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ORIGINAL PAPER

Energy decay analysis for Porous elastic system with microtemperature: Classical vs second spectrum approach

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Abstract

The stability features of the dissipative porous elastic systems have piqued the interest of several researchers. The desired exponential decay property of the energy is obtained unless the nonphysical equal speed condition is imposed. This work analyzes the porous elastic system with micro-temperature. First, the exponential stability is obtained in case where there is an assumption on physical constants. Then from a second-spectrum viewpoint, the system's global well-posedness is proved using the Faedo–Galerkin method. Later, we prove that the microtemperature effect is enough to get the exponential stability of the solution without any assumption on the physical constants. A numerical scheme is introduced. Finally, we present some numerical results which demonstrates the exponential behavior of the solution.

Keywords Exponential decay · Porous system · Micro-temperature · Finite element analysis

Mathematics Subject Classification 35-XX · 93B05

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1 Introduction

In later a long time, some so numerous mathematical researchers have considered pondering the asymptotic behavior of solutions to the equations proposed to study different flexible materials with voids [1–4], which have decent physical properties, are utilized broadly in engineering, such as vehicles, airplanes, expansive space structures and so on. Due to their broad applications, many researchers' interest comes from the need to establish results concerning the existence and stabilization of elasticity problems.

In addition to the conventional elastic effects, materials with voids have a microstructure in which the mass at each place is calculated by multiplying the material matrix mass density by the volume fraction. Nunziato and Cowin [4] pioneered the latter concept in their groundbreaking work on elastic materials with voids. Iesan [5–7] and Iesan an Quintanilla [8] expanded the hypothesis by including temperature and microtemperatures [9–11].

According to our knowledge, evaluating the temporal decay in one-dimensional porous-elastic substances was pioneered with the aid of Quintanilla [12], where he proved that porous-viscosity becomes not robust sufficient to stabilize the system exponentially. Interestingly, Casas and Quintanilla [13] proved that the mixture of porous-viscosity and temperature additionally lacks exponential stability [14–16]. However, the identical authors [17] confirmed that the mixture of porous-viscosity and thermal effects (each temperature and microtemparatures) stabilized the system exponentially. Similarly, Magana and Quintanilla [18] proved that viscoelasticity collectively with microtemperatures produced exponential stability, while viscoelasticity collectively with temperature lacks exponential stability [19, 20].

It is natural to think that a porous-elastic system with dissipation due to only microtemperatures will lack exponential stability. However Apalara [21] establishes the contrary, he proved that a porous system with microtempearture decays exponentially if and only if $\chi = 0$ where $\chi = \frac{\mu}{\rho} - \frac{\delta}{J}$, otherwise the system is polynomially stable [22–40].

The equations for a one-dimensional porous elastic system with microtempeartureare of the form

$$\begin{cases} \rho u_{tt} = T_x, \\ J\phi_{tt} = H_x + G, \\ \rho E_t = P_x + q - Q, \end{cases} \tag{1.1}$$

where $(x, t) \in (0, l) \times (0, \infty)$, t is the time, x is the distance along the center line of the beam structure and l is the length of the beam, T is the stress, H is the equilibrated stress, G is the equilibrated body force, q is the heat flux vector, P is the first heat flux moment, Q is the mean heat flux and E is the first moment of energy. The functions $u(x, t)$ and $\phi(x, t)$ are the displacement of the solid elastic material and the volume fraction. The constitutive equations are given by

$$\begin{cases} T = \mu u_x + b\phi, \\ H = \delta\phi_x - dw, \\ G = -bu_x - \xi\phi, \\ \rho E = -\alpha w - d\phi_x, \\ P = -\kappa w_x, \\ q = k_1 w, \\ Q = k_2 w, \end{cases} \tag{1.2}$$

where w is the microtemperature, k_1, k_2, d, μ, δ, α, b, κ and ξ are positive constants such that $\mu\xi - b^2 \geq 0$. Substitute system (1.2) in (1.1) we get the porous elastic system with microtemperature

$$\rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \quad (x,t) \in (0,l) \times (0,\infty), \tag{1.3}$$

$$J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + dw_x = 0, \quad (x,t) \in (0,l) \times (0,\infty), \tag{1.4}$$

$$\alpha w_t - \kappa w_{xx} + d\phi_{tx} + kw = 0, \quad (x,t) \in (0,l) \times (0,\infty), \tag{1.5}$$

with Neumann boundary conditions

$$u(0,t) = u(l,t) = \phi_x(0,t) = \phi_x(l,t) = w(0,t) = w(l,t) = 0, t > 0, \tag{1.6}$$

and the initial conditions are

$$\begin{aligned} &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \phi(x,0) = \phi_0(x), \ \ \phi_t(x,0) = \phi_1(x), \\ &w(x,0) = w_0(x), \quad x \in (0,l), \end{aligned} \tag{1.7}$$

where $k = k_1 - k_2 > 0$.

The presence of Neumann boundary conditions for ϕ hinders the application of Poincaré inequality. In order to overcome this obstacle, we introduce modifications to ϕ in the following manner. Using equation (1.4) and the boundary conditions (1.6), we obtain

$$J\frac{d^2}{dt^2}\int_0^l \phi(x,t)dx + \xi\int_0^l \phi(x,t)dx = 0,$$

which is solved by

$$\int_0^l \phi(x,t)dx = \left(\int_0^l \phi_0(x)dx\right)\cos\left(\frac{\sqrt{\xi}}{\sqrt{J}}t\right) + \left(\int_0^l \phi_1(x)dx\right)\sin\left(\frac{\sqrt{\xi}}{\sqrt{J}}t\right).$$

Hence, if we define

$$\overline{\phi}(x,t) = \phi(x,t) - \left(\int_0^l \phi_0(x)dx\right)\cos\left(\frac{\sqrt{\xi}}{\sqrt{J}}t\right) - \frac{\sqrt{\xi}}{\sqrt{J}}\left(\int_0^l \phi_1(x)dx\right)\sin\left(\frac{\sqrt{\xi}}{\sqrt{J}}t\right),$$

then, $(u, \overline{\phi}, w)$ satisfies (1.3)–(1.7), with the following initial condition for ϕ

$$\overline{\phi}_0(x) = \phi_0(x) - \left(\int_0^l \phi_0(x)dx\right).$$

Moreover

$$\int_0^l \overline{\phi}(x,t)dx = 0,$$

which allows the application of Poincaré inequality for ϕ. In the subsequent analysis, we will utilize $(u, \overline{\phi}, w)$ for our calculations, but for the sake of convenience, we will represent it as (u, ϕ, w). Note that the stabilization of the Porous system and the Bresse-Timoshenko system was studied by different researchers with different damping mechanisms (for example see [41–46]).

Apalara [7] study system (1.3)–(1.7) in the case where $\mu\xi > b^2$. This paper will deal with the case where $\mu\xi = b^2$.

The rest of the paper is as follows: In Sect. 2 we will establish exponential stability when $\mu\xi = b^2$. In Sect. 3 we will study the system with a second spectrum free. First, we study well-posedness using the Faedo Galerkin approximation and then prove the exponential stability without any assumption on parameters. In Sect. 4 we present some numerical results which demonstrate the exponential behavior of the solution.

2 Exponential stabilty

This section aims to show that the energy of system (1.3)–(1.7) decays exponentially when $\mu\xi = b^2$ and under the condition of equal speed limits.

First we state the well-posedness theorem that is proved by Apalara [7].

Theorem 1 *For $U_0 \in \mathcal{H}$ there exists a unique solution $U \in C(\mathbb{R}_+, \mathcal{H})$ to system (1.3)–(1.7). Moreover if $U_0 \in \mathcal{D}(\mathcal{A})$ then $U \in C(\mathbb{R}_+, \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H})$, where $\mathcal{H}$ is the Hilbert space defined by*

$$\mathcal{H} := H_0^1(0,l) \times L^2(0,l) \times H_*^1(0,l) \times L_*^2(0,l) \times L^2(0,l),$$

where the space $L_*^2(0,l)$ is defined as:

$$L_*^2(0,l) := \{u \in L^2(0,l) | \int_0^l u(x)dx = 0\},$$

and the space $H_*^1(0,l)$ is defined as:

$$H_*^1(0,l) := H^1(0,l) \cap L_*^2(0,l).$$

Let us first define the energy of system (1.3)–(1.7). Multiply equations (1.3), (1.4) and (1.6) by u_t, ϕ_t and w respectively we get

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\left(\rho||u_t||^2 + \mu||u_x||^2\right) + b\int_0^l \phi u_{xt} dx = 0, \\ \frac{1}{2}\frac{d}{dt}\left(J||\phi_t||^2 + \delta||\phi_x||^2 + \xi||\phi||^2\right) + b\int_0^l \phi_t u_x dx + d\int_0^l w_x \phi_t dx = 0, \\ \frac{1}{2}\frac{d}{dt}\left(\alpha||w||^2\right) + \kappa\int_0^l w_x^2 + d\int_0^l \phi_{tx} w + k\int_0^l w^2 = 0. \end{cases} \tag{2.1}$$

Sum the equations of system (2.1) we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\rho||u_t||^2 + \mu||u_x||^2 + J||\phi_t||^2 + \delta||\phi_x||^2 + \xi||\phi||^2 + \alpha||w||^2 + 2b(u_x, \phi)\right)$$
$$= -\kappa\int_0^l w_x^2 - k\int_0^l w^2.$$

Define

$$\mathbb{E}(t) = \frac{1}{2}\left(\rho||u_t||^2 + \mu||u_x||^2 + J||\phi_t||^2 + \delta||\phi_x||^2 + \xi||\phi||^2 + \alpha||w||^2 + 2b(u_x, \phi)\right).$$

Add then subtract $\frac{b^2}{2\xi}\|u_x\|^2$ to the right side of the above equation we arrive at

$$\mathbb{E}(t) = \frac{1}{2}\left(\rho||u_t||^2 + J||\phi_t||^2 + \delta||\phi_x||^2 + ||\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi||^2 + \alpha||w||^2\right)$$

where $||.||$ denotes the L^2–norm.

The dissipation law is given by

$$\frac{d}{dt}\mathbb{E}(t) = -\kappa\int_0^l w_x^2 - k\int_0^l w^2.$$

Now the exponential stability result is stated in the following theorem.

Theorem 2 *If* $\chi = 0$, *the energy* $\mathbb{E}(t)$ *of the system* (1.3)–(1.7) *decays exponentially as time* t *tends to infinity. That is, there exist two positive constants* M_1*and* ω_1*such that*

$$\mathbb{E}(t) \le M_1\mathbb{E}(0)e^{-\omega_1 t}, \quad \forall t \ge 0,$$

where

$$\chi = \frac{\mu}{\rho} - \frac{\delta}{J}.$$

The proof of Theorem 2 will be established through the following technical lemmas. First, we set

$$\mathcal{F}_1(t) = -\rho\int_0^l u_t u dx.$$

Lemma 1 *Let* (u, ϕ, w) *be a solution of the system* (1.3)–(1.7). *Then we have*

$$\frac{d}{dt}\mathcal{F}_1(t) \le -\rho\int_0^l |u_t|^2 dx + \frac{3}{2}\int_0^l \left|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right|^2 dx \\ +\xi c_p\int_0^l |\phi_x|^2 dx. \tag{2.2}$$

Proof Multiply equation (1.3) by u and integrate by parts over $(0, l)$ we get

$$\rho\int_0^l u_{tt}u dx + \mu\int_0^l |u_x|^2 dx + b\int_0^l u_x\phi dx = 0,$$

add then subtract the term $\frac{b^2}{\xi}\int_0^l |u_x|^2$ from the above equation we obtain

$$\rho\int_0^l u_{tt}u dx + \frac{b}{\sqrt{\xi}}\int_0^l \left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)u_x dx = 0,$$

taking into account that $\frac{d}{dt}(u_t u) = u_{tt}u + |u_t|^2$ we arrive at

$$\frac{d}{dt}\left(-\rho\int_0^l u_t u dx\right) = -\rho\int_0^l |u_t|^2 dx + \frac{b}{\sqrt{\xi}}\int_0^l \left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)u_x dx. \tag{2.3}$$

Apply Young's inequality and note that

$$\int_0^l |u_x|^2 dx \le 2\int_0^l \left|u_x + \frac{\xi}{b}\phi\right|^2 dx + 2c_p\int_0^l \left|\frac{\xi}{b}\phi_x\right|^2 dx \tag{2.4}$$

we get the desired result. □

Set

$$\mathcal{F}_2(t) = J\int_0^l \phi_t\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)dx - \delta\frac{\rho b}{\mu\sqrt{\xi}}\int_0^l u_{tx}\phi dx.$$

Lemma 2 *Let* (u, ϕ, w) *be a solution of the system* (1.3)–(1.7). Then we have

$$\frac{d}{dt}\mathcal{F}_2(t) \leq J\sqrt{\xi}\int_0^l |\phi_t|^2 - \frac{\sqrt{\xi}}{2}\int_0^l \left|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right|^2 dx + K_1\int_0^l |w_x|^2 dx, \tag{2.5}$$

where $K_1 = \frac{d^2}{2\sqrt{\xi}}$.

Proof Multiply equation (1.4) by $\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)$ we get:

$$\begin{aligned} &J\int_0^l \phi_{tt}\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)dx + \delta\int_0^l \phi_x\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)_x dx \\ &= -\sqrt{\xi}\int_0^l \left|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right|^2 dx - d\int_0^l w_x\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)dx. \end{aligned} \tag{2.6}$$

Using Young's inequality we obtain

$$\begin{aligned} &J\int_0^l \phi_{tt}\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)dx + \delta\int_0^l \phi_x\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)_x dx \\ &\leq -\frac{\sqrt{\xi}}{2}\int_0^l \left|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right|^2 dx + \frac{d^2}{2\sqrt{\xi}}\int_0^l |w_x|^2 dx. \end{aligned} \tag{2.7}$$

Add then subtract the term $\frac{\mu\xi}{b}\phi_x$ to equation (1.3) we get

$$\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)_x = \frac{\rho b}{\mu\sqrt{\xi}}u_{tt}. \tag{2.8}$$

Substitute $\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)_x$ in equation (2.7) we arrive at

$$\begin{aligned} &J\int_0^l \phi_{tt}\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)dx - \delta\frac{\rho b}{\mu\sqrt{\xi}}\int_0^l \phi u_{ttx}dx \\ &\leq -\frac{\sqrt{\xi}}{2}\int_0^l \left|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right|^2 dx + \frac{d^2}{2\sqrt{\xi}}\int_0^l |w_x|^2 dx. \end{aligned} \tag{2.9}$$

Taking into account that $u_{ttx}\phi = \frac{d}{dt}(u_{tx}\phi) - u_{tx}\phi_t$ and $\phi_{tt}\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right) = \frac{d}{dt}\left[\phi_t\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)\right] - \phi_t\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)_t$ we obtain

$$\begin{aligned} &\frac{d}{dt}\left(J\int_0^l \phi_t\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)dx - \delta\frac{\rho b}{\mu\sqrt{\xi}}\int_0^l \phi u_{tx}dx\right) \\ &\leq J\sqrt{\xi}\int_0^l |\phi_t|^2 + \frac{\delta b}{\sqrt{\xi}}\left(\frac{J}{\delta} - \frac{\rho}{\mu}\right)\int_0^l \phi_t u_{tx}dx \\ &\quad - \frac{\sqrt{\xi}}{2}\int_0^l \left|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right|^2 dx + K_1\frac{d^2}{2\sqrt{\xi}}\int_0^l |w_x|^2 dx. \end{aligned} \tag{2.10}$$

Set

$$\mathcal{F}_3(t) = -J\alpha\int_0^l w\left(\int_0^x \phi_t(y)dy\right)dx.$$

□

Lemma 3 *Let* (u, ϕ, w) *be a solution of the system* (1.3)–(1.7). *Then we have*

$$\frac{d}{dt}\mathcal{F}_3(t) \leq \frac{-Jd}{4}\int_0^l |\phi_t|^2 dx + (1+c_p)\epsilon_1\int_0^l |\phi_x|^2 dx + \epsilon_2 c_p\int_0^l |u_x|^2 dx$$
$$+ K_2\int_0^l |w|^2 dx + K_3\int_0^l |w_x|^2 dx. \tag{2.11}$$

Proof Multiply equation (1.5) by $J\int_0^x \phi_t(y)dy$ and integrate by parts by parts over $(0, l)$ we get

$$J\alpha\int_0^l w_t\left(\int_0^x \phi_t(y)dy\right)dx + J\kappa\int_0^l w_x\phi_t dx - Jd\int_0^l |\phi_t|^2 dx$$
$$+Jk\int_0^l w\left(\int_0^x \phi_t(y)dy\right)dx = 0. \tag{2.12}$$

Knowing that

$$\frac{d}{dt}\left[w\left(\int_0^x \phi_t(y)dy\right)\right] = w_t\left(\int_0^x \phi_t(y)dy\right) + w\left(\int_0^x \phi_{tt}(y)dy\right)$$

and by substituting ϕ_{tt} from equation (1.4) we obtain

$$\frac{d}{dt}\left[-J\alpha\int_0^l w\left(\int_0^x \phi_t(y)dy\right)dx\right] = -\delta\alpha\int_0^l \phi_x w dx + b\alpha\int_0^l uw dx + d\alpha\int_0^l |w|^2 dx + J\kappa\int_0^l w_x\phi_t dx$$
$$- Jd\int_0^l |\phi_t|^2 dx + \xi\alpha\int_0^l w\left(\int_0^x \phi(y)dy\right)dx$$
$$+ Jk\int_0^l w\left(\int_0^x \phi_t(y)dy\right)dx. \tag{2.13}$$

Apply Young's and Poincare's inequalities, we get

$$\frac{d}{dt}\left[-J\alpha\int_0^l w\left(\int_0^x \phi_t(y)dy\right)dx\right] \leq \frac{-Jd}{2}\int_0^l \Big|\phi_t|^2 dx + \epsilon_1\int_0^l \Big|\phi_x|^2 dx$$
$$+ \left(d\alpha + \frac{\delta^2\alpha^2}{\epsilon_1} + \frac{b^2\alpha^2}{\epsilon_2} + \frac{\xi^2\alpha^2}{\epsilon_3} + \frac{J^2k^2}{\epsilon_4}\right)\int_0^l |w|^2 dx$$
$$+ \frac{J\kappa^2}{2d}\int_0^l |w_x|^2 dx + \epsilon_3\int_0^l \left(\int_0^x \phi(y)dy\right)^2 dx$$
$$+ \epsilon_2 c_p\int_0^l |u_x|^2 dx + \epsilon_4\int_0^l \left(\int_0^x \phi_t(y)dy\right)^2 dx. \tag{2.14}$$

Using Cauchy–Shwarz inequality we obtain

$$\frac{d}{dt}\left[-J\alpha\int_0^l w\left(\int_0^x \phi_t(y)dy\right)dx\right] \leq \frac{-Jd}{2}\int_0^l |\phi_t|^2 dx + {}_{K_3}\frac{J\kappa^2}{2d}\int_0^l |w_x|^2 dx + \epsilon_1\int_0^l |\phi_x|^2 dx$$
$$+ {}_{K_2}\left(d\alpha + \frac{\delta^2\alpha^2}{\epsilon_1} + \frac{b^2\alpha^2}{\epsilon_2} + \frac{\xi^2\alpha^2}{\epsilon_1} + \frac{J^2k^2}{\epsilon_3}\right)\int_0^l |w|^2 dx$$
$$+ \epsilon_2 c_p\int_0^l |u_x|^2 dx + \epsilon_1\int_0^l |\phi|^2 dx$$
$$+ \epsilon_3\int_0^l |\phi_t|^2 dx. \tag{2.15}$$

Applying Poincare's inequality and taking $\epsilon_3 = \frac{Jd}{4}$ we get the desired result. $\square$

Set

$$\mathcal{F}_4(t) = J\int_0^l \phi_t \phi dx.$$

Lemma 4 *Let (u, ϕ, w) be a solution of the system (1.3)–(1.7). Then we have*

$$\frac{d}{dt}\mathcal{F}_4(t) \leq J\int_0^l |\phi_t|^2 dx - \frac{\delta}{4}\int_0^l |\phi_x|^2 dx + \frac{\xi c_p}{\delta}\int_0^l \left|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right|^2 dx + K_4\int_0^l |w|^2 dx. \tag{2.16}$$

Proof Now Multiply equation (1.4) by ϕ and integrate by parts over $(0, l)$we get

$$J\int_0^l \phi_{tt}\phi dx + \delta\int_0^l |\phi_x|^2 dx + \sqrt{\xi}\int_0^l \left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)\phi dx - d\int_0^l w\phi_x dx = 0, \tag{2.17}$$

taking into account that $\frac{d}{dt}(\phi_t\phi) = \phi_{tt}\phi + |\phi_t|^2$ we arrive at

$$\frac{d}{dt}\left(J\int_0^l \phi_t\phi dx\right) = J\int_0^l |\phi_t|^2 dx - \delta\int_0^l |\phi_x|^2 dx - \sqrt{\xi}\int_0^l \left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)\phi dx + d\int_0^l w\phi_x dx. \tag{2.18}$$

Apply Young's and Poincaré inequalities we arrive at

$$\frac{d}{dt}\left(J\int_0^l \phi_t\phi dx\right) \leq J\int_0^l |\phi_t|^2 dx - \frac{\delta}{4}\int_0^l |\phi_x|^2 dx + \frac{\xi c_p}{\delta}\int_0^l \left|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right|^2 dx + {}_{K_4}\frac{d^2}{\delta}\int_0^l |w|^2 dx. \tag{2.19}$$

$\square$

Set

$$\mathcal{F}_5(t) = \frac{\rho\delta}{b}\int_0^l u_t\phi_x dx + \frac{J\mu}{b}\int_0^l \phi_t u_x dx.$$

Lemma 5 *Let (u, ϕ, w) be a solution of the system (1.3)–(1.7). Then we have*

$$\frac{d}{dt}\mathcal{F}_5(t) \leq -\frac{\mu}{2}\int_0^l |u_x|^2 dx + (K_5 + \delta)\int_0^l |\phi_x|^2 dx + K_6\int_0^l |w_x|^2 dx. \tag{2.20}$$

Proof Differentiating $\mathcal{F}_5(t)$ with respect to t we get

$$\begin{aligned}&\frac{d}{dt}\left[\frac{\rho\delta}{b}\int_0^l u_t\phi_x dx + \frac{J\mu}{b}\int_0^l \phi_t u_x dx\right]\\ &\quad = \frac{\rho\delta}{b}\int_0^l u_{tt}\phi_x dx + \frac{J\mu}{b}\int_0^l \phi_{tt}u_x dx - \frac{J\rho}{b}\left(\frac{\mu}{\rho} - \frac{\delta}{J}\right)\int_0^l u_{tx}\phi_t dx.\end{aligned}$$

Substitute $u_{tt}$and ϕ_{tt} from equations (1.3) and (1.4) respectively we obtain

$$\frac{d}{dt}\left[\frac{\rho\delta}{b}\int_0^l u_t\phi_x dx + \frac{J\mu}{b}\int_0^l \phi_t u_x dx\right] = \frac{\mu\delta}{b}\int_0^l u_{xx}\phi_x dx + \delta\int_0^l |\phi_x|^2 dx + \frac{\mu\delta}{b}\int_0^l \phi_{xx}u_x dx$$

$$
\begin{aligned}
&-\mu\int_0^l|u_x|^2dx-\frac{\mu\xi}{b}\int_0^l u_x\phi dx-\frac{\mu d}{b}\int_0^l u_x w_x dx\\
&-\frac{J\rho}{b}\chi\int_0^l u_{tx}\phi_t dx. \qquad (2.21)
\end{aligned}
$$

Apply Young's and Poincaré inequalities we arrive at

$$
\begin{aligned}
\frac{d}{dt}\left[\frac{\rho\delta}{b}\int_0^l u_t\phi_x dx+\frac{J\mu}{b}\int_0^l \phi_t u_x dx\right]\le &-\frac{\mu}{2}\int_0^l|u_x|^2dx+\delta\int_0^l|\phi_x|^2dx\\
&+\kappa_5\frac{\mu\xi^2}{b}c_p\int_0^l|\phi_x|^2dx+\kappa_6\frac{\mu d^2}{b}\int_0^l|w_x|^2dx. \qquad (2.22)
\end{aligned}
$$

□

Let

$$
\mathbb{L}(t)=\mathbf{N}_1\mathbb{E}(t)+\mathcal{F}_1(t)+\mathbf{N}_2\mathcal{F}_2(t)+\mathbf{N}_3\mathcal{F}_3(t)+\mathbf{N}_4\mathcal{F}_4(t)+\mathcal{F}_5(t),
$$

Where $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ and $\mathbf{N}_4$ are positive constants to be fixed.

Theorem 3 *There exists positive constants $\sigma_1$and σ_2 such that*

$$
\sigma_1\mathbb{E}(t)\le\mathbb{L}(t)\le\sigma_2\mathbb{E}(t),\quad \forall t\ge 0.
$$

Proof We have

$$
\begin{aligned}
\mathbb{L}(t)-\mathbf{N}_1\mathbb{E}(t)=&-\rho\int_0^l u_t u dx+\mathbf{N}_2J\int_0^l\phi_t\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi\right)dx-\mathbf{N}_2\delta\frac{\rho b}{\mu\sqrt{\xi}}\int_0^l u_{tx}\phi dx\\
&-\mathbf{N}_3J\alpha\int_0^l w\left(\int_0^x\phi_t(y)dy\right)dx+\mathbf{N}_4J\int_0^l\phi_t\phi dx+\frac{\rho\delta}{b}\int_0^l u_t\phi_x dx\\
&+\frac{J\mu}{b}\int_0^l\phi_t u_x dx.
\end{aligned}
$$

By positivity of the constant $\mathbf{N}_2\delta\frac{\rho b}{\mu\sqrt{\xi}}$ we obtain

$$
\begin{aligned}
\mathbb{L}(t)-\mathbf{N}_1\mathbb{E}(t)\le&-\rho\int_0^l u_t u dx+\mathbf{N}_2J\int_0^l\phi_t\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi\right)dx\\
&-\mathbf{N}_3J\alpha\int_0^l w\left(\int_0^x\phi_t(y)dy\right)dx+\mathbf{N}_4J\int_0^l\phi_t\phi dx\\
&+\frac{\rho\delta}{b}\int_0^l u_t\phi_x dx+\frac{J\mu}{b}\int_0^l\phi_t u_x dx.
\end{aligned}
$$

Applying Young's and Poincaré inequalities, we obtain

$$
\begin{aligned}
|\mathbb{L}(t)-\mathbf{N}_1\mathbb{E}(t)|\le&\left(\frac{\rho}{2}+\frac{\rho\delta}{2b}\right)\int_0^l|u_t|^2dx+\left(\frac{\rho c_p}{2}+\frac{J\mu}{2b}\right)\int_0^l|u_x|^2dx\\
&+\frac{\mathbf{N}_2J}{2}\int_0^l\left|\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi\right|^2dx+\left(\frac{\mathbf{N}_4Jc_p}{2}+\frac{\rho\delta}{2b}\right)\int_0^l|\phi_x|^2dx\\
&+\left(\frac{\mathbf{N}_2J}{2}+\frac{\mathbf{N}_4J}{2}+\frac{J\mu}{2b}\right)\int_0^l|\phi_t|^2dx+\frac{\mathbf{N}_3J\alpha}{2}\int_0^l\left(\int_0^x\phi_t(y)dy\right)^2dx
\end{aligned}
$$

$$+\frac{\mathbf{N}_3 J\alpha}{2}\int_0^l |w|^2 dx.$$

Knowing that $\mu = \frac{b^2}{\xi}$ we get

$$\left(\frac{\rho c_p}{2}+\frac{J\mu}{2b}\right)\int_0^l |u_x|^2 dx = \left(\frac{\rho c_p}{2\mu}+\frac{J}{2b}\right)\int_0^l \left|\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi-\sqrt{\xi}\phi\right|^2 dx$$
$$\leq \left(\frac{\rho c_p}{\mu}+\frac{J}{b}\right)\int_0^l \left|\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi\right|^2 dx+\left(\frac{\rho c_p}{\mu}+\frac{J}{b}\right)\int_0^l |\sqrt{\xi}\phi|^2 dx, \tag{2.23}$$

Using (2.23) and Cauchy–Shwarz inequality we get

$$\begin{aligned}|\mathbb{L}(t)-\mathbf{N}_1\mathbb{E}(t)| \leq& \left(\frac{\rho}{2}+\frac{\rho\delta}{2b}\right)\int_0^l |u_t|^2 dx+\left(\frac{\mathbf{N}_2 J}{2}+\frac{\rho c_p}{\mu}+\frac{J}{b}\right)\int_0^l \left|\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi\right|^2 dx\\ &+\left(\frac{\mathbf{N}_2 J}{2}+\frac{\mathbf{N}_4 J}{2}+\frac{J\mu}{2b}\right)\int_0^l |\phi_t|^2 dx+\frac{\mathbf{N}_3 J\alpha}{2}\int_0^l |\phi_t|^2 dx\\ &+\left(\frac{\mathbf{N}_4 J c_p}{2}+\frac{\rho\delta}{2b}+\frac{\xi\rho c_p^2}{\mu}+\frac{\xi J c_p}{b}\right)\int_0^l |\phi_x|^2 dx+\frac{\mathbf{N}_3 J\alpha}{2}\int_0^l |w|^2 dx.\end{aligned}$$

Define

$$\begin{aligned}\mathbf{N}_0 :=\max\Bigg\{&\frac{\rho}{2}+\frac{\rho\delta}{2b};\frac{\mathbf{N}_2 J}{2}+\frac{\rho c_p}{\mu}+\frac{J}{b};\frac{\mathbf{N}_2 J}{2}+\frac{\mathbf{N}_4 J}{2}+\frac{J\mu}{2b};\frac{\mathbf{N}_3 J\alpha}{2};\frac{\mathbf{N}_4 J c_p}{2}\\ &+\frac{\rho\delta}{2b}+\frac{\xi\rho c_p^2}{\mu}+\frac{\xi J c_p}{b}\Bigg\}.\end{aligned}$$

Hence

$$|\mathbb{L}(t)-\mathbf{N}_1\mathbb{E}(t)|\leq \mathbf{N}_0\mathbb{E}(t),$$

which implies that

$$\sigma_1\mathbb{E}(t)\leq \mathbb{L}(t)\leq \sigma_2\mathbb{E}(t),$$

where $\sigma_1=\mathbf{N}_1-\mathbf{N}_0$ and $\sigma_2=\mathbf{N}_1+\mathbf{N}_0$ and $\mathbf{N}_1>\mathbf{N}_0$. □

Proof of Theorem 2 It follows from Lemmas 1, 2 3, 4, 5 that

$$\begin{aligned}\frac{d}{dt}\mathbb{L}(t)\leq& -\rho\int_0^l |u_t|^2 dx-{}_{\eta_1}\left(\frac{\mu}{2}-\mathbf{N}_3\epsilon_2 c_p\right)\int_0^l |u_x|^2 dx\\ &-{}_{\eta_2}\left(\mathbf{N}_3\frac{Jd}{4}-\mathbf{N}_4 J-\mathbf{N}_2 J\sqrt{\xi}\right)\int_0^l |\phi_t|^2 dx-{}_{\eta_3}\left(\mathbf{N}_2\frac{\sqrt{\xi}}{2}-\mathbf{N}_4\frac{\xi c_p}{\delta}-\frac{3}{2}\right)\int_0^l \left|\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi\right|^2 dx\\ &-{}_{\eta_4}\left(\mathbf{N}_4\frac{\delta}{4}-\mathbf{N}_3(1+c_p)\epsilon_1-(\xi c_p+\delta+K_5)\right)\int_0^l |\phi_x|^2 dx\\ &-{}_{\eta_5}(\mathbf{N}_1 k-\mathbf{N}_4 K_4-\mathbf{N}_3 K_2)\int_0^l |w|^2 dx-{}_{\eta_6}(\mathbf{N}_1\kappa-\mathbf{N}_2 K_1-\mathbf{N}_3 K_3-K_6)\int_0^l |w_x|^2 dx\end{aligned} \tag{2.24}$$

Choose $\epsilon_2=\frac{b^2}{2c_p\xi\mathbf{N}_3}$, we get that $\eta_1=0$,
now take

$$N_2>\frac{2\left(\mathbf{N}_4\frac{\xi c_p}{\delta}+\frac{3}{2}\right)}{\sqrt{\xi}},$$

which implies that $\eta_3 > 0$.

Choose

$$\epsilon_1 = \frac{1}{\mathbf{N}_3(1 + c_p)},$$

then take

$$N_4 > \frac{4(1 + \xi c_p + \delta + K_5)}{\delta},$$

then we obtain that $\eta_4 > 0$.

Now take

$$\mathbf{N}_3 > \frac{4(\mathbf{N}_4 + \mathbf{N}_2\sqrt{\xi})}{d},$$

hence we obtain that $\eta_2 > 0$.

Take

$$\mathbf{N}_1 > \max\left\{\frac{\mathbf{N}_4 K_4 + \mathbf{N}_3 K_2}{k}; \frac{\mathbf{N}_2 K_1 + \mathbf{N}_3 k_3 + K_6}{\kappa}\right\},$$

from where we obtain that $\eta_5, \eta_6 > 0$. Therefore we can conclude that there exists a positive constant $\boldsymbol{\eta} = 2\min\{1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6\}$ such that

$$\frac{d}{dt}\mathbb{L}(t) \leq -\boldsymbol{\eta}\mathbb{E}(t),$$

by equivalence between $\mathbb{E}(t)$ and $\mathbb{L}(t)$ according to Theorem 3 we get:

$$\frac{d}{dt}\mathbb{L}(t) \leq -\omega_1\mathbb{L}(t),$$

where $\omega_1 = \frac{\boldsymbol{\eta}}{\sigma_2}$. Now integrate the above inequality over $(0, t)$ we obtain

$$\mathbb{L}(t) \leq \mathbb{L}(0)e^{-\omega_1 t},$$

again by equivalence between $\mathbb{E}(t)$ and $\mathbb{L}(t)$ according to Theorem 3 we arrive at

$$\mathbb{E}(t) \leq M_1\mathbb{E}(0)e^{-\omega_1 t},$$

where $M_1 = \frac{\sigma_2}{\sigma_1}$. □

3 Second spectrum approach

In this section, we consider the porous system with second spectrum free and microtemperature:

$$\rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \quad (x, t) \in (0, l) \times (0.\infty), \tag{3.1}$$

$$-J u_{ttx} - \delta\phi_{xx} + bu_x + \xi\phi + dw_x = 0, \quad (x, t) \in (0, l) \times (0.\infty), \tag{3.2}$$

$$\alpha w_t - \kappa w_{xx} + d\phi_{tx} + kw = 0, \quad (x, t) \in (0, l) \times (0.\infty), \tag{3.3}$$

with Dirichlet boundary conditions

$$u(0, t) = u(l, t) = \phi(0, t) = \phi(l, t) = w(0, t) = w(l, t) = 0, t > 0, \tag{3.4}$$

and the initial conditions are

$$\begin{aligned} &u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad u_{tt}(x,0)=u_2(x),\\ &\phi(x,0)=\phi_0(x), \quad w(x,0)=w_0(x), \quad x\in(0,l). \end{aligned} \tag{3.5}$$

This system (3.1)–(3.5) is obtained by following the procedure of Elishakoff [28], which involves replacing the term ϕ_{tt} in (1.4) by $-u_{xtt}$ based on d' Alembert's principle for dynamic equilibrium. This eliminates the second spectrum of frequency and its damaging consequences for wave propagation speed. This work aims to get exponential decay without assuming any conditions on the physical parameters.

The dissipation of system (3.1)–(3.5) is obtained from the definition of energy. Indeed, multiply equation (3.1) by u_t, integrate by parts over $(0,l)$ and using boundary conditions (3.4) we get

$$\frac{1}{2}\frac{d}{dt}\left(\rho||u_t||^2+\mu||u_x||^2\right)+b\int_0^l \phi u_{xt}dx=0. \tag{3.6}$$

Multiply equation (3.2) by ϕ_t, integrate by parts over $(0,l)$ and using boundary conditions (3.4) we get

$$\frac{1}{2}\frac{d}{dt}\left(\delta||\phi_x||^2+\xi||\phi||^2\right)+b\int_0^l \phi_t u_x dx+J\int_0^l u_{tt}\phi_{tx}dx+d\int_0^l w_x\phi_t dx=0. \tag{3.7}$$

From equation (3.1) we obtain that

$$\phi_{xt}=\frac{\rho}{b}u_{ttt}-\frac{\mu}{b}u_{xxt},$$

substitute ϕ_{xt} in equation (3.7) we arrive at

$$\frac{1}{2}\frac{d}{dt}\left(\delta||\phi_x||^2+\xi||\phi||^2+\frac{J\rho}{b}\|u_{tt}\|^2+\frac{J\mu}{b}\|u_{xt}\|^2\right)+b\int_0^l \phi_t u_x dx+d\int_0^l w_x\phi_t dx=0. \tag{3.8}$$

Now multiply equation (3.3) by w, integrate by parts over $(0,l)$ and using boundary conditions (3.4) we get

$$\frac{1}{2}\frac{d}{dt}\left(\alpha||w||^2\right)+\kappa\int_0^l w_x^2+d\int_0^l \phi_{tx}w+k\int_0^l w^2=0. \tag{3.9}$$

Add equations (3.6), (3.8) and (3.9) we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\rho||u_t||^2+\mu||u_x||^2+\delta||\phi_x||^2+\xi||\phi||^2+\frac{J\rho}{b}\|u_{tt}\|^2+\frac{J\mu}{b}\|u_{xt}\|^2+\alpha||w||^2+2b(u_x,\phi)\right)\\ &\quad=-\kappa\int_0^l w_x^2-k\int_0^l w^2. \end{aligned}$$

Define

$$E(t)=\frac{1}{2}\left(\rho||u_t||^2+\mu||u_x||^2+\delta||\phi_x||^2+\xi||\phi||^2+\frac{J\rho}{b}\|u_{tt}\|^2+\frac{J\mu}{b}\|u_{xt}\|^2+\alpha||w||^2+2b(u_x,\phi)\right).$$

Add then subtract $\frac{b^2}{2\xi}\|u_x\|^2$ to the right side of the above equation we arrive at

$$E(t)=\frac{1}{2}\left(\rho||u_t||^2+\frac{J\rho}{b}||u_{tt}||^2+(\mu-b^2/\xi)||u_x||^2+\frac{J\mu}{b}||u_{xt}||^2+\delta||\phi_x||^2+||\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi||^2+\alpha||w||^2\right)$$

where $||.||$ denotes the L^2–norm.

The dissipation law is given by

$$\frac{d}{dt}E(t) = -\kappa \int_0^l w_x^2 - k \int_0^l w^2.$$

3.1 Well-Posedness

This section aims to show the existence and uniqueness of the weak solution of system (3.1)–(3.5). Therefore, we will use the classical Faedo–Galerkin approximation and a priori estimates, then pass the limits using compactness arguments.

Define the Hilbert space

$$\mathcal{H} := H_0^1(0,l) \times H_0^1(0,l) \times L^2(0,l) \times H_0^1(0,l) \times L^2(0,l).$$

Now multiply the equations (3.1), (3.2) and (3.3) by $\overline{u}, \overline{\phi}, \overline{w} \in H_0^1(0,l)$ respectively and integrate by parts over $(0,l)$ we get using boundary conditions (3.4)

$$\begin{cases} \rho(u_{tt}, \overline{u}) + \mu(u_x, \overline{u}_x) + b(\phi, \overline{u}_x) = 0, \\ J(u_{tt}, \overline{\phi}_x) + \delta(\phi_x, \overline{\phi}_x) + b(u_x, \overline{\phi}) + \xi(\phi, \overline{\phi}) + d(w_x, \overline{\phi}) = 0, \\ \alpha(w_t, \overline{w}) + \kappa(w_x, \overline{w}_x) + d(\phi_{tx}, \overline{w}) + k(w, \overline{w}) = 0. \end{cases} \tag{3.10}$$

Definition 1 Let the initial data $(u_0, u_1, u_2, \phi_0, w_0) \in \mathcal{H}$ then a function $V = (u, u_t, u_{tt}, \phi, w) \in C(0, T; \mathcal{H})$ is said to be a weak solution of (3.1)–(3.5) if it is a solution of the weak problem (3.10) for almost $t \in [0, T]$.

Theorem 4 *Suppose that the initial data $(u_0, u_1, u_2, \phi_0, w_0) \in \mathcal{H}$ then system (3.1)–(3.5) have a weak solution satisfying*

$$u \in L^\infty\big(0, T; H_0^1(0,l)\big), u_t \in L^\infty\big(0, T; H_0^1(0,l)\big),$$

$$u_{tt} \in L^\infty\big(0, T; L^2(0,l)\big), \phi \in L^\infty\big(0, T; H_0^1(0,l)\big),$$

$$w \in L^\infty\big(0, T; L^2(0,l)\big) \cap L^2\big(0, T; H_0^1(0,l)\big),$$

where the solution $V = (u, u_t, u_{tt}, \phi, w)$ depends continuously on the initial data in $\mathcal{H}$. In particular V is unique solution of system (3.1)–(3.5).

Proof We will use the Faedo-Galerkin method to prove the above theorem and proceed in five steps. **Step 1. Approximated solution** Let $(u_0, u_1, u_2, \phi_0, w_0) \in \mathcal{H}$. Let $\{\eta_i\}_{i=1}^\infty \subset C^\infty([0,l])$ be basis for $H_0^1(0,l)$, and let $V^m = span\{\eta_i\}_{i=1}^m$. Now we introduce

$$u^m = \sum_{i=0}^m a_i(t)\eta_i(x), \ \phi^m = \sum_{i=0}^m b_i(t)\eta_i(x), \ w^m = \sum_{i=0}^m c_i(t)\eta_i(x), \tag{3.11}$$

which solves the following approximated problem for $\overline{u}, \overline{\phi}, \overline{w} \in V^m$

$$\begin{cases} \rho(u_{tt}^m, \overline{u}) + \mu(u_x^m, \overline{u}_x) + b(\phi^m, \overline{u}_x) = 0, \\ J(u_{tt}^m, \overline{\phi}_x) + \delta(\phi_x^m, \overline{\phi}_x) + b(u_x^m, \overline{\phi}) + \xi(\phi^m, \overline{\phi}) + d(w_x^m, \overline{\phi}) = 0, \\ \alpha(w_t^m, \overline{w}) + \kappa(w_x^m, \overline{w}_x) + d(\phi_{tx}^m, \overline{w}) + k(w^m, \overline{w}) = 0. \end{cases} \tag{3.12}$$

with initial conditions

$$\big(u^m(0), u_t^m(0), u_{tt}^m(0), \phi^m(0), w^m(0)\big) = (u_0^m, u_1^m, u_2^m, \phi_0^m, w_0^m),$$

such that

$$(u_0^m, u_1^m, u_2^m, \phi_0^m, w_0^m) \to (u_0, u_1, u_2, \phi_0, w_0) strongly in \mathcal{H}.$$

By using the Carathoedory theorem for standard ordinary differential equations theory, system (3.12) has a local solution $\big(u^m(t), u_t^m(t), u_{tt}^m(t), \phi^m(t), w^m(t)\big)$ on the maximal interval $[0, t_m)$ with $0 < t_m \leq T$ for every $m \in \mathbb{N}$

Step 2. A priori estimates

Let $\overline{u} = u_t^m, \overline{\phi} = \phi_t^m$ and $\overline{w} = w^m$ and taking into consideration from equation (3.1) that

$$\phi_{xt} = \frac{\rho}{b} u_{ttt} - \frac{\mu}{b} u_{xxt},$$

then system (3.12) becomes

$$\begin{cases} \rho(u_{tt}^m, u_t^m) + \mu(u_x^m, u_{tx}^m) + b(\phi^m, u_{tx}^m) = 0 \\ \dfrac{J\rho}{b}\big(u_{tt}^m, u_{ttt}^m\big) + \dfrac{J\mu}{b}\big(u_{ttx}^m, u_{tx}^m\big) + \delta(\phi_x^m, \phi_{tx}^m) + b(u_x^m, \phi_t^m) + \xi(\phi^m, \phi_t^m) + d(w_x^m, \phi_t^m) = 0 \\ \alpha(w_t^m, w^m) + \kappa(w_x^m, w_x^m) + d(\phi_{tx}^m, w^m) + k(w^m, w^m) = 0, \end{cases} \tag{3.13}$$

which is equivalent to

$$\begin{cases} \dfrac{d}{2dt}\big(\rho||u_t^m||^2\big) + \dfrac{d}{2dt}\big(\mu||u_x^m||^2\big) + b(\phi^m, u_{tx}^m) = 0 \\ \dfrac{d}{2dt}\left(\dfrac{J\rho}{b}\|u_{tt}^m\|^2\right) + \dfrac{d}{2dt}\left(\dfrac{J\mu}{b}\|u_{tx}^m\|^2\right) + \dfrac{d}{2dt}\big(\delta||\phi_x^m||^2\big) + b(u_x^m, \phi_t^m) \\ + \dfrac{d}{2dt}\big(\xi||\phi^m||^2\big) + d(w_x^m, \phi_t^m) = 0 \\ \dfrac{d}{2dt}\big(\alpha||w^m||^2\big) + \kappa||w_x^m||^2 + d(\phi_{tx}^m, w^m) + k||w^m||^2 = 0, \end{cases} \tag{3.14}$$

where $||.||$ denotes the norm in $L^2(0, l)$.

Add the above two equations we get

$$\begin{aligned} &\frac{d}{2dt}\Bigg(\rho||u_t^m||^2 + \mu||u_x^m||^2 + \frac{J\rho}{b}\|u_{tt}^m\|^2 + \frac{J\mu}{b}\|u_{tx}^m\|^2 + \delta||\phi_x^m||^2 \\ &+ \xi||\phi^m||^2 + 2b(u_x^m, \phi^m) + \alpha||w^m||^2\Bigg) + \kappa||w_x^m||^2 + k||w^m||^2 = 0. \end{aligned} \tag{3.15}$$

Let

$$E^m(t) = \frac{1}{2}\left(\rho||u_t^m||^2 + \mu||u_x^m||^2 + \frac{J\rho}{b}\|u_{tt}^m\|^2 + \frac{J\mu}{b}\|u_{tx}^m\|^2 + \delta||\phi_x^m||^2 + \xi||\phi^m||^2 + 2b(u_x^m, \phi^m) + \alpha||w^m||^2\right).$$

Then equation (3.15) becomes

$$\frac{d}{dt}E^m(t) + \kappa||w_x^m||^2 + k||w^m||^2 = 0. \tag{3.16}$$

Now integrate (3.16) from 0 to $t < t_m$, we obtain from the choice of the initial data that for all $t \in [0, T]$ and for every $m \in \mathbb{N}$ that

$$E^m(t) + \kappa\int_0^t ||w_x^m(s)||^2 ds + k\int_0^t ||w^m(s)||^2 ds \leq C_0, \tag{3.17}$$

where C_0 is a positive constant depending on the initial data.

Step 3. Passing to the limit. Using (3.17) and by the definition of $E^m(t)$ we obtain that

$$\begin{cases} \{u^m\} \quad is\, bounded\, in \quad L^\infty\big(0,T;H_0^1(0,l)\big), \\ \{u_t^m\} \quad is\, bounded\, in \quad L^\infty\big(0,T;H_0^1(0,l)\big), \\ \{u_{tt}^m\} \quad is\, bounded\, in \quad L^\infty\big(0,T;L^2(0,l)\big), \\ \{\phi^m\} \quad is\, bounded\, in \quad L^\infty\big(0,T;H_0^1(0,l)\big), \\ \{w^m\} \quad is\, bounded\, in \quad L^\infty\big(0,T;L^2(0,l)\big)\cap L^2\big(0,T;H_0^1(0,l)\big). \end{cases}$$

Then we can extract a subsequence of $\{u^m\}$, $\{\phi^m\}$ and $\{w^m\}$ and still denoted by $\{u^m\}$, $\{\phi^m\}$ and $\{w^m\}$, such that

$$\begin{cases} u^m \to u\, weakly\, star\, in\, L^\infty\big(0,T;H_0^1(0,l)\big), \\ u_t^m \to u_t \quad weakly\, star\, in \quad L^\infty\big(0,T;H_0^1(0,l)\big), \\ u_{tt}^m \to u_{tt} \quad weakly\, star\, in \quad L^\infty\big(0,T;L^2(0,l)\big), \\ \phi^m \to \phi \quad weakly\, star\, in \quad L^\infty\big(0,T;H_0^1(0,l)\big), \\ w^m \to w \quad weakly\, star\, in \quad L^\infty\big(0,T;L^2(0,l)\big), \\ w^m \to w \quad weakly\, in \quad L^2\big(0,T;H_0^1(0,l)\big). \end{cases}$$

Now pass to the limits in the approximate variational problem (3.12) we get a weak solution satisfying

$$u \in L^\infty\big(0,T;H_0^1(0,l)\big), \quad u_t \in L^\infty\big(0,T;H_0^1(0,l)\big),$$

$$u_{tt} \in L^\infty\big(0,T;L^2(0,l)\big), \quad \phi \in L^\infty\big(0,T;H_0^1(0,l)\big),$$

$$w \in L^\infty\big(0,T;L^2(0,l)\big)\cap L^2\big(0,T;H_0^1(0,l)\big).$$

Step 4. Initial data. Knowing that

$$H_0^1(0,l) \subset L^2(0,l) \subset H^{-1}(0,l),$$

where $H^{-1}(0,l)$ is the dual space of $H_0^1(0,l)$.

By using Aubin-Lions lemma, see [39], we obtain that $L^\infty\big(0,T;H_0^1(0,l)\big)$ is compactly embedded in $C\big(0,T;L^2(0,l)\big)$. This implies that

$$u^m \to u \quad strongly\, in \quad C\big(0,T;L^2(0,l)\big),$$

$$u_t^m \to u_t \quad strongly\, in \quad C\big(0,T;L^2(0,l)\big),$$

Hence,

$$(u(0),u_t(0)) = (u_0,u_1).$$

Now differentiate with respect to t the first equation of system (3.12) we get

$$\rho(u_{ttt}^m,\overline{u}) + \mu(u_{tx}^m,\overline{u}_x) + b(\phi_t^m,\overline{u}_x) = 0,$$

for all $\overline{u} \in H_0^1(0,l)$.

Multiply the above equation by a test function

$$\lambda \in H_0^1(0,T), \quad such\, that \quad \lambda(0)=1, \lambda(T)=0,$$

and then integrate by parts over $[0, T]$

$$-\rho(u_2^m,\overline{u})-\rho\int_0^T(u_{tt}^m,\overline{u})\lambda_t dt+\mu\int_0^T(u_{tx}^m,\overline{u}_x)\lambda dt+b\int_0^T(\phi_t^m,\overline{u}_x)\lambda dt=0.$$

Take the limit $m\to\infty$, we arrive at

$$-\rho(u_2,\overline{u})-\rho\int_0^T(u_{tt},\overline{u})\lambda_t dt+\mu\int_0^T(u_{tx},\overline{u}_x)\lambda dt+b\int_0^T(\phi_t,\overline{u}_x)\lambda dt=0. \tag{3.18}$$

Now differentiate the first equation of system (3.10) with respect to time, then multiply the result by λ under the same conditions above and integrate by parts over $[0, T]$ we get

$$-\rho(u_{tt}(0),\overline{u})-\rho\int_0^T(u_{tt},\overline{u})\lambda_t dt+\mu\int_0^T(u_{tx},\overline{u}_x)\lambda dt+b\int_0^T(\phi_t,\overline{u}_x)\lambda dt=0. \tag{3.19}$$

Combine the two equations (3.18) and (3.19) we obtain that $u_{tt}(0)=u_2$. In the same way we can get that $(\phi(0), w(0))=(\phi_0, w_0)$.

Step 5. Continuous dependence on initial data. Let $V_1(t)=(u, u_t, u_{tt}, \phi, w)$ and $V_2(t)=(\widetilde{u}, \widetilde{u}_t, \widetilde{u}_{tt}, \widetilde{\phi}, \widetilde{w})$ be two solutions of the system (3.1)–(3.4) with initial data $V_1(0)=(u_0, u_1, u_2, \phi_0, w_0)$ and $V_2(0)=(\widetilde{u}_0, \widetilde{u}_1, \widetilde{u}_1, \widetilde{\phi}_0, \widetilde{w}_0)$ such that $V_1(0), V_2(0)\in\mathcal{H}$. Then $(U, U_t, U_{tt}, \Phi, W)=V_1(t)-V_2(t)$ satisfies the following equations

$$\rho U_{tt}-\mu U_{xx}-b\Phi_x=0, in(0,l)\times(0.\infty), \tag{3.20}$$

$$-JU_{ttx}-\delta\Phi_{xx}+bU_x+\xi\Phi+dW_x=0, in(0,l)\times(0.\infty), \tag{3.21}$$

$$\alpha W_t-\kappa W_{xx}+d\Phi_{tx}+kW=0, in(0,l)\times(0.\infty), \tag{3.22}$$

with initial data $(U_0, U_1, U_2, \Phi_0, W_0)=V_1(0)-V_2(0)$.

Now multiply (3.20) by U_t, (3.21) by Φ_t and (3.22) by W then integrate the result over $(0, l)$ we arrive at

$$\frac{d}{dt}\widehat{E}(t)=-\kappa\int_0^l W_x^2 dx-k\int_0^l W^2 dx, \tag{3.23}$$

where $\widehat{E}(t)$ is the energy related to $V_1(t)-V_2(t)$ and defined by

$$\widehat{E}(t)=\frac{1}{2}\bigg(\rho||U_t||^2+\frac{J\rho}{b}||U_{tt}||^2+(\mu-b^2/\xi)||U_x||^2+\frac{J\mu}{b}||U_{xt}||^2+\delta||\Phi_x||^2+||\frac{b}{\sqrt{\xi}}U_x \\ +\sqrt{\xi}\Phi||^2+\alpha||W||^2\bigg).$$

Integrate (3.23) over $(0, t)$, w get that there exists a positive constant C_T such that for any $t\in[0, T]$,

$$\widehat{E}(t)\le C_T\widehat{E}(0),$$

which implies that the weak solution depends continuously on the initial data. Consequently the weak solution of system (3.1)–(3.5) is unique. □

3.2 Exponential stability

This section will prove the exponential decay of the system's energy (3.1)–(3.5). We will study two cases, first, the case when $\mu\xi>b^2$, and second, the case when $\mu\xi=b^2$. The method of proof is based on multipliers techniques.

3.2.1 First case:

We will study the exponential decay when $\mu\xi > b^2$, and our result is stated in the following theorem.

Theorem 5 *The energy $E(t)$ of the system (3.1)–(3.5) decays exponentially as time t tends to infinity. That is, there exist two positive constants M_2 and ω_2 independent of the initial data and independent of any relationship between coefficients such that.*

$$E(t) \le M_2 E(0) e^{-\omega_2 t}, \forall t \ge 0.$$

The proof of Theorem 5 will be established through two lemmas. First, we set

$$\mathcal{F}(t) = \rho \int_0^l u_t u dx + \frac{J\mu}{b}\int_0^l u_{xt} u_x dx.$$

Lemma 6 *Let (u, ϕ, w) be a solution of the system (3.1–3.5). Then we have*

$$\begin{aligned} \frac{d}{dt}\mathcal{F}(t) \le \left(\frac{J\mu}{b} + 2\rho c_p\right)\int_0^l |u_{xt}|^2 dx - \rho\int_0^l |u_t|^2 dx - (\mu - b^2/\xi)\int_0^l |u_x|^2 dx - \frac{\delta}{2}\int_0^L |\phi_x|^2 dx \\ - \frac{J\rho}{b}\int_0^l |u_{tt}|^2 dx - \int_0^l \left|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right|^2 dx + \frac{d^2}{2\delta}\int_0^l |w|^2 dx. \end{aligned} \tag{3.1}$$

Proof Multiply equation (3.1) by u and integrate by parts over $(0, l)$ we get

$$\rho\int_0^l u_{tt} u dx + \mu\int_0^l |u_x|^2 dx + b\int_0^l u_x \phi dx = 0,$$

add then subtract the term $\frac{b^2}{\xi}\int_0^l |u_x|^2$ from the above equation we obtain

$$\rho\int_0^l u_{tt} u dx + (\mu - b^2/\xi)\int_0^l |u_x|^2 dx + \frac{b}{\sqrt{\xi}}\int_0^l \left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right) u_x dx = 0,$$

taking into account that $\frac{d}{dt}(u_t u) = u_{tt}u + |u_t|^2$ we arrive at

$$\frac{d}{dt}\left(\rho\int_0^l u_t u dx\right) = \rho\int_0^l |u_t|^2 dx - (\mu - b^2/\xi)\int_0^l |u_x|^2 dx - \frac{b}{\sqrt{\xi}}\int_0^l \left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right) u_x dx. \tag{3.2}$$

Multiply equation (3.2) by ϕ and integrate by parts over $(0, l)$ we get

$$J\int_0^l u_{tt}\phi_x dx + \delta\int_0^l |\phi_x|^2 dx + \sqrt{\xi}\int_0^l \left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)\phi dx + d\int_0^l w_x \phi dx = 0 \tag{3.3}$$

From equation (3.1) we get that $\phi_x = \frac{\rho}{b}u_{tt} - \frac{\mu}{b}u_{xx}$, then substitute ϕ_x in equation (3.3) and taking into account that $\frac{d}{dt}(u_{tx}u_x) = u_{ttx}u_x + |u_{tx}|^2$ we obtain

$$\begin{aligned} \frac{d}{dt}\left(\frac{J\mu}{b}\int_0^l u_{xt} u_x dx\right) - \frac{J\mu}{b}\int_0^l |u_{tx}|^2 dx + \frac{J\rho}{b}\int_0^l |u_{tt}|^2 dx + \delta\int_0^l |\phi_x|^2 dx \\ + \sqrt{\xi}\int_0^l \left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)\phi dx - d\int_0^l w\phi_x dx = 0, \end{aligned}$$

Using Poincaré and Young's inequality, we get

$$\frac{d}{dt}\left(\frac{J\mu}{b}\int_0^l u_{xt}u_x dx\right) \leq \frac{J\mu}{b}\int_0^l |u_{tx}|^2 dx - \frac{J\rho}{b}\int_0^l |u_{tt}|^2 dx - \frac{\delta}{2}\int_0^l |\phi_x|^2 dx$$
$$- \sqrt{\xi}\int_0^l \left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)\phi dx + \frac{d^2}{2\delta}\int_0^l |w|^2 dx. \tag{3.4}$$

Add the two equations (3.2) and (3.4) we obtain

$$\frac{d}{dt}\left(\rho\int_0^l u_t u dx + \frac{J\mu}{b}\int_0^l u_{xt}u_x dx\right) \leq \frac{J\mu}{b}\int_0^l |u_{xt}|^2 dx + \rho\int_0^l |u_t|^2 dx$$
$$- (\mu - b^2/\xi)\int_0^l |u_x|^2 dx$$
$$- \frac{J\rho}{b}\int_0^l |u_{tt}|^2 dx - \int_0^l \left|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right|^2 dx$$
$$+ \frac{d^2}{2\delta}\int_0^l |w|^2 dx - \frac{\delta}{2}\int_0^L |\phi_x|^2 dx.$$

Add and subtract the term $\rho\int_0^l |u_t|^2 dx$ to the right side of the above inequality, then use Poincaré inequality we get the desired result. □

Set

$$\mathcal{G}(t) = -J\int_0^l u_{tx}\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)dx - \delta\frac{\rho b}{\mu\sqrt{\xi}}\int_0^l u_{tx}\phi dx + \frac{\alpha C_1}{d}\int_0^l w u_t dx.$$

Lemma 7 *Let (u, ϕ, w)be a solution of the system (3.1)–(3.5). Then we have*

$$\frac{d}{dt}\mathcal{G}(t) \leq -\frac{Jb}{2\sqrt{\xi}}\int_0^l |u_{xt}|^2 dx + \frac{\varepsilon_3}{2}\int_0^l |u_t|^2 dx + \frac{\varepsilon_1}{2}\int_0^l |u_{tt}|^2 dx - \delta(\mu - b^2/\xi)\frac{\sqrt{\xi}}{\mu}\int_0^L |\phi_x|^2 dx$$
$$- \frac{\sqrt{\xi}}{2}\int_0^l \left|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right|^2 dx + C_2\int_0^l |w|^2 dx + C_3\int_0^l |w_x|^2 dx. \tag{3.5}$$

where C_1, C_2 and C_3 are positive constant to be determined.

Proof Multiply equation (3.2) by $\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)$ we get:

$$- J\int_0^l u_{ttx}\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)dx + \delta\int_0^l \phi_x\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)_x dx$$
$$= -\sqrt{\xi}\int_0^l \left|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right|^2 dx - d\int_0^l w_x\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)dx. \tag{3.6}$$

Using Young's inequality we obtain

$$- J\int_0^l u_{ttx}\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)dx + \delta\int_0^l \phi_x\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)_x dx$$
$$\leq -\frac{\sqrt{\xi}}{2}\int_0^l \left|\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right|^2 dx + \frac{d^2}{2\sqrt{\xi}}\int_0^l |w_x|^2 dx. \tag{3.7}$$

Add then subtract the term $\frac{\mu\xi}{b}\phi_x$ to equation (3.1) we get

$$\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi\right)_x=\frac{\rho b}{\mu\sqrt{\xi}}u_{tt}+(\mu-b^2/\xi)\frac{\sqrt{\xi}}{\mu}\phi_x. \tag{3.8}$$

Substitute $\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi\right)_x$ in equation (3.7) we arrive at

$$\begin{aligned}&-J\int_0^l u_{ttx}\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi\right)dx-\delta\frac{\rho b}{\mu\sqrt{\xi}}\int_0^l \phi u_{ttx}dx\\&\quad\le-\delta(\mu-b^2/\xi)\frac{\sqrt{\xi}}{\mu}\int_0^L|\phi_x|^2dx-\frac{\sqrt{\xi}}{2}\int_0^l\left|\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi\right|^2dx+\frac{d^2}{2\sqrt{\xi}}\int_0^l|w_x|^2dx.\end{aligned} \tag{3.9}$$

Taking into account that $u_{ttx}\phi=\frac{d}{dt}(u_{tx}\phi)-u_{tx}\phi_t$ and $u_{ttx}\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi\right)=\frac{d}{dt}\left[u_{tx}\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi\right)\right]-u_{tx}\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi\right)_t$ we obtain

$$\begin{aligned}&\frac{d}{dt}\left(-J\int_0^l u_{tx}\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi\right)dx-\delta\frac{\rho b}{\mu\sqrt{\xi}}\int_0^l\phi u_{tx}dx\right)\\&\quad\le-\frac{Jb}{\sqrt{\xi}}\int_0^l|u_{tx}|^2-c_1\left(J\sqrt{\xi}+\delta\frac{\rho b}{\mu\sqrt{\xi}}\right)\int_0^l\phi_t u_{tx}dx-\delta(\mu-b^2/\xi)\frac{\sqrt{\xi}}{\mu}\int_0^L|\phi_x|^2dx\\&\qquad-\frac{\sqrt{\xi}}{2}\int_0^l\left|\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi\right|^2dx+\frac{d^2}{2\sqrt{\xi}}\int_0^l|w_x|^2dx.\end{aligned} \tag{3.10}$$

Now multiply equation (3.3) by $\frac{C_1}{d}u_t$, integrate by parts over $(0,l)$we get and using boundary conditions (3.4) we have

$$\frac{\alpha C_1}{d}\int_0^l w_t u_t dx+\frac{\kappa C_1}{d}\int_0^l w_x u_{tx}dx-C_1\int_0^l\phi_t u_{tx}dx+\frac{kC_1}{d}\int_0^l wu_t dx=0.$$

Taking into account that $w_tu_t=\frac{d}{dt}(wu_t)-wu_{tt}$we obtain

$$\begin{aligned}\frac{d}{dt}\left(\frac{\alpha C_1}{d}\int_0^l wu_t dx\right)&=\frac{\alpha C_1}{d}\int_0^l wu_{tt}dx-\frac{\kappa C_1}{d}\int_0^l w_x u_{tx}dx\\&\quad+C_1\int_0^l\phi_t u_{tx}dx-\frac{kC_1}{d}\int_0^l wu_t dx.\end{aligned} \tag{3.11}$$

Add the two equations (3.10) and (3.11) and apply Young's inequality then for all ε_1, ε_2 and $\varepsilon_3>0$ we have

$$\begin{aligned}&\frac{d}{dt}\left(-J\int_0^l u_{ttx}\left(\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi\right)dx-\delta\frac{\rho b}{\mu\sqrt{\xi}}\int_0^l\phi u_{tx}dx+\frac{\alpha C_1}{d}\int_0^l wu_t dx\right)\\&\quad\le-\frac{Jb}{\sqrt{\xi}}\int_0^l|u_{tx}|^2-\delta(\mu-b^2/\xi)\frac{\sqrt{\xi}}{\mu}\int_0^L|\phi_x|^2dx-\frac{\sqrt{\xi}}{2}\int_0^l\left|\frac{b}{\sqrt{\xi}}u_x+\sqrt{\xi}\phi\right|^2dx\\&\qquad c_2\left(\frac{\alpha^2C_1^2}{2d^2\varepsilon_1}+\frac{k^2C_1^2}{2d^2\varepsilon_3}\right)\int_0^l|w|^2dx+\frac{\varepsilon_1}{2}\int_0^l|u_{tt}|^2dx\\&\qquad+c_3\left(\frac{\kappa^2C_1^2}{2d^2\varepsilon_2}+\frac{d^2}{2\sqrt{\xi}}\right)\int_0^l|w_x|^2dx+\frac{\varepsilon_2}{2}\int_0^l|u_{tx}|^2dx+\frac{\varepsilon_3}{2}\int_0^l|u_t|^2dx.\end{aligned} \tag{3.12}$$

Take $\varepsilon_2 = \frac{Jb}{2\sqrt{\xi}}$ we get the desired result.□

Let

$$\mathcal{L}(t) = N_1 E(t) + \mathcal{F}(t) + N_2 \mathcal{G}(t),$$

where $N_1$and $N_2$are positive constants to be fixed.

Theorem 6 *There exists positive constants $\nu_1$and ν_2 such that*

$$\nu_1 E(t) \leq \mathcal{L}(t) \leq \nu_2 E(t), \forall t \geq 0.$$

Proof We have

$$\begin{aligned}|\mathcal{L}(t) - N_1 E(t)| \leq \rho \int_0^l u_t u dx + \frac{J\mu}{b}\int_0^l u_{xt} u_x dx + N_2 J \int_0^l u_{tx}\left(\frac{b}{\sqrt{\xi}} u_x + \sqrt{\xi}\phi\right) dx \\ + N_2 \delta \frac{\rho b}{\mu\sqrt{\xi}} \int_0^l u_{tx}\phi dx + N_2 \frac{\alpha C_1}{d}\int_0^l w u_t dx.\end{aligned}$$

Apply Young's and Poincaré inequalities we obtain

$$\begin{aligned}|\mathcal{L}(t) - N_1 E(t)| \leq & \left(\frac{\rho}{2} + N_2\frac{\alpha C_1}{2d}\right)\int_0^l |u_t|^2 dx + \left(\frac{\rho c_p}{2} + \frac{J\mu}{2b}\right)\int_0^l |u_x|^2 dx \\ & + \left(\frac{J\mu}{2b} + \frac{N_2 J}{2} + N_2\delta\frac{\rho b}{2\mu\sqrt{\xi}}\right)\int_0^l |u_{tx}|^2 dx + \frac{N_2 J}{2}\int_0^l \left|\frac{b}{\sqrt{\xi}} u_x + \sqrt{\xi}\phi\right|^2 dx \\ & + N_2\delta c_p \frac{\rho b}{2\mu\sqrt{\xi}}\int_0^l |\phi_x|^2 dx + N_2\frac{\alpha C_1}{2d}\int_0^l |w|^2 dx\end{aligned}$$

Define

$$\begin{aligned}N_0 := \max\Bigg\{ & \frac{1}{\rho}\left(\rho + N_3\frac{\alpha C_1}{d}\right); \frac{1}{\mu - b^2/\xi}\left(\rho c_p + \frac{J\mu}{b}\right); \\ & \frac{b}{J\mu}\left(\frac{J\mu}{b} + N_2 J + N_2\delta\frac{\rho b}{\mu\sqrt{\xi}}\right); N_2 J; N_2 c_p \frac{\rho b}{\mu\sqrt{\xi}}; N_2\frac{C_1}{d}\Bigg\}\end{aligned}$$

Hence

$$|\mathcal{L}(t) - N_1 E(t)| \leq N_0 E(t),$$

which implies that

$$\nu_1 E(t) \leq \mathcal{L}(t) \leq \nu_2 E(t),$$

where $\nu_1 = N_1 - N_0$ and $\nu_2 = N_1 + N_0$ and $N_1 > N_0$.

Proof of Theorem 5 It follows from Lemmas 6 and 7 that

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \leq & -\left(\rho - \frac{N_2\varepsilon_3}{2}\right)\int_0^l |u_t|^2 dx - \left(\frac{J\rho}{b} - \frac{N_2\varepsilon_1}{2}\right)\int_0^l |u_{tt}|^2 dx - (\mu - b^2/\xi)\int_0^l |u_x|^2 dx \\ & - \left(N_2\frac{Jb}{2\sqrt{\xi}} - (\frac{J\mu}{b} + 2\rho c_p)\right)\int_0^l |u_{xt}|^2 dx - \left(1 + \frac{N_2\sqrt{\xi}}{2}\right)\int_0^l \left|\frac{b}{\sqrt{\xi}} u_x + \sqrt{\xi}\phi\right|^2 dx \\ & - \left(\frac{\delta}{2} + N_2\delta(\mu - b^2/\xi)\frac{\sqrt{\xi}}{\mu}\right)\int_0^l |\phi_x|^2 dx\end{aligned}$$

$$- (N_1k - N_2C_2 - \frac{d^2}{2\delta})\int_0^l |w|^2dx - (N_1\kappa - N_2C_3)\int_0^l |w_x|^2dx \tag{3.13}$$

Choose $\varepsilon_3 = \frac{\rho}{N_2}$, $\varepsilon_1 = \frac{J\rho}{bN_2}$, $N_2 > \frac{2\sqrt{\xi}}{Jb}(\frac{J\mu}{b} + 2\rho c_p)$ and $N_1 > \max\left\{\frac{N_2C_2+\frac{d^2}{2\delta}}{k}; \frac{N_2C_3}{\kappa}\right\}$, from where we obtain that $\zeta_1 = \rho - \frac{N_2\varepsilon_3}{2} > 0$, $\zeta_2 = \frac{J\rho}{b} - \frac{N_2\varepsilon_1}{2} > 0$, $\zeta_3 = N_2\frac{Jb}{2\sqrt{\xi}} - (\frac{J\mu}{b} + 2\rho c_p) > 0$, $\zeta_4 = 1+\frac{N_2\sqrt{\xi}}{2} > 0$, $\zeta_5 = \frac{\delta}{2}+N_2\delta(\mu-b^2/\xi)\frac{\sqrt{\xi}}{\mu} > 0$, $\zeta_6 = N_1k-N_2C_2-\frac{d^2}{2\delta} > 0$ and $\zeta_7 = N_1\kappa - N_2C_3 > 0$ and from where we can conclude that there exists a positive constant $\beta = 2\min\{1, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7\}$ such that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\beta E(t),$$

by equivalence between $E(t)$ and $\mathcal{L}(t)$ according to Theorem 6 we get:

$$\frac{d}{dt}\mathcal{L}(t) \leq -\omega_2\mathcal{L}(t),$$

where $\omega_2 = \frac{\beta}{\nu_2}$. Now integrate the above inequality over $(0, t)$ we obtain

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\omega_2 t},$$

again by equivalence between $E(t)$ and $\mathcal{L}(t)$ according to Theorem 3 we arrive at

$$E(t) \leq M_2E(0)e^{-\omega_2 t},$$

where $M_2 = \frac{\nu_2}{\nu_1}$. □

3.2.2 Second case

We will prove exponential stability for $\mu\xi = b^2$; our result is stated in the following theorem.

Theorem 7 *The energy $E(t)$ of the system (3.1–3.5) decays exponentially as time t tends to infinity. That is, there exist two positive constants M_3 and ω_3 independent of the initial data and independent of any relationship between coefficients such that*

$$E(t) \leq M_3E(0)\ e^{-\omega_3 t}, \forall t \geq 0.$$

Proof We will use the same multipliers used in Theorem 5, and define the Lyapunov functional.

$$\mathbf{L}(t) = N_3E(t) + \mathcal{F}(t) + N_4\mathcal{G}(t),$$

where N_3 and N_4 are positive constants to be fixed. We proceed similarly as the proof of Theorem 5 to get the desired result.

4 Numerical simulations

First we denote by $\widehat{u} = u_t, \widehat{\phi} = \phi_t, \widehat{w} = w_t$ and we introduce the following weak form after multiplying the equations (3.1), (3.2), (3.3) by $\overline{u}, \overline{\phi}, \overline{w} \in H_0^1(0,l)$

$$(WP)\begin{cases} \rho(\widehat{u}_t, \overline{u}) + \mu(u_x, \overline{u}_x) + b(\phi, \overline{u}_x) = 0 \\ J(\widehat{u}_t, \overline{\phi}_x) + \delta(\phi_x, \overline{\phi}_x) + b(u_x, \overline{\phi}) + \xi(\phi, \overline{\phi}) + d(w_x, \overline{\phi}) = 0 \\ \alpha(\widehat{w}, \overline{w}) + \kappa(w_x, \overline{w}_x) + d(\widehat{\phi}_x, \overline{w}) + k(w, \overline{w}) = 0 \end{cases} \tag{4.1}$$

Let us partition the interval $(0;l)$ into subintervals $I_j = (x_{j-1}; x_j)$ of length $h = \frac{1}{s}$ with $0 = x_0 < x_1 < \cdots < x_s = l$ and define the associated finite element spaces by

$$S_h^0 = \{u \in H_0^1(0,l); u \in C([0,l]), u|_{I_j} \in P_1(K)\}.$$

For a given final time T and a positive integer$N$, define the time step $\Delta t = \frac{T}{N}$ and the nodes$t_n = n\Delta t, n = 0, ..., N$. By using the Implicit Euler scheme in time and the finite element variational approximation in space, we introduce the following scheme. For $\overline{u}_h, \overline{\phi}_h, \overline{w}_h \in S_h^0$, find $u_h^n, \phi_h^n, w_h^n \in S_h^0$ such that,

$$(NP)\begin{cases} \dfrac{\rho}{\Delta t}(\widehat{u}_h^n - \widehat{u}_h^{n-1}, \overline{u}_h) + \mu(u_{hx}^n, \overline{u}_{hx}) + b(\phi_h^n, \overline{u}_{hx}) = 0 \\ \dfrac{J}{\Delta t}(\widehat{u}_h^n - \widehat{u}_h^{n-1}, \overline{\phi}_{hx}) + \delta(\phi_{hx}^n, \overline{\phi}_{hx}) + b(u_{hx}^n, \overline{\phi}_h) + \xi(\phi_h^n, \overline{\phi}_h) + d(w_{hx}^n, \overline{\phi}_h) = 0 \\ \dfrac{\alpha}{\Delta t}(w_h^n - w_h^{n-1}, \overline{w}_h) + \kappa(w_{hx}^n, \overline{w}_{hx}) + d(\widehat{\phi}_{hx}^n, \overline{w}_h) + k(w_h^n, \overline{w}_h) = 0, \end{cases} \tag{4.2}$$

where $\widehat{u}_h^n = \frac{u_h^n - u_h^{n-1}}{\Delta t}$, $\widehat{\phi}_h^n = \frac{\phi_h^n - \phi_h^{n-1}}{\Delta t}$ and $\widehat{w}_h^n = \frac{w_h^n - w_h^{n-1}}{\Delta t}$.

Plugging $\widehat{u}_h^n$, $\widehat{\phi}_h^n$ and $\widehat{w}_h^n$ in system (4.2) we get,

$$\begin{cases} \dfrac{\rho}{(\Delta t)^2}\Big(u_h^n - 2u_h^{n-1} + u_h^{n-2}, \overline{u}_h\Big) + \mu\big(u_{hx}^n, \overline{u}_{hx}\big) + b\big(\phi_h^n, \overline{u}_{hx}\big) = 0, \\ \dfrac{J}{(\Delta t)^2}\Big(\phi_h^n - 2\phi_h^{n-1} + \phi_h^{n-2}, \overline{\phi}_h\Big) + \delta\big(\phi_{hx}^n, \overline{\phi}_{hx}\big) + b\big(u_{hx}^n, \overline{\phi}_h\big) + \xi\big(\phi_h^n, \overline{\phi}_h\big) + d\big(w_{hx}^n, \overline{\phi}_h\big) = 0, \\ \dfrac{\alpha}{\Delta t}\Big(w_h^n - w_h^{n-1}, \overline{w}_h\Big) + \kappa\big(w_{hx}^n, \overline{w}_{hx}\big) + \dfrac{d}{\Delta t}\Big(\phi_{hx}^n - \phi_{hx}^{n-1}, \overline{w}_h\Big) + k\big(w_h^n, \overline{w}_h\big) = 0, \end{cases}$$

note that by the finite element theory, $u_h^n = \sum_{i=1}^{s} a_i^n \psi_i$, $\phi_h^n = \sum_{i=1}^{s} b_i^n \psi_i$ and $w_h^n = \sum_{i=1}^{s} c_i^n \psi_i$ where ψ_i are bases of the finite space S_h^0. Taking $\overline{u}_h = \psi_j$ we get

$$\begin{cases} \dfrac{\rho}{(\Delta t)^2} ZA^n + \mu TA^n + bYB^n = \dfrac{2\rho}{(\Delta t)^2} ZA^{n-1} - \dfrac{\rho}{(\Delta t)^2} ZA^{n-2}, \\ \dfrac{J}{(\Delta t)^2} ZA^n + \delta TB^n + bXA^n + \xi ZB^n + dXC^n = \dfrac{2J}{(\Delta t)^2} YA^{n-1} - \dfrac{J}{(\Delta t)^2} YA^{n-2} \\ \dfrac{\alpha}{\Delta t} ZC^n + \kappa TC^n + \dfrac{d}{\Delta t} XB^n + kZC^n = \dfrac{\alpha}{\Delta t} ZC^{n-1} + \dfrac{d}{\Delta t} XB^{n-1} \end{cases} \tag{4.3}$$

where the vectors A^n, B^n and C^n are given by

$$A^n = \left\{a_i^n\right\}$$

$$B^n = \left\{b_i^n\right\}$$

$$C^n = \{c_i^n\}$$

and the matrices Z, X, Y, and T are given by

$$X = (\psi_{ix}, \psi_j)$$

$$Y = (\psi_i, \psi_{jx})$$

$$Z = (\psi_i, \psi_j)$$

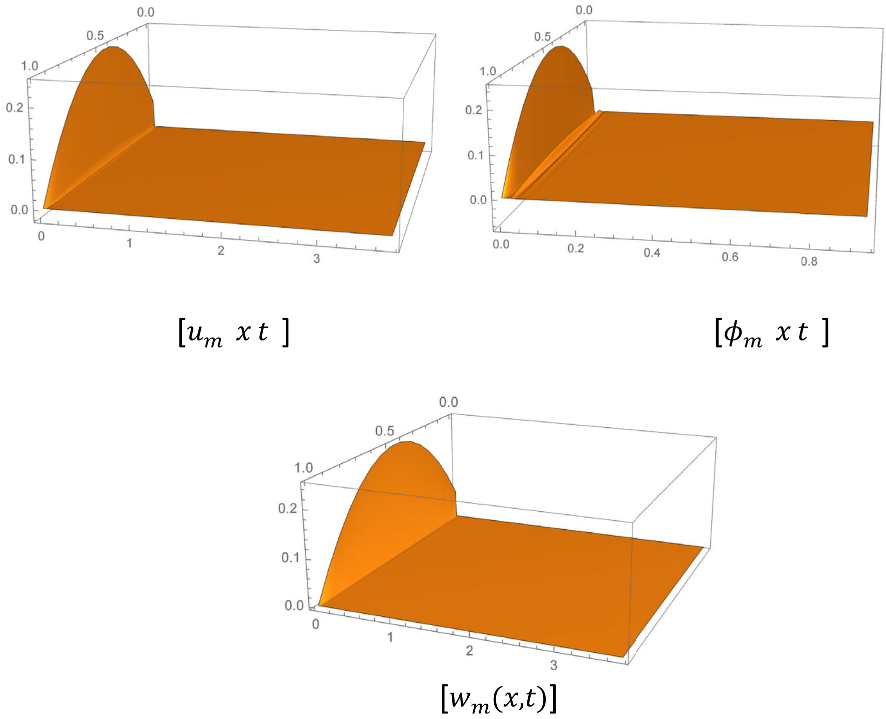
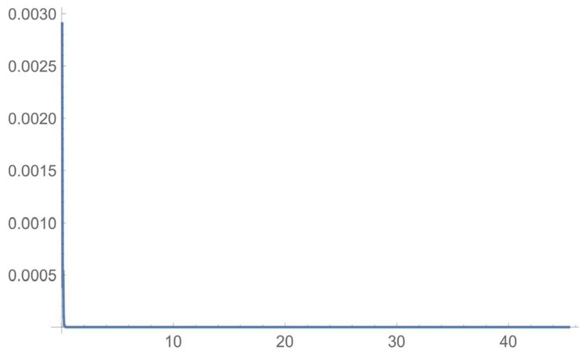


Fig. 1 Graphs-Classical system with equal speed limits

Fig. 2 Energy as a function of time



$$T = \left(\psi_{ix}, \psi_{jx}\right)$$

We solve (4.3) using the following initial and physical data, $\rho = d = \alpha = b = \xi = j = 0.001$, $k = 1$ and $\mu = 0.01$. The space discritization $\Delta x = \frac{1}{11}$ and the time dicritization $\Delta t = \frac{1}{22}$ with total time $T = 25$. The initial data $u_h^0 = \phi_h^0 = w_h^0 = u_h^1 = \phi_h^1 = (1 - x)x$.

4.1 Classical system with equal speed limit

See Figs. 1, 2 and 3.

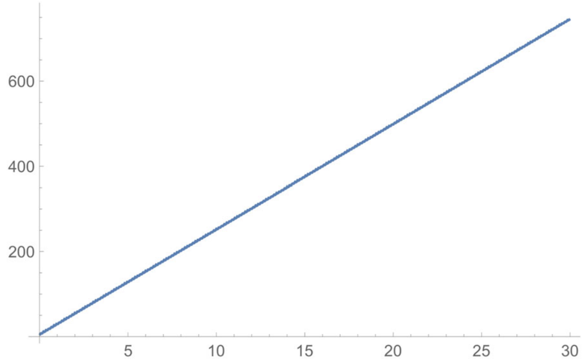


Fig. 3 $-log[energy]$ as function of time

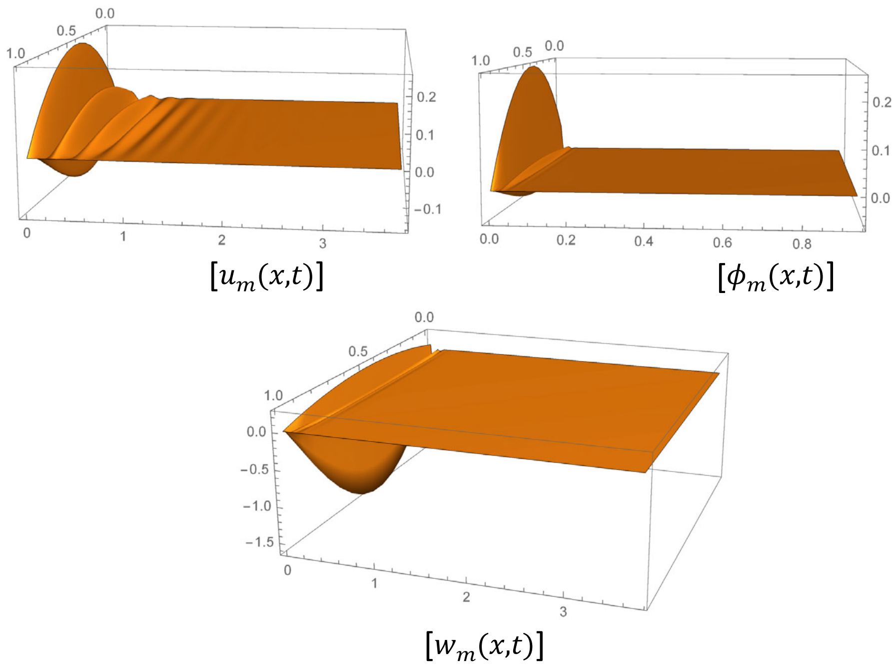


Fig. 4 Graphs-Classical system with non equal speed limits

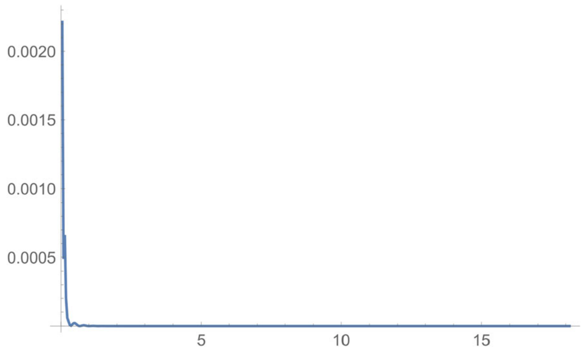


Fig. 5 Energy as a function of time

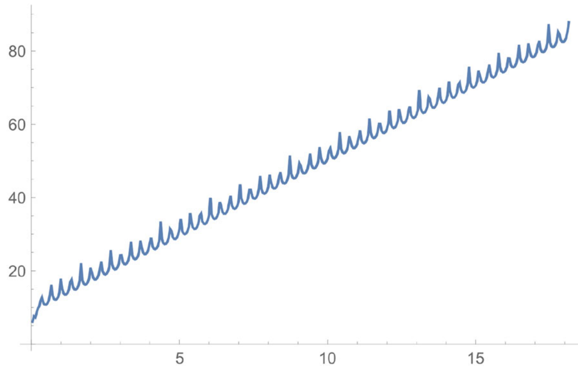


Fig. 6 $-log[energy]$ as function of time

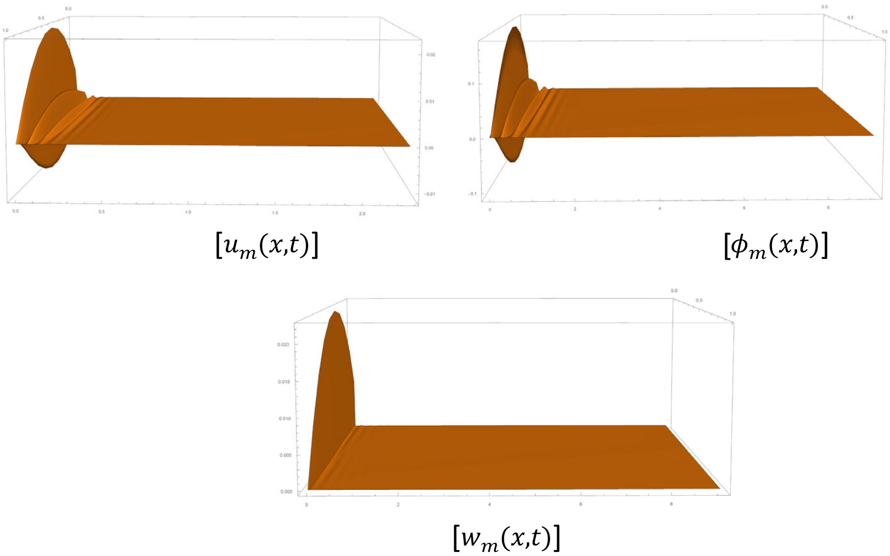


Fig. 7 Graphs-Second spectrum free system

4.2 Classical system with non-equal speed limit

See Figs. 4, 5 and 3.

4.3 Second spectrum free system

See Figs. 7, 8 and 9.

4.4 Graphical analysis

The case of classical system with equal Speed condition: In this case, the energy decays so fast to zero which shows that the decay type is exponential, and what proves this is the graph of $log(E(t))$ that shows a straight. This is graphical evidence of exponential behavior. The fast decay of u, ϕ, and w also show the exponential decay.

The case of classical system with non-equal Speed condition: Starting with the graphs of the functions u, ϕ, and w, it is clear that they have much more vibratory behavior than the previous case, and this is a direct indication that the decay is slower than the previous case. Regarding the graph of the energy, although the graph shows a decay to zero the graph of $log(E(t))$ proves that we lost the exponential decay.

The second spectrum free case: In this case also the energy decays so fast to zero which shows that the decay type is exponential, and what proves this is the graph of $log(E(t))$ that shows a straight. This is graphical evidence of exponential behavior.

Fig. 8 Energy as a function of time

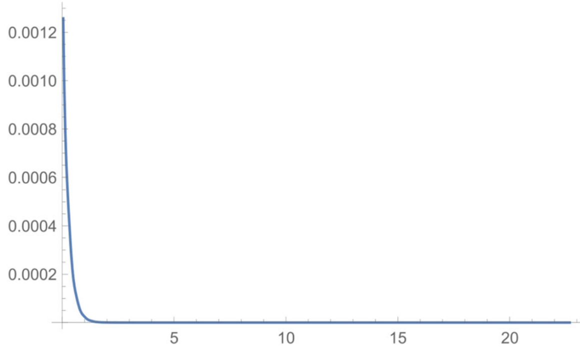
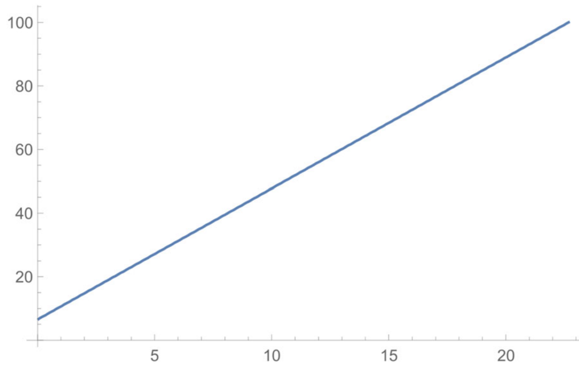


Fig. 9 $-log[energy]$ as function of time



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Data Availability Statement The manuscript has no associated real data.

Declarations

Conflict of Interest The authors have not disclosed any competing interests.

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