Annals of Functional Analysis (2020) 11:227–243
https://doi.org/10.1007/s43034-019-00024-1

Tusi Mathematical Research Group

Characterizing the metric compactification of L_p spaces by random measures

Armando W. Gutiérrez[1]

Received: 7 March 2019 / Accepted: 18 June 2019 / Published online: 1 January 2020

Abstract

We present a complete characterization of the metric compactification of L_p spaces for $1 \leq p < \infty$. Each element of the metric compactification of L_p is represented by a random measure on a certain Polish space. By way of illustration, we revisit the L_p-mean ergodic theorem for $1 < p < \infty$, and Alspach's example of an isometry on a weakly compact convex subset of L_1 with no fixed points.

Keywords Metric compactification · Horofunction compactification · Metric functional · Random measure · Banach spaces

Mathematics Subject Classification 54D35 · 46E30 · 46B20 · 60G57

1 Introduction

The metric compactification, which is known as the horofunction compactification in the setting of proper geodesic metric spaces, has been extensively studied during the last 20 years. Remarkably, the foundations of this compactification can be traced back to the early 1980s when Gromov [7,20] introduced a technique to attach certain boundary points at infinity of every proper geodesic metric space. In 2002, Rieffel [35] identified the metric compactification of every complete locally compact metric space with the maximal ideal space of a unital commutative C^*-algebra.

The modern procedure for constructing the metric compactification of general (not necessarily proper) metric spaces was discussed in [16,34]. Every metric space is continuously injected into its metric compactification which becomes metrizable provided that the metric space is separable. Moreover, surjective isometries between metric

Communicated by Yong Jiao.

✉ Armando W. Gutiérrez
wladimir.gutierrez@aalto.fi

[1] Department of Mathematics and Systems Analysis, Aalto University, Otakaari 1, Espoo, Finland

spaces can be extended continuously to homeomorphisms between the corresponding metric compactifications.

Banach spaces form an important class of metric spaces for which the metric compactification deserves further study. Complete characterizations of the metric compactification of finite- and infinite-dimensional ℓ_p spaces were presented by the author in [22] and [21], respectively. More studies in this context can be found in [24,33,36,40].

The purpose of this paper is to give a complete characterization of the metric compactification of the Banach space $L_p(\Omega, \Sigma, \mathbf{P})$ for all $1 \leq p < \infty$, where $(\Omega, \Sigma, \mathbf{P})$ is a non-atomic standard probability space.

Several works have confirmed the importance of the metric compactification as a topological and geometric tool for the study of isometry groups [30,32,37–39], random walks on hyperbolic groups [8,18], random product of semicontractions [17,19,28,29], Denjoy–Wolff theorems [1,9,25,26,31], Teichmüller spaces [4], limit graphs [12], random triangulations [11], and first-passage percolation [6]. This list is by no means exhaustive, but it gives the reader a brief overview of the variety of applications where the metric compactification or some of its elements appear. The notion of the metric compactification plays an essential role in the development of a metric spectral theory proposed by Karlsson [27].

2 Preliminaries

Let (X, d) be a metric space with an arbitrary base point $x_0 \in X$. Consider the mapping $y \mapsto \mathbf{h}_y$ from X into $\mathbb{R}^X$ defined by

$$\mathbf{h}_y(\cdot) := d(\cdot, y) - d(x_0, y). \tag{2.1}$$

Denote by $\mathrm{Lip}^1_{x_0}(X; \mathbb{R})$ the space of 1-Lipschitz real-valued functions on X vanishing at x_0. It is straightforward to verify that

$$\{\mathbf{h}_y \mid y \in X\} \subset \mathrm{Lip}^1_{x_0}(X; \mathbb{R}) \subset \prod_{x \in X} [-d(x_0, x), d(x_0, x)]. \tag{2.2}$$

Tychonoff's theorem asserts that the Cartesian product in (2.2) is compact in the topology of pointwise convergence, which in turn implies that the space $\mathrm{Lip}^1_{x_0}(X; \mathbb{R})$ is also compact as it is closed in this topology.

Definition 2.1 The *metric compactification* of (X, d), denoted by $\overline{X}^h$, is defined to be the pointwise closure of the family $\{\mathbf{h}_y \mid y \in X\}$. Every element $\mathbf{h} \in \overline{X}^h$ is called a *metric functional* on X. Metric functionals of the form (2.1) are called *internal*.

For every metric functional $\mathbf{h} \in \overline{X}^h$, there exists a net of points $(y_\alpha)_\alpha$ in X such that the net of internal metric functionals $(\mathbf{h}_{y_\alpha})_\alpha$ converges pointwise on X to $\mathbf{h}$. If the metric space is separable, the topology in $\overline{X}^h$ is metrizable. Hence, sequences are sufficient to describe metric funcionals on separable metric spaces. Also, the choice of the basepoint x_0 is irrelevant as different basepoints produce homeomorphic metric compactifications. See [16,34] for more details.

In general, the mapping $y \mapsto \mathbf{h}_y$ given by (2.1) is always a continuous injection. It becomes a homeomorphism onto its image whenever the metric space (X, d) is proper[1] and geodesic.[2] In this particular case, the metric compactification coincides with Gromov's original definition of horofunction compactification which is obtained as the closure of $\{\mathbf{h}_y \mid y \in X\}$ with respect to the topology of uniform convergence on bounded subsets of X; see [27].

Example 2.2 The set $\mathbb{R}$ of real numbers with the metric induced by the absolute value is proper and geodesic. Its metric/horofunction compactification $\overline{\mathbb{R}}^h$ contains the internal metric functionals

$$s \mapsto |s - r| - |r| \text{ with } r \in \mathbb{R}, \tag{2.3}$$

and the two additional "points at infinity" given by

$$s \mapsto -s, \quad s \mapsto s. \tag{2.4}$$

The elements of the compact space $\overline{\mathbb{R}}^h$ are used throughout the following sections, so it is convenient to establish the following special notations: functions of the form (2.3) are denoted by η_r with $r \in \mathbb{R}$, and the two functions in (2.4) are denoted by $\eta_{+\infty}$ and $\eta_{-\infty}$, respectively. Therefore, the metric compactification of $(\mathbb{R}, |\cdot|)$ is the compact Polish space

$$\overline{\mathbb{R}}^h = \{\eta_r \mid r \in \mathbb{R}\} \cup \{\eta_{+\infty}, \eta_{-\infty}\}. \tag{2.5}$$

Throughout the paper, we assume that $(\Omega, \Sigma, \mathbf{P})$ is a non-atomic standard probability space. We adopt the convention that all equalities involving measurable sets or measurable functions are assumed to hold modulo $\mathbf{P}$-null sets. For every $1 \leq p < \infty$, we denote by $L_p(\Omega, \Sigma, \mathbf{P})$, or simply L_p when no confusion arises, the Banach space of measurable functions $f : \Omega \to \mathbb{R}$ with finite L_p-norm

$$\|f\|_{L_p} := \left(\mathbf{E}[|f|^p]\right)^{1/p} = \left(\int_\Omega |f(\omega)|^p \, d\,\mathbf{P}(\omega)\right)^{1/p}.$$

For simplicity we choose the zero function as the basepoint. For each $g \in L_p$, the internal metric functional $\mathbf{h}_g$ in (2.1) becomes

$$\mathbf{h}_g(\cdot) = \|\cdot - g\|_{L_p} - \|g\|_{L_p}. \tag{2.6}$$

[1] Every closed and bounded subset of X is compact.

[2] For every pair of points $x, y \in X$, there is an isometry from the interval $[0, d(x, y)]$ into X.

The metric compactification $\overline{L_p}^h$ of the Banach space L_p is the set of all pointwise accumulation points of (2.6). We present explicit formulas for these limits by means of random measures on $\overline{\mathbb{R}}^h$.

The notion of random measure was introduced by Aldous [3]; however, there are various equivalent approaches to random measures [10,15,23]. Let $\mathcal{M}(\overline{\mathbb{R}}^h)$ denote the space of all signed Borel measures μ on the compact space $\overline{\mathbb{R}}^h$ such that the total variation $|\mu|\,(\overline{\mathbb{R}}^h)$ is finite. The linear space $\mathcal{M}(\overline{\mathbb{R}}^h)$ becomes a Banach space with respect to the norm $\|\mu\|_{\mathcal{M}(\overline{\mathbb{R}}^h)} = |\mu|\,(\overline{\mathbb{R}}^h)$. Let $C(\overline{\mathbb{R}}^h)$ denote the space of all real-valued continuous functions on the compact space $\overline{\mathbb{R}}^h$. The linear space $C(\overline{\mathbb{R}}^h)$ endowed with the norm

$$\|\varphi\|_{C(\overline{\mathbb{R}}^h)} = \sup_{\eta\in\overline{\mathbb{R}}^h} |\varphi(\eta)|$$

becomes a Banach space. The Riesz representation theorem (see e.g., [2, p. 78]) asserts that the dual space $C(\overline{\mathbb{R}}^h)^*$, equipped with the usual dual norm, can be identified with $\mathcal{M}(\overline{\mathbb{R}}^h)$ under the isometric isomorphism $\iota : \mathcal{M}(\overline{\mathbb{R}}^h) \to C(\overline{\mathbb{R}}^h)^*$, $\mu \mapsto \iota(\mu)$ given by

$$\langle \varphi, \iota(\mu)\rangle = \int_{\overline{\mathbb{R}}^h} \varphi(\eta)d\mu(\eta) \;\text{ for all } \varphi \in C(\overline{\mathbb{R}}^h).$$

We denote by $\mathcal{P}(\overline{\mathbb{R}}^h)$ the set of all Borel probability measures on $\overline{\mathbb{R}}^h$, i.e.,

$$\mathcal{P}(\overline{\mathbb{R}}^h) = \left\{\mu \in \mathcal{M}(\overline{\mathbb{R}}^h) \;\middle|\; \begin{array}{l}\langle\varphi,\mu\rangle \geq 0 \quad \forall\varphi \in C(\overline{\mathbb{R}}^h), \varphi \geq 0\\ \mu(\overline{\mathbb{R}}^h) = 1\end{array}\right\}.$$

Let $\Lambda_\infty(\Omega, \mathcal{M}(\overline{\mathbb{R}}^h))$ denote the space of mappings $\omega \mapsto \xi_\omega$ from Ω to $\mathcal{M}(\overline{\mathbb{R}}^h)$ with the following properties

1. the function $\omega \mapsto \langle\varphi, \xi_\omega\rangle$ is measurable for all $\varphi \in C(\overline{\mathbb{R}}^h)$,
2. the function $\omega \mapsto \|\xi_\omega\|_{\mathcal{M}(\overline{\mathbb{R}}^h)}$ is essentially bounded.

We say that an element ξ of $\Lambda_\infty(\Omega, \mathcal{M}(\overline{\mathbb{R}}^h))$ is a *random measure* on $\overline{\mathbb{R}}^h$ whenever $\xi_\omega \in \mathcal{P}(\overline{\mathbb{R}}^h)$ for $\mathbf{P}$-almost every $\omega \in \Omega$. Additionally, if $\xi_\omega(\{\eta_{-\infty}, \eta_{+\infty}\}) = 0$ for $\mathbf{P}$-almost every $\omega \in \Omega$, we say that ξ is a random measure on $\mathbb{R}$.

3 Main results

Theorem 3.1 *If* $\mathbf{h} \in \overline{L_1}^h$ *then there exists a random measure* ξ *on* $\overline{\mathbb{R}}^h$ *such that*

$$\mathbf{h}(f) = \mathbf{E}\left[\int_{\overline{\mathbb{R}}^h} \eta(f)d\xi(\eta)\right], \tag{3.1}$$

for all $f \in L_1$. Conversely, if ξ is a random measure on $\overline{\mathbb{R}}^h$ then (3.1) defines an element of $\overline{L_1}^h$.

Remark 3.2 Since ξ and f are defined on Ω, the expectation operator in the representation formula (3.1) should be interpreted as

$$\mathbf{E}\left[\int_{\overline{\mathbb{R}}^h} \eta(f) d\xi(\eta)\right] = \int_\Omega \int_{\overline{\mathbb{R}}^h} \eta(f(\omega)) d\xi_\omega(\eta) d\,\mathbf{P}(\omega).$$

Example 3.3 Let A and B be two disjoint measurable subsets of Ω. Suppose that ξ is a random measure on $\overline{\mathbb{R}}^h$ defined by

$$\xi_\omega := \begin{cases} \delta_{\eta_{+\infty}}, & \text{if } \omega \in A, \\ \delta_{\eta_{-\infty}}, & \text{if } \omega \in B, \\ \delta_{\eta_{g(\omega)}}, & \text{otherwise,} \end{cases}$$

where $g : \Omega \to \mathbb{R}$ is a measurable function. Then the metric functional on L_1 given by the formula (3.1) becomes

$$\mathbf{h}(f) = \mathbf{E}\left[-\mathbf{1}_A f\right] + \mathbf{E}\left[\mathbf{1}_B f\right] + \mathbf{E}\left[\mathbf{1}_{\Omega\setminus(A\cup B)}(|f-g| - |g|)\right], \tag{3.2}$$

for all $f \in L_1$. The metric functionals of the form (3.2) can be compared with those describing the metric compactification of the sequence space ℓ_1; see [21]. Furthermore, if A and B are sets of measure zero, and $g \in L_1$ then (3.2) becomes the internal metric functional $\mathbf{h}_g(\cdot) = \|\cdot - g\|_{L_1} - \|g\|_{L_1}$.

Example 3.4 Let Ω be the open interval $[0, 1]$ and let $\mathbf{P}$ be the Lebesgue measure. For each $n \in \mathbb{N}$, let A_n and B_n be the intervals

$$A_n = \left[\frac{1}{2} - \frac{1}{n+1}, \frac{1}{2}\right], \quad B_n = \left]\frac{1}{2}, \frac{1}{2} + \frac{1}{n+1}\right],$$

and define $g_n \in L_1$ by $g_n := -n^2\,\mathbf{1}_{A_n} + n^2\,\mathbf{1}_{B_n}$. It follows that

$$\|g_n\|_{L_1} = 2n^2/(n+1) \to \infty \text{ as } n \to \infty.$$

In particular, the sequence $(g_n)_{n\in\mathbb{N}}$ does not converge to zero in L_1-norm. However, for every $f \in L_1$ we obtain

$$\begin{aligned} \mathbf{h}_{g_n}(f) &= \|f - g_n\|_{L_1} - \|g_n\|_{L_1} \\ &= \mathbf{E}\left[|f - g_n| - |g_n|\right] \\ &= \mathbf{E}\left[\mathbf{1}_{A_n}\left(\left|f + n^2\right| - n^2\right)\right] + \mathbf{E}\left[\mathbf{1}_{B_n}\left(\left|f - n^2\right| - n^2\right)\right] \\ &\quad + \mathbf{E}\left[\mathbf{1}_{\Omega\setminus(A_n\cup B_n)}\,|f|\right] \\ &\xrightarrow[n\to\infty]{} \mathbf{E}\left[|f|\right] = \mathbf{h}_0(f). \end{aligned}$$

This shows that although the sequence $(g_n)_{n\in\mathbb{N}}$ does not converge to zero in L_1-norm, it does converge to zero in the metric compactification $\overline{L_1}^h$. The internal metric functional $\mathbf{h}_0$ has the representation (3.1) with the (constant) random measure $\omega \mapsto \xi_\omega = \delta_{\eta_0}$ with $\eta_0 \in \overline{\mathbb{R}}^h$.

Example 3.5 For each $n \in \mathbb{N}$, let A_n and B_n be the intervals defined in the previous example. Now, we define $g_n \in L_1$ by $g_n := -n\,\mathbf{1}_{A_n} + n\,\mathbf{1}_{B_n}$. Then

$$\|g_n\|_{L_1} = 2n/(n+1) < 2 \text{ for all } n \in \mathbb{N}.$$

The sequence $(g_n)_{n\in\mathbb{N}}$ is bounded in L_1, but it does not converge to zero in L_1-norm. However, as in the previous example, we have $\mathbf{h}_{g_n} \to \mathbf{h}_0$ as $n \to \infty$.

Example 3.6 Let A be a measurable subset of Ω with $\mathbf{P}(A) > 0$. Suppose that $g : \Omega \to \mathbb{R}$ is an element of L_1. For each $n \in \mathbb{N}$, we define $g_n \in L_1$ by $g_n := n\,\mathbf{1}_A + g\,\mathbf{1}_{\Omega\setminus A}$. It follows that

$$\|g_n\|_{L_1} = n\,\mathbf{P}(A) + \mathbf{E}\left[\mathbf{1}_{\Omega\setminus A}\,|g|\right] \to \infty \text{ as } n \to \infty.$$

On the other hand, for every $f \in L_1$ we obtain

$$\begin{aligned}\mathbf{h}_{g_n}(f) &= \mathbf{E}\left[|f - g_n| - |g_n|\right] \\ &= \mathbf{E}\left[\mathbf{1}_A\,(|f-n| - n)\right] + \mathbf{E}\left[\mathbf{1}_{\Omega\setminus A}\,(|f-g| - |g|)\right] \\ &\underset{n\to\infty}{\longrightarrow} \mathbf{E}\left[-\mathbf{1}_A\,f\right] + \mathbf{E}\left[\mathbf{1}_{\Omega\setminus A}\,(|f-g| - |g|)\right] = \mathbf{h}(f).\end{aligned}$$

Hence $\mathbf{h}_{g_n}$ converges to the metric functional $\mathbf{h}$ which has the representation formula (3.1), where ξ is the random measure on $\overline{\mathbb{R}}^h$ given by

$$\xi = \mathbf{1}_A\,\delta_{\eta_{+\infty}} + \mathbf{1}_{\Omega\setminus A}\,\delta_{\eta_g}.$$

Example 3.7 Let Ω be the interval $[0, 1]$ and let $\mathbf{P}$ be the Lebesgue measure. Consider the Rademacher sequence $(g_n)_{n\in\mathbb{N}}$ defined by $g_n(\omega) := \operatorname{sign}(\sin(2^n\pi\omega))$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$. It is not difficult to verify that $\mathbf{h}_{g_n}$ converges to the metric functional $\mathbf{h}$ with the representation formula (3.1), where ξ is the constant random measure $\omega \mapsto \xi_\omega = \frac{1}{2}\delta_{\eta_{-1}} + \frac{1}{2}\delta_{\eta_{+1}}$.

The above examples are merely to demonstrate some of the different cases that can appear when determining the metric functionals on L_1. In Sect. 5 we present some applications of the metric compactification of L_p spaces.

Theorem 3.8 *Let $p \in]1, \infty[$. Every element of $\overline{L_p}^h$ has exactly one of the following forms: either*

$$f \mapsto \mathbf{h}(f) = \left(\mathbf{E}\left[\int_{\mathbb{R}} |f-r|^p\,d\xi(r)\right] - \mathbf{E}\left[\int_{\mathbb{R}} |r|^p\,d\xi(r)\right] + c^p\right)^{1/p} - c, \quad (3.3)$$

where ξ is a random measure on $\mathbb{R}$ and $c^p \geq \mathbf{E}\left[\int_{\mathbb{R}} |r|^p \, d\xi(r)\right]$; or

$$f \mapsto \mathbf{h}(f) = -\mathbf{E}\left[f\zeta\right], \tag{3.4}$$

where ζ is an element of the closed unit ball of $L_{p/(p-1)}$.

Remark 3.9 If the metric functional (3.3) is represented by the random measure ξ on $\mathbb{R}$ given by $\omega \mapsto \xi_\omega = \delta_{g(\omega)}$ with $g \in L_p$, then the representation formula (3.3) becomes

$$\begin{aligned} f \mapsto \mathbf{h}(f) &= \left(\mathbf{E}\left[|f-g|^p\right] - \mathbf{E}\left[|g|^p\right] + c^p\right)^{1/p} - c \\ &= \left(\|f-g\|_{L_p}^p - \|g\|_{L_p}^p + c^p\right)^{1/p} - c \;\text{ with } c^p \geq \|g\|_{L_p}^p . \end{aligned} \tag{3.5}$$

In [21], the author showed that bounded nets in the infinite-dimensional ℓ_p space with $p \in]1, \infty[$ can only produce metric functionals represented by formulas analogous to (3.5). In the present paper, bounded nets in non-atomic L_p spaces with $p \in]1, \infty[$ yield metric functionals represented by the general formulas (3.3).

4 Proofs

Throughout this section we use well-known facts about representations of certain dual spaces. These results can be found in standard functional analysis books such as [13] or [14]. Recall that $\Lambda_\infty(\Omega, \mathcal{M}(\overline{\mathbb{R}}^h))$ is the space of mappings $\omega \mapsto \xi_\omega$ from Ω to $\mathcal{M}(\overline{\mathbb{R}}^h)$ such that for each $\varphi \in C(\overline{\mathbb{R}}^h)$ the real-valued function

$$\omega \mapsto \langle \varphi, \xi_\omega \rangle = \int_{\overline{\mathbb{R}}^h} \varphi(\eta) d\xi_\omega(\eta)$$

is measurable, and the function $\omega \mapsto \|\xi_\omega\|_{\mathcal{M}(\overline{\mathbb{R}}^h)}$ is essentially bounded. The linear space $\Lambda_\infty(\Omega, \mathcal{M}(\overline{\mathbb{R}}^h))$ is a Banach space with respect to the norm

$$\|\xi\|_{\Lambda_\infty} = \operatorname*{ess\,sup}_{\omega \in \Omega} \|\xi_\omega\|_{\mathcal{M}(\overline{\mathbb{R}}^h)} .$$

Let $L_1(\Omega, C(\overline{\mathbb{R}}^h))$ denote the linear space of all mappings $\psi : \Omega \to C(\overline{\mathbb{R}}^h)$ such that the real-valued function $\omega \mapsto \|\psi_\omega\|_{C(\overline{\mathbb{R}}^h)}$ is measurable and $\mathbf{E}\left[\|\psi\|_{C(\overline{\mathbb{R}}^h)}\right]$ is finite. The real-valued function $\psi \mapsto \mathbf{E}\left[\|\psi\|_{C(\overline{\mathbb{R}}^h)}\right]$ defines a norm on $L_1(\Omega, C(\overline{\mathbb{R}}^h))$. The dual space $L_1(\Omega, C(\overline{\mathbb{R}}^h))^*$, equipped with the usual dual norm, has the representation

$$L_1(\Omega, C(\overline{\mathbb{R}}^h))^* \cong \Lambda_\infty(\Omega, \mathcal{M}(\overline{\mathbb{R}}^h)) \tag{4.1}$$

under the isometric isomorphism $\vartheta : \Lambda_\infty(\Omega, \mathcal{M}(\overline{\mathbb{R}}^h)) \to L_1(\Omega, C(\overline{\mathbb{R}}^h))^*$, $\xi \mapsto \vartheta(\xi)$ defined by

$$[\vartheta(\xi)](\psi) := \mathbf{E}\left[\int_{\overline{\mathbb{R}}^h} \psi(\eta) d\xi(\eta)\right] \text{ for all } \psi \in L_1(\Omega, C(\overline{\mathbb{R}}^h)).$$

Lemma 4.1 *If $(g_\alpha)_\alpha$ is a net in L_1, then there exists a random measure ξ on $\overline{\mathbb{R}}^h$ and a subnet $(g_\beta)_\beta$ such that the net of random measures $(\delta_{\eta_{g_\beta}})_\beta$ on $\overline{\mathbb{R}}^h$ converges weakly-star to ξ in $L_1(\Omega, C(\overline{\mathbb{R}}^h))^*$. That is,*

$$\mathbf{E}\left[\psi(\eta_{g_\beta})\right] \underset{\beta}{\longrightarrow} \mathbf{E}\left[\int_{\overline{\mathbb{R}}^h} \psi(\eta) d\xi(\eta)\right] \textit{ for all } \psi \in L_1(\Omega, C(\overline{\mathbb{R}}^h)). \tag{4.2}$$

Moreover, if the net $(g_\beta)_\beta$ is bounded in L_1, i.e., $\sup_\beta \|g_\beta\|_{L_1} < \infty$, then ξ in (4.2) is a random measure on $\mathbb{R}$.

Proof For each α, the mapping $\omega \mapsto \delta_{\eta_{g_\alpha(\omega)}}$ defines a random measure on $\overline{\mathbb{R}}^h$, i.e., $\delta_{\eta_{g_\alpha}} \in \Lambda_\infty(\Omega, \mathcal{M}(\overline{\mathbb{R}}^h))$ and $\delta_{\eta_{g_\alpha(\omega)}} \in \mathcal{P}(\overline{\mathbb{R}}^h)$ for $\mathbf{P}$-almost every $\omega \in \Omega$. By the representation (4.1), the net $(\delta_{\eta_{g_\alpha}})_\alpha$ lies in the closed unit ball of the dual space $L_1(\Omega, C(\overline{\mathbb{R}}^h))^*$. By the Banach-Alaouglu theorem, there exists a subnet $(\delta_{\eta_{g_\beta}})_\beta$ and an element $\xi \in \Lambda_\infty(\Omega, \mathcal{M}(\overline{\mathbb{R}}^h))$ with $\|\xi\|_{\Lambda_\infty} \le 1$ such that $\delta_{\eta_{g_\beta}}$ converges weakly-star to ξ in $L_1(\Omega, C(\overline{\mathbb{R}}^h))^*$. Hence (4.2) holds.

Next, we will show that ξ is a random measure on $\overline{\mathbb{R}}^h$, i.e., $\xi_\omega \in \mathcal{P}(\overline{\mathbb{R}}^h)$ for $\mathbf{P}$-almost every $\omega \in \Omega$. Since $C(\overline{\mathbb{R}}^h)$ is separable, the space $C(\overline{\mathbb{R}}^h)_+$ of non-negative continuous functions on $\overline{\mathbb{R}}^h$ contains a dense countable subset, say $(\varphi_k)_{k\in\mathbb{N}}$. Fix $k \in \mathbb{N}$. For every non-negative integrable function $u : \Omega \to \mathbb{R}$, we apply (4.2) with the mapping $\psi \in L_1(\Omega, C(\overline{\mathbb{R}}^h))$, $\omega \mapsto \psi_\omega = u(\omega)\varphi_k$ to obtain

$$\mathbf{E}\left[u \int_{\overline{\mathbb{R}}^h} \varphi_k(\eta) d\xi(\eta)\right] = \lim_\beta \mathbf{E}\left[u\varphi_k(\eta_{g_\beta})\right] \ge 0.$$

Hence there is $N_k \in \Sigma$ with $\mathbf{P}(N_k) = 0$ such that $\int_{\overline{\mathbb{R}}^h} \varphi_k(\eta) d\xi_\omega(\eta) \ge 0$ for all $\omega \in \Omega \setminus N_k$. By letting $N = \cup_{k\in\mathbb{N}} N_k$, it follows that $\mathbf{P}(N) = 0$ and

$$\int_{\overline{\mathbb{R}}^h} \varphi_k(\eta) d\xi_\omega(\eta) \ge 0 \text{ for all } \omega \in \Omega \setminus N \text{ and all } k \in \mathbb{N}. \tag{4.3}$$

Let $\varphi : \overline{\mathbb{R}}^h \to \mathbb{R}$ be a non-negative continuous function. By density, there exists a sequence $(\varphi_{k_i})_{i\in\mathbb{N}}$ satisfying (4.3) and such that $\|\varphi_{k_i} - \varphi\|_{C(\overline{\mathbb{R}}^h)} \to 0$ as $i \to \infty$. Therefore,

$$\int_{\overline{\mathbb{R}}^h} \varphi(\eta) d\xi_\omega(\eta) \ge 0 \text{ for all } \omega \in \Omega \setminus N.$$

This shows that ξ_ω is a positive measure for all $\omega \in \Omega \setminus N$ with $\mathbf{P}(N) = 0$. Furthermore, if we apply (4.2) with the mapping $\omega \mapsto \psi_\omega = \mathbf{1}_{\overline{\mathbb{R}}^h}$, we obtain

$$\|\xi\|_{\Lambda_\infty} \geq \mathbf{E}\left[\|\xi\|_{\mathcal{M}(\overline{\mathbb{R}}^h)}\right] = \mathbf{E}\left[\int_{\overline{\mathbb{R}}^h} \psi(\eta) d\xi(\eta)\right] = \lim_\beta \mathbf{E}\left[\psi(\eta_{g_\beta})\right] = \mathbf{P}(\Omega) = 1.$$

Hence $\|\xi\|_{\Lambda_\infty} = 1$, and therefore ξ is a random measure on $\overline{\mathbb{R}}^h$.

For the last part of the lemma we need to prove that $\xi_\omega(\{\eta_{+\infty}, \eta_{-\infty}\}) = 0$ for $\mathbf{P}$-almost every $\omega \in \Omega$. Let $C = \sup_\beta \left\|g_\beta\right\|_{L_1}$ and let ϵ be a small positive real number. Define the compact interval

$$K_\epsilon := \left[-(C+1)\epsilon^{-1}, \ (C+1)\epsilon^{-1}\right].$$

Let φ^ϵ denote a continuous function on $\mathbb{R}$ with compact support such that $\varphi^\epsilon = 1$ on K_ϵ and $0 \leq \varphi^\epsilon \leq 1$ on $\mathbb{R}$. For each $\omega \in \Omega$ define the mapping $\psi^\epsilon_\omega \in C(\overline{\mathbb{R}}^h)$ by

$$\psi^\epsilon_\omega(\eta) := \begin{cases} \varphi^\epsilon(r), & \text{if } \eta = \eta_r, \\ 0, & \text{if } \eta = \eta_{\pm\infty}. \end{cases}$$

Then, for every β we have

$$\begin{aligned} 0 \leq \mathbf{E}\left[1 - \psi^\epsilon(\eta_{g_\beta})\right] &= \mathbf{E}\left[\mathbf{1}_{\Omega \setminus g_\beta^{-1}(K_\epsilon)}\left(1 - \psi^\epsilon(\eta_{g_\beta})\right)\right] \\ &\leq \mathbf{P}(\Omega \setminus g_\beta^{-1}(K_\epsilon)) \\ &= \mathbf{P}\left(\{\omega \in \Omega \mid \left|g_\beta(\omega)\right| > (C+1)\epsilon^{-1}\}\right) \\ &\leq (C+1)^{-1}\epsilon \, \mathbf{E}\left[\left|g_\beta\right|\right] \\ &\leq (C+1)^{-1}\epsilon C < \epsilon. \end{aligned}$$

Hence, by applying (4.2) with $\psi = \psi^\epsilon$, we obtain

$$0 \leq \mathbf{E}\left[1 - \int_{\overline{\mathbb{R}}^h} \psi^\epsilon(\eta) d\xi(\eta)\right] < \epsilon.$$

However, since $\mathbb{R}$ is identified in (2.5) with the set $\{\eta_r \mid r \in \mathbb{R}\}$, it follows that

$$\mathbf{E}\left[1 - \int_{\overline{\mathbb{R}}^h} \psi^\epsilon(\eta) d\xi(\eta)\right] = \mathbf{E}\left[1 - \int_{\mathbb{R}} \varphi^\epsilon(r) d\xi(r)\right] \underset{\epsilon \to 0}{\longrightarrow} \mathbf{E}\left[1 - \xi(\mathbb{R})\right].$$

Therefore $\xi_\omega(\mathbb{R}) = 1$ for $\mathbf{P}$-almost every $\omega \in \Omega$. This completes the proof of the lemma. □

Proof of Theorem 3.1 If $\mathbf{h}$ is an element of $\overline{L_1}^h$, then there exists a net $(g_\alpha)_\alpha$ in L_1 such that $\mathbf{h}_{g_\alpha}$ converges pointwise on L_1 to $\mathbf{h}$. By Lemma 4.1, there exists a subnet $(g_\beta)_\beta$

and a random measure ξ on $\overline{\mathbb{R}}^h$ such that the net of random measures $(\delta_{\eta_{g_\beta}})_\beta$ on $\overline{\mathbb{R}}^h$ converges weakly-star to ξ in $L_1(\Omega, C(\overline{\mathbb{R}}^h))^*$.

Let f be an element of L_1. For each β, the internal metric functional $\mathbf{h}_{g_\beta}$ on L_1 can be written as

$$\mathbf{h}_{g_\beta}(f) = \mathbf{E}\left[\left|f - g_\beta\right| - \left|g_\beta\right|\right] = \mathbf{E}\left[\int_{\overline{\mathbb{R}}^h} \eta(f) d\delta_{\eta_{g_\beta}}(\eta)\right] = \mathbf{E}\left[\eta_{g_\beta}(f)\right]. \quad (4.4)$$

Consider the mapping $\omega \mapsto \psi_\omega^f$ from Ω to $C(\overline{\mathbb{R}}^h)$ defined by $\psi_\omega^f(\eta) = \eta(f(\omega))$ for all $\eta \in \overline{\mathbb{R}}^h$. We claim that $\psi^f \in L_1(\Omega, C(\overline{\mathbb{R}}^h))$. Indeed, if $(\eta^{(k)})_{k\in\mathbb{N}}$ is a sequence in $\overline{\mathbb{R}}^h$ converging to some $\eta \in \overline{\mathbb{R}}^h$, then for $\mathbf{P}$-almost every $\omega \in \Omega$ we have $\eta^{(k)}(f(\omega)) \to \eta(f(\omega))$ as $k \to \infty$. Moreover, the function $\omega \mapsto \left\|\psi_\omega^f\right\|_{C(\overline{\mathbb{R}}^h)}$ is measurable and

$$\mathbf{E}\left[\left\|\psi^f\right\|_{C(\overline{\mathbb{R}}^h)}\right] = \mathbf{E}\left[\sup_{\eta\in\overline{\mathbb{R}}^h} |\eta(f)|\right] = \int_\Omega \sup_{\eta\in\overline{\mathbb{R}}^h} |\eta(f(\omega))| \, d\,\mathbf{P}(\omega) \leq [|f|].$$

Note now that the formula (4.4) becomes $\mathbf{h}_{g_\beta}(f) = \mathbf{E}\left[\psi^f(\eta_{g_\beta})\right]$. By applying the limit (4.2), we obtain

$$\mathbf{h}(f) = \lim_\beta \mathbf{h}_{g_\beta}(f) = \mathbf{E}\left[\int_{\overline{\mathbb{R}}^h} \eta(f) d\xi(\eta)\right].$$

Conversely, assume that ξ is a random measure on $\overline{\mathbb{R}}^h$. We need to find a net $(g_\gamma)_\gamma$ in L_1 such that $\mathbf{h}_{g_\gamma}$ converges pointwise on L_1 to the functional $\mathbf{h}$ given by the formula (3.1). For this purpose, we introduce first some notations: let γ denote a finite measurable partition $\left[S_\gamma^1, \ldots, S_\gamma^{|\gamma|}\right]$ of Ω, i.e., $\Omega = \cup_{j=1}^{|\gamma|} S_\gamma^j$ with $S_\gamma^j \in \Sigma$ and $\mathbf{P}(S_\gamma^j) > 0$ for all $j = 1, \ldots, |\gamma|$, and also $S_\gamma^j \cap S_\gamma^k = \emptyset$ for $j \neq k$. Here $|\gamma|$ denotes the number of elements of γ. Denote by $\mathbb{T}$ the collection of all finite measurable partitions of Ω. The set $\mathbb{T}$ becomes a directed set with the partial order $\succcurlyeq$ defined by $\gamma \succcurlyeq \tilde{\gamma}$ if and only if γ is a refinement of $\tilde{\gamma}$, i.e., for each $S_\gamma^j \in \gamma$ there exists $S_{\tilde{\gamma}}^k \in \tilde{\gamma}$ such that $\mathbf{P}(S_\gamma^j \setminus S_{\tilde{\gamma}}^k) = 0$.

Let us first suppose that the random measure ξ is constant of the form

$$\xi_\omega = \sum_{k=1}^N \theta_k \delta_{\eta^{(k)}}, \quad (4.5)$$

where $N \in \mathbb{N}$, $\sum_{k=1}^N \theta_k = 1$, $\theta_k \in [0, 1] \cap \mathbb{Q}$ and $\eta^{(k)} \in \overline{\mathbb{R}}^h$ for all $k = 1, \ldots, N$. Since the probability measure $\mathbf{P}$ is non-atomic, for each finite measurable partition $\gamma = \left[S_\gamma^1, \ldots, S_\gamma^{|\gamma|}\right]$ we can divide each S_γ^j into further N pairwise disjoint subsets

$\{S_\gamma^{j,1}, \ldots, S_\gamma^{j,N}\}$ such that $\mathbf{P}(S_\gamma^{j,k}) = \theta_k \, \mathbf{P}(S_\gamma^j)$ for all $k = 1, \ldots, N$. Now, we can define the net $(g_\gamma)_{\gamma \in \mathbb{T}}$ in L_1 by

$$g_\gamma(\omega) := \begin{cases} -|\gamma|, & \text{if } \eta^{(k)} = \eta_{-\infty}, \\ +|\gamma|, & \text{if } \eta^{(k)} = \eta_{+\infty}, \\ r_k, & \text{if } \eta^{(k)} = \eta_{r_k}, \end{cases} \tag{4.6}$$

for all $\omega \in S_\gamma^{1,k} \cup \cdots \cup S_\gamma^{|\gamma|,k}$ with $k = 1, \ldots, N$. Then the net of internal metric functionals $(\mathbf{h}_{g_\gamma})_{\gamma \in \mathbb{T}}$ converges pointwise on L_1 to the functional $\mathbf{h}$ given by the formula (3.1) and represented by the random measure (4.5).

Next, suppose that ξ is a general random measure on $\overline{\mathbb{R}}^h$. For each finite measurable partition $\gamma = \left[S_\gamma^1, \ldots, S_\gamma^{|\gamma|}\right]$ of Ω define $\xi^{(\gamma)} \in \Lambda_\infty(\Omega, \mathcal{M}(\overline{\mathbb{R}}^h))$ by

$$\left\langle \varphi, \xi_\omega^{(\gamma)} \right\rangle := \sum_{j=1}^{|\gamma|} \frac{1}{\mathbf{P}(S_\gamma^j)} \mathbf{E}\left[\mathbf{1}_{S_\gamma^j} \int_{\overline{\mathbb{R}}^h} \varphi(\eta) d\xi(\eta)\right] \mathbf{1}_{S_\gamma^j}(\omega), \tag{4.7}$$

for all $\varphi \in C(\overline{\mathbb{R}}^h)$. It follows that $\xi_\omega^{(\gamma)} \in \mathcal{P}(\overline{\mathbb{R}}^h)$ for $\mathbf{P}$-almost every $\omega \in \Omega$, and hence $\xi^{(\gamma)}$ is a random measure on $\overline{\mathbb{R}}^h$. We proceed to prove that

$$\xi^{(\gamma)} \xrightarrow[\gamma]{w*} \xi \text{ in } L_1(\Omega, C(\overline{\mathbb{R}}^h))^*. \tag{4.8}$$

Let ψ be an element of $L_1(\Omega, C(\overline{\mathbb{R}}^h))$ and let ϵ be a small positive real number. Then there exists a measurable finite partition $\gamma_\epsilon = \left[S_{\gamma_\epsilon}^1, \ldots, S_{\gamma_\epsilon}^{|\gamma_\epsilon|}\right]$ of Ω and a $C(\overline{\mathbb{R}}^h)$-valued simple function

$$\omega \mapsto \psi_\omega^{(\gamma_\epsilon)} = \sum_{j=1}^{|\gamma_\epsilon|} \varphi_j \, \mathbf{1}_{S_{\gamma_\epsilon}^j}(\omega) \text{ with } \varphi_j \in C(\overline{\mathbb{R}}^h) \text{ for } j = 1, \ldots, |\gamma_\epsilon|$$

such that $\mathbf{E}\left[\left\|\psi^{(\gamma_\epsilon)} - \psi\right\|_{C(\overline{\mathbb{R}}^h)}\right] < \epsilon/2$. Therefore, for every $\gamma \succcurlyeq \gamma_\epsilon$ we have

$$\begin{aligned} &\left| \mathbf{E}\left[\int_{\overline{\mathbb{R}}^h} \psi(\eta) d\xi^{(\gamma)}(\eta)\right] - \mathbf{E}\left[\int_{\overline{\mathbb{R}}^h} \psi(\eta) d\xi(\eta)\right] \right| \\ &\leq \mathbf{E}\left[\int_{\overline{\mathbb{R}}^h} \left|\psi(\eta) - \psi^{(\gamma_\epsilon)}(\eta)\right| d\xi^{(\gamma)}(\eta)\right] \\ &\quad + \left| \mathbf{E}\left[\int_{\overline{\mathbb{R}}^h} \psi^{(\gamma_\epsilon)}(\eta) d\xi^{(\gamma)}(\eta) - \int_{\overline{\mathbb{R}}^h} \psi^{(\gamma_\epsilon)}(\eta) d\xi(\eta)\right] \right| \\ &\quad + \mathbf{E}\left[\int_{\overline{\mathbb{R}}^h} \left|\psi^{(\gamma_\epsilon)}(\eta) - \psi(\eta)\right| d\xi(\eta)\right] \\ &< \epsilon/2 + 0 + \epsilon/2 = \epsilon. \end{aligned}$$

Hence (4.8) holds. Finally, we observe that the random measure $\xi^{(\gamma)}$ given by (4.7) is constant on each S_γ^j of the finite partition $\gamma = \left[S_\gamma^1, \ldots, S_\gamma^{|\gamma|}\right]$ of Ω. More precisely,

$$\xi^{(\gamma)}_{|S_\gamma^j} \in \mathcal{P}(\overline{\mathbb{R}}^h) \text{ for each } j = 1, \ldots, |\gamma|. \tag{4.9}$$

Furthermore, each probability measure (4.9) can be approximated by measures of the form (4.5) with respect to the weak-star topology $\sigma(\mathcal{M}(\overline{\mathbb{R}}^h), C(\overline{\mathbb{R}}^h))$. This permits us to construct a net of the form (4.6) on each S_γ^j. By a simple diagonal argument with respect to the directed set $\mathbb{T}$ of measurable finite partitions, we can construct a net $(g_\gamma)_{\gamma\in\mathbb{T}}$ in L_1 such that $\mathbf{h}_{g_\gamma}$ converges pointwise on L_1, as the partition γ gets finer and finer, to the functional $\mathbf{h}$ given by the formula (3.1). $\square$

Proof of Theorem 3.8 If $\mathbf{h} \in \overline{L_p}^h$ then there exists a net $(g_\alpha)_\alpha$ in L_p such that $\mathbf{h}_{g_\alpha}$ converges pointwise on L_p to $\mathbf{h}$. The net $(g_\alpha)_\alpha$ is either bounded or unbounded with respect to the L_p-norm.

Let us first suppose that the net is bounded in L_p, i.e., $\sup_\alpha \|g_\alpha\|_{L_p} < \infty$. By taking a subnet if necessary, we may assume that

$$\|g_\alpha\|_{L_p} \underset{\alpha}{\longrightarrow} c.$$

Due to the inclusion $L_p \subset L_1$, the net $(g_\alpha)_\alpha$ is bounded with respect to the L_1-norm. By Lemma 4.1, there exists a subnet $(g_\beta)_\beta$ and a random measure ξ on $\mathbb{R}$ for which the limit (4.2) holds. Now, let f be an element of L_p. The internal metric functional $\mathbf{h}_{g_\beta}$ on L_p given by (2.6) becomes

$$\mathbf{h}_{g_\beta}(f) = \left(\mathbf{E}\left[\int_{\mathbb{R}} (|f-r|^p - |r|^p)\, d\delta_{g_\beta}(r)\right] + \|g_\beta\|_{L_p}^p\right)^{1/p} - \|g_\beta\|_{L_p}. \tag{4.10}$$

Let $\psi^f : \Omega \to C(\overline{\mathbb{R}}^h)$ be the mapping defined by

$$\psi^f_\omega(\eta) := \begin{cases} |f(\omega) - r|^p - |r|^p & \text{if } \eta = \eta_r, \\ -\operatorname{sign}(f(\omega))\infty & \text{if } \eta = \eta_{+\infty}, \\ \operatorname{sign}(f(\omega))\infty & \text{if } \eta = \eta_{-\infty}. \end{cases}$$

Then (4.10) becomes $\mathbf{h}_{g_\beta}(f) = \left(\mathbf{E}\left[\psi^f(\eta_{g_\beta})\right] + \|g_\beta\|_{L_p}^p\right)^{1/p} - \|g_\beta\|_{L_p}$. Finally, due to the limit (4.2) we obtain

$$\mathbf{h}_{g_\beta}(f) \underset{\beta}{\longrightarrow} \left(\mathbf{E}\left[\int_{\mathbb{R}} \left(|f - r|^p - |r|^p \right) d\xi(r) \right] + c^p \right)^{1/p} - c$$
$$= \left(\mathbf{E}\left[\int_{\mathbb{R}} |f - r|^p \, d\xi(r) \right] - \mathbf{E}\left[\int_{\mathbb{R}} |r|^p \, d\xi(r) \right] + c^p \right)^{1/p} - c,$$

with $c^p - \mathbf{E}\left[\int_{\mathbb{R}} |r|^p \, d\xi(r)\right] \geq 0$.

On the other hand, if the net $(g_\alpha)_\alpha$ is unbounded in L_p, by taking a subnet, we may assume that

$$\|g_\alpha\|_{L_p} \underset{\alpha}{\longrightarrow} \infty.$$

By uniform convexity of the dual space $L_p^* \cong L_{p/(p-1)}$ and [21, Lemma 5.3], there exists a subnet $(g_\beta)_\beta$ and an element ζ of the closed unit ball of $L_{p/(p-1)}$ such that

$$\mathbf{h}_{g_\beta}(f) \underset{\beta}{\longrightarrow} -\mathbf{E}[f\zeta] \text{ for all } f \in L_p.$$

Conversely, assume that ζ is an arbitrary element of the closed unit ball of $L_{p/(p-1)}$. Pick an increasing sequence $\{A_n\}_{n\in\mathbb{N}}$ of measurable subsets of Ω with $A_1 \neq \emptyset$ and $\cup_{n\in\mathbb{N}} A_n = \Omega$. Define now the sequence $(\zeta_n)_{n\in\mathbb{N}}$ in $L_{p/(p-1)}$ by

$$\zeta_n := \mathbf{1}_{A_n}\zeta + \mathbf{1}_{\Omega\setminus A_n}\left(\frac{1 - \|\zeta\|_{L_{p/(p-1)}}^{p/(p-1)}}{\mathbf{P}(\Omega \setminus A_n)} + |\zeta|^{p/(p-1)} \right)^{(p-1)/p}. \tag{4.11}$$

Hence $\|\zeta_n\|_{L_{p/(p-1)}} = 1$ for all $n \in \mathbb{N}$. Furthermore, we observe that ζ_n converges weakly to ζ in $L_{p/(p-1)}$. Due to the $L_{p/(p-1)}/L_p$–duality, for each $n \in \mathbb{N}$ there exists $\tilde{g}_n \in L_p$ with $\|\tilde{g}_n\|_{L_p} = 1$ such that $\mathbf{E}\left[\tilde{g}_n\zeta_n\right] = 1$. By letting $g_n = n\tilde{g}_n$ for each $n \in \mathbb{N}$ and proceeding as in the proof of [21, Lemma 5.3], we can show that $\lim_{n\to\infty} \mathbf{h}_{g_n}(f) = -\mathbf{E}[f\zeta]$ for all $f \in L_p$. □

5 Applications

5.1 The L_p-mean ergodic theorem

Let $p \in]1, \infty[$. Assume that $(\Omega, \mathbf{P})$ is a standard probability space. Let T be a linear operator on $L_p = L_p(\Omega, \mathbf{P})$ such that $\|Tf\|_{L_p} \leq \|f\|_{L_p}$ for all $f \in L_p$. For an arbitrary element $g \in L_p$ define the mapping $F_g : L_p \to L_p$ by $F_g(f) := Tf + g$ for all $f \in L_p$. Hence F_g defines a 1-Lipschitz self-mapping of L_p. We observe that

$$F_g^n(0) = \sum_{k=0}^{n-1} T^k g \quad \text{for all } n \geq 1.$$

Karlsson's *metric spectral principle* [25,27] asserts that there exists a metric functional $\mathbf{h} \in \overline{L_p}^h$ such that

$$\lim_{n\to\infty} -\frac{1}{n}\mathbf{h}(F_g^n(0)) = \tau, \tag{5.1}$$

where the *escape rate* $\tau := \lim_{n\to\infty} n^{-1} \left\| F_g^n(0) \right\|_{L_p}$ is well-defined due to subadditivity of the sequence $\left(\left(\left\| F_g^n(0) \right\|_{L_p}\right)\right)_{n\geq 1}$.

If $\tau = 0$ then we obtain the trivial strong limit

$$\lim_{n\to\infty} \frac{1}{n} F_g^n(0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k g = 0.$$

Suppose now that $\tau > 0$. Then the metric functional $\mathbf{h}$ in (5.1) must be unbounded from below. Hence it is neither of the form (3.3) nor the zero functional in (3.4). Therefore, there exists $\zeta \in L_{p/(p-1)}$ with $0 < \|\zeta\|_{L_{p/(p-1)}} \leq 1$ such that $\mathbf{h}(f) = -\mathbf{E}[f\zeta]$ for all $f \in L_p$. We now proceed to show that ζ must be an element of the unit sphere of $L_{p/(p-1)}$. Indeed, for every $n \geq 1$ we have

$$-\frac{1}{n}\mathbf{h}(F_g^n(0)) = \frac{1}{n}\mathbf{E}\left[F_g^n(0)\zeta\right] \leq \frac{1}{n}\left\| F_g^n(0) \right\|_{L_p} \|\zeta\|_{L_{p/(p-1)}}.$$

By (5.1), it follows that $\tau \leq \tau \|\zeta\|_{L_{p/(p-1)}}$. Hence $\|\zeta\|_{L_{p/(p-1)}} = 1$. On the other hand, by $L_p/L_{p/(p-1)}$–duality, there exists $g^* \in L_p$ with $\|g^*\|_{L_p} = 1$ such that $\mathbf{E}\left[g^*\zeta\right] = 1$. Next, we claim that

$$\lim_{n\to\infty} \frac{1}{n} F_g^n(0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k g = \tau g^*. \tag{5.2}$$

Indeed, for every $n \geq 1$ we have

$$\begin{aligned} \frac{1}{n}\left\| F_g^n(0) \right\|_{L_p} + \|\tau g^*\|_{L_p} \geq \left\| \frac{1}{n} F_g^n(0) + \tau g^* \right\|_{L_p} &\geq \mathbf{E}\left[\frac{1}{n}F_g^n(0)\zeta\right] + \mathbf{E}\left[\tau g^* \zeta\right] \\ &= -\frac{1}{n}\mathbf{h}(F_g^n(0)) + \tau. \end{aligned}$$

Due to (5.1) we obtain $\left\| \frac{1}{n} F_g^n(0) + \tau g^* \right\|_{L_p} \to 2\tau$ as $n \to \infty$. Since L_p is uniformly convex, it follows that $\left\| \frac{1}{n} F_g^n(0) - \tau g^* \right\|_{L_p} \to 0$ as $n \to \infty$. Hence (5.2) holds.

5.2 Alspach's fixed-point free isometry

Alspach [5] presented an example of an isometry on a weakly compact convex subset of L_1 with no fixed points. More precisely, let Ω be the interval [0, 1] and let $\mathbf{P}$ be the Lebesgue measure. Consider the subset K of L_1 defined by

$$K = \{f \in L_1 \mid 0 \leq f \leq 2 \text{ a.e. and } \mathbf{E}[f] = 1\}.$$

The set K is a weakly compact convex subset of L_1. The mapping $F : K \to K$ defined by

$$F(f)(\omega) := \begin{cases} \min\{2\,,\ 2f(2\omega)\}, & \text{if } 0 \leq \omega \leq 1/2, \\ \max\{0\,,\ 2f(2\omega - 1) - 2\}, & \text{if } 1/2 < \omega \leq 1, \end{cases}$$

is an isometry, i.e., $\|F(f) - F(g)\|_{L_1} = \|f - g\|_{L_1}$ for all $f, g \in K$. Alspach proved that F has no fixed points in K. We verify this fact by using metric functionals on L_1.

First, we denote $g_0 = \mathbf{1}_\Omega$ and $g_n = F(g_{n-1})$ for all $n \in \mathbb{N}$. More precisely, for every $n \in \mathbb{N}$ we have

$$g_n(\omega) = \begin{cases} 2 & \text{if } \omega \in \bigcup_{j=0}^{2^{n-1}-1} \left[2j2^{-n}, (2j+1)2^{-n}\right], \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently, we can write $g_n = \mathbf{1}_\Omega + r_n$, where $r_n(\omega) = \text{sign}(\sin(2^n \pi \omega))$ is the n-th Rademacher function. Therefore, for every $f \in L_1$ we obtain

$$\lim_{n\to\infty} \mathbf{h}_{g_n}(f) = \mathbf{h}(f) = \mathbf{E}\left[\int_{\overline{\mathbb{R}}^h} \eta(f) d\xi(\eta)\right],$$

where ξ is the constant random measure on $\overline{\mathbb{R}}^h$ given by $\xi_\omega = \frac{1}{2}\delta_{\eta_0} + \frac{1}{2}\delta_{\eta_2}$ with $\eta_0, \eta_2 \in \overline{\mathbb{R}}^h$. That is,

$$\mathbf{h}(f) = \mathbf{E}\left[\frac{1}{2}|f| + \frac{1}{2}(|f - 2| - 2)\right] \text{ for all } f \in L_1. \tag{5.3}$$

In particular, we observe that the metric functional (5.3) vanishes on K, i.e., $\mathbf{h}(f) = 0$ for all $f \in K$.

Now, suppose that F has a fixed point g^* in K. Hence

$$\begin{aligned} 0 = \mathbf{h}(g^*) &= \lim_{n\to\infty} \mathbf{h}_{g_n}(g^*) \\ &= \lim_{n\to\infty} \left(\left\|F^n(g^*) - F^n(\mathbf{1}_\Omega)\right\|_{L_1} - \left\|F^n(\mathbf{1}_\Omega)\right\|_{L_1}\right) \\ &= \left\|g^* - \mathbf{1}_\Omega\right\|_{L_1} - 1. \end{aligned}$$

The only solutions on K to $\|g^* - \mathbf{1}_\Omega\|_{L_1} = 1$ are of the form $g^* = 2\,\mathbf{1}_A$, where A is a set of measure $1/2$. However, this is not possible because we would have

$$\mathbf{h} = \lim_{n\to\infty} \mathbf{h}_{F^n(g^*)} = \lim_{n\to\infty} \mathbf{h}_{g^*} = \mathbf{h}_{g^*},$$

which is a contradiction. Therefore F has no fixed points in K.

Acknowledgements Open access funding provided by Aalto University. The author is very grateful to Prof. Anders Karlsson, Prof. Kalle Kytölä, and Prof. Olavi Nevanlinna for many valuable discussions and suggestions. The author is also thankful to the anonymous referees for valuable suggestions that improved the presentation of this paper. This work was supported by the Academy of Finland, Grant No. 288318.

References

1. Abate, M., Raissy, J.: Wolff-Denjoy theorems in nonsmooth convex domains. Ann. Mat. Pura Appl. **193**(5), 1503–1518 (2014). (4)
2. Albiac, F., Kalton, N.J.: Topics in Banach Space Theory 2nd Ed., volume 233 of Graduate Texts in Mathematics. Springer, Cham (2016)
3. Aldous, D.J.: Subspaces of L^1, via random measures. Trans. Am. Math. Soc. **267**(2), 445–463 (1981)
4. Alessandrini, D., Liu, L., Papadopoulos, A., Su, W.: The horofunction compactification of Teichmüller spaces of surfaces with boundary. Topol. Appl. **208**, 160–191 (2016)
5. Alspach, D.E.: A fixed point free nonexpansive map. Proc. Am. Math. Soc. **82**(3), 423–424 (1981)
6. Auffinger, A., Damron, M., Hanson, J.: Limiting geodesics for first-passage percolation on subsets of $\mathbb{Z}^2$. Ann. Appl. Probab. **25**(1), 373–405 (2015)
7. Ballmann, W., Gromov, M., Schroeder, V.: Manifolds of Nonpositive Curvature, Volume 61 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston (1985)
8. Björklund, M.: Central limit theorems for Gromov hyperbolic groups. J. Theor. Probab. **23**(3), 871–887 (2010)
9. Budzyńska, M., Kuczumow, T., Reich, S.: Theorems of Denjoy-Wolff type. Ann. Math. Pura Appl. **192**(4), 621–648 (2013). (4)
10. Crauel, H.: Random Probability Measures on Polish spaces, Volume 11 of Stochastics Monographs. Taylor & Francis, London (2002)
11. Curien, N., Ménard, L.: The skeleton of the UIPT, seen from infinity. arXiv:1803.05249 (2018)
12. D'Angeli, D., Donno, A.: Metric compactification of infinite Sierpiński carpet graphs. Discrete Math. **339**(11), 2693–2705 (2016)
13. Dunford, N., Schwartz, J.T.: Linear operators. I. General theory. In: Bade, W.G., Bartle, R.G. (eds.) Pure and Applied Mathematics, vol. 7. Interscience Publishers, New York (1958). (Interscience Publishers, Ltd., London)
14. Edwards, R.E.: Functional analysis: theory and applications. Corrected reprint of the 1965 original. Dover Publications, Inc., New York (1995)
15. Garling, D.J.H.: Stable Banach spaces, random measures and Orlicz function spaces. In: Heyer H. (ed) Probability Measures on Groups. Lecture Notes in Mathematics, pp. 121–175. Springer, Berlin, Heidelberg (1982)
16. Gaubert, S., Vigeral, G.: A maximin characterisation of the escape rate of non-expansive mappings in metrically convex spaces. Math. Proc. Camb. Philos. Soc. **152**(2), 341–363 (2012)
17. Gouëzel, S.: Subadditive cocycles and horofunctions. In: Proceedings of the International Congress of Mathematicians (ICM 2018)
18. Gouëzel, S.: Analyticity of the entropy and the escape rate of random walks in hyperbolic groups. Discrete Anal., pages Paper No. 7, 37, (2017)

19. Gouëzel, S., Karlsson, A.: Subadditive and multiplicative ergodic theorems. arXiv:1509.07733 (2015)
20. Gromov, M.: Hyperbolic manifolds, groups and actions. In: Kra, I., Maskit, B. (eds.) Riemann Surfaces and Related Topics, vol. 97, pp. 183–213. Ann. of Math. Stud. Princeton Univ. Press, Princeton (1981)
21. Gutiérrez, A.W.: On the metric compactification of infinite–dimensional ℓ_p spaces. Canad. Math. Bull. 62(3), 491–507 (2019)
22. Gutiérrez, A.W.: The horofunction boundary of finite-dimensional ℓ_p spaces. Colloq. Math. **155**(1), 51–65 (2019)
23. Kallenberg, O.: Random Measures, Theory and Applications, volume 77 of Probability Theory and Stochastic Modelling. Springer, Cham (2017)
24. Karlsson, A., Metz, V., Noskov, G.A.: Horoballs in simplices and Minkowski spaces. Int. J. Math. Math. Sci. **20**, 23656 (2006)
25. Karlsson, A.: Non-expanding maps and Busemann functions. Ergodic Theory Dyn. Syst. **21**(5), 1447–1457 (2001)
26. Karlsson, A.: Dynamics of Hilbert nonexpansive maps. In: Papadopoulos, A., Troyanov, M. (eds.) Handbook of Hilbert geometry, pp. 263–273. Eur. Math. Soc., Zürich (2014)
27. Karlsson, A.: Elements of a metric spectral theory. arXiv:1904.01398, (2018)
28. Karlsson, A., Ledrappier, F.: Noncommutative ergodic theorems. In: Farb, B., Fisher, D. (eds.) Geometry, Rigidity, and Group Actions, Chicago Lectures in Math., pp. 396–418. Univ. Chicago Press, Chicago (2011)
29. Karlsson, A., Margulis, G.A.: A multiplicative ergodic theorem and nonpositively curved spaces. Commun. Math. Phys. **208**(1), 107–123 (1999)
30. Klein, T., Nicas, A.: The horofunction boundary of the Heisenberg group. Pac. J. Math. **242**(2), 299–310 (2009)
31. Lemmens, B., Lins, B., Nussbaum, R., Wortel, M.: Denjoy-Wolff theorems for Hilbert's and Thompson's metric spaces. J. Anal. Math. **134**(2), 671–718 (2018)
32. Lemmens, B., Walsh, C.: Isometries of polyhedral Hilbert geometries. J. Topol. Anal. **3**(2), 213–241 (2011)
33. Ji, L., Schilling, A.: Toric varieties vs. horofunction compactifications of polyhedral norms. Enseign. Math. **63**(3–4), 375–401 (2017)
34. Maher, J., Tiozzo, G.: Random walks on weakly hyperbolic groups. Journal für die reine und angewandte Mathematik (Crelles Journal) **2018**(742), 187–239 (2016)
35. Rieffel, M.A.: Group C^*-algebras as compact quantum metric spaces. Doc. Math. **7**, 605–651 (2002)
36. Walsh, C.: The horofunction boundary of finite-dimensional normed spaces. Math. Proc. Camb. Philos. Soc. **142**(3), 497–507 (2007)
37. Walsh, C.: The action of a nilpotent group on its horofunction boundary has finite orbits. Groups Geom. Dyn. **5**(1), 189–206 (2011)
38. Walsh, C.: The horoboundary and isometry group of Thurston's Lipschitz metric. In: Papadopoulos, A. (ed.) Handbook of Teichmüller Theory, vol. IV, pp. 327–353. Eur. Math. Soc., Zürich (2014)
39. Walsh, C.: The horofunction boundary and isometry group of the Hilbert geometry. In: Papadopoulos, A., Troyanov, M. (eds.) Handbook of Hilbert Geometry, pp. 127–146. Eur. Math. Soc., Zürich (2014)
40. Walsh, C.: Hilbert and Thompson geometries isometric to infinite-dimensional Banach spaces. Annales de l'Institut Fourier **68**(5), 1831–1877 (2018)