \mathbb{M}_n$ *such that* $X_i \geq 0$ *for* $i = 1, 2, ..., n$,

$$K = \begin{bmatrix} A_1^*A_1 + B_1^*B_1 & \cdots & A_1^*A_n + B_1^*B_n \\ A_2^*A_1 + B_2^*B_1 & \cdots & A_2^*A_n + B_2^*B_n \\ \vdots & \ddots & \vdots \\ A_n^*A_1 + B_n^*B_1 & \cdots & A_n^*A_n + B_n^*B_n \end{bmatrix}$$

and

$$X = \begin{bmatrix} X_1 & 0 & \dots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_n \end{bmatrix},$$

then

$$2s_j\left(\sum_{i=1}^{n} A_iX_iB_i^*\right) \le s_j\left(K^{1/2}XK^{1/2}\right) \tag{33}$$

for $j = 1, 2, ..., n$.

Proof Replace $A = \begin{bmatrix} A_1 & A_2 & \dots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$, $B = \begin{bmatrix} B_1 & B_2 & \dots & B_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ and $X = \begin{bmatrix} X_1 & 0 & \dots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_n \end{bmatrix}$ in inequality (29), we get inequality (33). □

Remark 3.5 Substituting $A_i = B_i = X_i = 0$ for $i = 2, 3, ..., n$ in inequality (33), leads to inequality (29), the way to show that inequalities (29) and (33) are equivalent.

Corollary 3.6 *Let* $A_i, B_i, X_i \in \mathbb{M}_n$ *such that* $X_i \ge 0$ *for* $i = 1, 2,$

$$L = \begin{bmatrix} A_1^*A_1 + B_1^*B_1 & A_1^*A_2 + B_1^*B_2 \\ A_2^*A_1 + B_2^*B_1 & A_2^*A_2 + B_2^*B_2 \end{bmatrix}$$

and

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}.$$

Then

$$2s_j\left(\sum_{i=1}^{2} A_iX_iB_i^*\right) \le s_j\left(L^{1/2}XL^{1/2}\right). \tag{34}$$

for $j = 1, 2, ..., n$.

Proof Specifies inequality (33) to $n = 2$, we give inequality (34). □

Remark 3.7 Substituting $X_1 = X_2 = I$ in inequality (34), we give inequality (8).

Depending on inequality (33), we now present our next inequality.

Corollary 3.8 *Let* $A, B, X_1, X_2 \in \mathbb{M}_n$ *such that* $X_1, X_2 \geq 0$, *where*

$$L = \begin{bmatrix} A^*A + B^*B & A^*B + B^*A \\ A^*B + B^*A & A^*A + B^*B \end{bmatrix} \text{ and } X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}.$$

Then

$$2s_j\left(AX_1B^* + BX_2A^*\right) \leq s_j\left(L^{1/2}XL^{1/2}\right) \tag{35}$$

for $j = 1, 2, ..., n$.

Proof Inequality (35) follows by substituting $n = 2$, $A_1 = B_2 = A$ and $A_2 = B_1 = B$ in inequality (33). □

Depending on inequality (35), we now present our next result, which is a refinement of inequality (28).

Corollary 3.9 *Let* $A, B \in \mathbb{M}_n$, $Q_1 = AA^* + BB^* + AB^* + BA^*$, $Q_2 = AA^* + BB^* - AB^* - BA^*$. *Then*

$$2s_j(AB^* + BA^*) \leq s_j(Q_1 \oplus Q_2) \tag{36}$$

for $j = 1, 2, ...n$.

Proof Substituting $X = I$ in inequality (35), we give

$$\begin{aligned} 2s_j(AB^* + BA^*) \leq& s_j\left(\begin{bmatrix} A^*A + B^*B & A^*B + B^*A \\ B^*A + A^*B & A^*A + B^*B \end{bmatrix}\right) \\ =& s_j\left(\begin{bmatrix} A^* & B^* \\ B^* & A^* \end{bmatrix}\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \\ =& s_j^2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \\ =& s_j^2\left(\begin{bmatrix} A^* & B^* \\ B^* & A^* \end{bmatrix}\right) \\ =& s_j^2\left(\begin{bmatrix} A^* + B^* & 0 \\ 0 & A^* - B^* \end{bmatrix}\right) \\ & \text{(Since unitarily equivalent matrices} \\ & \text{have the same singular values)} \\ =& s_j\left(\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}\right), \end{aligned}$$

which is precisely inequality (36). □

Remark 3.10 In view of the fact that $\pm(AB^* + BA^*) \leq AA^* + BB^*$ and Weyl's monotonicity principle, one can see that inequality (36) is sharper than inequality (28).

Depending on inequality (35), we now present our next result, which is another refinement of inequality (28).

Corollary 3.11 *Let* $A, B \in \mathbb{M}_n, Q = A^*A + B^*B + |A^*B + B^*A|$. *Then*

$$2s_j(AB^* + BA^*) \leq s_j(Q \oplus Q) \tag{37}$$

for $j = 1, 2, ...n$.

Proof Throughout this proof let $T = \begin{bmatrix} A^*A + B^*B & 0 \\ 0 & A^*A + B^*B \end{bmatrix}$, $Z = \begin{bmatrix} 0 & A^*B + B^*A \\ B^*A + A^*B & 0 \end{bmatrix}$. Substituting $X = I$ in inequality (35), we give

$$\begin{aligned} 2s_j(AB^* + BA^*) \leq & s_j\left(\begin{bmatrix} A^*A + B^*B & A^*B + B^*A \\ B^*A + A^*B & A^*A + B^*B \end{bmatrix}\right) \\ = & s_j(T + S) \\ \leq & s_j(|(T + S)|) \\ \leq & s_j(|T| + |S|) \\ = & s_j\left(T + \begin{bmatrix} |A^*B + B^*A| & 0 \\ 0 & |A^*B + B^*A| \end{bmatrix}\right) \\ = & s_j\left(\begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}\right). \end{aligned}$$

which is precisely inequality (37). □

Remark 3.12 By the fact that $|A^*B + B^*A| \leq A^*A + B^*B$ and by applying Weyl's monotonicity principle, one can see that inequality (37) is sharper than inequality (28).

The following result is an application of inequality (33).

Corollary 3.13 *Let* $A, B, \ X_1, X_2 \in \mathbb{M}_n \geq 0$. *Then*

$$s_j\left(A^{1/2}X_1A^{1/2} + B^{1/2}X_2B^{1/2}\right) \leq s_j(X^{1/2}JX^{1/2}) \tag{38}$$

for $j = 1, 2, ..., n$ where

$$J = \begin{bmatrix} A & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & B \end{bmatrix} \text{ and } X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}.$$

Proof Substituting $n = 2$, $A_1 = B_1 = A^{1/2}$, $A_2 = B_2 = B^{1/2}$ in inequality (33), leads to

$$2s_j(A^{1/2}X_1A^{1/2} + B^{1/2}X_2B^{1/2}) \le s_j\big((2J)^{1/2}X(2J)^{1/2}\big),$$

which implies that

$$\begin{aligned} s_j\big(A^{1/2}X_1A^{1/2} + B^{1/2}X_2B^{1/2}\big) \le& s_j(J^{1/2}XJ^{1/2})\\ =&\lambda_j(J^{1/2}XJ^{1/2})\\ =&\lambda_j(JX)\\ =&\lambda_j(X^{1/2}JX^{1/2})\\ =&s_j(X^{1/2}JX^{1/2}), \end{aligned}$$

as required. □

Corollary 3.14 *Let* $A, B,\ X_1, X_2 \in \mathbb{M}_n \ge 0$, *where*

$$\begin{aligned} &S_1 = X_1^{1/2}AX_1^{1/2}, S_2 = X_2^{1/2}B^{1/2}A^{1/2}X_1^{1/2},\\ &T_1 = X_2^{1/2}BX_2^{1/2} \text{ and } T_2 = X_1^{1/2}A^{1/2}B^{1/2}X_2^{1/2}. \end{aligned}$$

Then

$$2s_j(A^{1/2}X_1A^{1/2} + B^{1/2}X_2B^{1/2}) \le s_j\big(\big(S_1 + |S_2|\big) \oplus \big(T_1 + |T_2|\big)\big) \tag{39}$$

for $j = 1, 2, ...n$.

Proof Spreading inequality (38), leads to

$$\begin{aligned} s_j\big(A^{1/2}X_1A^{1/2} + B^{1/2}X_2B^{1/2}\big) \le& s_j(X^{1/2}JX^{1/2})\\ =&s_j\begin{bmatrix} S_1 & T_2\\ S_2 & T_1\end{bmatrix}\\ =&s_j\left|\begin{bmatrix} S_1 & T_2\\ S_2 & T_1\end{bmatrix}\right| \ (\text{Since } \begin{bmatrix} S_1 & T_2\\ S_2 & T_1\end{bmatrix} \ge 0)\\ =&s_j\left(\left|\begin{bmatrix} S_1 & 0\\ 0 & T_1\end{bmatrix} + \begin{bmatrix} 0 & T_2\\ S_2 & 0\end{bmatrix}\right|\right)\\ \le& s_j\left(\left|\begin{bmatrix} S_1 & 0\\ 0 & T_1\end{bmatrix}\right| + \left|\begin{bmatrix} 0 & T_2\\ S_2 & 0\end{bmatrix}\right|\right)\\ =&s_j\left(\begin{bmatrix} S_1 & 0\\ 0 & T_1\end{bmatrix} + \begin{bmatrix} |S_2| & 0\\ 0 & |T_2|\end{bmatrix}\right)\\ =&s_j\left(\begin{bmatrix} S_1 + |S_2| & 0\\ 0 & T_1 + |T_2|\end{bmatrix}\right), \end{aligned}$$

which is precisely inequality (39). □

Remark 3.15 Substituting $X_1 = X_2 = I$ in inequality (39), we give the following result which was proved in [5].

$$s_j(A+B) \le s_j\left(\left(A + \left|B^{1/2}A^{1/2}\right|\right) \oplus (B + \left|A^{1/2}B^{1/2}\right|)\right)$$

for $j = 1, 2, ...n$.

The following inequality is a generalization of inequality (7).

Corollary 3.16 *Let* $A,\ B,\ X_1, X_2 \in \mathbb{M}_n$ *such that* $X_1, X_2 \ge 0$. *Then*

$$s_j\left(AX_1A^* - BX_2B^*\right) \le s_j(X_1^{1/2}A^*AX_1^{1/2} \oplus X_2^{1/2}A^*AX_2^{1/2}) \tag{40}$$

for $j = 1, 2, ..., n$. *If* $A = B = I$, *we obtain inequality* (7), *and if* $X_1 = X_2 = I$, *then*

$$s_j(AA^* - BB^*) \le s_j(A^*A \oplus B^*B) \tag{41}$$

for $j = 1, 2, ..., n$.

Proof Substituting $n = 2,\ A_1 = B_1 = A,\ A_2 = -B_2 = B$, in inequality (33), where $Z = \begin{bmatrix} A^*A & 0 \\ 0 & B^*B \end{bmatrix}$ and $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$, leads to

$$\begin{aligned} 2s_j\left(AX_1A^* - BX_2B^*\right) &\le s_j\left(Z^{1/2}XZ^{1/2}\right) \\ &= \lambda_j\left(Z^{1/2}XZ^{1/2}\right) \\ &= \lambda_j(ZX) \\ &= \lambda_j\left(X^{1/2}ZX^{1/2}\right) \\ &= \lambda_j\left(X^{1/2}ZX^{1/2}\right) \\ &= s_j\left(X^{1/2}ZX^{1/2}\right), \end{aligned}$$

which is inequality (40). □

By making use of inequality (40) incites, we here by present the following theorem which has been proven in completely different technique in [13].

Theorem 3.17 *Let* $A, B, X \in \mathbb{M}_n$ *such that* $X \ge 0$. *Then*

$$2s_j(AXB^*) \le s_j\left((A^*A + B^*B)^{1/2}X(A^*A + B^*B)^{1/2}\right) \tag{42}$$

for $j = 1, 2, ..., n$.

Proof Let $C = \begin{bmatrix} A \\ B \end{bmatrix}$, $D = \begin{bmatrix} A \\ -B \end{bmatrix}$, $X_1 = X_2 = X$, and $W = X^{1/2}(A^*A + B^*B)X^{1/2}$. Then

$$CXC^* - DXD^* = \begin{bmatrix} 0 & 2AXB^* \\ 2BXA^* & 0 \end{bmatrix},$$

and

$$X^{1/2}C^*CX^{1/2} \oplus X^{1/2}D^*DX^{1/2} = W \oplus W,$$

Now, applying inequality (40), leads to

$$2s_j \begin{bmatrix} BXA^* & 0 \\ 0 & AXB^* \end{bmatrix} \le s_j((W) \oplus (W)).$$

This gives
$2s_j(AXB^*) \le s_j\left(X^{1/2}(A^*A + B^*B)X^{1/2}\right)$ for $j = 1, 2, ..., n$. as required. □

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References

1. Audeh, W.: Some generalizations for singular value inequalities of compact operators. Adv. Oper. Theory 14 (2021)
2. Audeh, W.: Singular value and norm inequalities of Davidson-Power type. J. Math. Inequal. **15**, 1311–1320 (2021)
3. Audeh, W.: Singular value inequalities and applications. Positivity **25**, 843–852 (2020)
4. Audeh, W.: Generalizations for singular value and arithmetic–geometric mean inequalities of operators. J. Math. Anal. Appl. 489 (2020)
5. Audeh, W.: Generalizations for singular value inequalities of operators. Adv. Oper. Theory **5**, 371–381 (2020)
6. Audeh, W., Kittaneh, F.: Singular value inequalites for compact operators. Linear Algebra Appl. **437**, 2516–2522 (2012)
7. Bhatia, R.: Matrix Analysis, GTM169. Springer, New York (1997)
8. Bhatia, R., Kittaneh, F.: The matrix arithmetic–geometric mean inequality revisited. Linear Algebra Appl. **428**, 2177–2191 (2008)
9. Bhatia, R., Kittaneh, F.: On the singular values of a product of operators. SIAM J. Matrix Anal. Appl. **11**, 272–277 (1990)
10. Gohberg, I.C., Krein, M.G.: Introduction to the Theory of Linear Nonselfadjoint Operators. Amer. Math. Soc., Providence (1969)
11. Hirzallah, O.: Inequalities for sums and products of operators. Linear Algebra Appl. **407**, 32–42 (2005)
12. Zhan, X.: Singular values of differences of positive semidefinite matrices. SIAM J. Matrix Anal. Appl. **22**(3), 819–823 (2000)
13. Zou, L.: An arithmetic–geometric mean inequality for singular values and its applications. Linear Algebra Appl. **528**, 25–32 (2017)