Annals of Functional Analysis (2020) 11:523–539
https://doi.org/10.1007/s43034-019-00035-y

Tusi Mathematical Research Group

ORIGINAL PAPER

# On normed spaces with the Wigner Property

**Ruidong Wang**[1] · **Dariusz Bugajewski**[2]

Received: 22 July 2019 / Accepted: 4 October 2019 / Published online: 4 December 2019


**Abstract**

The aim of this paper is to generalize the Wigner Theorem to real normed spaces. A normed space is said to have the Wigner Property if the Wigner Theorem holds for it. We prove that every two-dimensional real normed space has the Wigner Property. We also study the Wigner Property of real normed spaces of dimension at least three. It is also shown that strictly convex real normed spaces possess the Wigner Property.

**Keywords** Isometry · Phase equivalence · Phase isometry · The Wigner Property · The Wigner Theorem

**Mathematics Subject Classification** 46B03 · 46B04

## 1 Introduction

The well-known Wigner Theorem, which plays an important role in quantum mechanics, states that any transformations of the states of a physical system which preserve the transition probability associated to any pair of states are induced either by a unitary or by an anti-unitary operator on the Hilbert space associated with the physical system (see [5,23]). Since Wigner's proof was incomplete from the mathematical point of view, many papers have appeared to prove the Wigner Theorem (see [4,10,12,14,18,19,21]).

Communicated by Jacek Chmieliński.

✉ Dariusz Bugajewski
ddbb@amu.edu.pl

Ruidong Wang
wangruidong@tjut.edu.cn

1 College of Science, Tianjin University of Technology, Tianjin 300384, China

2 Department of Mathematics and Computer Science, Adam Mickiewicz University Poznań, Uniwersytetu Poznańskiego 4, 61-614 Poznań, Poland

The Wigner Theorem describes those transformations of $P_1(\mathcal{H})$ (all rank-one projections on the Hilbert space $\mathcal{H}$), which preserve the transition probability. Gehér and Šemrl [11] described the surjective isometries on the Grassmann space $P_n(\mathcal{H})$ of all rank $n$ projections with respect to the gap metric ($n \in \mathbb{N}$), which generalize the Wigner Theorem to the Grassmann space $P_n(\mathcal{H})$. We would also like to draw the readers' attention to the very recent paper [16] in which the author establishes a Wigner's type theorem for linear operators which map projections of a fixed rank to projections of other fixed rank.

Bargmann [3] proved the Wigner Theorem which is very close to Wigner's original statement.

**Theorem 1** [3] *Let* $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ *be an inner product space with* $dim(\mathcal{H}) \geq 2$ *and let* $T : \mathcal{H} \to \mathcal{H}$ *satisfy the equality*

$$|\langle T(x), T(y)\rangle| = |\langle x, y\rangle| \tag{1}$$

*for any* $x, y \in \mathcal{H}$. *Then there exists an isometry or an anti-isometry* $A$ *on* $\mathcal{H}$ *and a phase function* $\varepsilon : \mathcal{H} \to \mathbb{C}$ *with* $|\varepsilon(x)| = 1$ *such that* $T(x) = \varepsilon(x)A(x)$ *for any* $x \in \mathcal{H}$. *Moreover, if* $T$ *is surjective and* $\mathcal{H}$ *is a Hilbert space, then* $A$ *is a unitary or an anti-unitary operator.*

In connection with (1) let us draw the readers' attention to rich references concerning the Wigner equation. In particular, in [6] the authors proved the existence of a solution (satisfying some additional condition) to the equation

$$|\langle I(x), I(y)\rangle| = |\langle x, y\rangle|,$$

where $I : \mathcal{M} \to \mathcal{N}$ is a mapping between inner product modules $\mathcal{M}$ and $\mathcal{N}$ over certain $C^*$-algebras.

It is natural to ask whether the Wigner Theorem would still be true if $\mathcal{H}$ is a normed space. Unfortunately, there is no inner product in general normed spaces. However, since in an inner product space, the Eq. (1) is equivalent to the equality

$$\{\|T(x) + \alpha T(y)\| : |\alpha| = 1\} = \{\|x + \alpha y\| : |\alpha| = 1\} \tag{2}$$

for any $x, y \in \mathcal{H}$, one can raise the following problem. We call an operator $T$ between two normed spaces a *phase isometry* if $T$ satisfies the equality (2). If $E$ and $F$ are both normed spaces over the field $\mathbb{K}$, two mappings $T_1 : E \to F$ and $T_2 : E \to F$ are said to be *differ by a phase factor* or to be *phase equivalent* if there exists $\varepsilon : E \to \mathbb{K}$ with $|\varepsilon(x)| = 1$ such that $T_1(x) = \varepsilon(x)T_2(x)$ for any $x \in E$.

**Problem 1** Let $E$ and $F$ be normed spaces over the field $\mathbb{K}$ and let $T : E \to F$ be a surjective phase isometry. Is $T$ phase equivalent to a linear isometry $L$ from $E$ to $F$?

Recall that in 1932, Mazur and Ulam [15] showed that any surjective isometry between two real normed spaces is an affine map, that is it is a translation of a linear isometry. Bourgain [2] gave an example which shows that the complex version of the Mazur–Ulam Theorem is not valid.

In this paper we are going to consider the above problem in the case when $\mathbb{K} = \mathbb{R}$. Notice that by the Wigner Theorem, this problem is solved when both $E$ and $F$ are real inner product spaces.

Now, we are going to recall the following

**Definition 1** Let $E$ be a real normed space. $E$ is said to have the Wigner Property if for any real normed space $F$, and any surjective phase isometry $T : E \to F$, $T$ is phase equivalent to a linear isometry from $E$ to $F$.

Recently, Tan and Huang [20] proved that smooth real normed spaces have the Wigner Property. They also proved that some classical real Banach spaces, such as $\mathcal{L}^\infty(\Gamma)$-type space and an $\ell^1(\Gamma)$-space for some index set $\Gamma$, have the Wigner Property.

Let us briefly summarize the contents of this paper. In Sect. 2, we will study the property of a surjective phase isometry between two real normed spaces and will give some properties of a surjective phase isometry operator, which will be used in the sequel. In Sect. 3, we will show that any two-dimensional real normed space has the Wigner Property. Section 4 deals with the Wigner Property in real finite dimensional normed spaces of dimension at least three. In particular, we will prove that any strictly convex real normed space has the Wigner Property.

In this paper, we will use the standard notations. $E^*$ denotes the dual space of the normed space $E$. $S_E$ and $B_E$ denote the unit sphere and the closed unit ball of the normed space $E$, respectively. $w^* - \exp(B_{E^*})$ denotes the set of $w^*$ exposed points of the unit ball $B_{E^*}$ while $\text{ext}(B_{E^*})$ denotes the set of extremal points of that ball. $\text{sm}(S_E)$ denotes the set of smooth points of the sphere $S_E$. $[x, y] := \{\lambda x + (1-\lambda)y : \lambda \in [0, 1]\}$ for any $x, y \in E$. Finally, by $|A|$ we will denote the cardinality of the set $A$ and $\text{span}\{A\}$ will denote the linear subspace generated by the set $A$.

## 2 Preliminary results

In this section, we will study the general properties of surjective phase isometries between two real normed spaces. Lemmas 1 and 2 were given by Tan and Huang in the unpublished paper [20], so we include also their proofs for the readers' convenience.

**Lemma 1** [20] *Let $E$ and $F$ be real normed spaces and let $T : E \to F$ be a phase isometry. Then $\|T(x)\| = \|x\|$ and $T(-x) \in \{T(x), -T(x)\}$ for all $x \in E$. Moreover, if $T$ is surjective, then $T$ is injective and $T(-x) = -T(x)$ for all $x \in E$.*

***Proof*** Putting $y = x$ in the equality (2) we see that $T$ preserves the norm. Next, putting $y = -x$ in the equality (2), we get

$$\{\|T(x) + T(-x)\|, \|T(x) - T(-x)\|\} = \{2\|x\|, 0\},$$

which implies that $T(-x) \in \{T(x), -T(x)\}$.

Now, let $T(x) = T(y)$ for some $x, y \in E$, $x \neq y$. Since $\|x\| = \|y\| = \|T(x)\|$, it follows that $T(x) = T(y) \neq 0$. Because $T$ is surjective, there exists $z \in E$ such that $T(z) = -T(x)$. Using the equality (2) for $x, y, z$, we obtain

$$\{\|x + y\|, \|x - y\|\} = \{\|T(x) + T(y)\|, \|T(x) - T(y)\|\} = \{2\|x\|, 0\},$$
$$\{\|x + z\|, \|x - z\|\} = \{\|T(x) + T(z)\|, \|T(x) - T(z)\|\} = \{2\|x\|, 0\}.$$

This yields $y, z \in \{x, -x\}$. If $z = x$, then $T(x) = -T(x) = 0$, which contradicts to $T(x) \neq 0$, so we obtain $z = -x$. Now we will show that $y = x$. If not, we would get $y = -x = z$ and

$$T(x) = T(y) = T(z) = -T(x).$$

This leads to the contradiction that $T(x) \neq 0$. □

**Lemma 2** [20] *Let $E$ and $F$ be real normed spaces and let $T : E \to F$ be a phase isometry (that is not necessarily surjective). Then for every $w^*$ exposed point $x^*$ of $B_{E^*}$, there exists a linear functional $\varphi \in F^*$ of the norm one such that $x^*(x) \in \{\varphi(T(x)), -\varphi(T(x))\}$ for all $x \in E$.*

***Proof*** First, we will prove that if $E = \mathbb{R}$, then there exists a linear functional $\varphi \in F^*$ of the norm one such that $\varphi(T(t)) \in \{t, -t\}$ for all $t \in \mathbb{R}$. For every positive integer $n$, using the norm preserving property from Lemma 1, we have $\|T(n)\| = n$. The Hahn-Banach theorem guarantees the existence of a linear functional $\varphi_n \in S_{F^*}$ such that $\|\varphi_n\| = 1$ and $\varphi_n(T(n)) = n$. For every $t \in [-n, n]$, we get

$$\begin{aligned} 2n &= \varphi_n(T(n) - T(t)) + \varphi_n(T(n) + T(t)) \\ &\leq \|T(n) - T(t)\| + \|T(n) + T(t)\| \\ &= (n - t) + (n + t) = 2n, \end{aligned}$$

or, alternatively,

$$\begin{aligned} &\{\varphi_n(T(n) - T(t)), \varphi_n(T(n) + T(t))\} \\ &\quad = \{\|T(n) - T(t)\|, \|T(n) + T(t)\|\} \\ &\quad = \{n - t, n + t\}. \end{aligned}$$

Then $\varphi_n(T(t)) \in \{t, -t\}$ for all $t \in [-n, n]$. It follows from Alaoglu's theorem that the sequence $\{\varphi_n\}$ has a cluster point $\varphi$ in view of the $w^*$ topology. This entails that $\|\varphi\| \leq 1$ and $\varphi(T(t)) \in \{t, -t\}$ for every $t \in \mathbb{R}$. Clearly, $\|\varphi\| = 1$ and $\varphi$ is the desired mapping.

Now suppose that $\dim(E) > 1$ and $u \in S_E$ is a smooth point such that $x^*(u) = 1$. Let $G : \mathbb{R} \to F$ be defined by $G(t) = T(tu)$ for $t \in \mathbb{R}$ . Then $G$ is a phase isometry. By the proof above, there exists $\varphi \in F^*$ with $\|\varphi\| = 1$ such that

$$\varphi(T(tu)) = \varphi(G(t)) \in \{t, -t\}.$$

Since $u$ is a smooth point, it follows that $x^*$ is the only one supporting functional at $u$. Therefore, for every $x \in X$,

$$x^*(x) = \lim_{t \to 0^+} \frac{\|u + tx\| - \|u\|}{t} = \lim_{t \to +\infty} (\|tu + x\| - t).$$

From the Eq. (2), we get

$$\{\|T(tu) + T(x)\|, \|T(tu) - T(x)\|\} = \{\|tu + x\|, \|tu - x\|\}$$

for all $t > 0$ and $x \in E$. For a fixed nonzero vector $x \in E$, the set $(0, +\infty)$ will be divided into four parts:

$$\begin{aligned}
A_1 &:= \{t > 0 : \|T(tu) + T(x)\| = \|tu - x\|,\\
&\quad \|T(tu) - T(x)\| = \|tu + x\|,\ \varphi(T(tu)) = t\};\\
A_2 &:= \{t > 0 : \|T(tu) + T(x)\| = \|tu + x\|,\\
&\quad \|T(tu) - T(x)\| = \|tu - x\|,\ \varphi(T(tu)) = t\};\\
A_3 &:= \{t > 0 : \|T(tu) + T(x)\| = \|tu - x\|,\\
&\quad \|T(tu) - T(x)\| = \|tu + x\|,\ \varphi(T(tu)) = -t\};\\
A_4 &:= \{t > 0 : \|T(tu) + T(x)\| = \|tu + x\|,\\
&\quad \|T(tu) - T(x)\| = \|tu - x\|,\ \varphi(T(tu)) = -t\}.
\end{aligned}$$

Obviously, at least one of the sets $\{A_i : i = 1, 2, 3, 4\}$ is unbounded. We shall prove that if $A_i$ is unbounded, then

$$x^*(x) = (-1)^i \varphi(T(x))$$

for all $i = 1, 2, 3, 4$. Without loss of generality we can assume that $A_1$ is unbounded. Then, for every $t \in A_1$, we get

$$\begin{aligned}
\|tu + x\| - t &= \|T(tu) - T(x)\| - t \geq \varphi(T(tu) - T(x)) - t = -\varphi(T(x)),\\
\|tu - x\| - t &= \|T(tu) + T(x)\| - t \geq \varphi(T(tu) + T(x)) - t = \varphi(T(x)).
\end{aligned}$$

Leting $t \to +\infty$ in the two inequalities above, we get

$$x^*(x) = -\varphi(T(x)).$$

This completes the proof. □

**Lemma 3** *Let $E$ and $F$ be real normed spaces, $T : E \to F$ be a surjective phase isometry. For all $x, y \in E$ and $a, b \in \mathbb{R}$, if $T(x_1) = aT(x) + bT(y)$, $T(x_2) = aT(x) - bT(y)$ and $T(x_3) = aT(x)$ for some $x_1, x_2, x_3 \in E$, then $x^*(ax \pm by) \in \{\pm x^*(x_1), \pm x^*(x_2)\}$ and $x^*(ax) \in \{\pm x^*(x_3)\}$, for all $x^* \in w^* - exp(B_{E^*})$.*

***Proof*** By Lemma 2, for all $x^* \in w^*-\exp(B_{E^*})$ there exists a linear functional $\varphi \in F^*$ of the norm one such that $x^*(z) \in \{\varphi(T(z)), -\varphi(T(z))\}$ for all $z \in E$. Thus

$$\begin{aligned} x^*(ax \pm by) &\in \{\pm\varphi(aT(x) + bT(y)), \pm\varphi(aT(x) - bT(y))\} \\ &= \{\pm\varphi(T(x_1)), \pm\varphi(T(x_2))\} \\ &= \{\pm x^*(x_1), \pm x^*(x_2)\}, \end{aligned}$$

and

$$x^*(ax) = ax^*(x) \in \{\pm a\varphi(T(x))\} = \{\pm\varphi(aT(x))\} = \{\pm\varphi(T(x_3))\} = \{\pm x^*(x_3)\}.$$

This completes the proof. □

**Lemma 4** *Let $E$ be a real normed space with $dim(E) = 2$. If $|w^* - exp(B_{E^*})| = 4$, then $E$ is isometric to $l_1^{(2)}$, that is $\mathbb{R}^2$ with the $l_1-$norm.*

***Proof*** Let $\{x_1^*, x_2^*, -x_1^*, -x_2^*\} = w^* - \exp(B_{E^*})$. Since $\dim(E^*) = 2$, the set of $w^*$ exposed points of $B_{E^*}$ is dense in the set of extreme points [ [17], p.167, Th.18.6]. Thus $\text{ext}(B_{E^*}) = \{x_1^*, x_2^*, -x_1^*, -x_2^*\}$. By the Krein–Milman Theorem, $B_{E^*} = \text{co}\{x_1^*, x_2^*, -x_1^*, -x_2^*\}$. Therefore

$$\|x_1^* + x_2^*\| = \|x_1^* - x_2^*\| = 2.$$

Define the linear operator $V : E^* \to l_\infty^{(2)}$, by $V(x_1^*) = (1, 0)$ and $V(x_2^*) = (0, 1)$, where $l_\infty^{(2)}$ is $\mathbb{R}^2$ with the $l_\infty$-norm. It is obvious that $V$ is a linear isometry. Therefore $E$ is isometric to $l_1^{(2)}$. □

The following lemma is a simple case in dimension two of the fact that two linear functionals are linearly dependent if and only if they have the same kernel space.

**Lemma 5** *Let $E$ be a real normed space with $dim(E) = 2$, $x, y$ be linearly independent elements of $E$ and $x_1^*, x_2^* \in S_{E^*}$. If $x_1^*(x) = x_1^*(y)$ and $x_2^*(x) = x_2^*(y)$, then $x_1^* = x_2^*$ or $x_1^* = -x_2^*$.*

***Proof*** Since $\dim(E) = 2$, $x, y$ are linearly independent elements of $E$ and $x_1^*(x - y) = x_2^*(x - y) = 0$, it follows that $\ker(x_1^*) = \ker(x_2^*)$, where $\ker(x_i^*)$ denotes the kernel (null-space) of $x_i^*$ for $i = 1, 2$. Thus there exists $k \in \mathbb{R}$ such that $x_1^* = kx_2^*$ (see [13], the corollary on p.5). Because $x_1^*, x_2^* \in S_{E^*}$, we infer that $k = 1$ or $k = -1$. □

**Theorem 2** *Let $E$ and $F$ be real normed spaces and let $T : E \to F$ be a surjective phase isometry. If for any two linear independent elements $x, y \in E$, $T(\text{span}\{x, y\}) \subset \text{span}\{T(x), T(y)\}$, then $T(tx) \in \{tT(x), -tT(x)\}$ for all $x \in E$ and $t \in \mathbb{R}$. Moreover, $T$ is phase equivalent to a homogeneous surjective phase isometry.*

***Proof*** By Lemma 1, we only have to show that $T(tx) = tT(x)$ or $T(tx) = -tT(x)$ for all $x \in S_E$ and $t > 0$. If not, there would exist $x_0 \in S_E$ and $t_0 > 0$ such that neither $T(t_0x_0) = t_0T(x_0)$ nor $T(t_0x_0) = -t_0T(x_0)$. Because $T$ is surjective, there exists $x_1 \in E$ with $\|x_1\| = t_0$ such that $T(x_1) = t_0 T(x_0)$. Since $T$ is injective and odd, it follows that neither $x_1 = t_0 x_0$ nor $x_1 = -t_0 x_0$, so $x_0$ and $x_1$ are linearly independent. Let $E_0 = \mathrm{span}\{x_0, x_1\}$. By Lemma 3, we have $x^*(t_0 x_0) \in \{\pm x^*(x_1)\}$ for any $x^* \in w^* - \exp(B_{E_0^*})$.

If $|w^* - \exp(B_{E_0^*})| > 4$, because $\dim(E_0) = 2$, by Lemma 5, there exists $x_0^* \in w^* - \exp(B_{E_0^*})$ such that $x_0^*(tx_0) \notin \{\pm x_0^*(x_1)\}$, which contradicts to the fact that $x^*(tx) \in \{\pm x^*(x_1)\}$ for any $x^* \in w^* - \exp(B_{E_0^*})$.

If $|w^* - \exp(B_{E_0^*})| = 4$, by Lemma 4, $E_0$ is isometric to $l_1^{(2)}$. Thus there exist $e_1, e_2 \in B_{E_0}$ such that $\|e_1 + e_2\| = \|e_1 - e_2\| = 2$. Then

$$\{\|T(e_1) \pm T(e_2)\|\} = \{\|e_1 \pm e_2\|\} = \{2\}.$$

Firstly, assume that $0 < t_0 \leq 1$. Since $T(\mathrm{span}\{e_1, e_2\}) \subset \mathrm{span}\{T(e_1), T(e_2)\}$, putting $T(t_0 e_1) = aT(e_1) + bT(e_2)$, we obtain

$$t_0 = \|T(t_0 e_1)\| = \|aT(e_1) + bT(e_2)\| = |a| + |b|$$

and

$$\begin{aligned}
\{1 + |a| + |b|, |1 - |a| - |b||\} &= \{1 + t_0, |1 - t_0|\} \\
&= \{\|T(t_0 e_1) \pm T(e_1)\|\} \\
&= \{\|(aT(e_1) + bT(e_2)) \pm T(e_1)\|\} \\
&= \{|1 + a| + |b|, |1 - a| + |b|\} \\
&= \{1 + t_0, |1 - |a| + |b||\},
\end{aligned}$$

so we get $b = 0$. Thus $T(t_0 e_1) = t_0 T(e_1)$ or $T(t_0 e_1) = -t_0 T(e_1)$. Secondly, if $t_0 > 1$, then $T(e_0) \in \{\pm \frac{1}{t_0} T(t_0 e_0)\}$, so $T(t_0 e_0) \in \{\pm t_0 T(e_0)\}$.

Without loss of generality, we assume that $x_0 = \alpha e_1 + \beta e_2$ with $\alpha > 0, \beta > 0$ and $\alpha + \beta = 1$. Let $x_1 = \alpha_1 e_1 + \beta_1 e_2$. Since for any $x^* \in w^* - \exp(B_{E_0^*})$ we have $x^*(t_0 x_0) \in \{\pm x^*(x_1)\}$, we may assume that $\alpha_1 \geq 0, \beta_1 \geq 0$. Then

$$\alpha_1 + \beta_1 = t_0 \quad \text{and} \quad t_0\alpha - t_0\beta = -\alpha_1 + \beta_1. \tag{3}$$

Since $T(x_1) = t_0 T(x_0)$, we obtain

$$\begin{aligned}
&\{\|T(x_1) + t_0 T(e_1)\|, \|T(x_1) - t_0 T(e_1)\|\} \\
&\quad = \{t_0 \|T(x_0) + T(e_1)\|, t_0 \|T(x_0) - T(e_1)\|\} \\
&\quad = \{t_0 \|x_0 + e_1\|, t_0 \|x_0 - e_1\|\} \\
&\quad = \{t_0(1 + \alpha + \beta), t_0(1 - \alpha + \beta)\}
\end{aligned}$$

and

$$\begin{aligned}&\{\|T(x_1)+t_0T(e_1)\|,\|T(x_1)-t_0T(e_1)\|\}\\&\quad=\{\|T(x_1)+T(t_0e_1)\|,\|T(x_1)-T(t_0e_1)\|\}\\&\quad=\{\|x_1+t_0e_1\|,\|x_1-t_0e_1\|\}\\&\quad=\{t_0+\alpha_1+\beta_1,t_0-\alpha_1+\beta_1\}.\end{aligned}$$

By (4) and $\{t_0(1+\alpha+\beta), t_0(1-\alpha+\beta)\} = \{t_0+\alpha_1+\beta_1, t_0-\alpha_1+\beta_1\}$, we conclude that $t_0(1-\alpha+\beta) = t_0-\alpha_1+\beta_1$. Hence $\alpha_1 = t_0\alpha$ and $\beta_1 = t_0\beta$. that is $x_1 = t_0x_0$ and finally $T(t_0x_0) = t_0T(x_0)$ or $T(t_0x_0) = -t_0T(x_0)$. Thus $T(tx) \in \{tT(x), -tT(x)\}$ for all $x \in E$ and $t \in \mathbb{R}$.

To prove the last part of the theorem, by the axiom of choice, there is a set $L$ such that for every $0 \neq x \in E$ there exists a unique element $y \in E$ such that $x = sy$ for some $s \in \mathbb{R}$. Define $H : E \to F$ by

$$H(x) = sT(y) \quad \text{for every} \qquad x = sy \in E.$$

Then $H$ is well defined, homogeneous and phase equivalent to $T$. □

Next result shows that a surjective phase isometry preserves the strong convexity of its domain.

**Theorem 3** *Let $E$ and $F$ be two real normed spaces, and let $E$ be strictly convex. If there exists a surjective phase isometry $T : E \to F$, then $F$ is also a strictly convex real normed space.*

***Proof*** If $F$ is not strictly convex, then there exist $x, y \in S_F$ such that $[x, y] \subset S_F$. Since $T$ is surjective, there exist $x', y' \in S_E$ such that $T(x') = x$ and $T(y') = y$.

Because $T$ is a phase isometry, we have

$$\begin{aligned}\{\|x'+y'\|,\|x'-y'\|\}&=\{\|T(x')+T(y')\|,\|T(x')-T(y')\|\}\\&=\{\|x+y\|,\|x-y\|\}=\{2,\|x-y\|\}.\end{aligned}$$

Therefore $[x', y'] \subset S_E$ or $[-x', y'] \subset S_E$, which contradicts to the strict convexity of $E$. □

## 3 Two-dimensional normed spaces with the Wigner Property

In this section, we will show that any two-dimensional real normed spaces have the Wigner Property. First, we recall some definitions and notations.

**Definition 2** [8] Let $E$ be a real normed space. For any $x, y \in S_E$, $x \neq -y$, we define the arc of $x$ and $y$ to be the set

$$A(x,y)=\left\{z \mid z=\frac{\lambda x+(1-\lambda)y}{\|\lambda x+(1-\lambda)y\|},\quad \lambda\in[0,1]\right\}.$$

Freese et al. [8, Theorem 2.1, Theorem 2.2] gave the following important theorem describing a certain property of a unit sphere of a two-dimensional real normed space.

**Theorem 4** [8] *Let $E$ be a two-dimensional real normed space and let $x, y$ be linearly independent elements of $S_E$. If $z \in A(x, y) \setminus \{x, y\}$, then $\|x - z\| \leq \|x - y\|$, and either $\|z - x\| < \|y - x\|$ or $\|y + x\| < \|z + x\|$. Moreover, if $w \in A(x, y)$ is such that $\|x - z\| = \|x - w\|$ and $\|y - z\| = \|y - w\|$, then $w = z$.*

Let $E$ be a normed space. Suppose that $x, y \in E$. Then $x$ is said to be *isosceles orthogonal* to $y$, denoted by $x \perp_I y$, if $\|x + y\| = \|x - y\|$. Alonso [1] proved the existence and uniqueness of an isosceles orthogonal element of a unit sphere to any element of the unit sphere under consideration.

**Lemma 6** [1] *Let $E$ be a two dimensional normed space, Suppose that $x \in S_E$. Then there exists a unique (up to the sign) element $y \in S_E$ such that $x \perp_I y$.*

**Lemma 7** *Let $E$ and $F$ be real normed spaces, $dim(E) = 2$ and let $T : E \to F$ be a surjective phase isometry. Then $dim(F) = 2$.*

***Proof*** Suppose that $\dim(F) > 2$. Fix an arbitrary element $x \in S_E$, and set $I_x := \{y \in S_E : y \perp_I x\}$ and $I_{T(x)} := \{w \in S_F : w \perp_I T(x)\}$. Since $T : E \to F$ is a surjective phase isometry, it follows that $T(I_x) = I_{T(x)}$.

Since $\dim(E) = 2$, by Lemma 6, we know that $|I_x| = 2$. Because $\dim(F) > 2$, so there exist $x_1, x_2 \in S_F$ such that $\{T(x), x_1, x_2\}$ is a linearly independent set of $F$. For any $\alpha \in [0, \pi)$, let $F_\alpha = \text{span}\{T(x), (cos\alpha)x_1 + (sin\alpha)x_2\}$. By Lemma 6, there exists $x_\alpha \in S_{F_\alpha} \subset S_F$ such that $x_\alpha \in I_{T(x)}$. Since $S_{F_{\alpha_1}} \bigcap S_{F_{\alpha_2}} = \{T(x), -T(x)\}$, it follows that $x_{\alpha_1} \neq x_{\alpha_2}$, if $\alpha_1 \neq \alpha_2$. Thus $|I_{T(x)}| = +\infty$, which contradicts to $T(I_x) = I_{T(x)}$. □

**Theorem 5** *Let $E$ and $F$ be real normed spaces, $dim(E) = 2$ and let $T : E \to F$ be a surjective phase isometry. Then $T$ is phase equivalent to a homogeneous surjective phase isometry.*

***Proof*** By Lemma 7, we infer that $\dim(F) = 2$. Using Theorem 2 we deduce that $T$ is phase equivalent to a homogeneous surjective phase isometry. □

**Theorem 6** *Let $E$ and $F$ be real normed spaces, $dim(E) = 2$ and let $T : E \to F$ be a surjective phase isometry. If there exist two linearly independent vectors $x_0, y_0 \in S_E$ such that the following conditions holds for any $a, b \in \mathbb{R}$:*

(i) *there are two real numbers $\alpha(a, b), \beta(a, b)$ with $|\alpha(a, b)| = |\beta(a, b)| = 1$ such that $T(ax_0 + by_0) = \alpha(a, b)T(ax_0) + \beta(a, b)T(by_0)$;*
(ii) $\|ax_0 + by_0\| \geq \max\{a, b\}$ *for $a \geq 0$ and $b \geq 0$.*

*Then $T$ is phase equivalent to a linear isometry.*

***Proof*** Since phase equivalence is an equivalence relationship between all surjective phase isometries, by Theorem 5, we may assume that $T$ is a homogeneous surjective phase isometry.

For every $a, b \in \mathbb{R}$ with $ab \neq 0$, since $T$ is homogeneous, it follows that

$$\begin{aligned}T(ax_0 + by_0) &= bT(\frac{a}{b}x_0 + y_0)\\ &= b\left[\alpha\left(\frac{a}{b}, 1\right)T\left(\frac{a}{b}x_0\right) + \beta\left(\frac{a}{b}, 1\right)T(y_0)\right]\\ &= \alpha\left(\frac{a}{b}, 1\right)T(ax_0) + \beta\left(\frac{a}{b}, 1\right)T(by_0).\end{aligned}$$

Define a mapping $L : E \to F$ as follows:

$$L(ax_0 + by_0) = \begin{cases} 0, & \text{if } a = 0 \text{ and } b = 0,\\ \alpha(a, 1)\beta(a, 1)T(ax_0), & \text{if } a \neq 0 \text{ and } b = 0,\\ T(by_0), & \text{if } a = 0 \text{ and } b \neq 0,\\ \alpha(\frac{a}{b}, 1)\beta(\frac{a}{b}, 1)T(ax_0) + T(by_0), & \text{if } a \neq 0 \text{ and } b \neq 0.\end{cases}$$

It is obvious that $L$ is a homogeneous phase isometry and it is phase equivalent to $T$. Now, we will show that $\alpha(a, 1)\beta(a, 1) = \alpha(1, 1)\beta(1, 1)$ for all $a \neq 0$.

Since $a \neq 0$ and

$$\begin{aligned}\|L(ax_0 + y_0) + L(ax_0 - y_0)\| = |\alpha(a, 1)\beta(a, 1) + \alpha(-a, 1)\\ \beta(-a, 1)|\|T(ax_0)\| \in \{|2a|, 2\},\end{aligned}$$

it follows that

$$\alpha(a, 1)\beta(a, 1) = \alpha(-a, 1)\beta(-a, 1).$$

If $0 < a < 1$, because

$$\{\|L(ax_0 + y_0) \pm L(x_0 + y_0)\|\} = \{\|(a + 1)x_0 + 2y_0\|, 1 - a\},$$

we get

$$\begin{aligned}&|a\alpha(a, 1)\beta(a, 1) - \alpha(1, 1)\beta(1, 1)| = \|L(ax_0 + y_0) - L(x_0 + y_0)\|\\ &\in \{\|(a + 1)x_0 + 2y_0\|, 1 - a\}.\end{aligned}$$

Condition (ii) implies that

$$\|(a + 1)x_0 + 2y_0\| \geq 2 > a + 1 \geq |a\alpha(a, 1)\beta(a, 1) - \alpha(1, 1)\beta(1, 1)|,$$

so

$$|a\alpha(a, 1)\beta(a, 1) - \alpha(1, 1)\beta(1, 1)| = 1 - a,$$

that is

$$\alpha(a, 1)\beta(a, 1) = \alpha(1, 1)\beta(1, 1) \quad \text{for all } 0 < a < 1.$$

If $a > 1$, because

$$\{\|L(ax_0 + ay_0) \pm L(ax_0 + y_0)\|\} = \{\|2ax_0 + (a+1)y_0\|, a-1\},$$

we get

$$\|(\alpha(1,1)\beta(1,1) + \alpha(a,1)\beta(a,1))T(ax_0) + (a+1)T(y_0)\| \\ \in \{\|2ax_0 + (a+1)y_0\|, a-1\}.$$

The inequality $\|2ax_0 + (a+1)y_0\| \geq 2a > a+1$ implies that

$$\alpha(a,1)\beta(a,1) = \alpha(1,1)\beta(1,1) \quad \text{for all } a > 1.$$

The equation $\alpha(a,1)\beta(a,1) = \alpha(1,1)\beta(1,1)$ for all $a \neq 0$, and the definition of $L$ show that

$$L(ax_0 + by_0) = L(ax_0) + L(by_0) \quad (a, b \in \mathbb{R}).$$

This means that $L$ is a linear isometry from $E$ onto $F$. The proof is complete. □

Theorem 6 is important in the study of surjective phase isometry operators between two dimensional real normed spaces. We will use it to show first that any two dimensional strictly convex real normed space and next to show that any two dimensional non-strictly convex real normed space has the Wigner Property.

**Theorem 7** *Let $E$ be a two dimensional strictly convex real normed space and $F$ be a real normed space, and let $T : E \to F$ be a surjective phase isometry. Then $T$ is phase equivalent to a linear isometry.*

***Proof*** Since the set of smooth points of $S_E$ is dense in $S_E$, we can choose $x_0, y_0 \in S_E$ and $0 < \lambda_0 < 1$ such that $\|x_0 - y_0\| < 1$ and $\frac{\lambda_0 x_0 + (1-\lambda_0)y_0}{\|\lambda_0 x_0 + (1-\lambda_0)y_0\|}$, $x_0, y_0$ are smooth points of $S_E$. Let $z_0 = \frac{\lambda_0 x_0 + (1-\lambda_0)y_0}{\|\lambda_0 x_0 + (1-\lambda_0)y_0\|}$. Then, by Theorem 4 we obtain $\|z_0 - x_0\| \leq \|x_0 - y_0\| < 1$ and $\|z_0 - y_0\| \leq \|x_0 - y_0\| < 1$. Let $x_0^*, y_0^*, z_0^* \in S_{E^*}$ such that $x_0^*(x_0) = 1$, $y_0^*(y_0) = 1$ and $z_0^*(z_0) = 1$. We have

$$x_0^*(y_0) = x_0^*(x_0) - x_0^*(x_0 - y_0) \geq 1 - \|x_0 - y_0\| > 0.$$

Similarly, we obtain $y_0^*(x_0) > 0$, $z_0^*(x_0) > 0$ and $z_0^*(y_0) > 0$. For any $a > 0$, $b > 0$, we have

$$\|ax_0 + by_0\| \geq \max\{x_0^*(ax_0 + by_0), y_0^*(ax_0 + by_0)\} \geq \max\{a, b\}.$$

Let $0 < \lambda < 1$. Since

$$\{\|T(x_0) + T(y_0)\|, \|T(x_0) - T(y_0)\|\} = \{\|x_0 + y_0\|, \|x_0 - y_0\|\},$$

without loss of generality, we may assume that $\|T(x_0) - T(y_0)\| = \|x_0 - y_0\|$. We have

$$\{\|T(\lambda x_0 + (1-\lambda)y_0) \pm T(x_0)\|\} = \{(1-\lambda)\|x_0 - y_0\|, \|(1+\lambda)x_0 + (1-\lambda)y_0\|\}$$

and

$$\{\|T(\lambda x_0 + (1-\lambda)y_0) \pm T(y_0)\|\} = \{\lambda\|x_0 - y_0\|, \|\lambda x_0 + (2-\lambda)y_0\|\}.$$

We assume that $\|T(\lambda x_0 + (1-\lambda)y_0) + T(x_0)\| = (1-\lambda)\|x_0 - y_0\|$; Other cases can be discussed similarly. If

$$\|T(\lambda x_0 + (1-\lambda)y_0) - T(y_0)\| = \lambda\|x_0 - y_0\|,$$

then

$$\begin{aligned}\|x_0 + y_0\| &= \|T(x_0) + T(y_0)\| \\ &\le \|T(\lambda x_0 + (1-\lambda)y_0) + T(x_0)\| + \|T(\lambda x_0 + (1-\lambda)y_0) - T(y_0)\| \\ &= \|x_0 - y_0\|,\end{aligned}$$

which contradicts to $\|x_0 - y_0\| < \|x_0 + y_0\|$. Thus $\|T(\lambda x_0 + (1-\lambda)y_0) + T(y_0)\| = \lambda\|x_0 - y_0\|$ and

$$\begin{aligned}\|x_0 - y_0\| &= \|T(x_0) - T(y_0)\| \\ &\le \|T(\lambda x_0 + (1-\lambda)y_0) + T(x_0)\| + \|T(\lambda x_0 + (1-\lambda)y_0) + T(y_0)\| \\ &= \|x_0 - y_0\|.\end{aligned}$$

By Theorem 3, the normed space $F$ is strictly convex, so we obtain $T(\lambda x_0 + (1-\lambda)y_0) = -\lambda T(x_0) - (1-\lambda)T(y_0)$.

Since $T$ is surjective, let $x_1, x_2 \in E$ such that $T(x_1) = \lambda T(x_0) + (1-\lambda)T(y_0)$ and $T(x_2) = \lambda T(x_0) - (1-\lambda)T(y_0)$.

By Lemma 1, we have $-x_1 = \lambda x_0 + (1-\lambda)y_0$. Since $x^*(x) > 0$ for all $x^* \in \{x_0^*, y_0^*, z_0^*\}$ and all $x \in \{x_0, y_0\}$. It is obvious that $x^*(\lambda x_0 - (1-\lambda)y_0) \notin \{\pm x^*(x_1)\}$ for all $x^* \in \{x_0^*, y_0^*, z_0^*\}$. By Lemma 3, we infer that $x^*(\lambda x_0 - (1-\lambda)y_0) \in \{\pm x^*(x_2)\}$. Thus Lemma 5 shows that $\lambda x_0 - (1-\lambda)y_0 \in \{\pm x_2\}$.

By Theorem 6, we infer that $T$ is phase equivalent to a linear isometry. □

**Lemma 8** *Let $E$ be a two-dimensional non-strictly convex real normed space, $T : E \to F$ be a surjective phase-isometry. If a segment $[x, y] \subset S_E$, then for any $0 < \lambda < 1$, there are two real numbers $\alpha(\lambda), \beta(\lambda)$ with $|\alpha(\lambda)| = |\beta(\lambda)| = 1$ such that $T(\lambda x + (1-\lambda)y) = \alpha(\lambda)\lambda T(x) + \beta(\lambda)(1-\lambda)T(y)$.*

***Proof*** For any $0 < \lambda < 1$, because $[x, y] \subset S_E$, by Lemma 1, we have $\|T(\lambda x + (1-\lambda)y)\| = 1$.

If $\|x-y\| < \|x+y\| = 2$, since $\{\|T(x) \pm T(y)\|\} = \{2, \|x-y\|\}$, without loss of generality, we may assume that $\|T(x) - T(y)\| = \|x-y\|$. We have

$$\{\|T(\lambda x + (1-\lambda)y) \pm T(x)\|\} = \{(1-\lambda)\|x-y\|, 2\}.$$

We will consider the case $\|T(\lambda x+(1-\lambda)y)+T(x)\| = (1-\lambda)\|x-y\|$, the other case can be discussed similarly. If $T(\lambda x + (1-\lambda)y) \in A(-T(x), T(y))$, by Theorem 4, we get

$$\|T(\lambda x + (1-\lambda)y) + T(y)\| \geq \| - T(x) + T(y)\| > \lambda\|x-y\|.$$

Since

$$\{\|T(\lambda x + (1-\lambda)y) \pm T(y)\|\} = \{\lambda\|x-y\|, 2\},$$

if $\|T(\lambda x + (1-\lambda)y) - T(y)\| = \lambda\|x-y\|$ , then

$$\begin{aligned}\|x+y\| &= \|T(x) + T(y)\| \\ &\leq \|T(\lambda x + (1-\lambda)y) + T(x)\| + \|T(\lambda x + (1-\lambda)y) - T(y)\| \\ &= \|x-y\|,\end{aligned}$$

which contradicts to $\|x-y\| < \|x+y\|$. Thus $T(\lambda x + (1-\lambda)y) \in [-T(x), -T(y)]$ and $T(\lambda x + (1-\lambda)y) = -\lambda T(x) - (1-\lambda)T(y)$.

If $\|x-y\| = \|x+y\| = 2$, then $\|T(x) + T(y)\| = \|T(x) - T(y)\| = 2$. Without loss of generality, we assume that $\|T(\lambda x + (1-\lambda)y) + T(x)\| = 2(1-\lambda)$ and $\|T(\lambda x + (1-\lambda)y) + T(y)\| = 2\lambda$. Then $T(\lambda x + (1-\lambda)y) \in [-T(x), -T(y)]$ and $T(\lambda x + (1-\lambda)y) = -\lambda T(x) - (1-\lambda)T(y)$. This completes the proof. □

**Theorem 8** *Let $E$ be a two-dimensional non-strictly convex real normed space and $F$ be a real normed space, let $T : E \to F$ be a surjective phase isometry. Then $T$ is a phase equivalent to a linear isometry.*

***Proof*** We will divide the proof into two cases. Case one, if $E$ is isometric to $l_1^{(2)}$. Let $x_0 = e_1 = (1, 0)$ and $y_0 = e_2 = (0, 1)$. Then, by Lemma 8 and Theorem 6, we infer that $T$ is a phase equivalent to a linear isometry.

Case two, if $E$ is not isometric to $l_1^{(2)}$, then there exist $x_1^*, x_2^*, x_3^* \in \exp(B_{E^*})$ such that $x_i^* \neq \pm x_j^*$ for $i \neq j$. Since $E$ is non-strictly convex, there exist $x_0, y_0 \in S_E$ such that $[x_0, y_0] \subset S_E \cap \mathrm{sm}(S_E)$ and $\|x_0 - y_0\| < 1$. $x_i^*(x_0) \neq 0$, and $x_i^*(y_0) \neq 0$ for $i = 1, 2, 3$. For any $0 < \lambda < 1$, Since $T$ is surjective, let $x_1, x_2 \in E$ be such that $T(x_1) = \lambda T(x_0) + (1-\lambda)T(y_0)$ and $T(x_2) = \lambda T(x_0) - (1-\lambda)T(y_0)$. By Lemma 8, we get $\lambda x_0 + (1-\lambda)y_0 \in \{\pm x_1, \pm x_2\}$. Without loss of generality, we assume that $\lambda x_0 + (1-\lambda)y_0 \in \{\pm x_1\}$. Since $x_i^*(x_0) \neq 0$ and $x_i^*(y_0) \neq 0$, it follows that

$$x_i^*(\lambda x_0 - (1-\lambda)y_0) \neq \pm x_i^*(\lambda x_0 + (1-\lambda)y_0)$$

for $i = 1, 2, 3$.

Lemma 3 implies that $x_i^*(\lambda x_0 - (1-\lambda)y_0) \in \{\pm x_i^*(x_2)\}$, for $i = 1, 2, 3$. By Lemma 5, we obtain that $\lambda x_0 - (1-\lambda)y_0 \in \{\pm x_2\}$. Then, by Lemma 8 and Theorem 6, we infer that $T$ is a phase equivalent to a linear isometry. This completes the proof. □

Now, we can obtain the main result of this section using Theorems 7 and 8.

**Theorem 9** *If $E$ is a two-dimensional real normed space, then $E$ has the Wigner Property.*

## 4 Wigner Property on real Banach spaces of dimension at least three

In the proof of the Wigner Theorem from [22], Uhlhorn highlights the connection between the Wigner Theorem and the First Fundamental Theorem of projective geometry. The First Fundamental Theorem of projective geometry says that an abstract automorphism of the set of lines in vector spaces which preserves "incidence relations" must have a simple algebraic form (see [7]). In this section, we will show that the First Fundamental Theorem of projective geometry also plays an important role in the study of real normed spaces with the Wigner Property.

Let $f$ be a mapping from a set $X$ into a set $Y$ and let $D$ be a subset of $2^X$ - the power set of $X$. A mapping $F : D \to 2^Y$ is said to be *induced* or *generated* by $f$ if for every $M \in D$, $F(M) = \{f(m) : m \in M\}$. As usual, this last set is also noted $f(M)$. If $X$ is a real vector space, we denote the *projectivised space* (that is the set of all one-deminsional subspaces) by $P(X)$. The element of $P(X)$ generated by $0 \neq x \in X$ will be denoted by $[x] := \mathbb{R} \cdot x$.

Gehér [9, Theorem 3] proved the following special case of the First Fundamental Theorem of projective geometry for real vector spaces, which will be used later.

**Proposition 1** [9] *Let $E_1$ and $E_2$ be two real vector spaces of dimensions at least three. If $g : P(E_1) \to P(E_2)$ satisfies the following conditions:*

(i) *the range of $g$ is not contained in a two-dimensional subspace of $Y$;*
(ii) $0 \neq z \in \operatorname{span}\{x, y\}$ $(x \neq 0 \neq y)$ *implies* $g([z]) \subset \operatorname{span}\{g([x]), g([y])\}$;

*then there exists an injective linear transformation $A : E_1 \to E_2$ inducing $g$.*

**Theorem 10** *Let $E$ and $F$ be real normed spaces, $dim(E) \geq 3$ and let $T : E \to F$ be a surjective phase isometry. If for any two linearly independent elements $x, y \in E$, $T(\operatorname{span}\{x, y\}) = \operatorname{span}\{T(x), T(y)\}$, then $T$ is a phase equivalent to a linear isometry.*

***Proof*** Since $T$ is a surjective phase isometry and $T(\operatorname{span}\{x, y\}) = \operatorname{span}\{T(x), T(y)\}$, by Lemmas 1, 7 and Theorem 9, we deduce that $T$ is injective, $T([x]) = [T(x)]$ and $T(x + y) = \alpha T(x) + \beta T(y)$ for any $x, y \in E$ with $|\alpha| = |\beta| = 1$. Hence the function $g : P(E) \to P(F)$, $g([x]) = T([x])$ is well defined. Similarly to the proof of Lemma 7, we infer that the range of $g$ is not contained in a two-dimensional subspace of $F$. For any $0 \neq z \in \operatorname{span}\{x, y\}$ $(x \neq 0 \neq y)$, because $T(\operatorname{span}\{x, y\}) = \operatorname{span}\{T(x), T(y)\}$, so by the definition of the function $g$, we deduce that $g([z]) \subset \operatorname{span}\{g([x]), g([y])\}$.

By Proposition 1, there exists an injective linear map $A : E \to F$ such that $g([x]) = [A(x)]$, that is $[T(x)] = [A(x)]$ for every $x \in E$. Consequently, there exists a function $\lambda : E \to \mathbb{R}$ such that $T(x) = \lambda(x)A(x)$ for every $x \in X$. Since $T$ is homogeneous, $\lambda(tx) = \lambda(x)$ for every $x \in E$ and $0 \neq t \in \mathbb{R}$. Moreover, suppose that $x, y \in E$ are two linearly independent vectors. Let us write $T(x+y) = \alpha T(x)+\beta T(y)$ for some real numbers $\alpha$ and $\beta$ with $|\alpha| = |\beta| = 1$. We immediately obtain

$$\alpha\lambda(x)A(x) + \beta\lambda(y)A(y) = T(x + y) = \lambda(x + y)A(x) + \lambda(x + y)A(y),$$

so $\lambda(x + y) = \alpha\lambda(x) = \beta\lambda(y)$. As a consequence, $|\lambda(x)|$ is a constant for any $x \in E$, which we denote by $\lambda$. Hence we can define a desired phase function $\varepsilon : E \to \{-1, 1\}$ such that $T = \varepsilon\lambda A$. Thus $T$ is phase equivalent to the linear isometry $\lambda A$. □

Using the above result we can prove the following

**Theorem 11** *Let $E$ be a real strictly convex normed space. Then $E$ has the Wigner Property.*

***Proof*** If $\dim(E) = 2$, then $E$ has the Wigner Property by Theorem 9. Let us assume that $\dim(E) \geq 3$, $F$ is any real normed space and $T : E \to F$ is a surjective phase isometry. We will show that $T$ is a phase equivalent to a linear isometry. By Theorem 10, we need only to prove that $T(\text{span}\{x, y\}) = \text{span}\{T(x), T(y)\}$ for any two linear independent elements $x, y \in E$.

Since $E$ is a real strictly convex normed space, we can prove that $T(tx) = tT(x)$ or $T(tx) = -tT(x)$ for any $x \in E$ and $t \in \mathbb{R}$ in the same way as in the proof of Theorem 2. For any two linear independent elements $x, y \in E$, let $E_0 = \text{span}\{x, y\}$. Then there exist $x_0, y_0 \in S_{E_0}$ such that $\|x_0 - y_0\| = \|x_0 + y_0\|$ and $E_0 = \text{span}\{x_0, y_0\}$. To show that $T(\text{span}\{x_0, y_0\}) = \text{span}\{T(x_0), T(y_0)\}$, we only have to show that for any $0 < \lambda < 1$, $T(\lambda x_0 \pm (1-\lambda)y_0) = \alpha\lambda T(x_0) + \beta(1-\lambda)T(y_0)$ with $|\alpha| = |\beta| = 1$.

Firstly, we will show that $T(\lambda x_0 + (1 - \lambda)y_0) = \alpha\lambda T(x_0) + \beta(1 - \lambda)T(y_0)$ with $|\alpha| = |\beta| = 1$. Since

$$\{\|T(\lambda x_0 + (1 - \lambda)y_0) \pm T(x_0)\|\} = \{(1 - \lambda)\|x_0 - y_0\|, \|(1 + \lambda)x_0 + (1 - \lambda)y_0\|\}$$

and

$$\{\|T(\lambda x_0 + (1 - \lambda)y_0) \pm T(y_0)\|\} = \{\lambda\|x_0 - y_0\|, \|\lambda x_0 + (2 - \lambda)y_0\|\},$$

without loss of generality, we may assume that $\|T(\lambda x_0 + (1 - \lambda)y_0) + T(x_0)\| = (1 - \lambda)\|x_0 - y_0\|$ and $\|T(\lambda x_0 + (1 - \lambda)y_0) - T(y_0)\| = \lambda\|x_0 - y_0\|$. Then

$$\begin{aligned}\|T(x_0) + T(y_0)\| &\leq \|T(\lambda x_0 + (1 - \lambda)y_0) + T(x_0)\| \\ &\quad + \|T(\lambda x_0 + (1 - \lambda)y_0) - T(y_0)\| \\ &= \|x_0 - y_0\| \\ &= \|T(x_0) + T(y_0)\|.\end{aligned}$$

The last equality holds because $\|x_0 + y_0\| = \|x_0 - y_0\|$ and

$$\{\|T(x_0) \pm T(y_0)\|\} = \{\|x_0 + y_0\|, \|x_0 - y_0\|\}.$$

By Theorem 3, $F$ is a strictly convex normed space and we get $T(\lambda x_0 + (1-\lambda)y_0) = -\lambda T(x_0) - (1-\lambda)T(y_0)$.

Similarly, we can prove that $T(\lambda x_0 - (1-\lambda)y_0) = \alpha\lambda T(x_0) + \beta(1-\lambda)T(y_0)$ with $|\alpha| = |\beta| = 1$. This completes the proof. □

Since every inner product space is a strictly convex normed space, by Theorem 11, one can easily get the following

**Corollary 1** *Let $E$ be a real inner product space and $F$ be a real normed space. Let $T : E \to F$ be a surjective phase isometry. Then $F$ is a real inner product space.*

**Theorem 12** *Let $E$ be a real normed space. If for any three different points $x, y, z \in E$ with $\|x\| \in \{\|y\|, \|z\|\}$ there exists $x^* \in w^* - exp(B_{E_0^*})$ such that $x^*(x) \notin \{\pm x^*(y), \pm x^*(z)\}$, where $E_0 = \text{span}\{x, y, z\}$, then $E$ has the Wigner Property.*

***Proof*** Let $F$ be any real normed space and $T : E \to F$ be a surjective phase isometry. By Theorem 10, we need only to prove that for any two linearly independent elements $x, y \in E$ there exist $\alpha, \beta \in \mathbb{R}$ with $|\alpha| = |\beta| = 1$ such that $T(x + y) = \alpha T(x) + \beta T(y)$. If not, there would exist $x_0, y_0 \in E$ such that $T(x_0 + y_0) \notin \{\pm(T(x_0) + T(y_0)), \pm(T(x_0) - T(y_0))\}$. Since $T$ is surjective, there exist $x_1, x_2 \in E$ such that $T(x_1) = T(x_0) + T(y_0)$ and $T(x_2) = T(x_0) - T(y_0)$. Because

$$\{\|x_1\|, \|x_2\|\} = \{\|T(x_0) + T(y_0)\|, \|T(x_0) - T(y_0)\|\} = \{\|x_0 + y_0\|, \|x_0 - y_0\|\},$$

so $\|x_0 + y_0\| \in \{\|x_1\|, \|x_2\|\}$. Then there exists $x^* \in w^* - \exp(B_{E_0^*})$ such that $x^*(x_0 + y_0) \notin \{\pm x^*(x_1), \pm x^*(x_2)\}$, where $E_0 = \text{span}\{x_0 + y_0, x_1, x_2\}$, which leads to a contradiction by Lemma 3. This completes the proof. □

It is obvious that the dimension of the subspace $E_0$ in Theorem 12 is less or equal to three. Thus a three-dimensional subspace plays a very important role in the study of real normed spaces with the Wigner Property. Hence it seems to be natural to rise the following two problems.

**Problem 2** Does every real normed space have the Wigner Property if and only if its every subspace of the dimension three has the Wigner Property?

**Problem 3** Does every real normed space of the dimensional three has the Wigner Property?

**Acknowledgements** The authors would like to thank the anonymous referee for all his/her valuable comments which allowed us to improve the quality of this paper. The first author is grateful to the Department of Mathematics and Computer Science, Adam Mickiewicz University, Poznań, for the excellent working conditions during the visit in February 2018 to August 2018. Moreover, he wishes also to express his appreciation to Prof. Guanggui Ding for many very helpful comments regarding the isometries theory in Banach spaces. Ruidong Wang is supported by the Natural Science Foundation of China (Grant nos. 11371201.11201337, 11201338, 11301384).

## References

1. Alonso, J.: Uniqueness properties of isosceles orthogonality in normed linear spaces. Ann. Sci. Math. Qué. **18**, 25–38 (1994)
2. Bourgain, J.: Real isomorphic complex Banach spaces need not be complex isomorphic. Proc. Am. Math. Soc. **96**, 221–226 (1986)
3. Bargmann, V.: Note on Wigner's theorem on symmetry operations. J. Math. Phys. **5**, 862–868 (1964)
4. Bracci, L., Morchio, G., Strocchi, F.: Wigner's theorem on symmetries in indefinite metric spaces. Commun. Math. Phys. **41**, 289–299 (1975)
5. Chevalier, G.: Wigner's theorem and its generalizations. In: Engesser, K., Gabbay, D.M., Lehmann, D. (eds.) Handbook of Quantum Logic and Quantum Structure: Quantum Structures, pp. 429–475. Elsevier, Amsterdam (2007)
6. Chmieliński, J., Ilišević, D., Moslehian, M.S., Sadeghi, G.: Perturbation of the Wigner equation in inner product $C^*$-modules. J. Math. Phys. **49**, 033519 (2008)
7. Faure, C.A.: An elementary proof of the fundamental theorem of projective geometry. Geometriae Dedicata **90**, 145–151 (2002)
8. Freese, R.W., Diminnie, C.R., Andalafte, E.Z.: Angle bisectors in normed linear spaces. Math. Nachr. **131**(1), 167–173 (1987)
9. Gehér, Gy.P.: On n-norm presevers and the Aleksandrov conservative $n$-distance problem. Aequat. Math. **91**(5), 933–943 (2017)
10. Gehér, Gy.P.: An elementary proof for the non-bijective version of Wigner's theorem. Phys. Lett. A. **378**, 2054–2057 (2014)
11. Gehér, Gy.P., Šemrl, P.: Isometries of Grassmann spaces. J. Funct. Anal. **270**, 1585–1601 (2016)
12. Györy, M.: A new proof of Wigner's theorem. Rep. Math. Phys. **54**, 159–167 (2004)
13. Holmes, R.B.: Geometric Functional Analysis and Its Applications. Springer, Berlin (1975)
14. Molnár, L.: Orthogonality preserving transformations on indefinite inner product spaces: generalization of Uhlhorn's version of Wigner's theorem. J. Funct. Anal. **194**(2), 248–262 (2002)
15. Mazur, S., Ulam, S.: Sur les transformationes isométriques despaces vectoriels normés. C. R. Math. Acad. Sci. Paris. **194**, 946–948 (1932)
16. Pankov, M.: Wigner's type theorem in terms of linear operators which send projections of a fixed rank to projections of other fixed rank. J. Math. Anal. Appl. **474**(2), 1238–1249 (2019)
17. Rockafellar, R.T.: Convex Analysis. Princeton University Press, Princeton (1972)
18. Sharma, C.S., Almeida, D.L.: A direct proof of Wigner's theorem on maps which preserve transition probabilities between pure states of quantum systems. Ann. Phys. **197**(2), 300–309 (1990)
19. Simon, R., Mukunda, N., Chaturvedi, S., Srinivasan, V.: Two elementary proofs of the Wigner theorem on symmetry in quantum mechanics. Phys. Lett. A. **372**(46), 6847–6852 (2008)
20. Tan, D., Huang, X.: Phase-isometries on real normed spaces. (**Preprint**)
21. Turnšek, A.: A variant of Wigner's functional equation. Aequat. Math. **89**(4), 949–956 (2015)
22. Uhlhorn, U.: Representation of symmetry transformations in quantum mechanics. Arkiv for Fysik **23**(30), 307–340 (1962)
23. Wigner, E.: Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren. Vieweg, Brunswick (1931)