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ORIGINAL PAPER

General decay of the solution to a nonlinear viscoelastic beam with delay

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Abstract
In this paper we consider a nonlinear viscoelastic beam with a linear delay term and infinite memory term. The well posedness of solutions is proved using the semigroup method. We establish a general decay results by using minimal and general conditions on the relaxation function, from which the usual exponential and polynomial decay rates are only special cases.

Keywords Viscoelastic · Nonlinear beam · Delay term · Decay rate · Lyapunov functionals

Mathematics Subject Classification 35B40 · 35L70 · 49K25 · 93D15 · 93D20

1 Introduction

In this paper, we consider the following one-dimensional version of a system which describes the vibrations of shallow shells with time delay and infinite memory term:

$$\begin{cases} u_{tt} - [g(u,w)]_x + \alpha_1 u_t + \alpha_2 u_t(t-\tau) & = 0 \text{ in } \mathscr{Q}, \\ \mathscr{D} w_{tt} + w_{xxxx} - [f(u,w)]_x + k(x)\, g(u,w) + h * w_{xxxx} & = 0 \text{ in } \mathscr{Q}, \end{cases} \tag{1}$$

where $\mathscr{Q} = I \times \mathbb{R}_+$, $I =]0, L[$ is an interval, ∂I its boundary ($\partial I = \{0\} \cup \{L\}$), $\Sigma = \{0\} \times \mathbb{R}_+ \cup \{L\} \times \mathbb{R}_+$, $\mathscr{D}$ is the operator $\mathscr{I} - \partial_x^2$. The functions f and g are defined by:

$$f(u,w) = \frac{2}{1-\mu}\left[w_x\left(u_x + \frac{1}{2}w^2 + k(x)w\right)\right], \tag{2}$$

$$g(u,w) = \frac{2}{1-\mu}\left[u_x + \frac{1}{2}w^2 + k(x)w\right]. \tag{3}$$

In (1), subscripts mean partial derivatives, the space variable x runs in the interval $0 < x < L$ and t denotes the positive time variable. The functions $u = u(x,t)$ and $w = w(x,t)$ are,

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respectively, the longitudinal and transversal displacements of the beam at the point x at time t. Additionally, μ is a constant, $0 < \mu < 1$ and $k = k(x)$ represents the curvature of the beam at the point x.

In the system (1), $\alpha_1 u_t$ represents a frictional damping. The time delay is given by $\alpha_2 u_t(t-\tau)$, where α_1, α_2, τ are positive constants.

In (1), $(h * u)(t)$ is defined by

$$(h * u)(t) = \int_0^{\infty} h(t-s)u(x,s)ds.$$

The viscoelastic damping term that appears in the equations describes the relationship between the stress and the history of the strain in the beam, according to Boltzmann theory. The function h represents the kernel of the memory term or the relaxation function.

The system (1) is subjected with the boundary conditions

$$\begin{aligned} w(0,t) &= w(L,t) = 0, \\ w_x(0,t) &= w_x(L,t) = 0, \\ u(0,t) &= u(L,t) = 0, \quad t \in \mathbb{R}_+^* \end{aligned} \tag{4}$$

and the initial conditions

$$(u(x,0), u_t(x,0), w(x,-s), w_t(x,0)) = (u_0(x), u_1(x), w_0(x), w_1(x)), (x,s) \in \mathcal{Q}. \tag{5}$$

The main purpose about problem (1)–(5) is to deal with the well posedness and asymptotic behavior of solutions. Before stating and proving our results, let us recall some other results related to our work.

Several authors have studied the Mindlin–Timoshenko system of equations (see, e.g., [16]). This Model is a widely used and fairly complete mathematical model for describing the transverse vibrations of beams. It is a more accurate model than the Euler-Bernoulli one, since it also takes into account transverse shear effects.

For a beam of length $L > 0$, this one-dimensional nonlinear system reads as

$$\begin{aligned} \frac{h_0^3}{12}\phi_{tt} - \phi_{xx} + k[\phi + \psi_x)] = 0, \quad &\text{in} \quad Q, \\ \rho h \psi_{tt} - k[\phi + \psi_x)]_x + \left[\psi_x\left(\eta_x + \frac{1}{2}\psi_x^2\right)\right]_x = 0, \quad &\text{in} \quad Q, \\ \rho h \eta_{tt} - \left[\eta_x + \frac{1}{2}(\psi_x)^2\right]_x = 0, \quad &\text{in} \quad Q, \end{aligned} \tag{6}$$

where $Q = (0, L) \times (0, T)$ and T is a given positive time. Here, the unknown $\phi = \phi(x,t)$ represent the angle of rotation. The parameter k is the so called modulus of elasticity in shear. It is given by the expression $k = \widehat{k} E h_0 / 2(1+\epsilon)$, where $\widehat{k}$ is a shear correction coefficient, E is the Young's modulus and ϵ is the Poisson's ratio, $0 < \epsilon < 1/2$.

For Mindlin–Timoshenko system, there is a large literature, addressing problems of existence, uniqueness and asymptotic behavior in time when some damping effects are considered, as well as some other important properties (see [13, 26, 28] and references therein).

When one assumes the linear filament of the beam to remain orthogonal to the deformed middle surface, the transverse shear effects are neglected, and one obtains, from the Mindlin–Timoshenko system of equations, the following von Kármán system (see [28]).

$$\begin{cases} \rho h_0 \mathscr{D}\psi_{tt} + \psi_{xxxx} - \left[\psi_x(\eta_x + \frac{1}{2}\psi_x^2)\right]_x = 0, \text{ in }]0, L[\times\mathbb{R}_+^*, \\ \rho h\eta_{tt} - [\eta_x + \frac{1}{2}(\psi_x)^2]_x = 0, \text{ in }]0, L[\times\mathbb{R}_+^*. \end{cases} \tag{7}$$

There is also an extensive literature about system (7) (see [13, 18, 26, 27, 38, 51, 53–56] and references therein).

Lagnese and Leugering [27] considered a one-dimensional version of the von Kármán system describing the planar motion of a uniform prismatic beam of length L. More precisely, in [27] the following system was considered:

$$\begin{cases} \psi_{tt} + \psi_{xxxx} - h_0\psi_{xxtt} - \left[\psi_x(\eta_x + \frac{1}{2}\psi_x^2)\right]_x = 0 \text{ in }]0, L[\times\mathbb{R}_+^* \\ \eta_{tt} - \left[\eta_x + \frac{1}{2}(\psi_x)^2\right]_x = 0 \text{ in }]0, L[\times\mathbb{R}_+^* \end{cases} \tag{8}$$

In [27], suitable dissipative boundary conditions at $x = 0$, $x = L$ and initial conditions at $t = 0$ were given and the stabilization problem was studied.

In [4], Araruna et al. have showed how the so called von Kármán model (8) can be obtained as a singular limit of a modified Mindlin–Timoshenko system (6) when the modulus of elasticity in shear k tends to infinity, provided a regularizing term through a fourth order dispersive operator is added. Introducing damping mechanisms, the authors also show that the energy of solutions for this modified Mindlin–Timoshenko system decays exponentially, uniformly with respect to the parameter k. As $k \longrightarrow \infty$, the authors obtain the damped von Kármán model with associated energy exponentially decaying to zero as well.

The subject of stability of von Kármán system has received a lot of attention in the last years, see [11, 12, 18, 18, 25, 26, 33, 39, 49, 50, 52] and references therein.

Delay effects are very important because most natural phenomena are in many cases very complicated and do not depend only on the current state but also on the past history of the system. The presence of delay can be a source of instability. In recent years, the stabilization of PDEs with delay effects has draw attention for many author and become an active area of research, see [11, 15, 24, 46–48, 57, 59–61].

For the stability of other kind of wave equation, let us mention the following problem:

$$\begin{aligned} u_{tt}(x,t) - \Delta u(x,t) + \int_0^t h(t-s)\Delta u(x,s)ds & \\ +\alpha_1 h_1(u_t(x,t)) + \alpha_2 h_2(u_t(x,t-\tau)) = 0, &\quad \text{in} \quad \Omega \times \mathbb{R}_+, \\ u(x,t) = 0, &\quad \text{in} \quad \Gamma \times \mathbb{R}_+, \\ u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x), &\quad \text{in} \quad \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), &\quad \text{in} \quad \Omega\times]0,\tau[, \end{aligned} \tag{9}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, with a smooth boundary $\partial\Omega = \Gamma$, h is a positive non-increasing function defined on $\mathbb{R}^n$, h_1 and h_2 are two functions, $\tau > 0$ is a time delay, α_1 and α_2 are positive real numbers and the initial data (u_0, u_1, f_0) belong to a suitable function space.

In the case $h \equiv 0$, problem (9) has been studied by many authors (see [6–8, 10, 46, 61]).

For a wider class of relaxation functions, Messaoudi [36, 37] considered

$$u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)ds = b \mid u \mid^{\gamma} u, \tag{10}$$

for $\gamma > 0$ and $b = 0$ or $b = 1$, and the relaxation function satisfies

$$h'(t) \leq -\zeta(t)h(t), \tag{11}$$

where ζ is a differentiable nonincreasing positive function. He established a more general decay result, from which the usual exponential and polynomial decay results are only special cases. Such a condition was then employed in a series of papers, see for instance [3, 22, 23, 41, 42, 52].

Recently, Mustafa and Messaoudi [45] studied the problem (10) with $b = 0$ for the relaxation functions satisfying

$$h'(t) \leq -H(h(t)), \tag{12}$$

where H is a nonnegative function, with $H(0) = H'(0) = 0$ and H is strictly increasing and strictly convex on $]0, k[$ for some $k_0 > 0$. The authors showed a general relation between the decay rate for the energy and that of the relaxation function h without imposing restrictive assumptions on the behavior of h at infinity. On the other hand, a condition of the form (12) where H is a convex function satisfying some smoothness properties, was introduced by Alabau-Boussouira and Cannarsa [2] and used then by several authors with different approaches. We refer to [32] where not only general but also optimal result was established by Lasiecka and Wang.

The main objective of this work is to investigate the problem (1) with the following very general class of relaxation functions

$$h'(t) \leq -\zeta(t)H(h(t)), \tag{13}$$

where H is increasing and convex without any additional constraints on H and the coefficients. We will establish a general decay rate for the energy associated to the system for linear damping, time delay terms and finite memory. We would like to see the influence of frictional and viscoelastic dampings on the rate of decay of solutions in the presence of linear degenerate delay term.

To prove decay estimates, we shall pursue a strategy based on an adaptation of non linear differential inequalities technique developed in [40, 43, 44] and we use a perturbed energy method and some properties of convex functions which were introduced and developed by many authors [1, 9, 14, 17, 30, 31, 34].

Our work is organized as follows. In the next section, we prepare some material needed in the proof of our result, like some lemmas (Poincaré's and Young's inequalities) and some useful notations. We introduce the different functionals by which we modify the classical energy to get an equivalent useful one. In Sect. 4, we state and prove the well-posedness of the problem. Finally, in Sect. 5, we will prove our main results concerning the exponential decay of the energy associated to the solutions of the problem.

2 Statement of results

In order to deal with the delay feedback term, motivated by [46, 47], we define the following new dependent variables η and z:

$$\begin{aligned} z(x, t, p) &= u_t(x, t - p\tau) \quad \text{in } \mathscr{Q} \times (0, 1), \\ \eta(x, t, s) &= w(x, t) - w(x, t - s) \quad \text{in} \mathscr{Q} \times \mathbb{R}_+, \end{aligned} \tag{14}$$

consequently, we obtain

$$\begin{aligned} \eta(x,t,0) &= 0 && \text{in } \mathscr{Q}, \\ \eta(x,t,s) &= 0 && \text{in } \Sigma\times\mathbb{R}_+^2, \\ \eta_0(x,s) := w_0(x) - w(x,-s) & && \text{in } \mathscr{Q}\times\mathbb{R}_+, \\ z(x,t,0) &= u_t(x,t) && \text{in } \mathscr{Q}, \\ z(x,0,p) &= z_0(x,-p\tau) && \text{in } I\times(0,1), \end{aligned} \tag{15}$$

clearly, (14) gives

$$\begin{aligned} \tau z_t(x,t,p) - z_p(x,t,p) &= 0 \text{ in } \mathscr{Q}\times(0,1), \\ \eta_t(x,t,s) + \eta_s(x,t,s) - w_t(x,t) &= 0 \quad \text{in} \mathscr{Q}\times\mathbb{R}_+, \end{aligned} \tag{16}$$

where $z_p = \partial_p z$ and $\eta_s = \partial_s \eta$.

Therefore, problem (1)–(5) is equivalent to

$$\begin{cases} u_{tt} - [g(u,w)]_x + \alpha_1 u_t + \alpha_2 z(1) = 0 & \text{in } \mathscr{Q}, \\ \mathscr{D} w_{tt} + l w_{xxxx} - [f(u,w)]_x + g(u,w) + \int_0^\infty h(s)\eta_{xxxx}(s)ds = 0 & \text{in} \mathscr{Q}, \\ \tau z_t(p) - z_p(p) = 0 & \text{in } \mathscr{Q}\times(0,1), \\ \eta_t(s) + \eta_s(s) - w_t = 0 & \text{in } \mathscr{Q}\times\mathbb{R}_+, \end{cases} \tag{17}$$

where $l = 1 - \int_0^\infty h(s)ds$, with boundary conditions

$$\begin{aligned} & u(x,t) = 0 && \text{in } \Sigma, \\ & w(x,t) = w_x(x,t) = 0 && \text{in } \Sigma, \\ & \eta(x,t,s) = \eta_x(x,t,s) = 0 && \text{in } \Sigma\times\mathbb{R}_+^2, \end{aligned} \tag{18}$$

and initial conditions

$$\begin{aligned} & (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)) && \text{on } I, \\ & (w(x,-s), w_t(x,0)) = (w_0(x,s), w_1(x)) && \text{in } \mathscr{Q}, \\ & z_0(x,p) = z(x,-p\tau) && \text{in } I\times(0,1), \\ & \eta_0(x,s) = w_0(x) - w(x,-s) && \text{in } \mathscr{Q}. \end{aligned} \tag{19}$$

In what follow, we assume that the function k belong to the sobolev space $H^1(0,L)$.

3 Preliminaries

In this section, we state our stability results for problem (17)–(19). For this purpose, we start with the following hypotheses:

$(\mathscr{H}_1)$ $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-increasing differentiable function such that $h(0) > 0$ and

$$l = 1 - \int_0^{+\infty} h(s)ds > 0. \tag{20}$$

Also assume that there exist a positive constant αsuch that

$$h'(s) \le \alpha h(s), \quad \forall s \ge 0, \tag{21}$$

$(\mathscr{H}_1)$ There exists an increasing strictly convex function $G : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0,+\infty[)$ satisfying

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t\to+\infty} G'(t) = +\infty$$

such that

$$\int_0^{+\infty} \frac{h(s)}{G^{-1}(-h'(s))} ds + \sup_{s\in\mathbb{R}_+} \frac{h(s)}{G^{-1}(-h'(s))} < +\infty. \tag{22}$$

Remark 1 The condition (22) introduced in [19] is satisfied by any positive function h of class $C^1(\mathbb{R}_+)$ with $h' < 0$ and h is integrable on $\mathbb{R}_+$ (see [19–21] for explicit examples).

C and c denote some general positive constants, which may be different in different estimates.

3.1 Functional setting and assumptions.

In order to prove the well-posedness of (17) by using the semigroups theory, we introduce some functional spaces.

Let us introduce the energy space $\mathscr{H}$ by

$$\mathscr{H} = H_0^1(I) \times L^2(I) \times H_0^2(I) \times H_0^1(I) \times L^2(I \times (0,1)) \times L_h(I)$$

where $L_h(I)$ is the weighted Sobolev space defined by

$$L_h(I) = \left\{ v \in L^2\left(0,\infty; H_0^2(I)\right) \mid \int_0^\infty h(s) \left\| v_{xx}(s) \right\|^2 ds < \infty \right\}.$$

The space $L_h(I)$ is endowed with the inner product

$$\langle v, \overline{v} \rangle_h = \int_0^\infty h(s) \left\langle v_{xx}(s), \overline{v}_{xx}(s) \right\rangle ds.$$

Also, we denote by $\langle\langle ., . \rangle\rangle$ the natural inner product on the space $L^2(I \times (0,1))$, we note that the norms

$$\int_0^L \int_0^1 z^2(x,t,p) dp dx \quad \text{and} \quad \int_0^L \int_0^1 e^{-2\tau p} z^2(x,t,p) dp dx$$

are equivalent in $L^2(I \times (0,1))$.

Let ξ be any positive number which satisfy

$$\alpha_2 \tau < \xi < (2\alpha_1 - \alpha_2)\tau.$$

Also, we define

$$L_h^*(0,L) = \left\{ v \in L^2\left(0,\infty; \left(H^4 \cap H_0^2\right)(I)\right) \mid \int_0^\infty h(s) \left\| v_{xx}(s) \right\|^2 ds < \infty \right\}.$$

$\mathscr{H}$ is endowed with the norm

$$\|\mathscr{U}\|_{\mathscr{H}} = \frac{2}{1-\mu} \|u_x\|^2 + \|u_t\|^2 + l\|w_{xx}\|^2 + \|w\|^2 + \|w_{tx}\|^2 + \xi |||z|||^2 + h \circ \eta_{xx} \tag{23}$$

where

$$\mathscr{U} = (u, u_t, w, w_t, z, \eta)$$

$$|||z|||^2 = \int_0^1 e^{-2\tau p} \|z(p)\|^2 dp$$

and

$$h \circ \eta_{xx} = \int_0^\infty h(s) \left\| \eta_{xx}(s) \right\|^2 ds.$$

3.2 Energy identity

We start by the following lemma:

Lemma 1 *For* $(h, \phi) \in \left(\mathscr{C}^1 \cap L^1\right)(\mathbb{R}_+) \times \mathscr{C}^1(\mathbb{R})$, *we have*

$$\begin{aligned} 2\,(h * \phi)\,\phi_t &= \frac{d}{dt}\left\{-\int_0^\infty h(s)\,(\phi(t) - \phi(t-s))^2\,ds + h_\infty\,|\phi(t)|^2\right\} \\ &\quad + \int_0^\infty h(s)\,(\phi(t) - \phi(t-s))^2\,ds \end{aligned} \tag{24}$$

Lemma 2 *Assume that* $(\psi, \psi_t, \eta, \eta_t, z)$ *is a strong solution of the problem (17)-(19). Then we have*

$$\begin{aligned} \frac{\xi}{\tau}\frac{d}{dt}\int_0^L\int_0^1 z^2(x,p,t)dpdx &= -\frac{\xi}{2\tau}\int_0^L\int_0^1 \frac{\partial}{\partial p} z^2(x,p,t)dpdx \\ &= \frac{\xi}{2\tau}\int_0^L \left\{z^2(x,0,t) - z^2(x,1,t)\right\}dx. \end{aligned} \tag{25}$$

Proof We multiply the third equation in (17) by $\frac{\xi}{\tau}z$ and integrate the result over $(0, L)\times(0,1)$ with respect to p and x, respectively, to get

$$\begin{aligned} \frac{\xi}{\tau}\frac{d}{dt}\int_0^L\int_0^1 z^2(x,p,t)d\rho dx &= -\frac{\xi}{2\tau}\int_0^L\int_0^1 \frac{\partial}{\partial p} z^2(x,p,t)dpdx \\ &= \frac{\xi}{2\tau}\int_0^L \left\{z^2(x,0,t) - z^2(x,1,t)\right\}dx \end{aligned}$$

which gives (25). □

We define the energy associated with the solution of system (17)–(19) by

$$\begin{aligned} 2\mathscr{E}(t) &= \int_0^L \left\{u_t^2 + \frac{2}{1-\mu}\left(u_x + \frac{1}{2}w^2 + k(x)w\right)^2 + l w_{xx}^2 + w_{xt}^2 dx\right\} \\ &\quad + h \circ \eta_{xx} + \xi \int_0^L\int_0^1 e^{-2\tau p} z^2\,(x,p)\,dpdx \end{aligned} \tag{26}$$

where ξ is a positive constant such that

$$\alpha_2\tau < \xi < (2\alpha_1 - \alpha_2)\tau \tag{27}$$

and α_1 and α_1 satisfying

$$\alpha_2 < \alpha_1. \tag{28}$$

Lemma 3 *Assume that* $(\psi, \psi_t, \eta, \eta_t, z)$ *is a strong solution of the problem (17)–(19). Then the derivative of* $\mathscr{E}(t)$ *satisfies*

$$2\frac{d\mathscr{E}(t)}{dt} = h' \circ \eta_{xx} + (\frac{\xi}{2\tau} - \alpha_1)\int_0^L u_t^2(s)dx \\ -\frac{\xi}{2\tau}\int_0^L z^2(1)dx - \alpha_2\int_0^L z(1)u_t dx. \tag{29}$$

Moreover, for all $t \geq 0$, we have

$$\begin{aligned} 2\frac{d}{dt}\mathscr{E}(t) &= h' \circ \eta_{xx} + (\frac{\xi}{2\tau} - \alpha_1)\int_0^L u_t^2(s)dx \\ &\quad -\frac{\xi}{2\tau}\int_0^L z^2(1)dx - \alpha_2\int_0^L z(1)u_t dx \\ &\leq h' \circ \eta_{xx} + (\frac{\alpha_2}{2} - \frac{\xi}{2\tau})\int_0^L z^2(1)dx \\ &\quad +(\frac{\alpha_2}{2} + \frac{\xi}{2\tau} - \alpha_1)\int_0^L u_t^2(s)dx \\ &\leq 0. \end{aligned} \tag{30}$$

Proof Multiplying the first equation in (17) by u_t, the second by w_t and the third by $\xi z(p)$, integrating by part and using boundary condition in (18) and Lemma 2 yields (30). □

Lemma 4 (Jensen inequality) *Let F be a convex function on $[a, b]$, $r_1 : \Omega \to [a, b]$ and r_2 are integrable functions on Ω, $r_2(x) \geq 0$, and $\int_\Omega r_2(x)dx = k_0 > 0$, then Jensen's inequality states that*

$$F\left[\frac{1}{k_0}\int_\Omega r_1(x)r_2(x)dx\right] \leq \frac{1}{k_0}\int_\Omega F[r_1(x)]r_2(x)dx. \tag{31}$$

4 Global well-posedness

In this section we show the existence and regularity of solutions of the one dimensional viscoelastic Marguerre–Vlasov system (17)–(19).

Then problem (17)–(19) is reduced to the following problem for an abstract first-order evolutionary equation:

$$\begin{aligned} \mathscr{U}_t &= \mathscr{A}\mathscr{U} + \mathscr{B}(\mathscr{U}) \\ \mathscr{U}(0) &= \mathscr{U}_0 = (u_0, u_1, w_0, w_1, z_0, \eta_0)^T, \end{aligned} \tag{32}$$

where $\mathscr{U} = (u, u_t, w, w_t, z, \eta)^T$, and

$$\mathscr{A}\mathscr{U} = \begin{bmatrix} u_t \\ \frac{2}{1-\mu}u_{xx} - \alpha_1 u_t - \alpha_2 z(1) \\ w_t \\ \mathscr{D}^{-1}\left[-l w_{xxxx} - \int_0^\infty h(s)\eta_{xxxx}(s)ds\right] \\ \frac{1}{\tau}z_p(p) \\ -\eta_s(s) + w \end{bmatrix},$$

$$\mathscr{B}\mathscr{U} = \begin{bmatrix} 0 \\ \frac{2}{1-\mu}\left[\frac{1}{2}w^2 + kw\right]_x \\ 0 \\ \mathscr{D}^{-1}\left[[f(u,w)]_x - kg(u,w)\right] \\ 0 \\ 0 \end{bmatrix}.$$

with the domain

$$D(\mathscr{A}) = \left(H^2 \cap H_0^1\right)(I) \times L^2(I) \times \left(H^4 \cap H_0^2\right)(I) \times H_0^2(I) \\ \times L^2(I \times H^1(0,1)) \times L_h^*(0,L)$$

Lemma 5 *The operator* $\mathscr{A}$ *defined in* (32) *is the infinitesimal generator of a* C_0*-semigroup in* $\mathscr{H}$.

Proof For all $\mathscr{U}(t) \in D(\mathscr{A})$, one has

$$\begin{aligned} \langle A\mathscr{U}, \mathscr{U}\rangle_{\mathscr{H}} &= -\alpha_1 \int_0^L u_t^2 dx - \alpha_2 \int_0^L u_t z(1) dx + l \int_0^L w_{xx} w_{txx} dx \\ &\quad - l \int_0^L w_t . \mathscr{D}\mathscr{D}^{-1} w_{xxxx} dx - \int_0^\infty h(s) \int_0^L w_t . \mathscr{D}\mathscr{D}^{-1} \eta_{xxxx}(s) ds dx \\ &\quad + \frac{\xi}{\tau}\left\langle\!\left\langle z_p, z\right\rangle\!\right\rangle - \langle \eta_s, \eta\rangle_h + \langle w, \eta\rangle_h . \end{aligned} \tag{33}$$

Integrating by parts, we have

$$-\langle \eta_s, \eta\rangle_h = h' \circ \eta_{xx} \tag{34}$$

and

$$2\left\langle\!\left\langle z_p, z\right\rangle\!\right\rangle = \int_0^L u_t^2 dx - \int_0^L z^2(1) dx. \tag{35}$$

Plugging (34) and (35) in (33), we get

$$\langle \mathscr{A}\mathscr{U}, \mathscr{U}\rangle_{\mathscr{H}} = 2\frac{d}{dt}\mathscr{E}(t) \le 0. \tag{36}$$

which implies that $\mathscr{A}$ is dissipative.

Next we will prove that the operator $(\mathscr{I} - \mathscr{A})$: $D(\mathscr{A}) \longrightarrow \mathscr{H}$ is onto, that is, given $\mathscr{F} = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathscr{H}$, we seek $\mathscr{U} = (u, u_t, w, w_t, z, \eta)^T \in D(\mathscr{A})$ such that

$$(\mathscr{I} - \mathscr{A})\mathscr{U} = \mathscr{F}. \tag{37}$$

Equivalently, one must consider the system given by

$$u - u_t = f_1, \quad (18.1) \tag{38}$$

$$u_t - \frac{2}{1-\mu}u_{xx} + \alpha_1 u_t + \alpha_2 z(1) = f_2, \quad (18.2) \tag{39}$$

$$w - w_t = f_3, \quad (18.3) \tag{40}$$

$$\mathscr{D}w_t + l w_{xxxx} + \int_0^\infty h(s)\eta_{xxxx}(s) ds = \mathscr{D} f_4, \quad (18.4) \tag{41}$$

$$\tau z - z_p(p) = \tau f_5, \quad (18.5) \tag{42}$$

$$\eta(s) + \eta_s(s) - w = f_6 \quad (18.6). \tag{43}$$

By integrating the Eqs. (42) and (43), we obtain

$$z(1) = u_t e^{-\tau} + \tau \int_0^1 e^{\tau(1-r)} f_4(r)\, dr \tag{44}$$

and

$$\mu(s) = (1 - e^{-s})(w - f_1) + \int_0^s e^{r-s} f_5(r) dr. \tag{45}$$

Plugging (44) and (45) in (39) and (41) and keeping in mind that $u_t = u - f_1$ and $w_t = w - f_3$ we find that u and w fulfil the equations

$$\begin{cases} \left(1+\alpha_1+\alpha_2 e^{-\tau}\right) u - \frac{2}{1-\mu} u_{xx} &= \beta_1, \\ \widetilde{l} w_{xxxx} + \mathscr{D} w &= \beta_2, \end{cases} \tag{46}$$

where

$$\widetilde{l} = l + \int_0^\infty h(s)(1-e^{-s}) ds > 0, \tag{47}$$

$$\beta_1 = f_2 + \left(1+\alpha_1+\alpha_2 e^{-\tau}\right) f_1 - \alpha_2 \tau \int_0^1 e^{\tau(1-r)} f_4(r)\, dr, \tag{48}$$

$$\begin{aligned} \beta_2 = \mathscr{D}(f_3 + f_4) &- \int_0^\infty h(s) \int_0^s e^{r-s} f_{5xxxx}(r) dr ds \\ &+ \int_0^\infty h(s)(1-e^{-s}) ds f_{1xxxx}. \end{aligned} \tag{49}$$

Since $x \mapsto (f_1, f_5) \in H_0^2(I) \times H_0^2(I)$ then $(f_{1xxxx}, f_{5xxxx}) \in H^{-2}(I) \times H^{-2}(I)$. To show that $\beta_1 \in H^{-2}(I)$, we have to show that

$$x \longmapsto \int_0^\infty h(s) \int_0^s e^{r-s} f_{5xxxx}(r) dr ds \in H^{-2}(I).$$

Applying Cauchy–Shwarz inequality and Fubini's theorem, we get

$$\begin{aligned} \left\| \int_0^\infty h(s) e^{-s} \int_0^s e^r f_{5xx}(r) dr ds \right\|^2 &= \int_0^1 \left(\int_0^\infty h(s) e^{-s} \int_0^s e^r f_{5xx}(r) dr ds \right)^2 dx \\ &\le \int_0^1 \left(\int_0^\infty h(s) e^{-s} \int_0^s e^r \left| f_{5xx}(r) \right| dr ds \right)^2 dx \\ &\le \int_0^1 \left(\int_0^\infty e^r \left| f_{5xx}(r) \right| \int_r^\infty h(s) e^{-s} dr ds \right)^2 dx \\ &\le \int_0^1 \left(\int_0^\infty h(r) e^r \left| f_{5xx}(r) \right| \int_r^\infty e^{-s} ds dr \right)^2 dx \\ &\le \int_0^1 \left(\int_0^\infty h(s) \left| f_{5xx}(r) \right| ds \right)^2 dx \\ &\le \left(\int_0^\infty h(s) ds \right) \int_0^1 \int_0^\infty h(s) f_{5xx}^2(r)(s) ds dx \\ &= h_\infty \left\| f_5 \right\|_h^2 < \infty. \end{aligned}$$

Thus

$$x \longmapsto \int_0^\infty h(s) \int_0^s e^{r-s} f_{5xxxx}(r) dr ds \in H^{-2}(0, L) \tag{50}$$

Therefore, by Lax–Milgram Theorem, system (38) admits a unique solution $\mathscr{U} \in D(\mathscr{A})$. This means that (37) holds and consequently $I - \mathscr{A}$ is onto. Thus, by using the Lumer–Phillips [35, Theorem 4.3], we deduce that the operator $\mathscr{A}$ generates a C_0 semigroup of contractions in $\mathscr{H}$. □

Lemma 6 *The operator $\mathscr{B}$ defined in (32) is locally Lipschitz in $\mathscr{H}$.*

Proof Let $\widetilde{\mathscr{U}} = (u, u_t, w, w_t, z, \eta)$ and $\widetilde{\mathscr{U}} = (\widetilde{u}, \widetilde{u}_t, \widetilde{w}, \widetilde{w}_t, \widetilde{z}, \widetilde{\eta})$ two elements of $\mathscr{H}$. A direct calculation shows that

$$\mathscr{B}(\mathscr{U}) - \mathscr{B}(\widetilde{\mathscr{U}}) = \frac{2}{1-\mu} \left(0, F, 0, \mathscr{D}^{-1} G, 0, 0\right)$$

where

$$F = \left[\frac{1}{2}\left(w^2 - \widetilde{w}^2\right) + k\,(w - \widetilde{w})\right]_x$$

and

$$G = \left[w_x \left(u_x + \frac{1}{2} w^2 + k(x) w\right) - \widetilde{w}_x \left(\widetilde{u}_x + \frac{1}{2}\widetilde{w}^2 + k(x)\widetilde{w}\right)\right]_x$$
$$-k\left[\frac{1}{2}\left(w^2 - \widetilde{w}^2\right) + k\,(w - \widetilde{w})\right].$$

So we have to estimate F and G in $L^2(I)$ and $H_0^1(I)$ norm respectively.

Since $k \in H^1(I)$, we can use the embedding $H^1(I) \hookrightarrow L^\infty(I)$ to prove

$$\begin{aligned}
\|F\| &= \left\|\frac{1}{2}\left(w^2 - \widetilde{w}^2\right) + k(x)\,(w - \widetilde{w})\right\|_{H_0^1} \\
&= \frac{1}{2}\|(w - \widetilde{w})\,(w + \widetilde{w})\|_{H_0^1} + \|k(x)\,(w - \widetilde{w})\|_{H_0^1} \\
&\le C\left(\|w_{xx}\| + \|\widetilde{w}_{xx}\| + \|k\|_{H^1}\right)\|w_x - \widetilde{w}_x\| \\
&\le C\left(\|\mathscr{U}\|_{\mathscr{H}}, \left\|\widetilde{\mathscr{U}}\right\|_{\mathscr{H}}\right)\left\|\mathscr{U} - \widetilde{\mathscr{U}}\right\|_{\mathscr{H}}.
\end{aligned} \tag{51}$$

Now, let

$$G_1 = \mathscr{D}^{-1}\partial_x \left[w_x \left(u_x + \frac{1}{2} w^2 + k(x) w\right) - \widetilde{w}_x \left(\widetilde{u}_x + \frac{1}{2}\widetilde{w}^2 + k(x)\widetilde{w}\right)\right] \tag{52}$$

Taking into account that the operator $\mathscr{D}^{-1}\partial_x$ is bounded from $L^2(I)$ into $H_0^1(I)$, we can write

$$\|G_1\|_{H_0^1} \le C\left\|w_x \left(u_x + \frac{1}{2} w^2 + k(x) w\right) - \widetilde{w}_x \left(\widetilde{u}_x + \frac{1}{2}\widetilde{w}^2 + k(x)\widetilde{w}\right)\right\|. \tag{53}$$

Adding and subtracting the term $\widetilde{w}_x \left(u_x + \frac{1}{2} w^2 + k(x) w\right)$ inside the norm on the right hand side of (27) and proceed with the same manner, we find that

$$\|G_1\|_{H0^1} \le C\left(\|\mathscr{U}\|_{\mathscr{H}}, \left\|\widetilde{\mathscr{U}}\right\|_{\mathscr{H}}\right)\left\|\mathscr{U} - \widetilde{\mathscr{U}}\right\|_{\mathscr{H}}.$$

Finally, let

$$G_2 = \left[\frac{1}{2} \left(w^2 - \widetilde{w}^2 \right) + k(x) \left(w - \widetilde{w} \right) \right]_x .$$

Similarly, have

$$\|G_2\| \le C \left(\|\mathscr{U}\|_{\mathscr{H}}, \|\widetilde{\mathscr{U}}\|_{\mathscr{H}} \right) \|\mathscr{U} - \widetilde{\mathscr{U}}\|_{\mathscr{H}} . \tag{54}$$

Then the operator $\mathscr{B}$ is locally Lipschitz in $\mathscr{H}$. So problem (17)–(19) admits a local solution. The proof is hence complete. □

The boundedness of the energy in (30) allows to extend the solution on $[0, T]$ for an any arbitrary $T > 0$, so we have shown:

Theorem 1 *Assume that* $(\mathbf{H}_1)$ *holds. Let* $\mathscr{U}_0 \in D(\mathscr{A})$, *then* (17)–(19) *has a unique solution*

$$\mathscr{U} \in \mathscr{C}(\mathbb{R}_+, D(\mathscr{A})) .$$

5 General decay

In this section we consider a wider class of kernel functions, and we establish a general decay result, where exponential and polynomial decay rates are special cases.

The main result of general decay is the following.

Theorem 2 *Assume that* (20) *hold such that* (21) *hold or there exists a positive constant* M *such that*

$$\sup_{s>0} \int_0^L \eta_{0xx}^2(s) dx \le M, \ \forall s \in \mathbb{R}_+, \tag{55}$$

then there exists positive constants c', c'' *and* ϵ_0 *for which* $\mathscr{E}$ *satisfies*

$$\mathscr{E}(t) \le c'' e^{-c't}, \forall t \in \mathbb{R}_+, \tag{56}$$

or

$$\mathscr{E}(t) \le c'' G_1^{-1}(c't), \forall t \in \mathbb{R}_+, \tag{57}$$

where

$$G_1(s) = \int_s^1 \frac{dr}{r G'(\epsilon_0 r)}, s \in (0, 1],$$

Remark 2 The previous theorem shows that exponential decay holds when (20) holds, otherwise, if (21) holds, we get a weak decay of energy. For precise examples illustrating (57) see [19–21].

To prove Theorem 2, we need some useful lemmas.

Lemma 7 *The following inequalities holds*

$$\left(\int_0^\infty h(s)\mu(s) ds \right)^2 \le h_\infty \int_0^\infty h(s)\mu^2(s) ds, \tag{58}$$

$$\left(\int_0^\infty h(s)\mu(s) ds \right)^2 \le -h(0) \int_0^\infty h'(s)\mu^2(s) ds. \tag{59}$$

Proof For inequality (58), we have

$$\left(\int_0^\infty h(s)\mu(s)ds\right)^2 = \left(\int_0^\infty \sqrt{h(s)}(\sqrt{h(s)}\mu(s)ds\right)^2.$$

Cauchy–Schwarz inequality leads to

$$\left(\int_0^\infty h(s)\mu(s)ds\right)^2 \le \int_0^\infty h(s)ds \int_0^\infty h(s)\mu^2(s)ds.$$

Similarly, we prove (59) by replacing $\sqrt{h(t-s)}$ by $\sqrt{-h'(t-s)}$. □

Let $\mathscr{F}$ be the functional defined by

$$\mathscr{F}(t) = \lambda\mathscr{E}(t) + \delta_1\mathscr{I}(t) + \delta_2\mathscr{I}_2(t) + \delta_3\mathscr{I}_3(t) \tag{60}$$

where

$$\mathscr{I}_1(t) == \int_0^L u.u_t + w.\mathscr{D}w_t dx, \tag{61}$$

$$\mathscr{I}_2(t) = -\int_0^L w_t \int_0^\infty h(s)\eta(s)dsdx, \tag{62}$$

$$\mathscr{I}_3(t) = \int_0^L \int_0^1 e^{-2\tau p}z(p)dpdx, \tag{63}$$

$\mathscr{E}(t)$ is defined in (26), $\lambda > 0$, δ_1, δ_2 and δ_3 are positive constants that will be chosen later.

The following proposition gives the equivalence between $\mathscr{E}(t)$ and the functional $\mathscr{F}(t)$.

Proposition 1 *Assume that* $(\mathbf{H}_1)$ *holds, then there exists two positive constants* β_1, β_2 *such that*

$$\beta_1\mathscr{E}(t) \le \mathscr{F}(t) \le \beta_2\mathscr{E}(t). \tag{64}$$

Proof To compare $\mathscr{F}(t)$ with $\mathscr{E}(t)$, we have to estimate the terms $\mathscr{I}(t)$, $\mathscr{J}(t)$ and $\mathscr{K}(t)$ of the right hand side of (60) and show that.

$$|\mathscr{F}(t) - \lambda\mathscr{E}(t)| \le c^*\mathscr{E}(t), \quad c^* > 0.$$

From (61), (62) and (63), we obtain

• *Estimate for* $\mathscr{I}_1(t)$

Using Poincaré's and (72, we obtain

$$\begin{aligned}
\delta_1 |\mathscr{I}_1(t)| &= \delta_1 \left|\int_0^L u_t u + \mathscr{D}w_t.wdx\right| \\
&\le \frac{\delta_1}{2}\int_0^L u_t^2 dx + \frac{\delta_1}{2}\int_0^L u^2 dx + \frac{\delta_1}{2}\int_0^L w_t^2 dx \\
&\quad + \frac{\delta_1}{2}\int_0^L w_{xt}^2 dx + \frac{\delta_1}{2}L^2\int_0^L w_{xx}^2 dx \\
&\le c_1\mathscr{E}(t).
\end{aligned} \tag{65}$$

where c_1 is a positive constant.

• *Estimate for* $\mathscr{I}_2(t)$

Using Poincaré's and (58, we obtain

$$\begin{aligned}\delta_2 \left|\mathscr{I}_2(t)\right| &= \delta_2 \left|\int_0^L \mathscr{D} w_t \int_0^\infty h(s)\mu(s) ds dx\right| \\ &\leq \frac{\delta_2}{2}\int_0^L w_t^2 dx + \frac{\delta_2}{2}\int_0^L w_{xt}^2 dx \\ &\quad + \frac{\delta_2}{2} L^2 \int_0^L w_{xx}^2 dx + Ch \circ \eta_{xx} \\ &\leq c_2 \mathscr{E}(t). \end{aligned} \tag{66}$$

where c_2 is a positive constant.

• *Estimate for* $\mathscr{I}_3(t) := \int_0^L \int_0^1 e^{-2\tau p} z^2 dp dx$

Since $\mathscr{I}_3(t)$ defines a norm in $L^2(0, L; L^2(0,1))$ which is equivalent to the one induced by $L^2(0, L; L^2(0,1))$, then we have

$$\delta_3 \left|\mathscr{I}_3(t)\right| \leq \delta_3 \int_0^L \int_0^1 e^{-2\tau p} z^2 dp dx \leq \delta_3 \int_0^L \int_0^1 z^2 dp dx \leq c_3 \mathscr{E}(t), \tag{67}$$

where c_3 is a positive constant.

According to (65), (66) and (67), we have

$$\mid \mathscr{F}(t) - \lambda \mathscr{E}(t) \mid \leq c^* \mathscr{E}(t), \quad c^* > 0,$$

where

$$c^* = \max\{c_1, c_2, c_3\}.$$

Therefore, we obtain

$$\mid \mathscr{F}(t) - \lambda \mathscr{E}(t) \mid \leq c^* \mathscr{E}(t),$$

that is

$$(\lambda - c^*)\mathscr{E}(t) \leq \mathscr{F}(t) \leq (\lambda + c^*)\mathscr{E}(t).$$

So, we choose λ large enough such that $\beta_1 = \lambda - c^* > 0$, $\beta_2 = \lambda + c^* > 0$. Then (64) holds true.

This completes the proof. □

In order to proof the main theorem, we need some additionals lemmas.

Lemma 8 *Suppose that* $(\psi, \psi_t, \eta, \eta_t, z)$ *is the solution of* (17)–(19)*. Then the derivative of the functional* $\mathscr{I}_1(t)$ *satisfies*

$$\begin{aligned}\frac{d}{dt}\mathscr{I}_1(t) \leq C_\varepsilon \int_0^L u_t^2 dx + \int_0^L w_t^2 dx + \int_0^L w_{xt}^2 dx - (l - \varepsilon C)\int_0^L w_{xx}^2 dx \\ - \left(\frac{8}{1-\mu} - \varepsilon C\right)\int_0^L \left(u_x + \frac{1}{2}w^2 + kw\right)^2 dx \\ + C_\varepsilon \int_0^L z^2(1) dx + C_\varepsilon h \circ \eta_{xx}. \end{aligned} \tag{68}$$

where ε *is an arbitrary positive constant.*

Proof Multiplying the first equation in (17) by u, the second by w and taking account that $\frac{d}{dt}\int_0^L h_t h dx = \int_0^L \left(h_{tt}h + h_t^2\right)dx$, then integrating by part, we get

$$\begin{aligned}0 = &-\frac{d}{dt}\mathscr{I}_1(t) + \int_0^L u_t^2 dx + \int_0^L w_t^2 dx + \int_0^L w_{xt}^2 dx - l\int_0^L w_{xx}^2 dx \\ &-\frac{8}{1-\mu}\int_0^L \left(u_x + \frac{1}{2}w^2 + kw\right)^2 dx + \int_0^L (\alpha_1 u_t + \alpha_2 z(1))\, u dx \\ &+\int_0^L w_{xx}\int_0^\infty h(s)\eta_{xx}(s)dsdx. \end{aligned} \tag{69}$$

Using Young's and Poincaré's inequalities, the two last terms are estimated by

$$\int_0^L (\alpha_1 u_t + \alpha_2 z(1))\, u dx \le \varepsilon\int_0^L u^2 dx + C_\varepsilon\int_0^L u_t^2 dx + C_\varepsilon\int_0^L z^2(1)dx \tag{70}$$

and using (58), we arrive at

$$\int_0^L w_{xx}\int_0^\infty h(s)\eta_{xx}(s)dsdx \le \varepsilon\int_0^L w_{xx}^2 dx + C_\varepsilon h\circ\eta_{xx}. \tag{71}$$

Now to estimate $\int_0^L u^2 dx$, we have

$$\begin{aligned}\int_0^L u^2 dx &\le L^2\int_0^L u_x^2 dx \le L^2\int_0^L \left(u_x + \frac{1}{2}w^2 + k(x)w\right)^2 dx \\ &\quad + L^6\int_0^L w_{xx}^4 dx + L^6\left\|k\right\|_{L^\infty}\int_0^L w_{xx}^2 dx \\ &\le L^2\int_0^L \left(u_x + \frac{1}{2}w^2 + k(x)w\right)^2 dx + \left(\frac{L^6 C\mathscr{E}(0)}{4} + L^6\left\|k\right\|_{L^\infty}\right)\int_0^L w_{xx}^2 dx \\ &\le C\left\{\int_0^L \left(u_x + \frac{1}{2}w^2 + k(x)w\right)^2 dx + \int_0^L w_{xx}^2 dx\right\}. \end{aligned} \tag{72}$$

Plugging (70), (71) and (72) in (69) this proves (68). □

Lemma 9 *Assume that* $(\psi, \psi_t, \eta, \eta_t, z)$ *is the solution of* (17)–(19). *Then the derivative of the functional* $\mathscr{I}_2(t)$ *satisfies*

$$\begin{aligned}\frac{d}{dt}\mathscr{I}_2(t) \le &-(h_\infty - 3\varepsilon)\int_0^L w_t^2 dx - (h_\infty - \varepsilon)\int_0^L w_{tx}^2 dx + \left(\varepsilon + \varepsilon^2\right)C\int_0^L w_{xx}^2 dx \\ &+C\varepsilon\int_0^L \left(u_x + \frac{1}{2}w^2 + k(x)w\right)^2 dx - C_\varepsilon h'\circ\eta_{xx} + C_\varepsilon h\circ\eta_{xx}. \end{aligned} \tag{73}$$

where ε *is an arbitrary positive constant.*

Proof Multiplying the second equation in (18) by $\int_0^\infty h(s)\mu(s)ds$ and integrating by parts, we get

$$
\begin{aligned}
0 &= \int_0^L \mathscr{D}w_{tt}\int_0^\infty h(s)\mu(s)dsdx + l\int_0^L w_{xx}\int_0^\infty h(s)\mu_{xx}(s)dsdx \\
&\quad + \int_0^L f(u,w)\int_0^\infty h(s)\eta_x(s)dsdx + \int_0^L k\langle(u,w)wdx + \int_0^L\left(\int_0^\infty h(s)\eta_{xx}(s)ds\right)^2 dx.
\end{aligned} \tag{74}
$$

For the first term in the right hand side of (74), we have

$$
\int_0^L \mathscr{D}w_{tt}\int_0^\infty h(s)\mu(s)ds = -\frac{d}{dt}\mathscr{J}(t) - \int_0^L \mathscr{D}w_t\frac{d}{dt}\int_0^\infty h(s)\mu(s)dsdx. \tag{75}
$$

In the other hand, we also have

$$
\begin{aligned}
\frac{d}{dt}\int_0^\infty h(s)\eta(s)ds &= \frac{d}{dt}\int_0^\infty h(s)\left(w(t)-w(t-s)\right)ds \\
&= \frac{d}{dt}\int_{-\infty}^t h(t-s)\left(w(t)-w(s)\right)ds \\
&= \int_{-\infty}^t h'(t-s)\left(w(t)-w(t-s)\right) + h_\infty w_t \\
&= \int_0^\infty h'(s)\eta(s)ds + h_\infty w_t.
\end{aligned} \tag{76}
$$

Plugging (76) in (75), we infer

$$
\begin{aligned}
\int_0^L \mathscr{D}w_t\int_0^\infty h(s)\mu(s)dsdx &= -\frac{d}{dt}\mathscr{J}(t) - h_\infty\int_0^L w_t^2dx - h_\infty\int_0^L w_{tx}^2dx \\
&\quad + \int_0^L \mathscr{D}w_t\int_0^\infty h'(s)\eta(s)dsdx.
\end{aligned} \tag{77}
$$

Integrating by parts and applying Young's inequality and (59), then Poincaré's inequality, we get

$$
\begin{aligned}
\int_0^L \mathscr{D}w_t\int_0^\infty h'(s)\mu(s)dsdx &= \int_0^L (w_t - w_{txx})\int_0^\infty h'(s)\eta(s)dsdx \\
&\le 2\varepsilon\int_0^L w_t^2dx - C_\varepsilon h'\circ\eta_{xx}.
\end{aligned} \tag{78}
$$

Using the embedding $H^1(I)\hookrightarrow L^\infty(I)$, we estimate $-\int_0^L [f(u,w)]_x\int_0^\infty h(s)\mu(s)dsdx$ as follows

$$
\begin{aligned}
-\int_0^L & [f(u,w)]_x\int_0^\infty h(s)\mu(s)dsdx \\
&= -\frac{2}{1-\mu}\int_0^L w_x\left(u_x + \frac{1}{2}w^2 + k(x)w\right)\int_0^\infty h(s)\eta_x(s)dsdx \\
&\le \varepsilon^2\int_0^L\left[w_x\left(u_x + \frac{1}{2}w^2 + k(x)w\right)\right]^2dx + C_\varepsilon h\circ\eta_{xx} \\
&\le C\varepsilon^2\|w_x\|_{L\infty}\int_0^L\left(u_x + \frac{1}{2}w^2 + k(x)w\right)^2dx + C_\varepsilon h\circ\eta_{xx}
\end{aligned}
$$

$$\leq C\varepsilon^3 \int_0^L w_{xx}^2 dx + C\varepsilon \int_0^L \left(u_x + \frac{1}{2}w^2 + k(x)w\right)^2 dx$$
$$+C_\varepsilon h \circ \eta_{xx}. \tag{79}$$

Similarly, we estimate $\int_0^L kg(u, w)wdx$ using the umbedding $H^1(I) \hookrightarrow L^\infty(I)$.

Gathering (77), (78) and (79), (73) is proven. □

Lemma 10 *Suppose that* $(\psi, \psi_t, \eta, \eta_t, z)$ *is the solution of* (17)–(19)*. Then the time derivative of the functional* $\mathscr{I}_3(t)$ *satisfies*

$$\frac{d}{dt}\mathscr{I}_3(t) = \frac{e^{-2\tau}}{\tau}\int_0^L z^2(1) - \frac{1}{\tau}\int_0^L \eta_t^2 dx - 2\mathscr{I}_3(t). \tag{80}$$

Proof Multiplying the third equation in (17) by $e^{-2\tau p}z$ and integrating over $I \times (0, 1)$, we arrive at

$$\begin{aligned} 0 &= \tau \int_0^L \int_0^1 e^{-2\tau p} z(p) z_t(p) dp dx - \int_0^L \int_0^1 e^{-2\tau p} z(p) z_p(p) dp dx \\ &= \frac{\tau}{2}\frac{d}{dt}\int_0^L \int_0^1 e^{-2\tau p} z^2(p) dp dx - \frac{1}{2}\int_0^L \int_0^1 e^{-2\tau p}\frac{d}{dp}\left[z^2(p)\right] dp dx \\ &= \frac{\tau}{2}\frac{d}{dt}\mathscr{I}_3(t) + \frac{1}{2}\int_0^L u_t^2 dx - \frac{e^{-2\tau}}{2}\int_0^L z(1)^2 dx + \tau \mathscr{I}_3(t), \end{aligned}$$

which gives (80) □

Proposition 2 *Assume that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold, then there exists two positive constants* β_1, β_2 *such that*

$$\frac{d}{dt}\mathscr{F}(t) \leq -\beta_1 \mathscr{E}(t) + \beta_2 h \circ \psi_{xx}. \tag{81}$$

Proof By using (60) and combining (30), (68), (73) and (80), we get

$$\begin{aligned} \frac{d}{dt}\mathscr{F}(t) \leq & \left\{-N_1(l - \varepsilon C) + N_2 C\left(\varepsilon + \varepsilon^2\right)\right\} \|\psi_{xx}\|^2 \\ & + \{N_1 - N_2(h_\infty - 3\varepsilon)\} \|\psi_t\|^2 + \left\{N_1 C_\varepsilon + \lambda(\frac{\xi}{2\tau} + \frac{\alpha_2}{2} - \alpha_1)\right\} \|\eta_t\|^2 \\ & + \{N_1 - N_2(h_\infty - \varepsilon)\} \\ & \times \|\psi_{tx}\|^2 + \left\{N_1\left(\varepsilon C - \frac{8}{1-\mu}\right) + C N_2 \varepsilon\right\} \left\|u_x + \frac{1}{2}w^2 + k(x)w\right\|^2 \\ & - \left(N_2 C_\varepsilon - \frac{\lambda}{2}\right) h' \circ \psi_{xx} + C_\varepsilon (N_1 + N_2) h \circ \psi_{xx} - 2\mathscr{I}_3(t) \\ & + \left\{N_2 C_\varepsilon - \frac{e^{-2\tau}\alpha_1}{\tau} + \lambda(\frac{\alpha_2}{2} - \frac{\xi}{2\tau})\right\} \|z(1)\|^2. \end{aligned} \tag{82}$$

We want to impose suitable conditions on the different parameters so that the coefficients on the right hand side of (82) are all strictly negative. That is to obtain the following inequalities

$$N_2 C\left(\varepsilon + \varepsilon^2\right) < N_1(l - \varepsilon C), \tag{83}$$
$$N_1 < N_2(h_\infty - 3\varepsilon), \tag{84}$$

$$N_1 < N_2\left(h_\infty - \varepsilon\right), \tag{85}$$

$$CN_2\varepsilon < N_1\left(\frac{8}{1-\mu} - \varepsilon C\right), \tag{86}$$

$$N_2 C_\varepsilon < \frac{\lambda}{2}, \tag{87}$$

$$N_2 C_\varepsilon < \frac{e^{-2\tau}\alpha_1}{\tau} - \lambda\left(\frac{\alpha_2}{2} - \frac{\xi}{2\tau}\right), \tag{88}$$

$$N_1 C_\varepsilon < -\lambda\left(\frac{\xi}{2\tau} + \frac{\alpha_2}{2} - \alpha_1\right). \tag{89}$$

We observe that (83) and (84) will be satisfied if we choose $\varepsilon > 0$ small enough and such that

$$\varepsilon < \max\left\{\frac{l}{C}, \frac{h_\infty}{3}, \frac{8}{C\left(1-\mu\right)}\right\}.$$

To make (84) and (85) hold we can choose

$$N_1 < N_2\left(h_\infty - 3\varepsilon\right).$$

Concerning (87), (88) and (89), we pick

$$\lambda = \max\left\{2N_2C_\varepsilon, -\frac{N_2C_\varepsilon}{\left(\frac{\alpha_2}{2} - \frac{\xi}{2\tau}\right)}, -\frac{N_1C_\varepsilon}{\left(\frac{\xi}{2\tau} + \frac{\alpha_2}{2} - \alpha_1\right)}, c^*\right\},$$

This completes the proof. □

We consider the following two cases.

Case I. $H(t)$ is linear:

By multiplying (81) by $\xi(t)$ and using (30), we get

$$\begin{aligned}\frac{d}{dt}\mathscr{F}(t)\xi(t) &\leq -\beta_1\xi(t)\mathscr{E}(t) + \beta_2\xi(t)h \circ \psi_{xx}\\ &\leq -\beta_1\xi(t)\mathscr{E}(t) + \beta_2 h\xi \circ \psi_{xx}\\ &\leq -\beta_1\xi(t)\mathscr{E}(t) - \beta_2 a h' \circ \psi_{xx}\\ &\leq -\beta_1\xi(t)\mathscr{E}(t) - c\mathscr{E}'(t)\end{aligned}$$

which gives, as ξ is nonincreasing,

$$\frac{d}{dt}\left(\mathscr{F}(t)\xi(t) + c\mathscr{E}(t)\right) \leq -\beta_1\xi(t)\mathscr{E}(t), \quad \forall t \geq t_1.$$

Hence, using the fact that $\mathscr{F}(t)\xi(t) + c\mathscr{E}(t)$ is equivalent to $\mathscr{E}(t)$, it is easy to see that

$$\frac{d}{dt}\left(\mathscr{F}(t)\xi(t) + c\mathscr{E}(t)\right) \leq -\beta_1\xi(t)(\mathscr{F}(t)\xi(t) + c\mathscr{E}(t)), \quad \forall t \geq t_1.$$

for some $\beta_1 > 0$. Then

$$(\mathscr{F}(t)\xi(t) + c\mathscr{E}(t)) \leq \gamma_2 e^{-\beta_1\int_{t_1}^t \xi(s)ds}, \quad \forall t \geq t_1$$

from which we deduce

$$\mathscr{E}(t) \leq \gamma_2 e^{-\beta_1\int_{t_1}^t \xi(s)ds}, \forall t \geq t_1$$

for some $\gamma_2 > 0$.

Furthermore, using the continuity and boundedness of $\mathscr{E}(t)$ in $[0, t_1]$, we get

$$\mathscr{E}(t) \leq \gamma_2 e^{-\beta_1 \int_0^t \xi(s)ds}, \quad \forall t \geq 0.$$

Case II. $H(t)$ is *nonlinear*:

Next, with $f(t) = \int_t^\infty h(s)ds$, we use the functional

$$\mathscr{K}(t) = \int_0^t f(t-s) \|\psi_{xx}(s)\|_I^2 \, ds. \tag{90}$$

Lemma 11 *Suppose that* $(\psi, \psi_t, \eta, \eta_t, z)$ *is the solution of* ((17))–(19). *The functional* $\mathscr{K}$ *defined by* (90) *satisfies, for any* $\varepsilon > 0$, *the estimate*

$$\frac{d}{dt}\mathscr{K}(t) \leq (2\varepsilon - 1)\, h \circ \psi_{xx} + (f(t) + C_\varepsilon) \|\psi_{xx}\|^2. \tag{91}$$

Proof By Young's inequality and the fact $f'(t) = -h(t)$, we see that

$$\begin{aligned} \frac{d}{dt}\mathscr{K}(t) &= f(0) \|\psi_{xx}\|^2 - \int_0^t h(t-s) \|\psi_{xx}(s)\|^2 \, ds \\ &= -h \circ \psi_{xx} - 2 \langle \psi_{xx}, h \diamond \psi_{xx} \rangle + f(t) \|\psi_{xx}\|^2. \end{aligned} \tag{92}$$

But

$$-2 \langle \psi_{xx}, h \diamond \psi_{xx} \rangle \leq C_\varepsilon \|\psi_{xx}\|^2 + 2\varepsilon h \circ \psi_{xx}, \tag{93}$$

Combining (92) and (93), we obtain (91). □

Let us introduce the functional

$$\widetilde{\mathscr{F}}(t) = \mathscr{F}(t) + \sigma \mathscr{K}(t),$$

where σ is a positive constant. Then we have

$$\widetilde{\mathscr{F}}(t) \sim \mathscr{E}(t).$$

Therefore, it is always possible to pick N_1 (in 82) and h large enough to get

$$\frac{d}{dt}\widetilde{\mathscr{F}}(t) \leq -C\mathscr{E}(t).$$

Integrating over (t_0, ∞), we get

$$C \int_{t_0}^\infty \mathscr{E}(s)ds \leq \widetilde{\mathscr{F}}(t_0) < \infty. \tag{94}$$

Next, let us define the functional $\mathscr{L}(t)$

$$\mathscr{L}(t) = q \int_{t_0}^t \|\psi_{xx}(s) - \psi_{xx}(t-s)\|_I^2 \, ds, \forall t \geq t_0.$$

where $q > 0$. Thanks to (94), we can always choose q such that

$$\mathscr{L}(t) < 1, \forall t \geq t_0. \tag{95}$$

Next we define

$$\mathscr{L}_h(t) = -\int_{t_0}^t h'(s) \|\psi_{xx}(t) - \psi_{xx}(t-s)\|^2 \, ds, \; \forall t \geq t_0.$$

Observe that

$$\mathscr{L}_h(t) \le -C\mathscr{E}'(t),$$

for some positive constant C.

Since H is strictly convexe on $(0, r]$ and $H(0) = 0$ we have

$$H(\theta x) \le \theta H(x), \quad (\theta, x) \in [0, 1] \times (0, r]. \tag{96}$$

Using $(\mathbf{H}_2)$, we get:

$$\begin{aligned}
\mathscr{L}_h(t) &= \frac{1}{q\mathscr{L}(t)} \int_{t_0}^{t} \mathscr{L}(t) \left(-h'(s)\right) q \left\|\psi_{xx}(t) - \psi_{xx}(t-s)\right\|^2 ds \\
&\ge \frac{1}{q\mathscr{L}(t)} \int_{t_0}^{t} \mathscr{L}(t)\xi(s) H\left(h(s)\right) q \left\|\psi_{xx}(t) - \psi_{xx}(t-s)\right\|^2 ds \\
&\ge \frac{\xi(t)}{q\mathscr{L}(t)} \int_{t_0}^{t} H\left(\mathscr{L}(t)h(s)\right) q \left\|\psi_{xx}(t) - \psi_{xx}(t-s)\right\|^2 ds,
\end{aligned}$$

Keeping in mind (95) and applying inequality (96) for $\theta := \mathscr{L}(t)$ and $x = h(s)$, yields

$$\mathscr{L}_h(t) \ge \frac{\xi(t)}{q\mathscr{L}(t)} \int_{t_0}^{t} H\left(\mathscr{L}(t)h(s)\right) q \left\|\psi_{xx}(t) - \psi_{xx}(t-s)\right\|^2 ds. \tag{97}$$

Applying Jensen's inequality in (31) for $r_1(t) = \mathscr{L}(t)h(s)$ and $r_2(s) = q \left\|\psi_{xx}(t) - \psi_{xx}(t-s)\right\|^2$, we obtain

$$\begin{aligned}
\mathscr{L}_h(t) &\ge \frac{\xi(t)}{q\mathscr{L}(t)} \int_{t_0}^{t} H\left(\mathscr{L}(t)h(s)\right) q \left\|\psi_{xx}(t) - \psi_{xx}(t-s)\right\|^2 ds \\
&\ge \frac{\xi(t)}{q} H\left(\frac{1}{\mathscr{L}(t)} \int_{t_0}^{t} \mathscr{L}(t)h(s)q \left\|\psi_{xx}(t) - \psi_{xx}(t-s)\right\|^2 ds\right) \\
&= \frac{\xi(t)}{q} H\left(\int_{t_0}^{t} h(s)q \left\|\psi_{xx}(t) - \psi_{xx}(t-s)\right\|^2 ds\right) \\
&= \frac{\xi(t)}{q} \overline{H}\left(\int_{t_0}^{t} h(s)q \left\|\psi_{xx}(t) - \psi_{xx}(t-s)\right\|^2 ds\right),
\end{aligned}$$

where $\overline{H}$ is an extention of H such that $\overline{H}$ is strictly increasing and strictly convexe $\mathscr{C}^2$ function on $(0, \infty)$ and this leads to

$$\int_{t_0}^{t} g(s)q \left\|\psi_{xx}(t) - \psi_{xx}(t-s)\right\|^2 ds \le \frac{1}{q}\overline{H}^{-1}\left(\frac{q\mathscr{L}_g(t)}{\xi(t)}\right).$$

So (81) becomes

$$\widetilde{\mathscr{F}}(t) \le -\beta_1\mathscr{E}(t) + \beta_2\frac{1}{q}\overline{H}^{-1}\left(\frac{q\mathscr{L}_g(t)}{\xi(t)}\right), \ \forall t > t_0. \tag{98}$$

Let $\epsilon_0 < r$, using the fact that $\mathscr{E}' \le 0$, $\overline{H}' > 0$, $\overline{H}'' > 0$, we observe that the functional $\mathscr{N}$ defined by

$$\mathscr{N}(t) := \overline{H}'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right)\mathscr{F}(t) + \mathscr{E}(t),$$

is equivalent to $\mathscr{E}$.

Using (98), we find that $\mathscr{N}$ satisfies

$$\begin{aligned}\frac{d}{dt}\mathscr{N}(t) &= \epsilon_0 \frac{\mathscr{E}'(t)}{\mathscr{E}(0)}\overline{H}''\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right)\mathscr{F}(t) + \overline{H}'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right)\mathscr{F}'(t) + \mathscr{E}'(t)\\ &\le \epsilon_0 \frac{\mathscr{E}'(t)}{\mathscr{E}(0)}\overline{H}''\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right)\mathscr{F}(t)\\ &\quad + \overline{H}'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right)\left[-\beta_1\mathscr{E}(t) + \beta_2\frac{1}{q}\overline{H}^{-1}\left(\frac{q\mathscr{L}_g(t)}{\xi(t)}\right)\right] + \mathscr{E}'(t)\\ &\le -\beta_1\mathscr{E}(t)\overline{H}'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right)\\ &\quad + \frac{\beta_2}{q}\overline{H}^{-1}\left(\frac{q\mathscr{L}_g(t)}{\xi(t)}\right)\overline{H}'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right) + \mathscr{E}'(t).\end{aligned} \tag{99}$$

Let us denote by G^* the conjugate function of the convex function G defined by $G^*(s) = Sup_{t\in\mathbb{R}^+}(st - G(t))$, then

$$st \le G^*(s) + G(t), \tag{100}$$

and, thanks to the arguments given in [5, 9, 14, 29, 30]

$$\overline{G}^*(s) = s(\overline{G}')^{-1}(s) - \overline{G}\left[(\overline{H}')^{-1}(s)\right], \quad \forall s \ge 0.$$

This and the definition of H give

$$\overline{H}^*(s) = s(\overline{H}')^{-1}(s) - \overline{H}\left[(\overline{H}')^{-1}(s)\right], \quad \forall s \ge 0. \tag{101}$$

Taking $s := \frac{C_2}{q}\overline{H}'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right)$ and $t := \overline{H}^{-1}\left(\frac{q\mathscr{L}_g(t)}{\xi(t)}\right)$ in (100), then making use of (99), (100) and (101), we arrive at

$$\begin{aligned}&\frac{d}{dt}\mathscr{N}(t)\\ &\le -\beta_1\mathscr{E}(t)\overline{H}'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right) + \overline{H}\left[\overline{H}^{-1}\left(\frac{q\mathscr{L}_g(t)}{\xi(t)}\right)\right] + \overline{H}^*\left[\frac{\beta_2}{q}\overline{H}'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right)\right] + \mathscr{E}'(t)\\ &\le -\beta_1\mathscr{E}(t)\overline{H}'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right) + \frac{q\mathscr{L}_g(t)}{\xi(t)} + \overline{H}^*\left[\frac{\beta_2}{q}\overline{H}'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right)\right] + \mathscr{E}'(t)\\ &\le -\beta_1\mathscr{E}(t)\overline{H}'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right) + \frac{q\mathscr{L}_g(t)}{\xi(t)} + \frac{\beta_2\epsilon_0}{q}\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\overline{H}'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right) + \mathscr{E}'(t).\end{aligned} \tag{102}$$

Next, multiplying (102) by $\xi(t)$ and using the fact that $\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)} < r$, $\overline{H}'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right) = H'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right)$, we get

$$\begin{aligned}\xi(t)\frac{d}{dt}\mathscr{N}(t) &\le -\beta_1\mathscr{E}(t)\xi(t)\overline{H}'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right) + q\mathscr{L}_g(t)\\ &\quad + \frac{\beta_2\epsilon_0}{q}\xi(t)\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\overline{H}'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right) + \xi(t)\mathscr{E}'(t)\\ &\le -\beta_1\mathscr{E}(t)\xi(t)\overline{H}'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right) + \frac{\beta_2\epsilon_0}{q}\xi(t)\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\overline{H}'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right) - c\mathscr{E}'(t),\end{aligned}$$

where c is a positive constant [58].

Now, let us define the functional $\widetilde{\mathscr{N}}$

$$\widetilde{\mathscr{N}}(t) = \mathscr{N}(t)\xi(t) + \mathscr{E}(t).$$

It is not difficult to see that there exist positive constants ρ_1 and ρ_2 for which we have

$$\rho_1\widetilde{\mathscr{N}}(t) \le \mathscr{E}(t) \le \rho_2\widetilde{\mathscr{N}}(t). \tag{103}$$

Consequently, with an appropriate choice of ϵ_0, then there exists a positive constant k such that

$$\frac{d}{dt}\widetilde{\mathscr{N}}(t) \le -k\xi(t)\overline{H}'\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right) = -k\xi(t)H_2\left(\epsilon_0\frac{\mathscr{E}(t)}{\mathscr{E}(0)}\right)\ ,\ \forall t \ge t_0 \tag{104}$$

where $H_2(s) = sH'(\epsilon_0 s)$.

Since $H_2'(s) = H'(\epsilon_0 s) + \epsilon_0 sH''(\epsilon_0 s)$, we use the strict convexity of H on $[0, r)$, we observe that $H_2 > 0$, $H_2' > 0$ on $(0, r]$.

Defining now

$$\mathscr{R}(t) = \frac{\delta_1\widetilde{\mathscr{N}}(t)}{\mathscr{E}(0)},$$

thanks to (103) and (104) we have $\mathscr{E} \sim \mathscr{R}$ and for a positive constant $\widetilde{k}$

$$\frac{d}{dt}\mathscr{R}(t) \le -\widetilde{k}\xi(t)H_2(\mathscr{R}(t))\ ,\ \forall t \ge t_0.$$

Then, integrating over (t_0, t) yields

$$\int_{t_0}^{t}\frac{\mathscr{R}'(s)}{H_2(\mathscr{R}(s))} \le -\int_{t_0}^{t}\widetilde{k}\xi(s)ds,$$

and this leads to

$$\int_{\epsilon_0\mathscr{R}(t)}^{\epsilon_0\mathscr{R}(t_0)}\frac{\mathscr{R}'(s)}{H'(\mathscr{R}(s))} \ge \widetilde{k}\int_{t_0}^{t}\xi(s)ds,$$

which gives us

$$\mathscr{R}(t) \le \tfrac{1}{\epsilon_0}H_1^{-1}\left(\widetilde{k}\textstyle\int_{t_0}^{t}\xi(s)ds\right),$$

where $H_1(t) = \int_t^r \frac{ds}{sH'(s)}$.

This completes the proof.

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References

1. Alabau-Boussouira, F.: On convexity and weighted integral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems. Appl. Math. Optim. **51**, 61–105 (2005)
2. Alabau-Boussouira, F., Cannarsa, P.: A general method for proving sharp energy decay rates for memory dissipative evolution equations. C. R. Acad. Sci. Paris Ser. **I**(347), 867–872 (2009)
3. Al-Mahdi, A.M., Al-Gharabli, M.M., Messaoudi, S.A.: New general decay of solutions in a porous thermoelastic system with infinite memory. J. Math. Anal. Appl. **500**, 125–136 (2021)
4. Araruna, F.D., Silva, P.B.E., Zuazua, E.: Asymptotic limits and stabilization for the 1-D nonlinear Mindlin–Timoshenko system. J. Syst. Sci. Complex. **23**, 1–17 (2010)
5. Arnold, V.I.: Mathematical Methods of Classical Mechanics. Springer, New York (1989)
6. Benaissa, A., Louhibi, N.: Global existence and energy decay of solutions to a nonlinear wave equation with a delay term. Georgian Math. J. **20**, 1–24 (2013)
7. Benaissa, A., Benguessoum, A., Messaoudi, S.A.: Global existence and energy decay of solutions to a viscoelastic wave equation with a delay term in the nonlinear internal feedback. Int. J. Dyn. Syst. Differ. Equ. **5**, 1–26 (2014)
8. Berrimi, S., Messaoudi, S.A.: Existence and decay of solutions of a viscoelastic equation with a nonlinear source. Nonlinear Anal. Theory Methods Appl. **64**, 2314–2331 (2006)
9. Cavalcanti, M.M., Cavalcanti, V.D., Lasiecka, I.: Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction. J. Differ. Equ. **236**, 407–459 (2007)
10. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Soriano, J.A.: Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping. Electron. J. Differ. Equ. 1–14 (2002)
11. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Martinez, P.: General decay rate estimates for viscoelastic dissipative systems. Nonlinear Anal. **68**, 177–193 (2008)
12. Chueshov, I., Lasiecka, I.: Global attractors for von Kármán evolutions with a nonlinear boundary dissipation. J. Differ. Equ. **198**, 196–231 (2004)
13. Chueshov, I., Lasiecka, I.: Global attractors for Mindlin–Timoshenko plates and for their Kirchhoff limits. Milan J. Math. **74**, 117–138 (2006)
14. Daoulatli, M., Lasiecka, I., Toundykov, D.: Uniform energy decay for a wave equation with partially supported nonlinear boundary dissipation without growth restrictions. Discrete Contin. Dyn. Syst. **2**, 67–94 (2009)
15. Datko, R.: Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks. SIAM J. Control. Optim. **26**, 697–713 (1988)
16. Doyle, J.F.: Wave Propagation in Structures. Springer, New York (1997)
17. Eller, M., Lagnese, J.E., Nicaise, S.: Decay rates for solutions of a Maxwell system with nonlinear boundary damping. Comput. Appl. Math. **21**, 135–165 (2002)
18. Favini, A., Horn, M.A., Lasiecka, I., Tartaru, D.: Global existence, uniqueness and regularity of solutions to a von Kármán system with nonlinear boundary dissipation. Differ. Integr. Eqn. **9**, 267–294 (1996)
19. Guesmia, A.: Asymptotic stability of abstract dissipative systems with infinite memory. J. Math. Anal. Appl. **382**, 748–760 (2011)
20. Guesmia, A., Messaoudi, S.: A general decay result for a viscoelastic equation in the presence of past and infinite history memories. Nonlinear Anal. **13**, 476–485 (2012)
21. Guesmia, A., Messaoudi, S., Soufyane, A.: On the stabilization for a linear Timoshenko system with infinite history and applications to the coupled Timoshenko-heat systems. Electron. J. Differ. Equ. **2012**, 1–45 (2012)
22. Han, X., Wang, M.: General decay of energy for a viscoelastic equation with nonlinear damping. Math. Methods Appl. Sci. **32**, 346–358 (2009)
23. Hassan, J.H., Messaoudi, S.A., Zahri, M.: Existence and new general decay results for a viscoelastic Timoshenko system. J. Anal. Appl. **39**, 185–222 (2020)
24. Kafini, M., Messaoudi, S.A., Mustafa, M.I.: Energy decay rates for a Timoshenko-type system of thermoelasticity of type III with constant delay. Appl. Anal. **93**, 1201–1216 (2014)
25. Khemmoudj, A., Mokhtari, Y.: General decay of the solution to a nonlinear viscoelastic modified Von-Karman system with delay. Discrete Contin. Dyn. Syst. Ser. A (2019). https://doi.org/10.3934/dcds.2019155
26. Lagnese, J.E.: Boundary Stabilization of Thin Plates. SIAM, Philadelphia (1989)
27. Lagnese, J.E., Leucering, G.: Uniform stabilization of a nonlinear beam by nonlinear boundary feedback. J. Differ. Equ. **91**, 355–388 (1991)
28. Lagnese, J.E., Lions, J.L.: Modelling Analysis and Control of Thin Plates, RMA 6. Masson, Paris (1988)
29. Lasiecka, I.: Mathematical Control Theory of Coupled PDE's (CBMS-NSF Regional Conference Series in Applied Mathematics). SIAM, Philadelphia (2002)

30. Lasiecka, I., Doundykov, D.: Energy decay rates for the semilinear wave equation with nonlinear localized damping and a nonlinear source, Nonlinear. Analysis **64**, 1757–1797 (2006)
31. Lasiecka, I., Tataru, D.: Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. Differ. Integr. Equ. **6**, 507–533 (1993)
32. Lasiecka, I., Wang, X.: Intrinsic decay rate estimates for semilinear abstract second order equations with memory. In: New Prospects in Direct, Inverse and Control Problems for Evolution Equations, Springer INdAM, vol. 10, pp. 271–303. Springer, Cham (2014)
33. Lee, M.J., Park, J.Y., Kang, Y.H.: Exponential decay rate for a quasilinear von Kármán equation of memory type with acoustic boundary conditions. Bound. Value Probl. **2015**, 14 (2015)
34. Liu, W.J., Zuazua, E.: Decay rates for dissipative wave equations. Ricerche Mat. **48**, 61–75 (1999)
35. Liu, Z., Zheng, S.: Semigroups associated with dissipative systems. Chapman and Hall/CRC Research Notes in Mathematics, vol. 398. Chapman and Hall/CRC, Boca Raton (1999)
36. Messaoudi, S.A.: General decay of the solution energy in a viscoelastic equation with a nonlinear source. Nonlinear Anal. **69**, 2589–2598 (2008)
37. Messaoudi, S.A.: General decay of solutions of a viscoelastic equation. J. Math. Anal. Appl. **341**, 1457–1467 (2008)
38. Munoz Rivera, J.E., Menzala, G.P.: Decay rates of solutions of a von Kármán system for viscoelastic plates with memory. Q. Appl. Math. **57**, 181–200 (1999)
39. Munoz Rivera, J.E., Portillo Oquendo, H., Santos, M.L.: Asymptotic behavior to a von Kármán plate with boundary memory conditions. Nonlinear Anal. **62**, 1183–1205 (2005)
40. Mustafa, M.I.: Optimal decay rates for the viscoelastic wave equation. Math. Methods Appl. Sci. **41**, 192–204 (2018)
41. Mustafa, M.I.: Well posedness and asymptotic behavior of a coupled system of nonlinear viscoelastic equations. Nonlinear Anal. Real World Appl. **13**, 452–463 (2012)
42. Mustafa, M.I.: Uniform decay rates for viscoelastic dissipative systems. J. Dyn. Control Syst. **22**, 101–116 (2016)
43. Mustapha, M.I.: General decay result for nonlinear viscoelastic equation. J. Math. Anal. Appl. **457**, 134–152 (2018)
44. Mustapha, M.I.: Laminated Timoshenko beams with viscoelastic damping. J. Math. Anal. Appl. **466**, 619–641 (2018)
45. Mustapha, M.I., Messaoudi, S.A.: General stability result for viscoelastic wave equations. J. Math. Phys. **53**, 053702, 14 (2012)
46. Nicaise, S., Pignotti, C.: Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. SIAM J. Control. Optim. **45**, 1561–1585 (2006)
47. Nicaise, S., Pignotti, C.: Stabilization of the wave equation with boundary or internal distributed delay. Differ. Integral Equ. **21**, 935–958 (2008)
48. Park, S.H.: General decay for a viscoelastic von Karman equation with delay and variable exponent nonlinearities. Bound. Value Probl. **2022**, Paper No. 23, 29 (2022). https://doi.org/10.1186/s13661-022-01602-4
49. Park, S.H.: A general decay result for a von Karman equation of memory type with acoustic boundary conditions. J. Appl. Anal. Comput. **12**(1), 17–30 (2022). https://doi.org/10.11948/20200460
50. Park, J.Y., Park, S.H.: Uniform decay for a von Kármán plate equation with a boundary memory condition. Math. Methods Appl. Sci. **28**, 2225–2240 (2005)
51. Park, S.H., Park, J.Y., Kang, Y.H.: General decay for a von Kármán equation of memory type with acoustic boundary conditions. Z. Angew. Math. Phys. **63**, 813–823 (2012)
52. Park, J.Y., Park, S.H.: General decay for quasilinear viscoelastic equations with nonlinear weak damping. J. Math. Phys. **50**, Article ID: 083505, 10 (2009)
53. Perla Menzala, G., Zuazua, E.: On a one-dimensional version of the dynamical Marguerre–Vlasov system. Bol. Soc. Bras. Mat. **32**, 303–319 (2001)
54. Perla Menzala, G., Zuazua, E.: The beam equation as a limit of a 1-D nonlinear von Kármán model. Appl. Math. Lett. **12**, 47–52 (1999)
55. Perla Menzala, G., Zuazua, E.: Timoshenko's plate equation as a singular limit of the dynamical von Kármán system. J. Math. Pures Appl. **79**, 73–94 (2000)
56. Perla Menzala, G., Zuazua, E.: Timoshenko's beam equation as a limit of a nonlinear one-dimentional von Kármán system. Proc. R. Soc. Edinb. Sect. A **130**, 855–875 (2000)
57. Shinskey, F.G.: Process Control Systems. McGraw-Hill Book Company, New York (1967)
58. Simon, J.: Compact sets in the space $L^p(0, T; B)$. Ann. Mat. Pura Appl. **146**, 65–96 (1987)
59. Suh, I.H., Bien, Z.: Use of time delay action in the controller design. IEEE Trans. Autom. Control **25**, 600–603 (1980)

60. Wu, S.T.: Asymptotic behavior for a viscoelastic wave equation with a delay term. Taiwan. J. Math. **17**, 765–784 (1980)
61. Xu, C.Q., Yung, S.P., Li, L.K.: Stabilization of the wave system with input delay in the boundary control. ESAIM Control Optim. Calc. Var. **12**, 770–785 (2006)