Annals of Functional Analysis (2020) 11:271–284
https://doi.org/10.1007/s43034-019-00002-7

Tusi Mathematical Research Group

ORIGINAL PAPER

A theorem of Brown–Halmos type for dual truncated Toeplitz operators

Yuanqi Sang[1] · Yueshi Qin[2] · Xuanhao Ding[3,4]

Received: 26 March 2019 / Accepted: 18 July 2019 / Published online: 1 January 2020

Abstract
In this paper, we investigate commuting dual truncated Toeplitz operators on the orthogonal complement of the model space K_u^2. Let $f, g \in L^\infty$, if two dual truncated Toeplitz operators D_f and D_g commute, we obtain similar conditions of Brown–Halmos Theorem for Hardy-Toeplitz operators, that is, both f and g are analytic, or both f and g are co-analytic, or a nontrivial linear combination of f and g is constant. However, the first two conditions are not sufficient, one can easily construct two non-commuting dual truncated Toeplitz operators with analytic or co-analytic symbols. We prove that two bounded dual truncated Toeplitz operators D_f and D_g commute if and only if $f, g, \bar{f}(u-\lambda)$ and $\bar{g}(u-\lambda)$ all belong to H^2 for some constant λ; or $\bar{f}, \bar{g}, f(u-\lambda)$ and $g(u-\lambda)$ all belong to H^2 for some constant λ; or a nontrivial linear combination of f and g is constant.

Keywords Harmonic function · Model space · Dual truncated Toeplitz operator

Mathematics Subject Classification Primary 47B35 · Secondary 31A05

Communicated by Kehe Zhu.

✉ Yuanqi Sang
yuanqisang@fudan.edu.cn

Yueshi Qin
yqin@cqu.edu.cn

Xuanhao Ding
xuanhaod@qq.com

1 School of Mathematical Sciences, Fudan University, Shanghai 200433, People's Republic of China

2 College of Mathematics and Statistics, Chongqing University, Chongqing 401331, People's Republic of China

3 School of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, People's Republic of China

4 Chongqing key Laboratory of Social Economy and Applied Statistics, Chongqing 400067, People's Republic of China

1 Introduction

Inspired by Sarason's seminal paper [11], many work in the study of truncated Toeplitz operators has been done over the past ten years [1,2,6,8]. In particular, the study of algebraic properties of Toeplitz operators is an active area of research [4,12]. This paper aims to study the commutativity of two dual truncated Toeplitz operators, where the dual truncated Toeplitz operator is a newly defined operator on a Hilbert space of harmonic functions that closely relates to truncated Toeplitz operators.

We start by recalling a few basic definitions and facts. Let H^2 be the classical Hardy space of open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $L^2 = L^2(\mathbb{T})$ be the usual Lebesgue space on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The space L^∞ is the collection of all essentially bounded measurable functions on $\mathbb{T}$, the space H^∞ consists of all the functions that are analytic and bounded on $\mathbb{D}$. Let P be the orthogonal projection from L^2 onto H^2. One defines for f and g in L^∞ the Toeplitz operator T_f and dual Toeplitz operator S_g on H^2 and $(H^2)^\perp$, respectively, as the following:

$$T_f x = P(fx), \quad x \in H^2,$$
$$S_g y = (I - P)(gy), \quad y \in (H^2)^\perp.$$

To each non-constant inner function u and $f \in L^2$, the truncated Toeplitz operator A_f is densely defined on model space $K_u^2 = H^2 \ominus uH^2$ by the formula

$$A_f x = P_u(fx), \quad x \in K_u^2.$$

Here $P_u = P - M_u P M_{\bar{u}}$ is the orthogonal projection from L^2 onto K_u^2. Then we define the dual truncated Toeplitz operator D_f on the orthogonal complement of K_u^2 by:

$$D_f y = (I - P_u)(fy), \quad y \in (K_u^2)^\perp.$$

Clearly, $D_f^* = D_{\bar{f}}$ and $\|D_f\| = \|f\|_\infty$ [7, Property 2.1.].

Brown and Halmos [3, Theorem 9.] give a necessary and sufficient condition for the commutativity of two Toeplitz operators. By anti-unitary equivalence [9] of T_f and $S_{\bar{f}}$, means $S_f S_g = S_g S_f$ if and only if $T_f T_g = T_g T_f$, one gets immediately that S_f and S_g are commuting if and only if either both f and g are analytic, or both f and g are co-analytic, or a nontrivial linear combination of f and g is constant.

Such equivalent property does not hold for truncated Toeplitz operators and dual truncated Toeplitz operators. It is easy to observe that two truncated Toeplitz operators with analytic symbols commute. I. Chalendar and D. Timotin [4] have a general criterion for the commutation of truncated Toeplitz operators. However, one can easily construct two non-commuting dual truncated Toeplitz operators with analytic symbols.

Example 1 Let u be an inner function with $u(0) = 0$. z and zu are analytic functions. It is easy to check that $D_z D_{uz} \bar{z} = zu$ and $D_{uz} D_z \bar{z} = 0$. Hence $D_z D_{uz} \neq D_{uz} D_z$.

It's not hard to understand the above example if one sees that D_f lives on $(K_u^2)^\perp = uH^2 \oplus \overline{zH^2}$, a Hilbert space of harmonic functions, while A_f is on K_u^2, a Hilbert space of analytic functions. Also, D_f behaves differently from the Toeplitz operators on the harmonic Bergman space [5].

For the commuting problem of dual truncated Toeplitz operators, an interesting aspect is that we only need to consider the dual truncated Toeplitz operators with analytic symbols (see Theorem 2). Here, we turn this problem into a mixed commuting problem of three Hankel operators ($H_{\bar{f}} H_{\bar{u}}^* H_{\bar{g}} = H_{\bar{g}} H_{\bar{u}}^* H_{\bar{f}}$) that still remains unsolved in classic Hankel operator theory.

We can now state our main result.

Theorem 3 *Let u be a nonconstant inner function and $f, g \in L^\infty$. Then*

$$D_f D_g = D_g D_f$$

if and only if one of the following cases holds:

1. $f, g, \bar{f}(u - \lambda)$ *and* $\bar{g}(u - \lambda)$ *all belong to* H^2 *for some constant* λ,
2. $\bar{f}, \bar{g}, f(u - \lambda)$ *and* $g(u - \lambda)$ *all belong to* H^2 *for some constant* λ,
3. *a nontrivial linear combination of f and g is constant.*

2 Necessary condition

For f and g in L^2, let

$$f_+ = Pf, \quad f_- = P_- f = (I - P)f.$$

Let A and B be bounded operators on a Hilbert space, then the commutator of A and B is define as

$$[A, B] = AB - BA.$$

Define an operator V on L^2 by

$$Vf(w) = \overline{w}\overline{f(w)}$$

for $f \in L^2$. It is easy to check that V is anti-unitary. The operator V satisfies the following properties:

$$V^2 = I, \quad VPV = I - P, \quad VT_f = S_{\bar{f}} V. \tag{1}$$

Let $K_\lambda = \frac{1}{1 - w\bar{\lambda}}$ denote the reproducing kernel of H^2 at λ and $k_\lambda = \frac{\sqrt{1-|\lambda|^2}}{1-w\bar{\lambda}}$ denote the normalized reproducing kernel of H^2 at λ. Let us state the well-known result of Brown and Halmos.

Theorem 1 *[3, Theorem 9.] (Brown-Halmos Theorem.) For ψ and φ in L^∞. Then*

$$T_\psi T_\varphi = T_\varphi T_\psi$$

if and only if at least one of the following holds:

1. *both ψ and φ are analytic,*
2. *both ψ and φ are co-analytic,*
3. *a nontrivial linear combination of ψ and φ is constant.*

If T_f and T_g satisfy

$$T_{\bar{z}}[T_f, T_g]T_z = [T_f, T_g],$$

it follows from [3, Theorem 6] and [13, Corollary 4.5] that T_f and T_g commute. The following lemma generalizes the above result for arbitrary inner function instead of function z.

Lemma 1 *Let u be a nonconstant inner function. On the Hardy space H^2, for $f, g \in L^\infty$, if*

$$[T_f, T_g] = T_{\bar{u}}[T_f, T_g]T_u,$$

then either

1. *both f and g are analytic, or*
2. *both f and g are co-analytic, or*
3. *a nontrivial linear combination of f and g is constant.*

Proof Assume $[T_f, T_g] = T_{\bar{u}}[T_f, T_g]T_u$, we have

$$T_f T_g - T_g T_f = T_{\bar{u}^n f} T_{u^n g} - T_{\bar{u}^n g} T_{u^n f}$$

for each positive integer n. In [10, Lemma 2.1], Guo and Wang obtained $T_{\bar{u}^n} \to 0$ (SOT), for $F \in L^2$, $P(\bar{u}^n F) = P(\bar{u}^n F_+)$, thus

$$\lim_{n\to\infty} \|P(\bar{u}^n F)\|_2 = 0,$$

and

$$\|(I-P)(u^n F)\|_2 = \|VPVu^n F\|_2 = \|VP\bar{u}^n(VF)\|_2 \to 0, \quad \text{as } n \to \infty. \tag{2}$$

Write

$$\begin{aligned}\left\langle T_{\bar{u}^n f} T_{u^n g} k_z, k_z\right\rangle &= \left\langle P[\bar{u}^n f P(u^n g k_z)], k_z\right\rangle \\ &= \left\langle \bar{u}^n f P(u^n g k_z), k_z\right\rangle \\ &= \left\langle f g k_z, k_z\right\rangle - \left\langle \bar{u}^n f (I-P)(u^n g k_z), k_z\right\rangle \\ &= \left\langle f g k_z, k_z\right\rangle - \left\langle (I-P)(u^n g k_z), u^n \bar{f} k_z\right\rangle.\end{aligned}$$

Note that the Cauchy-Schwarz inequality yields

$$|\langle (I-P)(u^n g k_z), u^n \bar{f} k_z\rangle| \leq ||(I-P)(u^n g k_z)||_2 ||\bar{f} k_z||_2,$$

Using (2), we have

$$\lim_{n\to\infty} \langle T_{\bar{u}^n f} T_{u^n g} k_z, k_z\rangle = \langle f g k_z, k_z\rangle.$$

Similarly,

$$\lim_{n\to\infty} \langle T_{\bar{u}^n g} T_{u^n f} k_z, k_z\rangle = \langle f g k_z, k_z\rangle.$$

Hence

$$\langle [T_f, T_g] k_z, k_z\rangle = 0.$$

Since Berezin transform is one-to-one, $[T_f, T_g] = 0$. By Brown-Halmos Theorem (see Theorem 1), it follows that either both f and g are all analytic, or f and g are all co-analytic, or a nontrivial linear combination of f and g is constant. □

Recall that the Hankel operator H_f with symbol $f \in L^2$ is densely defined by

$$H_f x = (I-P)(fx), \text{ for } x \in H^2, \tag{3}$$

and H_f^* is defined by

$$H_f^* y = P(\bar{f} y), \text{ for } y \in [H^2]^{\perp}.$$

If M_f is expressed as an operator matrix with respect to the decomposition $L^2 = H^2 \oplus \overline{zH^2}$, the result is of the form

$$M_f = \begin{pmatrix} T_f & H_{\bar{f}}^* \\ H_f & S_f \end{pmatrix}.$$

Since $M_f M_g = M_{fg}$, we have

$$T_{fg} = T_f T_g + H_{\bar{f}}^* H_g; \tag{4}$$

$$H_{fg} = H_f T_g + S_f H_g. \tag{5}$$

If $g \in H^\infty$, then $H_g = 0$, (5) becomes

$$H_{fg} = H_f T_g. \tag{6}$$

Similarly, if $f \in H^\infty$, (5) becomes

$$H_{fg} = S_f H_g. \tag{7}$$

Define the unitary operator

$$U : L^2 (= H^2 \oplus \overline{zH^2}) \to [K_u^2]^\perp (= uH^2 \oplus \overline{zH^2})$$

by

$$U = \begin{pmatrix} M_u & 0 \\ 0 & I \end{pmatrix}.$$

Clearly, U^* maps $[K_u^2]^\perp$ to L^2 and equals

$$U^* = \begin{pmatrix} M_{\bar{u}} & 0 \\ 0 & I \end{pmatrix}.$$

Next lemma gives a matrix representation of D_ϕ. The representation is useful in this paper and shows that the dual truncated Toeplitz operators on $[K_u^2]^\perp$ are closely related to the Toeplitz operators and Hankel operators on H^2.

Lemma 2 *On* $L^2 = H^2 \oplus \overline{zH^2}, \phi \in L^\infty$,

$$U^* D_\phi U = \begin{pmatrix} T_\phi & H^*_{u\bar{\phi}} \\ H_{u\phi} & S_\phi \end{pmatrix}. \tag{8}$$

Proof If f_1 is in H^2, by the definition of D_ϕ we have

$$D_\phi u f_1 = uP\bar{u}\phi u f_1 + (I - P)\phi u f_1.$$

If f_2 is in $\overline{zH^2}$, similarly we have that

$$D_\phi f_2 = uP\bar{u}\phi f_2 + (I - P)\phi f_2.$$

Therefore for given $[f_1, f_2]^T$ in $H^2 \oplus \overline{zH^2}$ the above calculation gives

$$\begin{aligned} U^* D_\phi U \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= U^* D_\phi \begin{pmatrix} u f_1 \\ f_2 \end{pmatrix} \\ &= U^* \begin{pmatrix} uP\bar{u}\phi u f_1 + uP\bar{u}\phi f_2 \\ (I - P)\phi u f_1 + (I - P)\phi f_2 \end{pmatrix} \\ &= \begin{pmatrix} P\bar{u}\phi u f_1 + P\bar{u}\phi f_2 \\ (I - P)\phi u f_1 + (I - P)\phi f_2 \end{pmatrix} \\ &= \begin{pmatrix} T_\phi & H^*_{u\bar{\phi}} \\ H_{u\phi} & S_\phi \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \end{aligned}$$

□

By Lemmas 1 and 2, we now obtain the necessary conditions similar to Theorem 1. By Example 1, the conditions are not sufficient.

Theorem 2 *Let $f, g \in L^\infty$. Assume $D_f D_g = D_g D_f$, then either*

1. *both f and g are analytic, or*
2. *both f and g are co-analytic, or*
3. *a nontrivial linear combination of f and g is constant.*

Proof Assume $D_f D_g = D_g D_f$. by the matrix representation (8) , we have

$$U^* D_f D_g U = \begin{pmatrix} T_f T_g + H^*_{u\bar{f}} H_{gu} & T_f H^*_{u\bar{g}} + H^*_{u\bar{f}} S_g \\ H_{fu} T_g + S_f H_{gu} & H_{fu} H^*_{\bar{g}u} + S_f S_g \end{pmatrix}$$

and

$$U^* D_g D_f U = \begin{pmatrix} T_g T_f + H^*_{u\bar{g}} H_{fu} & T_g H^*_{u\bar{f}} + H^*_{u\bar{g}} S_f \\ H_{gu} T_f + S_g H_{fu} & H_{gu} H^*_{\bar{f}u} + S_g S_f \end{pmatrix}.$$

Hence,

$$T_f T_g + H^*_{u\bar{f}} H_{gu} = T_g T_f + H^*_{u\bar{g}} H_{fu}.$$

By (4), we have

$$T_f T_g - T_g T_f = T_{\bar{u}f} T_{ug} - T_{\bar{u}g} T_{uf}.$$

By Lemma 1, then either both f and g are analytic, or f and g are co-analytic, or a nontrivial linear combination of f and g is constant. □

3 Necessary and sufficient condition

Since $D^*_f = D_{\bar{f}}$, Theorem 2 shows that the study of commuting dual truncated Toeplitz operators can be reduced to the following question.

Problem 1 For which bounded analytic functions f and g, $D_f D_g = D_g D_f$?

In the case of analytic symbols, we translate Problem 1 into solving an equation about a Toeplitz operator and a Hankel operator.

Lemma 3 *Let u be a nonconstant inner function and $f, g \in H^\infty$, the following statements are equivalent.*

1. $D_f D_g = D_g D_f$ *holds on* $[K^2_u]^\perp$;
2. $D_{\bar{f}} D_{\bar{g}} = D_{\bar{g}} D_{\bar{f}}$ *holds on* $[K^2_u]^\perp$;
3. $H_{u\bar{f}} T_{\bar{g}} + S_{\bar{f}} H_{u\bar{g}} = H_{u\bar{g}} T_{\bar{f}} + S_{\bar{g}} H_{u\bar{f}}$ $(H^2 \to [H^2]^\perp)$;

4. $H_{u\bar{f}}T_{\bar{g}} - H_{\bar{f}}T_{u\bar{g}} = H_{u\bar{g}}T_{\bar{f}} - H_{\bar{g}}T_{u\bar{f}}$ $(H^2 \to [H^2]^\perp)$;
5. $H_{\bar{f}}H_{\bar{u}}^*H_{\bar{g}} = H_{\bar{g}}H_{\bar{u}}^*H_{\bar{f}}$ $(H^2 \to [H^2]^\perp)$;

Proof (1) $\Leftrightarrow$ (2): Since $D_f^* = D_{\bar{f}}$, it is clear that (1) is equivalent to (2).

(2) $\Leftrightarrow$ (3): Assume $f, g \in H^\infty$, hence

$$U^*D_{\bar{f}}D_{\bar{g}}U = \begin{pmatrix} T_{\bar{f}}T_{\bar{g}} & 0 \\ H_{\bar{f}u}T_{\bar{g}} + S_{\bar{f}}H_{\bar{g}u} & S_{\bar{f}}S_{\bar{g}} \end{pmatrix}$$

and

$$U^*D_{\bar{g}}D_{\bar{f}}U = \begin{pmatrix} T_{\bar{g}}T_{\bar{f}} & 0 \\ H_{\bar{g}u}T_{\bar{f}} + S_{\bar{g}}H_{\bar{f}u} & S_{\bar{g}}S_{\bar{f}} \end{pmatrix}$$

By Brown-Halmos Theorem (see Theorem 1), we have

$$T_{\bar{f}}T_{\bar{g}} = T_{\bar{g}}T_{\bar{f}},$$
$$S_{\bar{f}}S_{\bar{g}} = S_{\bar{g}}S_{\bar{f}}.$$

Hence

$$D_{\bar{f}}D_{\bar{g}} = D_{\bar{g}}D_{\bar{f}}$$

if and only if

$$H_{u\bar{f}}T_{\bar{g}} + S_{\bar{f}}H_{u\bar{g}} = H_{u\bar{g}}T_{\bar{f}} + S_{\bar{g}}H_{u\bar{f}}.$$

(3) $\Leftrightarrow$ (4): By (5), we have

$$S_{\bar{f}}H_{u\bar{g}} = H_{u\overline{fg}} - H_{\bar{f}}T_{u\bar{g}}$$

and

$$S_{\bar{g}}H_{u\bar{f}} = H_{u\overline{fg}} - H_{\bar{g}}T_{u\bar{f}},$$

the result follows.

(4) $\Leftrightarrow$ (5): Since (4) and (6),

$$\begin{aligned} H_{u\bar{f}}T_{\bar{g}} - H_{\bar{f}}T_{u\bar{g}} &= H_{\bar{f}}T_uT_{\bar{g}} - H_{\bar{f}}T_{u\bar{g}} \\ &= H_{\bar{f}}(T_uT_{\bar{g}} - T_{u\bar{g}}) \\ &= -H_{\bar{f}}H_{\bar{u}}^*H_{\bar{g}}. \end{aligned}$$

Similarly,

$$H_{u\bar{g}}T_{\bar{f}} - H_{\bar{g}}T_{u\bar{f}} = -H_{\bar{g}}H_{\bar{u}}^*H_{\bar{f}}.$$

□

Example 2 Let $u = \theta\eta$, both θ and η are inner functions. For $\lambda \in \mathbb{D}$,

$$\begin{aligned} H_{\bar{\theta}} H_{\bar{u}}^{*} H_{\bar{\eta}} k_{\lambda} &= H_{\bar{\theta}} P\theta\eta(I-P)\bar{\eta}k_{\lambda} \\ &= H_{\bar{\theta}} P\theta k_{\lambda} - H_{\bar{\theta}} P\theta\eta P\bar{\eta}k_{\lambda} \\ &= 0 - \overline{\eta(\lambda)} H_{\bar{\theta}}\theta\eta k_{\lambda} = 0, \end{aligned}$$

Similarly,

$$H_{\bar{\eta}} H_{\bar{u}}^{*} H_{\bar{\theta}} k_{\lambda} = 0.$$

Since by $\{k_{\lambda}\}_{\lambda\in\mathbb{D}}$ is dense in H^2 and Lemma 3, we have $D_{\theta} D_{\eta} = D_{\eta} D_{\theta}$.

Lemma 4 *Let $f, g \in H^{\infty}$ and f is not constant. Assume there exists a constant λ such that $(u-\lambda)\bar{f} \in H^2$, then $D_f D_g = D_g D_f$ implies that $(u-\lambda)\bar{g} \in H^2$.*

Proof Assume $(u-\lambda)\bar{f} \in H^2$ and $D_f D_g = D_g D_f$. By Lemma 3, we have $H_{\bar{f}} H_{\bar{u}}^{*} H_{\bar{g}} = H_{\bar{g}} H_{\bar{u}}^{*} H_{\bar{f}}$. Also, $H_{\bar{f}} H_{\overline{u-\lambda}}^{*} H_{\bar{g}} = H_{\bar{g}} H_{\overline{u-\lambda}}^{*} H_{\bar{f}}$, Note that

$$H_{(u-\lambda)\bar{f}\bar{g}} = H_{\bar{f}} T_{(u-\lambda)\bar{g}} + S_{\bar{f}} H_{(u-\lambda)\bar{g}} = H_{\bar{g}} T_{(u-\lambda)\bar{f}} + S_{\bar{g}} H_{(u-\lambda)\bar{f}}.$$

An easy computation gives

$$\begin{aligned} H_{(u-\lambda)\bar{f}\bar{g}} - H_{\bar{f}} H_{\overline{u-\lambda}}^{*} H_{\bar{g}} &= H_{\bar{f}} T_{(u-\lambda)\bar{g}} + S_{\bar{f}} H_{(u-\lambda)\bar{g}} - H_{\bar{f}} H_{\bar{u}-\bar{\lambda}}^{*} H_{\bar{g}} \\ &= H_{\bar{f}} (T_{(u-\lambda)\bar{g}} - H_{\overline{u-\lambda}}^{*} H_{\bar{g}}) + S_{\bar{f}} H_{(u-\lambda)\bar{g}} \\ &= H_{\bar{f}} T_{(u-\lambda)} T_{\bar{g}} + S_{\bar{f}} H_{(u-\lambda)\bar{g}} \\ &= H_{(u-\lambda)\bar{f}} T_{\bar{g}} + S_{\bar{f}} H_{(u-\lambda)\bar{g}}. \end{aligned}$$

Similarly,

$$H_{(u-\lambda)\bar{f}\bar{g}} - H_{\bar{g}} H_{\overline{u-\lambda}}^{*} H_{\bar{f}} = H_{(u-\lambda)\bar{g}} T_{\bar{f}} + S_{\bar{g}} H_{(u-\lambda)\bar{f}}.$$

Thus we have

$$H_{(u-\lambda)\bar{f}} T_{\bar{g}} + S_{\bar{f}} H_{(u-\lambda)\bar{g}} = H_{(u-\lambda)\bar{g}} T_{\bar{f}} + S_{\bar{g}} H_{(u-\lambda)\bar{f}}. \tag{9}$$

Since $\bar{f}(u-\lambda) \in H^2$, $H_{(u-\lambda)\bar{f}} = 0$, (9) becoms

$$S_{\bar{f}} H_{(u-\lambda)\bar{g}} = H_{(u-\lambda)\bar{g}} T_{\bar{f}}.$$

Hence

$$S_{\bar{f}} H_{(u-\lambda)\bar{g}} 1 = H_{(u-\lambda)\bar{g}} T_{\bar{f}} 1.$$

This implies that

$$(\bar{f} - \overline{f(0)})P_{-}(u - \lambda)\bar{g} = 0.$$

Since f is not constant, $P_{-}(u - \lambda)\bar{g} = 0$. Hence $(u - \lambda)\bar{g} \in H^2$. □

Lemma 5 *Let u be a nonconstant inner function and $f, g \in H^{\infty}$. Then*

$$D_f D_g = D_g D_f$$

if and only if one of the following cases holds:

1. *$\bar{f}(u - \lambda)$ and $\bar{g}(u - \lambda)$ both belong to H^2 for some constant λ.*
2. *a nontrivial linear combination of f and g is constant.*

Proof Assume that $\bar{f}(u - \lambda)$ and $\bar{g}(u - \lambda)$ are both analytic for some constant λ. For $z \in \mathbb{D}$,

$$\begin{aligned} H_{\bar{f}} H_{\bar{u}}^{*} H_{\bar{g}} k_z &= H_{\bar{f}} H_{\overline{u-\lambda}}^{*} H_{\bar{g}} k_z \\ &= H_{\bar{f}} P(u - \lambda)[\bar{g} - \overline{g(z)}]k_z \\ &= H_{\bar{f}}(u - \lambda)[\bar{g} - \overline{g(z)}]k_z \\ &= (I - P)\overline{fg}(u - \lambda)k_z. \end{aligned}$$

Similarly,

$$H_{\bar{g}} H_{\bar{u}}^{*} H_{\bar{f}} k_z = (I - P)\overline{fg}(u - \lambda)k_z.$$

Since by $\{k_{\lambda}\}_{\lambda \in \mathbb{D}}$ is dense in H^2, $H_{\bar{f}} H_{\bar{u}}^{*} H_{\bar{g}} = H_{\bar{g}} H_{\bar{u}}^{*} H_{\bar{f}}$. Hence $D_f D_g = D_g D_f$ by Lemma 3.

Conversely, by Lemma 3, $D_f D_g = D_g D_f$ implies that

$$H_{\bar{f}} T_{u\bar{g}} - H_{u\bar{f}} T_{\bar{g}} = H_{\bar{g}} T_{u\bar{f}} - H_{u\bar{g}} T_{\bar{f}}.$$

Easy calculations give

$$\begin{aligned}
&\big(H_{\bar f}(1\otimes 1)T_{u\bar g} - H_{u\bar f}(1\otimes 1)T_{\bar g} - H_{\bar g}(1\otimes 1)T_{u\bar f} + H_{u\bar g}(1\otimes 1)T_{\bar f}\big)T_z\\
&\quad = H_{\bar f}(I - T_zT_{\bar z})T_{u\bar g}T_z - H_{u\bar f}(I - T_zT_{\bar z})T_{\bar g}T_z - H_{\bar g}(I - T_zT_{\bar z})T_{u\bar f}T_z\\
&\qquad + H_{u\bar g}(I - T_zT_{\bar z})T_{\bar f}T_z\\
&\quad = H_{\bar f}T_{u\bar g}T_z - H_{\bar f}T_zT_{\bar z}T_{u\bar g}T_z - H_{u\bar f}T_{\bar g}T_z + H_{u\bar f}T_zT_{\bar z}T_{\bar g}T_z - H_{\bar g}T_{u\bar f}T_z\\
&\qquad + H_{\bar g}T_zT_{\bar z}T_{u\bar f}T_z + H_{u\bar g}T_{\bar f}T_z - H_{u\bar g}T_zT_{\bar z}T_{\bar f}T_z\\
&\quad = H_{\bar f}T_{u\bar g}T_z - H_{\bar f}T_zT_{u\bar g} - H_{u\bar f}T_{\bar g}T_z + H_{u\bar f}T_zT_{\bar g} - H_{\bar g}T_{u\bar f}T_z + H_{\bar g}T_zT_{u\bar f}\\
&\qquad + H_{u\bar g}T_{\bar f}T_z - H_{u\bar g}T_zT_{\bar f}\\
&\quad = H_{\bar f}T_{u\bar g}T_z - S_zH_{\bar f}T_{u\bar g} - H_{u\bar f}T_{\bar g}T_z + S_zH_{u\bar f}T_{\bar g} - H_{\bar g}T_{u\bar f}T_z + S_zH_{\bar g}T_{u\bar f}\\
&\qquad + H_{u\bar g}T_{\bar f}T_z - S_zH_{u\bar g}T_{\bar f}\\
&\quad = (H_{\bar f}T_{u\bar g} - H_{u\bar f}T_{\bar g} - H_{\bar g}T_{u\bar f} + H_{u\bar g}T_{\bar f})T_z\\
&\qquad - S_z(H_{\bar f}T_{u\bar g} - H_{u\bar f}T_{\bar g} - H_{\bar g}T_{u\bar f} + H_{u\bar g}T_{\bar f})\\
&\quad = 0.
\end{aligned}$$

The first equality follows from $1\otimes 1 = I - T_zT_{\bar z}$, and the fourth equality follows from (6) and (7).

Thus we have

$$(H_{\bar f}1)\otimes(T_{\bar z\bar u g}1) - (H_{u\bar f}1)\otimes(T_{\bar z g}1) = (H_{\bar g 1})\otimes(T_{\bar z\bar u f}1) - (H_{u\bar g}1)\otimes(T_{\bar z f}1). \tag{10}$$

Since

$$T_{\bar z\bar u g}1 = P\bar z\bar u g1 = PVu\bar g = VP\_u\bar g = VH_{u\bar g}1,$$

Similarly,

$$\begin{aligned}
T_{\bar z g}1 &= VH_{\bar g}1,\\
T_{\bar z\bar u f}1 &= VH_{u\bar f}1,\\
T_{\bar z f}1 &= VH_{\bar f}1.
\end{aligned}$$

Now (10) becomes

$$(H_{\bar f}1)\otimes(VH_{u\bar g}1) - (H_{u\bar f}1)\otimes(VH_{\bar g}1) = (H_{\bar g}1)\otimes(VH_{u\bar f}1) - (H_{u\bar g}1)\otimes(VH_{\bar f}1). \tag{11}$$

Obviously, if f or g is constant, then condition (2) hold. Assume that neither of f and g is constant.

Case 1.

Assume that $\left\{H_{\bar{f}}1, H_{u\bar{f}}1\right\}$ is linearly dependent. Hence, there exist a complex number λ such that

$$H_{u\bar{f}}1 = \lambda H_{\bar{f}}1.$$

So, $\bar{f}(u-\lambda) \in H^2$. By Lemma 4, we have $\bar{g}(u-\lambda) \in H^2$. Condition (1) would hold.

Case 2.

Assume that $\left\{H_{\bar{f}}1, H_{u\bar{f}}1\right\}$ is linearly independent. Then $\left\{H_{\bar{g}}1, H_{u\bar{g}}1\right\}$ is also linearly independent. If $\left\{H_{\bar{g}}1, H_{u\bar{g}}1\right\}$ is linearly dependent, which contradicts to Lemma 4. Since V is anti-unitary, $\left\{VH_{\bar{g}}1, VH_{u\bar{g}}1\right\}$ is linearly independent, by Gram-Schmidt procedure, there exist a nonzero function x_0 in $span\left\{VH_{\bar{g}}1, VH_{u\bar{g}}1\right\}$ such that

$$\langle VH_{u\bar{g}}1, x_0\rangle = 1,$$
$$\langle VH_{\bar{g}}1, x_0\rangle = 0.$$

Applying operator Eq. (11) to x_0 gives

$$H_{\bar{f}}1 = \langle x_0, VH_{u\bar{f}}1\rangle H_{\bar{g}}1 - \langle x_0, VH_{\bar{f}}1\rangle H_{u\bar{g}}1.$$

Hence,

$$H_{\bar{f}}1 \in span\left\{H_{\bar{g}}1, H_{u\bar{g}}1\right\}.$$

Similarly,

$$H_{u\bar{f}}1 \in span\left\{H_{\bar{g}}1, H_{u\bar{g}}1\right\},$$
$$H_{\bar{g}}1 \in span\left\{H_{\bar{f}}1, H_{u\bar{f}}1\right\},$$
$$H_{u\bar{g}}1 \in span\left\{H_{\bar{f}}1, H_{u\bar{f}}1\right\}.$$

Therefore

$$span\left\{H_{\bar{g}}1, H_{u\bar{g}}1\right\} = span\left\{H_{\bar{f}}1, H_{u\bar{f}}1\right\},$$

there exist constants $a_{11}, a_{12}, a_{21}, a_{22}$ such that

$$\begin{aligned} H_{\bar{f}}1 &= a_{11}H_{\bar{g}}1 + a_{12}H_{u\bar{g}}1, \\ H_{u\bar{f}}1 &= a_{21}H_{\bar{g}}1 + a_{22}H_{u\bar{g}}1. \end{aligned} \tag{12}$$

Replacing above formulas in (11) yields

$$\begin{aligned}(H_{\bar{g}}1) \otimes (\overline{a_{11}} V H_{u\bar{g}}1 - \overline{a_{21}} V H_{\bar{g}}1 - V H_{u\bar{f}}1) \\ = (H_{u\bar{g}}1) \otimes (\overline{a_{12}} V H_{u\bar{g}}1 - \overline{a_{22}} V H_{\bar{g}}1 + V H_{\bar{f}}1).\end{aligned}$$

Since $\left\{H_{\bar{g}}1, H_{u\bar{g}}1\right\}$ is linearly independent,

$$\begin{aligned}V H_{\bar{f}}1 &= -\overline{a_{12}} V H_{u\bar{g}}1 + \overline{a_{22}} V H_{\bar{g}}1, \\ V H_{u\bar{f}}1 &= \overline{a_{11}} V H_{u\bar{g}}1 - \overline{a_{21}} V H_{\bar{g}}1,\end{aligned}$$

which simplifies to

$$\begin{aligned}H_{\bar{f}}1 &= a_{22} H_{\bar{g}}1 - a_{12} H_{u\bar{g}}1, \\ H_{u\bar{f}}1 &= -a_{21} H_{\bar{g}}1 + a_{11} H_{u\bar{g}}1.\end{aligned} \tag{13}$$

Combining (12) and (13) gives

$$\begin{aligned}a_{11} &= a_{22}, \\ a_{12} &= -a_{12}, \\ a_{21} &= -a_{21}.\end{aligned}$$

Then, $a_{12} = a_{21} = 0$. Let $a_{11} = a_{22} = c$, we have $H_{\bar{f}}1 = c H_{\bar{g}}1$, and hence

$$\bar{f} - c\bar{g} \in H^{\infty}.$$

Since $f, g \in H^{\infty}$, $\bar{f} - c\bar{g}$ is a constant, condition (2) would hold. □

Corollary 1 *If $f, g \in K_u^2 \cap H^{\infty}$, then $D_f D_g = D_g D_f$.*

Proof In fact, $K_u^2 = H^2 \cap u\overline{zH^2} \subseteq H^2 \cap u\overline{H^2}$. Since $f, g \in K_u^2$, there exist $f_1, g_1 \in H^2$ such that $f = u\bar{f}_1$, $g = u\bar{g}_1$. Thus $u\bar{f} = u(\bar{u} f_1) = f_1 \in H^2$ and $u\bar{g} = u(\bar{u} g_1) = g_1 \in H^2$. Hence $D_f D_g = D_g D_f$ by Lemma 5. □

Combining Theorem 2 and Lemma 3, we get our main result.

Theorem 3 *Let u be a nonconstant inner function and $f, g \in L^{\infty}$.Then*

$$D_f D_g = D_g D_f$$

if and only if one of the following cases holds:

1. *f, g, $\bar{f}(u - \lambda)$ and $\bar{g}(u - \lambda)$ all belong to H^2 for some constant λ,*
2. *$\bar{f}$, $\bar{g}$, $f(u - \lambda)$ and $g(u - \lambda)$ all belong to H^2 for some constant λ,*
3. *a nontrivial linear combination of f and g is constant.*

On classical Hardy space H^2, assume that any nontrivial linear combination of f and g is not constant, then $T_f T_g = T_g T_f$ implies that $T_f T_g = T_{fg}$. In general, $T_f T_g = T_{fg}$ does not implies that $T_f T_g = T_g T_f$. But on Hilbert spaces of harmonic functions $(K_u^2)^\perp = uH^2 \oplus \overline{zH^2}$, $D_f D_g = D_{fg}$ implies that $D_f D_g = D_g D_f$ by the following theorem. However, the converse is not true.

Theorem 4 *[7, Theorem 4.7] Let $f, g \in L^\infty$ and u be a nonconstant inner function. Then $D_f D_g = D_{fg}$ if and only if one of the following cases holds:*

1. *f, g, $\bar{f}(u-\lambda)$, $\bar{g}(u-\lambda)$ and $\bar{f}\bar{g}(u-\lambda)$ all belong to H^2 for some constant λ.*
2. *$\bar{f}$, $\bar{g}$, $f(u-\lambda)$, $g(u-\lambda)$ and $fg(u-\lambda)$ all belong to H^2 for some constant λ.*
3. *either f or g is constant.*

Example 3 Let $u = \theta\eta$, where θ and η are nonconstant inner functions. Let $f = u$,$g = \theta$. Since $u\bar{f} = 1 \in H^2$ and $u\bar{g} = \eta \in H^2$, $D_f D_g = D_g D_f$ by Theorem 3. But $u\bar{f}\bar{g} = \bar{\eta}$ is not analytic, by the above theorem, we have $D_f D_g \neq D_{fg}$.

Acknowledgements The authors thank the referee for several valuable suggestions which greatly improve the presentation of this paper. Yuanqi Sang is supported by China Postdoctoral Science Foundation(2019M651331). Yueshi Qin is supported by National Natural Science Foundation of China(11501059). Xuanhao Ding is supported by National Natural Science Foundation of China(11871122), Chong Qing National Natural Science Foundation of China(cstc2018jcyjAX0595).

References

1. Baranov, A., Bessonov, R., Kapustin, V.: Symbols of truncated Toeplitz operators. J. Funct. Anal. **261**(12), 3437–3456 (2011)
2. Baranov, A., Chalendar, I., Fricain, E., Mashreghi, J., Timotin, D.: Bounded symbols and reproducing Kernel thesis for truncated Toeplitz operators. J. Funct. Anal. **259**(10), 2673–2701 (2010)
3. Brown, A., Halmos, P.: Algebraic properties of Toeplitz operators. J. Reine Angew. Math. **213**, 89–102 (1964)
4. Chalendar, I., Timotin, D.: Commutation relations for truncated Toeplitz operators. Oper. Matrices. **8**(3), 877–888 (2014)
5. Choe, B.R., Lee, Y.J.: Commuting Toeplitz operators on the harmonic Bergman space. Michigan Math. J. **46**(1), 163–174 (1999)
6. Cima, J.A., Garcia, S.R., Ross, W.T., Wogen, W.R.: Truncated Toeplitz operators: spatial isomorphism, unitary equivalence, and similarity. Indiana Univ. Math. J. **59**(2), 595–620 (2010)
7. Ding, X., Sang, Y.: Dual truncated Toeplitz operators. J. Math. Anal. Appl. **461**(1), 929–946 (2018)
8. Garcia, S.R., Ross, W.T.: Recent Progress on Truncated Toeplitz Operators. Blaschke products and their applications, Fields Inst. Commun., vol. 65, pp. 275–319. Springer, New York (2013)
9. Guediri, H.: Dual Toeplitz operators on the sphere. Acta Math. Sin. (Engl. Ser.) **29**(9), 1791–1808 (2013)
10. Guo, K., Wang, K.: On operators which commute with analytic Toeplitz operators modulo the finite rank operators. Proc. Amer. Math. Soc. **134**(9), 2571–2576 (2006)
11. Sarason, D.: Algebraic properties of truncated Toeplitz operators. Oper. Matrices. **1**(4), 491–526 (2007)
12. Sedlock, N.: Algebras of truncated Toeplitz operators. Oper. Matrices. **5**(2), 309–326 (2010)
13. Stroethoff, K.: Algebraic properties of Toeplitz operators on the Hardy space via the Berezin transform, Function spaces (Edwardsville, IL, 1998), Contemp. Math. **232**, Amer. Math. Soc., Providence, RI (1999)