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# Singular value inequalities involving convex and concave functions of positive semidefinite matrices

**Fadi Alrimawi[1] · Omar Hirzallah[2] · Fuad Kittaneh[3]**

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## Abstract

Let $A$ and $B$ be $n \times n$ positive semidefinite matrices, and let $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta = 1$. Among other inequalities, it is shown that

(a) If $f$ is a non-negative concave function on $[0, \infty)$, then

$$s_j(\alpha f(A) + \beta f(B)) \leq s_j(f(\sqrt{2}|\alpha A + i\beta B|))$$

for $j = 1, \ldots, n$.

(b) If $f$ is a non-negative strictly increasing convex function on $[0, \infty)$ with $f(0) = 0$, *then*

$$s_j(f(\alpha A + \beta B)) \leq \sqrt{2} s_j(\alpha f(A) + i\beta f(B))$$

for $j = 1, \ldots, n$. Here $s_j(X)$ denotes the largest $j$th singular value of the matrix $X$.


**Keywords** Accretive-dissipative matrix · Positive semidefinite matrix · Singular value · Unitarily invariant norm · Convex function · Concave function · Inequality

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✉ Fuad Kittaneh
fkitt@ju.edu.jo

Fadi Alrimawi
f.rimawi@ammanu.edu.jo

Omar Hirzallah
o.hirzal@hu.edu.jo

[1] Department of Basic Sciences, Al-Ahliyyah Amman University, Amman, Jordan

[2] Department of Mathematics, The Hashemite University, Zarqa, Jordan

[3] Department of Mathematics, The University of Jordan, Amman, Jordan

## 1 Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. For a matrix $A \in \mathbb{M}_n(\mathbb{C})$, let $s_1(A), s_2(A), \dots, s_n(A)$ denote the singular values of $A$ (i.e., the eigenvalues of $|A| = (A^*A)^{1/2}$) arranged in decreasing order and repeated according to multiplicity.

A matrix $A \in \mathbb{M}_n(\mathbb{C})$ is called Hermitian if $A = A^*$. The notation $A \geq 0$ $(A > 0)$ is used to mean that $A$ is positive semidefinite (positive definite). If $A$ and $B$ are Hermitian and $A - B$ is positive semidefinite, then we write $A \geq B$.

A matrix $T \in \mathbb{M}_n(\mathbb{C})$ is called accretive-dissipative if in its Cartesian decomposition, $T = A + iB$, the matrices $A$ and $B$ are positive semidefinite, where $A = \mathrm{Re}T = \frac{T+T^*}{2}$ and $B = \mathrm{Im}T = \frac{T-T^*}{2i}$.

The spectral norm $||\cdot||$ is the norm defined on $\mathbb{M}_n(\mathbb{C})$ by $||A|| = \max\{ ||Ax|| : x \in \mathbb{C}^n, ||x|| = 1\}$. It is known (see, e.g., [2, p. 7]) that for every $A \in \mathbb{M}_n(\mathbb{C})$, we have

$$||A|| = s_1(A).$$

On $\mathbb{M}_n(\mathbb{C})$, a unitarily invariant norm $|||\cdot|||$ is a matrix norm that satisfies the invariance property $|||UAV||| = |||A|||$ for every $A \in \mathbb{M}_n(\mathbb{C})$ and for all unitary matrices $U, V \in \mathbb{M}_n(\mathbb{C})$.

If $A$ is a Hermitian matrix with eigenvalues $\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A)$, arranged in decreasing order and repeated according to multiplicity, then the minimax principle (see, e.g., [2, p. 58 ]) says that

$$\lambda_j(A) = \max_{\dim M = j} \min_{\substack{x \in M \\ \|x\| = 1}} \langle Ax, x\rangle \tag{1.1}$$

and if $A$ is any matrix, then

$$s_j(A) = \max_{\dim M = j} \min_{\substack{x \in M \\ \|x\| = 1}} \|Ax\|. \tag{1.2}$$

If $a$ and $b$ are real numbers, then we have

$$|a + b| \leq \sqrt{2}|a + ib|. \tag{1.3}$$

Matrix versions of this elementary and fundamental inequality have been given in [6]. It has been shown (see [6, Theorem 1.1]) that for positive semidefinite matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, we have

$$s_j(A + B) \leq \sqrt{2} s_j(A + iB) \tag{1.4}$$

for $j = 1, \dots, n$, which is stronger than the inequality

$$|||A + B||| \leq \sqrt{2}|||A + iB|||. \tag{1.5}$$

Many important results for the singular values of accretive-dissipative matrices have been discussed by several mathematicians. Some of these results have found interesting applications in physics and in the geometry of operator ideals. A useful reference for such results is the book [2]. Special results relating singular values and norms of $T = A + iB$ with those of $A$ and $B$ may be found in [1, 5, 7, 8], and in other papers cited therein.

## 2 A generalization of (1.4) and some related results

The aim of this section is to generalize the inequality (1.4). In order to do that, we start with the following two lemmas. The first lemma is a well-known result that can be proved by using the spectral theorem and Jensen's inequality. The inequalities in this lemma are of the Peierls-Bogoliubov type (see, e.g., [2, p. 281] or [10, p. 101–102]). The second lemma (see, e.g., [2, p. 291]) has an important role in our generalization of the inequality (1.4). Henceforth, we assume that every function is continuous.

**Lemma 2.1** *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a positive semidefinite matrix and $x \in \mathbb{C}^n$ be a unit vector. Then*

(a) $\langle f(A)x, x\rangle \le f(\langle Ax, x\rangle)$ *for every non-negative concave function $f$ on* $[0, \infty)$.
(b) $f(\langle Ax, x\rangle) \le \langle f(A)x, x\rangle$ *for every non-negative convex function $f$ on* $[0, \infty)$.

**Lemma 2.2** *Let $A \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite and let $f$ be a non-negative increasing function on* $[0, \infty)$. *Then*

$$s_j(f(A)) = f(s_j(A))$$

*for $j = 1, \dots, n$.*

Now, we have the following result.

**Theorem 2.1** *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite matrices, and let $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta = 1$.*

(a) *If $f$ is a non-negative concave function on* $[0, \infty)$, *then*

$$s_j(\alpha f(A) + \beta f(B)) \le s_j\left(f\left(\sqrt{2}|\alpha A + i\beta B|\right)\right)$$

for $j = 1, \dots, n$.
(b) *If $f$ is a non-negative strictly increasing convex function on* $[0, \infty)$ *with* $f(0) = 0$, *then*

$$s_j(f(\alpha A + \beta B)) \le \sqrt{2} s_j(\alpha f(A) + i\beta f(B))$$

for $j = 1, \dots, n$.

***Proof*** (a) Let $f$ be a non-negative concave function on $[0, \infty)$. Then $f$ is increasing on $[0, \infty)$. For every unit vector $x \in \mathbb{C}^n$, we have

$$\begin{aligned}
&\langle(\alpha f(A) + \beta f(B))x, x\rangle \\
=&\alpha\langle f(A)x, x\rangle + \beta\langle f(B)x, x\rangle \\
\le&\alpha f(\langle Ax, x\rangle) + \beta f(\langle Bx, x\rangle) \quad \text{(by Lemma 2.1(a))} \\
\le&f(\langle(\alpha A + \beta B)x, x\rangle) \quad \text{(since } f \text{ is concave)} \\
\le&f\left(\sqrt{2}|\langle(\alpha A + i\beta B)x, x\rangle|\right) \quad \text{(by the inequality (1.3))} \\
\le&f(\sqrt{2}\|(\alpha A + i\beta B)x\|) \text{ (by the Cauchy-Schwarz inequality).}
\end{aligned} \tag{2.1}$$

Consequently,

$$\begin{aligned}
s_j(\alpha f(A) + \beta f(B)) &= \max_{\dim M = j} \min_{\substack{x \in M \\ \|x\| = 1}} \langle(\alpha f(A) + \beta f(B))x, x\rangle \\
&\qquad \text{(by the relation (1.1))} \\
&\le \max_{\dim M = j} \min_{\substack{x \in M \\ \|x\| = 1}} f(\sqrt{2}\|(\alpha A + i\beta B)x\|) \\
&\qquad \text{(by the inequality (2.1))} \\
&= f(\max_{\dim M = j} \min_{\substack{x \in M \\ \|x\| = 1}} \sqrt{2}\ \|(\alpha A + i\beta B)x\|) \text{ (since } f \text{ is increasing)} \\
&= f\left(s_j\left(\sqrt{2}|\alpha A + i\beta B|\right)\right) \text{ (by the relation (1.2))} \\
&= s_j\left(f\left(\sqrt{2}|\alpha A + i\beta B|\right)\right) \text{ (by Lemma 2.2),}
\end{aligned}$$

as required.

(b) Let $f$ be a non-negative strictly increasing convex function on $[0, \infty)$ with $f(0) = 0$. Then $f^{-1}$ is a non-negative concave function on $[0, \infty)$. So, applying part (a) to the function $f^{-1}$, we have

$$\begin{aligned}
s_j(\alpha f^{-1}(A) + \beta f^{-1}(B)) &\le s_j(f^{-1}(\sqrt{2}|\alpha A + i\beta B|)) \\
&= f^{-1}\left(\sqrt{2} s_j(\alpha A + i\beta B)\right)
\end{aligned} \tag{2.2}$$

for $j = 1, \dots, n$. In the inequality (2.2), replacing $A$ and $B$ by $f(A)$ and $f(B)$, respectively, we have

$$\begin{aligned} s_j(\alpha A + \beta B) =& s_j\left(\alpha f^{-1}(f(A)) + \beta f^{-1}(f(B))\right) \\ \leq& f^{-1}\left(\sqrt{2}s_j(\alpha f(A) + i\beta f(B))\right) \end{aligned} \tag{2.3}$$

for $j = 1, \ldots, n$. Since $f$ is increasing, we have

$$\begin{aligned} s_j(f(\alpha A + \beta B)) =& f\left(s_j(\alpha A + \beta B)\right) \\ \leq& f\left(f^{-1}\left(\sqrt{2}s_j(\alpha f(A) + i\beta f(B))\right)\right) \text{ (by the inequality (2.3))} \\ =& \sqrt{2}s_j(\alpha f(A) + i\beta f(B)) \end{aligned}$$

for $j = 1, \ldots, n$, as required. □

To give our first application of Theorem 2.1, we need the following lemma, which has been given by Tao in [11].

**Lemma 2.3** *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ such that $X = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ is positive semidefinite. Then $2s_j(B) \leq s_j(X)$ for $j = 1, \ldots, n$.*

Our first application of Theorem 2.1 can be stated as follows.

**Corollary 2.1** *Let $A, B, C, D \in \mathbb{M}_n(\mathbb{C})$ such that $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is accretive-dissipative. If $f$ is a non-negative strictly increasing convex function on $[0, \infty)$ with $f(0) = 0$, then*

$$s_j\left(f\left(\frac{|(B - C^*) + i(B + C^*)|}{2}\right)\right) \leq \sqrt{2}s_j\left(\frac{f(\mathrm{Re}T) + if(\mathrm{Im}T)}{2}\right)$$

*for $j = 1, \ldots, n$.* □

***Proof*** In Theorem 2.1(b), letting $A = \mathrm{Re}T, B = \mathrm{Im}T$, and $\alpha = \beta = \frac{1}{2}$ imply that

$$\begin{aligned} s_j\left(f\left(\frac{\mathrm{Im}(1+i)T}{2}\right)\right) & \\ =& s_j\left(f\left(\frac{\mathrm{Re}T + \mathrm{Im}T}{2}\right)\right) \\ \leq& \sqrt{2}s_j\left(\frac{f(\mathrm{Re}T) + if(\mathrm{Im}T)}{2}\right) \quad \text{(by Theorem 2.1(b)).} \end{aligned} \tag{2.4}$$

Since $\mathrm{Re}T$ and $\mathrm{Im}T$ are positive semidefinite and $\mathrm{Im}(1 + i)T = \mathrm{Re}T + \mathrm{Im}T$, then $\frac{\mathrm{Im}(1+i)T}{2}$ is positive semidefinite. It follows from Lemma 2.3 that

$$s_j\left(\frac{B + C^*}{2} + \frac{B - C^*}{2i}\right) \leq s_j\left(\frac{\mathrm{Im}(1 + i)T}{2}\right),$$

and so

$$s_j\left(f\left(\frac{|(B-C^*)+i(B+C^*)|}{2}\right)\right) \leq s_j\left(f\left(\frac{\mathrm{Im}(1+i)T}{2}\right)\right). \quad (2.5)$$

Now, the result follows from the inequalities (2.4) and (2.5).

***Remark 2.1*** It should be mentioned here that Tao's inequality given in Lemma 2.3 can be inferred from Corollary 2.1. This can be demonstrated as follows: Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ such that $X = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ is positive semidefinite. Applying Corollary 2.1 to the accretive-dissipative matrix $T = X + iX$ and letting $f(t) = t$, it follows, by direct computations, that $2s_j(B) \leq s_j(X)$ for $j = 1, \ldots, n$.

Another applications of Theorem 2.1 can be seen in the following corollary.

**Corollary 2.2** *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite matrices, and let $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta = 1$. Then*

$$s_j(\alpha A^p + \beta B^p) \leq 2^{p/2} s_j^p(\alpha A + i\beta B), \quad 0 < p \leq 1$$

*and*

$$s_j^p(\alpha A + \beta B) \leq \sqrt{2} s_j(\alpha A^p + i\beta B^p), \quad 1 \leq p < \infty$$

*for $j = 1, \ldots, n$. In particular,*

$$s_j(A^p + B^p) \leq 2^{1-p/2} s_j^p(A + iB), \quad 0 < p \leq 1 \quad (2.6)$$

*and*

$$s_j^p(A + B) \leq 2^{p-1/2} s_j(A^p + iB^p), \quad 1 \leq p < \infty \quad (2.7)$$

*for $j = 1, \ldots, n$.*

According to the inequalities (2.6), (2.7), and using the fact that unitarily invariant norms are increasing functions of singular values, we have the following result.

**Corollary 2.3** *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite matrices. Then*

$$\left|\left|\left|(A^p + B^p)^{1/p}\right|\right|\right| \leq 2^{1/p-1/2} |||A + iB|||, 0 < p \leq 1 \quad (2.8)$$

*and*

$$|||(A + B)^p||| \leq 2^{p-1/2} |||A^p + iB^p|||, 1 \leq p < \infty \quad (2.9)$$

*for every unitarily invariant norm.*

***Remark 2.2*** If we put $p = 1$ in the inequality (2.8) or ( 2.9), the inequality (1.5) will be obtained. So, the inequality (1.5) can be considered as a special case of the inequalities (2.8) and (2.9).

Based on Theorem 2.1, we have the following corollary.

**Corollary 2.4** *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *be positive semidefinite matrices, and let* $\alpha, \beta \in (0, 1)$ *such that* $\alpha + \beta = 1$.

(a) *If f is a non-negative concave function on* $[0, \infty)$, *then*

$$|||\alpha f(A) + \beta f(B)||| \leq \left|\left|\left|f(\sqrt{2}|\alpha A + i\beta B|)\right|\right|\right|$$

for every unitarily invariant norm.

(b) *If f is a non-negative strictly increasing convex function on* $[0, \infty)$ *with* $f(0) = 0$, *then*

$$|||f(\alpha A + \beta B)||| \leq \sqrt{2}|||\alpha f(A) + i\beta f(B)|||$$

for every unitarily invariant norm.

Now, we have the following lemma.

**Lemma 2.4** *Let* $a, b \in [0, \infty)$, *and let* $\alpha, \beta \in (0, 1)$ *such that* $\alpha + \beta = 1$.

(a) *If f is a non-negative function on* $[0, \infty)$ *such that* $f(\sqrt{t})$ *is concave, then*

$$\alpha f(a) + \beta f(b) \leq f(\left|\sqrt{\alpha} a + i\sqrt{\beta} b\right|).$$

(b) *If f is a non-negative function on* $[0, \infty)$ *such that* $f(\sqrt{t})$ *is convex with* $f(0) = 0$, *then*

$$\alpha f(a) + \beta f(b) \geq f(\left|\sqrt{\alpha} a + i\sqrt{\beta} b\right|).$$

□

***Proof*** We prove part (a), the proof of part (b) is similar.

Let $g(t) = f(\sqrt{t})$. Then $g(t)$ is concave, and so

$$\begin{aligned}\alpha f(a)+\beta f(b) =&\alpha g(a^2)+\beta g(b^2)\\ \leq& g(\alpha a^2+\beta b^2)\\ =&g(\left|\sqrt{\alpha}a+i\sqrt{\beta}b\right|^2)\\ =&f(\left|\sqrt{\alpha}a+i\sqrt{\beta}b\right|).\end{aligned}$$

**Corollary 2.5** *Let* $a, b \in [0, \infty)$ *and* $\alpha, \beta \in (0, 1)$ *such that* $\alpha + \beta = 1$. *Then*

$$\alpha a^p+\beta b^p \leq \left|\sqrt{\alpha}a+i\sqrt{\beta}b\right|^p \text{ for } 0<p\leq 2$$

*and*

$$\alpha a^p+\beta b^p \geq \left|\sqrt{\alpha}a+i\sqrt{\beta}b\right|^p \text{ for } 2\leq p<\infty.$$

Based on Lemma 2.4, we have the following result.

**Theorem 2.2** *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *be positive semidefinite matrices, and let* $\alpha, \beta \in (0, 1)$ *such that* $\alpha + \beta = 1$.

(a) *If* $f$ *is a non-negative concave function on* $[0,\infty)$, *then*

$$s_j(\alpha f(A)+\beta f(B)) \leq s_j\left(f\left(\left|\sqrt{\alpha}A+i\sqrt{\beta}B\right|\right)\right) \tag{2.10}$$

for $j = 1, \ldots, n$.

(b) *If* $f$ *is a non-negative strictly increasing convex function on* $[0,\infty)$ *with* $f(0) = 0$, *then*

$$s_j(f(\alpha A+\beta B)) \leq s_j\left(\sqrt{\alpha}f(A)+i\sqrt{\beta}f(B)\right) \tag{2.11}$$

for $j = 1, \ldots, n$.

***Proof*** (a) Let $x \in \mathbb{C}^n$ be a unit vector. Then

$$\begin{aligned}&\langle(\alpha f(A)+\beta f(B))x,x\rangle\\ &\quad=\alpha\langle f(A)x,x\rangle+\beta\langle f(B)x,x\rangle\\ &\quad\leq\alpha f(\langle Ax,x\rangle)+\beta f(\langle Bx,x\rangle) \qquad \text{(by Lemma 2.1(a))}\\ &\quad\leq f(\left|\sqrt{\alpha}\langle Ax,x\rangle+i\sqrt{\beta}\langle Bx,x\rangle\right|) \quad \text{(by Lemma 2.4(a))}\\ &\quad=f\left(\left|\left\langle(\sqrt{\alpha}A+i\sqrt{\beta}B)x,x\right\rangle\right|\right)\\ &\quad\leq f\left(\left\|(\sqrt{\alpha}A+i\sqrt{\beta}B)x\right\|\right).\end{aligned} \tag{2.12}$$

So, the inequality (2.10) follows from the inequality (2.12) by an argument similar to that used in the proof of Theorem 2.1(a).

(b) Let $f$ be a non-negative strictly increasing convex function on $[0,\infty)$ with $f(0)=0$. Then $f^{-1}$ is a non-negative concave function on $[0,\infty)$. Now, the inequality (2.11) follows by applying part (a) to the function $f^{-1}$ and then using an argument similar to that used in the proof of part (b) of Theorem 2.1. □

Based on Theorem 2.2, we have the following two corollaries.

**Corollary 2.6** *Let $A,B\in\mathbb{M}_n(\mathbb{C})$ be positive semidefinite matrices, and let $\alpha,\beta\in(0,1)$ such that $\alpha+\beta=1$. Then*

$$s_j(\alpha A^p+\beta B^p)\le s_j^p(\sqrt{\alpha}A+i\sqrt{\beta}B),\, 0<p\le 2$$

*and*

$$s_j^p(\alpha A+\beta B)\le s_j\left(\sqrt{\alpha}A^p+i\sqrt{\beta}B^p\right),\, 2\le p<\infty$$

*for $j=1,\dots,n$.*

**Corollary 2.7** *Let $A,B\in\mathbb{M}_n(\mathbb{C})$ be positive semidefinite matrices, and let $\alpha,\beta\in(0,1)$ such that $\alpha+\beta=1$.*

(a) *If $f$ is a non-negative concave function on $[0,\infty)$, then*

$$|||\alpha f(A)+\beta f(B)|||\le\left|\left|\left|f\left(\left|\sqrt{\alpha}A+i\sqrt{\beta}B\right|\right)\right|\right|\right|$$

for every unitarily invariant norm.

(b) *If $f$ is a non-negative strictly increasing convex function on $[0,\infty)$ with $f(0)=0$, then*

$$|||f(\alpha A+\beta B)|||\le\left|\left|\left|\sqrt{\alpha}f(A)+i\sqrt{\beta}f(B)\right|\right|\right|$$

for every unitarily invariant norm.

Now, we have the following result.

**Theorem 2.3** *Let $A,B\in\mathbb{M}_n(\mathbb{C})$ be positive semidefinite matrices, and let $\alpha,\beta\in(0,1)$ such that $\alpha^2+\beta^2=1$. If $f$ is a non-negative function on $[0,\infty)$ such that $f^2$ is concave, then*

$$s_j(\alpha f(A)+i\beta f(B))\le\sqrt{2}s_j(f(\alpha^2A+\beta^2B))$$

*for $j=1,\dots,n$.*

***Proof*** Let $x \in \mathbb{C}^n$ be a unit vector. Then

$$\begin{aligned}\left\langle |\alpha f(A) + i\beta f(B)|^2 x, x \right\rangle =& \|(\alpha f(A)x + i\beta f(B)x\|^2 \\ \leq& (\alpha \|f(A)x\| + \beta \|f(B)x\|)^2 \\ \leq& 2(\alpha^2 \|f(A)x\|^2 + \beta^2 \|f(B)x\|^2) \\ =& 2(\alpha^2 \left\langle f^2(A)x, x \right\rangle + \beta^2 \left\langle f^2(B)x, x \right\rangle) \\ \leq& 2(\alpha^2 f^2(\langle Ax, x\rangle) + \beta^2 f^2(\langle Bx, x\rangle)) \\ & \quad \text{(by Lemma 2.1(a))} \\ \leq& 2f^2(\left\langle \alpha^2 Ax, x \right\rangle + \left\langle \beta^2 Bx, x \right\rangle) \\ & \quad \text{(since } f^2 \text{ is concave on } [0, \infty)) \\ =& 2f^2(\left\langle (\alpha^2 A + \beta^2 B)x, x \right\rangle) \\ \leq& 2f^2(\left\| (\alpha^2 A + \beta^2 B)x \right\|). \end{aligned} \tag{2.13}$$

So, the result follows from the inequality (2.13) by an argument similar to that used in the proof of Theorem 2.1(a). □

Based on Theorem 2.3, we have the following corollary.

**Corollary 2.8** *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite matrices, and let $\alpha, \beta \in (0, 1)$ such that $\alpha^2 + \beta^2 = 1$. Then*

$$s_j^2(\alpha A^{p/2} + i\beta B^{p/2}) \leq 2s_j^p(\alpha^2 A + \beta^2 B) \tag{2.14}$$

*for $j = 1, \dots, n$ and $0 < p \leq 1$. In particular,*

$$s_j^2(A^{p/2} + iB^{p/2}) \leq 2^{2-p} s_j^p(A + B) \tag{2.15}$$

*and*

$$s_j(A + iB) \leq \sqrt{2} s_j^{1/2}(A^2 + B^2). \tag{2.16}$$

***Proof*** The inequality (2.14) follows directly from Theorem 2.3 by taking $f(t) = t^{p/2}, 0 < p \leq 1$. Also, if we put $\alpha^2 = \beta^2$ in the inequality (2.14), we obtain the inequality (2.15). The inequality (2.16) is a special case of the inequality (2.15) by taking $p = 1$, and replacing $A$ and $B$ by $A^2$ and $B^2$, respectively. □

In their investigation of singular value inequalities on the sector matrices, Drury and Lin [9] proved that if $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite matrices, then

$$s_j(A + iB) \leq \sqrt{2} s_j(A + B) \tag{2.17}$$

for $j = 1, \dots, n$, which is closely related to our inequality (2.16). It should be mentioned here that for $j = 1$, which corresponds to the spectral norm $\|\cdot\|$, the inequality (2.16) is better than the inequality (2.17). In fact, it is known [4] that

$$s_1^r(A+B) \le s_1(A^r + B^r)$$

for $0 < r \le 1$. In particular, letting $r = \frac{1}{2}$ and replacing $A$, $B$ by $A^2, B^2$ respectively, we have

$$s_1^{1/2}(A^2 + B^2) \le s_1(A+B).$$

For $j > 1$, we may have

$$s_j^{1/2}(A^2 + B^2) > s_j(A+B),$$

as it can be demonstrated by considering $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. In this case, $A + B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $A^2 + B^2 = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$. Now, $s_2(A^2 + B^2) = \frac{5-\sqrt{17}}{2}$ and $s_2(A+B) = \frac{3-\sqrt{5}}{2}$. So, we have $s_2^{1/2}(A^2 + B^2) > s_2(A+B)$.

According to Theorem 2.3, we have the following two corollaries.

**Corollary 2.9** *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *be positive semidefinite matrices, and let* $\alpha, \beta \in (0,1)$ *such that* $\alpha^2 + \beta^2 = 1$. *If f is a non-negative function on* $[0, \infty)$ *such that* $f^2$ *is concave, then*

$$|||\alpha f(A) + i\beta f(B)||| \le \sqrt{2}\left|\left|\left|f\left(\alpha^2 A + \beta^2 B\right)\right|\right|\right|$$

*for every unitarily invariant norm.*

**Corollary 2.10** *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *be positive semidefinite matrices. Then, for every unitarily invariant norm, we have*

$$\left|\left|\left|A^{p/2} + iB^{p/2}\right|\right|\right| \le 2^{1-p/2}\left|\left|\left|(A+B)^{p/2}\right|\right|\right|$$

*for* $0 < p \le 1$. *In particular,*

$$|||A + iB||| \le \sqrt{2}\left|\left|\left|(A^2 + B^2)^{1/2}\right|\right|\right|.$$

Now, to give our next result, we need the following inequality, which follows from the arithmetic-geometric mean inequality for singular values [3].

**Lemma 2.5** *Let* $T \in \mathbb{M}_n(\mathbb{C})$. *Then*

$$2s_j(T^2) \le s_j(T^*T + TT^*)$$

*for* $j = 1, \dots, n$.

**Theorem 2.4** *Let $T \in \mathbb{M}_n(\mathbb{C})$ with the Cartesian decomposition $T = A + iB$. Then*

$$2^{1-p/2} s_j^{p/2}(T^2) \leq s_j(|A|^p + |B|^p)$$

*for $j = 1, \dots, n$ and $2 \leq p < \infty$.*

***Proof*** For every unit vector $x \in \mathbb{C}^n$, we have

$$\begin{aligned}
\langle (T^*T + TT^*)x, x\rangle =& 2\left(\left\langle A^2x, x\right\rangle + \left\langle B^2x, x\right\rangle\right) \\
=& 2\left(\left(\left\langle A^2x, x\right\rangle + \left\langle B^2x, x\right\rangle\right)^{p/2}\right)^{2/p} \\
\leq& 2\left(2^{p/2-1}\left(\left\langle A^2x, x\right\rangle^{p/2} + \left\langle B^2x, x\right\rangle^{p/2}\right)\right)^{2/p} \\
\leq& 2^{2-2/p}(\langle |A|^p x, x\rangle + \langle |B|^p x, x\rangle)^{2/p} \\
& \quad \text{(by Lemma 2.1(b))} \\
=& 2^{2-2/p}\langle (|A|^p + |B|^p)x, x\rangle^{2/p}.
\end{aligned} \tag{2.18}$$

It follows from the inequality (2.18) and the relation (1.1) that

$$s_j(T^*T + TT^*) \leq 2^{2-2/p} s_j^{2/p}(|A|^p + |B|^p) \tag{2.19}$$

for $j = 1, \dots, n$. Now, the result follows from Lemma 2.5 and the inequality (2.19). □

The following example shows that Theorem 2.4 is not true when $0 < p < 2$.

***Example 2.1*** Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $p = 1$. Then $A$ and $B$ are positive semidefinite matrices with $|A| + |B| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $T^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Now, $s_1^2(|A| + |B|) = s_1^2(T^2) = 1$. So, we have $\sqrt{2} s_1^{1/2}(T^2) > s_1(|A| + |B|)$.

According to Theorem 2.4, we have the following corollary.

**Corollary 2.11** *Let $T \in \mathbb{M}_n(\mathbb{C})$ with the Cartesian decomposition $T = A + iB$. Then, for every unitarily invariant norm, we have*

$$|||T^*T + TT^*||| \leq 2^{2-2/p} \left|\left|\left|(|A|^p + |B|^p)^{2/p}\right|\right|\right|$$

*and*

$$\left|\left|\left|T^2\right|\right|\right| \leq 2^{1-2/p} \left|\left|\left|(|A|^p + |B|^p)^{2/p}\right|\right|\right|$$

*for $2 \leq p < \infty$.*

## 3 Some related results for $A \geq B \geq 0$

In this section, we give some results for positive semidefinite matrices $A$, $B$ with $A \geq B$.

We start with the following result.

**Theorem 3.1** *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite matrices such that $A \geq B$. Then*

$$s_j^p(A^{1/p} - B^{1/p}) \leq s_j(A + iB)$$

*for $j = 1, \ldots, n$ and $1 \leq p < \infty$.*

***Proof*** Since $f(t) = t^{1/p}$ is a matrix monotone function for $1 \leq p < \infty$, we have $A^{1/p} \geq B^{1/p}$.

For every unit vector $x \in \mathbb{C}^n$, we have

$$\begin{aligned} \left\langle (A^{1/p} - B^{1/p})x, x \right\rangle^p &= \left( \left\langle A^{1/p}x, x \right\rangle - \left\langle B^{1/p}x, x \right\rangle \right)^p \\ &\leq \left\langle A^{1/p}x, x \right\rangle^p - \left\langle B^{1/p}x, x \right\rangle^p \\ &\leq \left| \left\langle A^{1/p}x, x \right\rangle^p + i \left\langle B^{1/p}x, x \right\rangle^p \right| \\ &\leq |\langle Ax, x \rangle + i \langle Bx, x \rangle| \quad \text{(by Lemma 2.1(b))} \\ &\leq \|(A + iB)x\|. \end{aligned} \tag{3.1}$$

So, the result follows from the inequality (3.1) by an argument similar to that used in the proof of Theorem 2.1(a). □

The following example shows that Theorem 3.1 is not true without the assumption $A \geq B$.

***Example 3.1*** Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $p = 1$. Then $A$ and $B$ are positive semidefinite matrices with $A \ngeq B$, $A - B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, and $A + iB = \begin{bmatrix} 1+i & 1 \\ 1 & 1 \end{bmatrix}$. Now, $s_2^2(A - B) = \frac{3-\sqrt{5}}{2}$ and $s_2^2(A + iB) = \frac{5-\sqrt{21}}{2}$. So, we have $s_2(A - B) > s_2(A + iB)$.

The following example shows that Theorem 3.1 is not true when $0 < p < 1$.

***Example 3.2*** Consider $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $p = 1/2$. Then $A$ and $B$ are positive semidefinite matrices with $A \geq B$, $A^2 - B^2 = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$, and $A + iB = \begin{bmatrix} 2+i & 1 \\ 1 & 1 \end{bmatrix}$. Now, $s_2^2(A^2 - B^2) = 19 - \sqrt{360}$ and $s_2^2(A + iB) = 30 - \sqrt{896}$. So, we have $s_2(A^2 - B^2) > s_2^2(A + iB)$.

According to Theorem 3.1, we have the following corollary.

**Corollary 3.1** *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite matrices such that $A \geq B$. Then, for every unitarily invariant norm, we have*

$$\left|\left|\left|(A^{1/p} - B^{1/p})^p\right|\right|\right| \leq |||A + iB|||$$

*for* $1 \leq p < \infty$.

Now, we need the following lemma to give our next result.

**Lemma 3.1** *Let $a, b \in \mathbb{R}$. Then*

$$|a + ib|^p \leq 2^{p/2-1}(|a|^p + |b|^p) \tag{3.2}$$

*for* $2 \leq p < \infty$, *and*

$$2^{p/2-1}(|a|^p + |b|^p) \leq |a + ib|^p \tag{3.3}$$

*for* $0 < p \leq 2$.

***Proof*** For $a, b \in \mathbb{R}$ and $2 \leq p < \infty$, we have

$$\begin{aligned} |a + ib|^p &= \left(|a + ib|^2\right)^{p/2} \\ &= (|a|^2 + |b|^2)^{p/2} \\ &\leq 2^{p/2-1}(|a|^p + |b|^p), \end{aligned}$$

which proves the inequality (3.2). Similarly, one can prove the inequality (3.3).

**Theorem 3.2** *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite matrices such that $A \geq B$. Then*

$$s_j^p(A - B) \leq 2^{p/2-1} s_j(A^p + B^p)$$

*for* $j = 1, \ldots, n$ *and* $2 \leq p < \infty$.

***Proof*** For every unit vector $x \in \mathbb{C}^n$, we have

$$\begin{aligned} &\langle (A - B)x, x\rangle^p \\ &\quad = (\langle Ax, x\rangle - \langle Bx, x\rangle)^p \\ &\quad \leq |\langle (A + iB)x, x\rangle|^p \\ &\quad = |\langle Ax, x\rangle + i\langle Bx, x\rangle|^p \\ &\quad \leq 2^{p/2-1}\left(\langle Ax, x\rangle^p + \langle Bx, x\rangle^p\right) \text{ (by the inequality (3.2))} \\ &\quad \leq 2^{p/2-1}(\langle A^p x, x\rangle + \langle B^p x, x\rangle) \text{ (by Lemma 2.1(b))} \\ &\quad = 2^{p/2-1}\langle (A^p + B^p)x, x\rangle \\ &\quad \leq 2^{p/2-1}\|(A^p + B^p)x\|. \end{aligned} \tag{3.4}$$

So, the result follows from the inequality (3.4) by an argument similar to that used in the proof of Theorem 2.1(a). □

The following example shows that Theorem 3.2 is not true when $0 < p < 2$.

***Example 3.3*** Consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $p = 1$. Then $A$ and $B$ are positive semidefinite matrices with $A \geq B, A + B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$, and $A - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Now, $s_2^2(A + B) = s_2^2(A - B) = 1$. So, we have $s_2(A - B) > 2^{-1/2} s_2(A + B)$.

According to Theorem 3.2, we have the following corollary.

**Corollary 3.2** *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be any positive semidefinite matrices such that $A \geq B$. Then, for every unitarily invariant norm, we have*

$$|||(A - B)^p||| \leq 2^{p/2-1} |||A^p + B^p|||$$

*for* $2 \leq p < \infty$.

## 4 A result for $T = A + iB$ when $A > 0$

In this section, we give a result for matrices with positive definite real parts based on the following two lemmas. The first lemma can be found in [2, p. 75] and the second one can be easily proved.

**Lemma 4.1** *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$. Then*

$$s_j(AXB) \leq \|A\| \|B\| s_j(X)$$

*for* $j = 1, \ldots, n$.

**Lemma 4.2** *Let $X \in \mathbb{M}_n(\mathbb{C})$ be Hermitian and let $Y = I + iX$. Then*

(a)
$$s_j^r(Y) \leq 1 + s_j^r(X)$$
for $j = 1, \ldots, n$ and $0 < r \leq 2$.

(b)
$$s_j^r(Y) \leq 2^{r/2-1}(1 + s_j^r(X))$$
for $j = 1, \ldots, n$ and $2 \leq r < \infty$.

The condition number of an invertible matrix $A \in \mathbb{M}_n(\mathbb{C})$ is defined by $k(A) = \|A\| \|A^{-1}\|$. Based on Lemmas 4.1 and 4.2, we have the following result.

**Theorem 4.1** *Let $T \in \mathbb{M}_n(\mathbb{C})$ with the Cartesian decomposition $T = A + iB$ such that $A$ is positive definite. Then*

$$s_j^r(T) \le s_1^r(A) + k^r(A) s_j^r(B)$$

*for $j = 1, \ldots, n$ and $0 < r \le 2$.*

***Proof*** Since $T = A + iB$, we have $A^{-1/2}TA^{-1/2} = I + iA^{-1/2}BA^{-1/2}$. By part (a) of Lemma 4.2, we have

$$s_j^r(A^{-1/2}TA^{-1/2}) \le 1 + s_j^r(A^{-1/2}BA^{-1/2}). \tag{4.1}$$

Now,

$$\begin{aligned} s_j^r(T) =& s_j^r(A^{1/2}A^{-1/2}TA^{-1/2}A^{1/2}) \\ \le& \|A\|^r s_j^r(A^{-1/2}TA^{-1/2}) \text{ (by Lemma 4.1)} \\ \le& \|A\|^r(1 + s_j^r(A^{-1/2}BA^{-1/2})) \text{ (by the inequality (4.1))} \\ \le& \|A\|^r + \|A\|^r \left\|A^{-1}\right\|^r s_j^r(B) \text{ (by Lemma 4.1)} \\ =& s_1^r(A) + k^r(A) s_j^2(B), \end{aligned}$$

as required. □

Using an argument similar to that used in the proof of Theorem 4.1 , one can prove the following related result.

**Theorem 4.2** *Let $T \in \mathbb{M}_n(\mathbb{C})$ be with the Cartesian decomposition $T = A + iB$ such that $A$ is positive definite. Then*

$$s_j^r(T) \le 2^{r/2-1}(s_1^r(A) + k^r(A) s_j^r(B))$$

*for $j = 1, \ldots, n$ and $2 \le r < \infty$.*

For a matrix $A \in \mathbb{M}_n(\mathbb{C})$ it is known that $k(A) \ge 1$. So, one might ask whether the following two inequalities hold:

$$s_j^r(T) \le s_1^r(A) + s_j^r(B)$$

for $j = 1, \ldots, n$ and $0 < r \le 2$, and

$$s_j^r(T) \le 2^{r/2-1}(s_1^r(A) + s_j^r(B))$$

for $j = 1, \ldots, n$ and $2 \le r < \infty$. In fact, the following example shows that they are false for $r = 2$.

***Example 4.1*** Consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then $A$ is a positive definite matrix with $s_1^2(A) = 4$ and $B$ is a Hermitian matrix with $s_1^2(B) = 2$. For $T = A + iB$, $s_1^2(T) = \frac{21-\sqrt{425}}{2}$ and $s_1^2(T) = \frac{9+\sqrt{29}}{2}$. So, we have $s_1^2(T) > s_1^2(A) + s_1^2(B)$.

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