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ORIGINAL PAPER

Mittag-leffler-type function of arbitrary order and their application in the fractional kinetic equation

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Abstract
In this paper, we stress the importance of the Mittag–Leffler function of two parameters and a single variable in the framework of mathematical physics and applied mathematics. We begin with pseudo hyperbolic and trigonometric functions and progress to introduce an arbitrary order Mittag–Leffler-type function. We study its properties, basic relations, integral representations, pure relations, and differential relations. We then justify the relevance of the arbitrary Mittag–Leffler-type function as a solution to the fractional kinetic equation. Also, we discuss the connection with known families of Mittag-Leffler functions and elementary functions and use operational tools to analyze all associated problems from a unified perspective.

Keywords Mittag–Leffer function · Recurrence relations · Integral relations · Fractional kinetic equations · Lapla-ce transforms

Mathematics Subject Classification 33C45 · 33E12

1 Introduction and definitions

The special functions

$$E_k(z; j) = \sum_{n=0}^{\infty} \frac{z^{nj+k}}{(nj+k)!} (j, k \in \mathbb{N}, j \geq 1, k \geq 0), \tag{1.1}$$

$$S_k(z; j) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{nj+k}}{(nj+k)!} (j, k \in \mathbb{N}, j \geq 1, k \geq 0), \tag{1.2}$$

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and their general form

$$F_{j,k}^{\lambda}(z) = \sum_{n=0}^{\infty} \frac{\lambda^n z^{nj+k}}{(nj+k)!}, (j, k \in \mathbb{N}, j \geq 1, k \geq 0, \lambda \in \mathbb{C}), \tag{1.3}$$

are called the pseudo-hyperbolic functions of order j, the pseudo-trigonometric functions of order j and the λ-hyperbolic functions of order λ respectively. The former (1.1) was introduced by [9]; see also [27]. The function defined by (1.2) appeared in the work of Erd́elyi et al.([9]; see also [4, 5] and [3]). The function $F_{j,k}^{\lambda}(z)$ was introduced by Muldoon and Ungar [23]. It should be mentioned the generalized λ-hyperbolic functions $F_{j,k}^{\lambda}(z)$ are related to the functions in (1.1) and (1.2) by the relations

$$F_{j,k}^{1}(z) = E_k(z; j), \tag{1.4}$$

$$F_{j,k}^{-1}(z) = S_k(z; j). \tag{1.5}$$

The importance of the pseudo-hyperbolic and pseudo-trigonometric functions in (1.1) and (1.2) in applications has been recognized recently within the context of problems involving arbitrary order coherent states (see [6, 7, 19, 24, 33], and [16]) and the emission of electromagnetic radiation by accelerated charges [6]. The concepts of [27] have opened a wider scenario on the possibility of employing larger classes of pseudo-type functions.
It is important to note that the functions in (1.1) to (1.3) are related to the Mittag–Leffler function of one parameter

$$E_a(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(an+1)!}, (\Re(a) > 0, z \in \mathbb{C}, \tag{1.6}$$

introduced and investigated by Mittag–Leffler [20–22], which is important in the theory of entire functions. The relations are

$$F_{j,0}^{1}(z) = E_j(z^j), \ E_0(z; j) = E_j(z^j), \ S_0(z; j) = E_j(-z^j).$$

The Mittag–Leffler function of two parameters $E_{a,b}(z)$ is defined by the power series

$$E_{a,b}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(an+b)}, (\Re(a) > 0, \Re(b) > 0, z \in \mathbb{C}), \tag{1.7}$$

first appeared in the work of Wiman [35]. For the Mittag-Leffler function of two parameters $E_{a,b}(z)$, we infer that

$$E_k(z; j) = z^k E_{j,k+1}(z^j), \ S_k(z; j) = z^k E_{j,k+1}(-z^j).$$

Besides Wiman [35], the function $E_{a,b}(z)$ has been studied by many other researchers, for example, by Agrawai [1], Humbert [14], and Humbert and Agrawal [15]. Ever since its introduction in 1905, the Mittag-Leffler function $E_{a,b}(z)$ has received considerable attention from several researchers. The fact that it not only furnishes an interesting generalization of the Mittag–Leffler function of one parameter $E_a(z)$ but it also naturally arises in the solution of fractional order integral or differential equations, and especially in the investigations of fractional generalization of the kinetic equation, random walks, Lévy flights, superdiffusive transport and in the study of complex systems, as illustrated in [11–13], [17] and [30]. These functions $E_{a,b}(z)$ interpolate between a purely exponential law and power-law-like behavior of phenomena governed by ordinary kinetic equations and their fractional counterparts (see e.g. [29]). The most essential properties of these entire functions $E_{a,b}(z)$, investigated by many mathematicians, can be found in [9, 10, 34], and [17]. Moreover, several authors have studied the properties of some important particular cases and slightly modified forms of the function $E_{a,b}(z)$. In this regard, Humbert and Agarwal [14] and [15] introduced the modified Mittag–Leffler function

$$\mathbb{E}_{a,b}(z) = z^{\frac{b-1}{a}} E_{a,b}(z) = \sum_{n=0}^{\infty} \frac{z^{n+\frac{b-1}{a}}}{\Gamma(an+b)}. \tag{1.8}$$

Rabotnov, in his works on viscoelasticity, see e.g.[9] and [25] introduced the function of time t, that he denoted by

$$R_a(b,t) = t^a \sum_{n=0}^{\infty} \frac{b^n t^{n(a+1)}}{\Gamma((a+1)(n+1))} = t^a E_{a+1,a+1}(bt^{a+1}). \tag{1.9}$$

Also, in a recent paper [2], the authors introduced the following normalization of the Mittag-Leffler function:

$$\mathbb{E}_{a,b}(z) = \Gamma(\beta) z E_{a,b}(z) = \Gamma(\beta) \sum_{n=0}^{\infty} \frac{z^{n+1}}{\Gamma(an+b)}. \tag{1.10}$$

Motivated by the aforementioned important connections between the Mittag–Leffler function $E_{a,b}(z)$ and hyperbolic and trigonometric functions, the role of all these functions in a variety of fields of physics and engineering, and the contributions in [2, 6, 7, 9, 12–14] and [15] toward the unification and generalization of the Mittag-Leffler function $E_{a,b}(z)$, this work aims at introducing and investigating several properties and representations of new Mittag–Leffler function-type of arbitrary order. We establish basic properties, integral representations, and differential and pure recurrence relations. As applications of our findings, we will investigate the solutions of six generalized forms of generalized fractional kinetic equations. Also, we discuss the link for the various results, which are presented in this paper, with known results. Throughout this paper, let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of natural numbers, integer numbers, real numbers, and complex numbers, respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}_0^- = \mathbb{Z} \backslash \mathbb{N}$.

2 The arbitrary order Mittag–Leffler-type function

Based on the previous definitions of the Mittag-Leffler function $E_{a,b}(z)$ including its interesting special cases as well as hyperbolic and trigonometric functions, we present the following new definition of arbitrary order Mittag-Leffler-type function $E_{a,b}^{j,k}(z)$.

Definition 2.1 The arbitrary order Mittag–Leffler-type function $E_{a,b}^{j,k}(z)$ is defined by the power series:

$$E_{a,b}^{j,k}(z) = \sum_{n=0}^{\infty} \frac{z^{nj+k}}{\Gamma(b+a(nj+k))},$$
$$(j \geq 1, k \geq 0, z \in \mathbb{C}, \Re(a) > 0; \Re(b) > 0). \tag{2.1}$$

Observe that, definition (2.1), is another rewriting (formulation) of definition (1.7). Indeed, we have

$$E_{a,b}^{j,k}(z) = z^k E_{aj,b+ak}(z^j) = \sum_{n=0}^{\infty} \frac{z^{nj+k}}{\Gamma(b+a(nj+k))}. \tag{2.2}$$

Clearly, for the function $E_{a,b}^{j,k}(z)$ we have the following relationships:
(i) $E_{1,1}^{j,k}(z) = E_k(z; j)$, (ii) $(-1)^{-\frac{k}{j}} E_{1,1}^{j,k}((-1)^{-\frac{1}{j}} z) = S_k(z; j)$
(iii) $\lambda^{-\frac{k}{j}} E_{1,1}^{j,k}(\lambda^{-\frac{1}{j}} z) = F_{j,k}^{\lambda}(z)$, (iv) $E_{a,1}^{1,0}(z) = E_a(z)$, (v)$E_{a,b}^{1,0}(z) = E_{a,b}(z)$

$$\text{(vi)} E_{a,0}^{j,0}(z) = \sum_{n=1}^{\infty} \frac{z^{nj}}{\Gamma(anj)} = \sum_{n=0}^{\infty} \frac{z^{nj+j}}{\Gamma(anj+aj)},$$

and hence

$$E_{a,0}^{j,0}(z) = z^j E_{aj,aj}(z^j),$$
$$\text{(vii)} E_{0,b}^{j,0}(z) = \sum_{n=0}^{\infty} \frac{z^{nj}}{\Gamma(b)} = \frac{1}{\Gamma(b)} \sum_{n=0}^{\infty} z^{nj}, \tag{2.3}$$

and hence

$$E_{0,b}^{j,0}(z) = \frac{1}{\Gamma(b)} \frac{1}{(1-z^j)}.$$

(viii) $E_{a,b}^{2,0}(z) = E_{2a,b}(z^2)$,
(ix)$E_{1,b}^{j,0}(z) = \frac{1}{\Gamma(b)} {}_1F_j \left[1; \frac{b}{j}, \cdots, \frac{b+j-1}{j} : \frac{z}{j^j}\right]$,
where ${}_1F_j$ is the generalized hypergeometric function (see e.g. [2]),
(x) $E_{1,1}^{3,0}(z) = \frac{1}{3}\left[e^z + 2e^{\frac{-z}{2}} \cos \frac{\sqrt{3}z}{2}\right]$,
(xi) $E_{1,1}^{3,1}(z) = \frac{1}{3}\left[e^z - 2e^{\frac{-z}{2}} \cos\left(\frac{\sqrt{3}z}{2} + \frac{1}{3}\pi\right)\right]$,
(xii) $E_{1,1}^{3,2}(z) = \frac{1}{3}\left[e^z - 2e^{\frac{-z}{2}} \cos\left(\frac{\sqrt{3}z}{2} - \frac{1}{3}\pi\right)\right]$,
(xiii) $E_{1,1}^{4,2}(z) = \frac{1}{2}\left[\cosh(z) - \cos(z)\right]$,
(xiv) $E_{1,1}^{4,3}(z) = \frac{1}{2}\left[\sinh(z) - \sin(z)\right]$.
The fractional forms of the sine and cosine functions have been suggested by Luchko and Srivastava [25, p. 19(1.69) and (1.70)]:

$$\sin_{a,b}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{\Gamma(2an+2a-b+1)},$$
$$\cos_{a,b}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{\Gamma(2an+a-b+1)},$$

which can be expressed in terms of the arbitrary order Mittag–Leffler-type function (2.1) as follows:

$$\sin_{a,b}(z) = (-1)^{-\frac{1}{2}} E_{a,1-b}^{2,2}((-1)^{\frac{1}{2}} z), \qquad \cos_{a,b}(z) = z^{-1} E_{a,1-b}^{2,1}(-z).$$

Proposition 2.1 *Let* $j \geq 1, k \geq 0, z \in \mathbb{C}, \Re(a) > 0, \Re(b) > 0,$ *then*

$$E_{a,b}^{j,k}(z) = \frac{1}{z^j}\left[E_{a,b-aj}^{j,k}(z) - \frac{z^k}{\Gamma(b-a(j-k))}\right]. \tag{2.4}$$

Proof We have

$$E_{a,b}^{j,k}(z) = \sum_{n=0}^{\infty} \frac{z^{jn+k}}{\Gamma(b + a(jn + k))} = \sum_{n=1}^{\infty} \frac{z^{jn+k-j}}{\Gamma(b + a(jn - j + k))}$$
$$= z^{-j} \left[\sum_{n=0}^{\infty} \frac{z^{jn+k}}{\Gamma(b + a(jn - j + k))} - \frac{z^k}{\Gamma(b - a(j - k))} \right],$$

which is the desired result (2.4). ∎

An interesting special case would occur when we set $b = b + aj$ in (2.4) of the form

$$E_{a,b+aj}^{j,k}(z) = \frac{1}{z^j} \left[E_{a,b}^{j,k}(z) - \frac{z^k}{\Gamma(b + ak)} \right], \tag{2.5}$$

which for $b = k = 0$, we find again (2.3). If $b = \beta + ma$ in (2.4), then we obtain recursion in m

$$E_{a,\beta+ma}^{j,k}(z) = \frac{1}{z^j} \left[E_{a,\beta+a(m-j)}^{j,k}(z) - \frac{z^k}{\Gamma(\beta + a(m + k - j))} \right].$$

Proposition 2.2 *Let* $m \in \mathbb{N}$, $j \geq 1$, $k \geq 0$, $z \in \mathbb{C}$, $\Re(a) > 0$; $\Re(b) > 0$, *then*

$$z^{mj} E_{a,\beta+maj}^{j,k}(z) = E_{a,\beta}^{j,k}(z) - \sum_{n=0}^{m-1} \frac{z^{nj+k}}{\Gamma(\beta + a(nj + k))}. \tag{2.6}$$

Proof We have

$$E_{a,\beta+maj}^{j,k}(z) = \sum_{n=0}^{\infty} \frac{z^{jn+k}}{\Gamma(\beta + maj + a(jn + k))}$$
$$= \sum_{n=m}^{\infty} \frac{z^{jn+k-mj}}{\Gamma(\beta + a(jn + k))}$$
$$= z^{-mj} \left[\sum_{n=0}^{\infty} \frac{z^{jn+k}}{\Gamma(\beta + a(jn + k))} - \sum_{r=0}^{m-1} \frac{z^{rj+k}}{\Gamma(\beta + a(rj + k))} \right],$$

which leads us to the desired result (2.6) ∎.

Similarly, one can show that

$$z^{-mj} E_{a,\beta-maj}^{j,k}(z) = E_{a,\beta}^{j,k}(z) - \sum_{n=1}^{m} \frac{z^{nj+k}}{\Gamma(\beta + a(nj + k))}. \tag{2.7}$$

Subtracting (2.7) from (2.6) yields

$$z^{mj} E_{a,\beta+maj}^{j,k}(z) - z^{-mj} E_{a,\beta-maj}^{j,k}(z)$$
$$= -\sum_{n=0}^{m-1} \frac{z^{nj+k}}{\Gamma(\beta + a(nj + k))} - \sum_{n=1}^{m} \frac{z^{nj+k}}{\Gamma(\beta + a(nj + k))}$$

$$z^{mj} E_{a,\beta+maj}^{j,k}(z) - z^{-mj} E_{a,\beta-maj}^{j,k}(z) = -\sum_{l=-m}^{m-1} \frac{z^{lj+k}}{\Gamma(\beta + a(lj + k))}, \tag{2.8}$$

illustrating for $\Re(a) \geq 0$, $z \geq 0$ and positive $\beta > ma$ that

$$z^{2mj} E^{j,k}_{a,\beta+maj}(z) < E^{j,k}_{a,\beta-maj}(z). \tag{2.9}$$

Corollary 2.1 *Let* $m = 4$, $j \geq 1$, $k \geq 0$, $z \in \mathbb{C}$, $\Re(a) > 0$; $\Re(b) > 0$, *then there holds the formulas:*

$$z^{4j} E^{j,k}_{a,\beta+4aj}(z) = E^{j,k}_{a,\beta}(z) + \frac{z^k}{\Gamma(\beta+aj)} + \frac{z^{j+k}}{\Gamma(\beta+a(j+k))} + \frac{z^{2j+k}}{\Gamma(\beta+a(2j+k))} + \frac{z^{3j+k}}{\Gamma(\beta+a(3j+k))}, \tag{2.10}$$

$$\frac{\partial}{\partial z} E^{j,0}_{m,1}(z^m) = z^{m-1} E^{j,j-1}_{m,m}(z^m). \tag{2.11}$$

Proof The proof of the assertion (2.10) is the direct use of the result (2.6). Using definition (2.1), we get

$$\frac{\partial}{\partial z} E^{j,0}_{m,1}(z^m) = \sum_{n=\frac{1}{mj}}^{\infty} \frac{z^{mnj-1}}{(mnj-1)!} = \sum_{n=0}^{\infty} \frac{z^{mnj+mj-1}}{\Gamma(mnj+mj)},$$

which leads us to the desired result (2.11). ■.

Note that, from

$$E^{j,k}_{a,b}(z) = \frac{z^k}{\Gamma(b+ak)} + \sum_{n=1}^{\infty} \frac{z^{jn+k}}{\Gamma(b+a(jn+k))},$$

we observe that

$$\lim_{a\to\infty} E^{j,k}_{a,b}(z) = 0.$$

The generalized arbitrary order Mittag-Leffler-type functions $E^{j,k}_{a,b}(z)$ have the following connections with the Wright generalized hypergeometric function ${}_p\Psi_q$ and Fox H-function $H^{m,n}_{r,s}$ [31]:

$$E^{j,k}_{a,b}(z) = z^k\, {}_1\Psi_1 \left[\begin{matrix} (1,1); \\ (b+ak, aj); \end{matrix} \; z^j \right], \tag{2.12}$$

$$= z^k H^{1,1}_{1,2} \left[-z^j \,\middle|\, \begin{matrix} (0,1); \\ (0,1), (1-b-ak, aj); \end{matrix} \right]. \tag{2.13}$$

3 Integrals

Several integrals associated with Mittag-Leffler-type function $E^{j,k}_{a,b}(z)$ are presented in this section, which can be easily established using Beta and Gamma function formulas and other techniques, (see e.g. [31, 32] and [28]). The Beta function $B(a,b)$ is a function of two complex variable a and b, where $\Re(a)>0$ and $\Re(b)>0$, defined by:

$$B(a,b)=\left\{\begin{array}{l}\text{(i) } \int_0^1 t^{a-1}(1-t)^{b-1}dt \\ \text{(ii) } 2\int_0^{\frac{\pi}{2}}(\sin\theta)^{2a-1}(\cos\theta)^{2b-1}d\theta \\ \text{(iii) } \int_0^\infty \frac{u^{a-1}}{(1+u)^{a+b}}du \end{array}\right\} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\ (a,b\in\mathbb{C}\setminus\mathbb{Z}_0^-). \tag{3.1}$$

Proposition 3.1 *Let* $j\geq 1, k\geq 0, z\in\mathbb{C}, \Re(a)>0; \Re(b)>0,$ *then*

$$E_{a,b+1}^{j,k}(z^a)=\frac{2}{\Gamma(b)}\int_0^{\frac{\pi}{2}}(\cos\theta)^{2b-1}\sin\theta\ E_{ak}\left(z(\sin\theta)^2;\ ja\right)d\theta, \tag{3.2}$$

$$E_{a,b+1}^{j,k}(z)=\frac{2}{(-1)^{\frac{k}{j}}\Gamma(b)}\int_0^{\frac{\pi}{2}}(\cos\theta)^{2b-1}\sin\theta\ S_{ak}\left((-1)^{\frac{1}{aj}}z(\sin\theta)^2;\ ja\right)d\theta. \tag{3.3}$$

Proof it follows from (1.1) that

$$\begin{aligned}&\frac{2}{\Gamma(b)}\int_0^{\frac{\pi}{2}}(\cos\theta)^{2b-1}\sin\theta\ E_{ak}\left(z(\sin\theta)^2;\ ja\right)d\theta\\ &=\frac{2}{\Gamma(b)}\int_0^{\frac{\pi}{2}}\sum_{n=0}^{\infty}\frac{z^{anj+ak}}{\Gamma(anj+ak+1)}(\cos\theta)^{2b-1}(\sin\theta)^{2(anj+ak+1)-1}.\end{aligned}$$

The desired result now follows by changing the order of integration and summation and employing the formula (ii) of equation (3.1) and this completes the proof of (3.2). Similarly, by employing (1.2) one can prove the formula (3.3). ■.

Now, other integral representations for the function $E_{a,b}^{j,k}(z^a)$ are based upon the integral formula (iii) in (3.1).

Proposition 3.2 *Let* $j\geq 1, k\geq 0, z\in\mathbb{C}, \Re(a)>0; \Re(b)>0,$ *then*

$$E_{a,b+1}^{j,k}(z^a)=\frac{1}{(-1)^{\frac{k}{j}}\Gamma(b)}\int_0^{\infty}(1+u)^{-(b+1)}\ E_{ak}\left(\frac{zu}{1+u};\ ja\right)du, \tag{3.4}$$

$$E_{a,b+1}^{j,k}(z^a)=\frac{1}{(-1)^{\frac{k}{j}}\Gamma(b)}\int_0^{\infty}(1+u)^{-(b+1)}\ S_{ak}\left(\frac{-zu}{1+u};\ ja\right)du. \tag{3.5}$$

Proof By employing definitions (1.1), (1.2), the integral relation (iii) of equation(3.1) and exploiting the same procedure leading to the results (3.2) and (3.3), one can derive the formulas (3.4) and (3.5). ■.

Next, it is not difficult to infer the following proposition.

Proposition 3.3 *Let* $j\geq 1, k\geq 0, z\in\mathbb{C}, \Re(a)>0; \Re(b)>0,$ *then*

$$E_{a,b+1}^{j,k}(z)=\frac{z^k}{\Gamma(b+ak)}\int_0^1\zeta^{ak+b-1}\ E_{aj}\left(z^j(1-\zeta)^{aj}\right)d\zeta, \tag{3.6}$$

$$E_{a,b}^{j,k}(z)=\frac{z^k}{\Gamma(ak)}\int_0^1\zeta^{ak-1}(1-\zeta)^{b-1}\ E_{aj,b}\left(z^j(1-\zeta)^{aj}\right)d\zeta, \tag{3.7}$$

$$E^{j,k}_{a,b+1}(z^a) = \frac{1}{\Gamma(b)} \int_0^1 (1-\zeta)^{b-1}\, E_{ak}\left(z\zeta^a; ja\right) d\zeta, \tag{3.8}$$

$$E^{j,k}_{a,b+1}(z^a) = \frac{1}{(-1)^{\frac{k}{j}}\Gamma(b)} \int_0^1 (1-\zeta)^{b-1}\, S_{ak}\left(-z\zeta^a; ja\right) d\zeta. \tag{3.9}$$

Proof From (1.1), we have

$$\frac{z^k}{\Gamma(b+ak)} \int_0^1 \zeta^{ak+b-1}\, E_{aj}\left(z^j(1-\zeta)^{aj}\right) d\zeta$$
$$= \frac{z^k}{\Gamma(b+ak)} \int_0^1 \sum_{n=0}^{\infty} \frac{z^{nj}}{\Gamma(anj+1)} (1-\zeta)^{(anj+1)-1} \zeta^{ak+b-1} d\zeta.$$

The desired result now follows by changing the order of integration and summation and employing the formula (i) of Eq. (3.1) and this completes the proof of (3.6). Similarly, one can prove the results (3.7) to (3.9). ■.

Additionally, as shown below, we can construct another integral kind for the function $E^{j,k}_{a,b}(z)$.

Proposition 3.4 *Let* $j \geq 1, k \geq 0, z \in \mathbb{C}, \Re(a) > 0, \Re(b) > 0,$ *then*

$$z^{b-1} E^{j,k}_{a,b}(z^a) = \frac{1}{\Gamma(ak)} \int_0^z (z+\zeta)^{ak-1} \zeta^{b-1}\, E_{aj,b}\left(\zeta^{aj}\right) d\zeta. \tag{3.10}$$

Proof Denote, for convenience, the right-hand side of (3.10) by I. Then by using (1.7) and changing the order of integration with the summation, we can write

$$I = \frac{1}{\Gamma(ak)} \sum_{n=0}^{\infty} \frac{1}{\Gamma(anj+b)} \int_0^z (z+\zeta)^{ak-1} \zeta^{anj+b-1} d\zeta.$$

By letting $\zeta = zt$ and rearranging, we obtain

$$I = \frac{1}{\Gamma(ak)} \sum_{n=0}^{\infty} \frac{z^{anj+ak+b-1}}{\Gamma(anj+b)} \int_0^1 (1-t)^{ak-1} t^{anj+b-1} dt.$$

Now, using (i) of Eq. (3.1) and considering the definition (2.1), we lead to the left-hand side of the assertion (3.10). ■.

Next, we first recall the definition of the well-known Hankelś integral for the reciprocal of the Gamma function (see e.g. [25]), namely

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi i} \int_{H_a} e^t t^{-x} dt. \tag{3.11}$$

Proposition 3.5 *Let* $j \geq 1, k \geq 0, z \in \mathbb{C}, \Re(a) > 0; \Re(b) > 0,$ *then*

$$E^{j,k}_{a,b}(z^a) = \frac{z^k}{2\pi i} \int_{H_a} \frac{e^t\, t^{-(b+a(j+k))}}{\left(t^{aj} - z^j\right)} dt. \tag{3.12}$$

Proof Let $x = b + a(nj+k)$ in (3.11), multiply by z^{nj+k} and sum to get

$$\sum_{n=0}^{\infty} \frac{z^{nj+k}}{\Gamma(b+a(nj+k))} = \frac{z^k}{2\pi i} \int_{H_a} e^t\, t^{-(b+ak)} \sum_{n=0}^{\infty} z^{nj}\, t^{-anj} dt$$
$$= \frac{z^k}{2\pi i} \int_{H_a} e^t\, t^{-(b+ak)} \left(1 - \frac{z^j}{t^{aj}}\right)^{-1} dt,$$

which gives the desired result (3.12). ■.

Proposition 3.6 *Let* $j \geq 1, k \geq 0, z \in \mathbb{C}, \Re(a) > 0, \Re(b) > 0, \Re(c) > 0$ *then*

$$\int_0^1 t^{b-1} E_{a,b}^{j,k}(zt^a)\,(1-t)^{c-1} E_{a,c}^{j,k}(\omega(1-t)^a)dt = \frac{\omega^k z^j E_{a,b+c+ak}^{j,k}(z) - z^k \omega^j E_{a,b+c+ak}^{j,k}(\omega)}{z^j - \omega^j}. \tag{3.13}$$

Proof Denote by I the first member. By using the definition of the arbitrary order Mittag–Leffler-type function with (2.1) and changing the order with the integration we obtain

$$I = \sum_{n=0}^{\infty} \frac{z^{nj+k}}{\Gamma(b+a(nj+k))} \sum_{m=0}^{\infty} \frac{\omega^{mj+k}}{\Gamma(c+a(mj+k))} \int_0^1 t^{a(nj+k)+b-1}(1-t)^{a(mj+k)+c-1}dt.$$

The remaining integral is nothing more than the definition of the beta function which on using the relation with the Gamma function (i) in equation (3.1) and simplifying, allows us to write

$$I = \sum_{n=0}^{\infty} \frac{z^{nj+k}\omega^{mj+k}}{\Gamma(b+c+a(nj+k)+a(mj+k))}.$$

Entering the index change $s = m + n$ that is $m = s - n$, and rearranging, we obtain

$$I = \sum_{s=0}^{\infty} \frac{z^k \omega^{sj+k}}{\Gamma(b+c+a(sj+2k))} \sum_{n=0}^{s} \left(\frac{z}{\omega}\right)^{nj}.$$

Noting that the second summation is a geometric series with a finite number of terms, we can write

$$I = \sum_{s=0}^{\infty} \frac{z^k \omega^k}{\Gamma(b+c+a(sj+2k))} \left\{ \frac{z^{(s+1)j} - \omega^{(s+1)j}}{z^i - \omega^j} \right\}.$$

Finally, using the definition of the arbitrary order Mittag-Leffler-type function (2.1), we lead to the desired result (3.13). ■.

In its special case when $k = 0$, $j = 1$, the assertion (3.13) would correspond to the known formula (see e.g. [8, p.88(13)]):

$$\int_0^1 t^{b-1} E_{a,b}(zt^a)\,(1-t)^{c-1} E_{a,c}(\omega(1-t)^a)dt = \frac{zE_{a,b+c}(z) - \omega E_{a,b+c}(\omega)}{z - \omega^j}.$$

Upon setting $\omega = z$, we have an indeterminacy on the second member. To raise it, we use the ÍHôpital rule from which we can write the elegant formula

$$\int_0^1 t^{b-1} E_{a,b}^{j,k}(zt^a)\,(1-t)^{c-1} E_{a,c}^{j,k}(z(1-t)^a)dt = \frac{x^k}{j}\left\{ jE_{a,b+c+ak}^{j,k}(z) + z\acute{E}_{a,b+c+ak}^{j,k}(z) - kE_{a,b+c+ak}^{j,k}(z) \right\},$$

which for $k = 0$ and $j = 1$ reduces to another known result [8, p.89(14)]:

$$\int_0^1 t^{b-1} E_{a,b}(zt^a)\ (1-t)^{c-1} E_{a,c}(z(1-t)^a) dt = E_{a,b+c}(z) + z\acute{E}_{a,b+c}(z).$$

In view of the relationship $E_{1,1}^{j,k}(z) = E_k(z; j)$, in (3.13), we set $a = b = c = 1$. We thus find for the pseudo-hyperbolic function $E_k(z; j)$ that

Corollary 3.1 *Let* $j \geq 1, k \geq 0, z \in \mathbb{C}$ *then*

$$\int_0^1 E_k(zt; j)\ E_k(\omega(1-t); j) dt = \frac{\omega^k z^j E_k(z; j) - z^k \omega^j E_k(\omega; j)}{z^j - \omega^j}. \tag{3.14}$$

4 Pure and differential relations

The arbitrary Mittag–Leffler-type functions $E_{a,b}^{j,k}(z)$ as a function satisfies some pure and differential recurrence relations. Fortunately, these properties of $E_{a,b}^{j,k}(z)$ can be developed directly from the definition (2.1). First, we establish a relation for the Mittag–Leffler-type relaxation function $E_{a,b}^{j,k}(-z)$ when z is a non-negative real variable.

Proposition 4.1 *If* $j \geq 1, k \geq 0, k$ *even number*, $z \in \mathbb{C}, \Re(a) > 0, \Re(b) > 0,$ *then there holds the formula*

$$E_{a,b}^{j,k}(-z) = (-1)^k E_{2a,b}^{j,k/2}(z^2) + (-1)^{j+k} z^j E_{2a,b+aj}^{j,k/2}(z^2). \tag{4.1}$$

Proof After splitting odd and even indices in the n-sum of (2.1), we obtain

$$\begin{aligned} E_{a,b}^{j,k}(-z) &= \sum_{n=0}^{\infty} \frac{(-z)^{nj+k}}{\Gamma(b+a(nj+k))} \\ &= (-1)^k \sum_{n=0}^{\infty} \frac{z^{2nj+k}}{\Gamma(b+a(2nj+k))} + (-1)^{j+k} z^j \sum_{n=0}^{\infty} \frac{z^{2nj+k}}{\Gamma(b+a(nj+j+k))}. \end{aligned}$$

Hence, we get the desired pure recurrence relation (4.1). ■.

Next, we deduce some useful results. In (4.1) replace z by $-z$, we obtain

$$E_{a,b}^{j,k}(z) = E_{2a,b}^{j,k/2}(z^2) + (-1)^j z^j E_{2a,b+aj}^{j,k/2}(z^2). \tag{4.2}$$

Adding (4.2) to (4.1) leads to

$$\begin{aligned} &E_{a,b}^{j,k}(z) + E_{a,b}^{j,k}(-z) \\ &= E_{2a,b}^{j,k/2}(z^2) + z^j E_{2a,b+aj}^{j,k/2}(z^2) + (-1)^k E_{2a,b}^{j,k/2}(z^2) + (-1)^{j+k} z^j E_{2a,b+aj}^{j,k/2}(z^2), \end{aligned} \tag{4.3}$$

which for $k \mapsto 2k$ yields the interesting result

$$E_{2a,b}^{j,k}(z^2) = \frac{E_{a,b}^{j,2k}(z) - E_{a,b}^{j,2k}(-z)}{2}, \tag{4.4}$$

and, similarly,

$$E^{j,k}_{2a,b+aj}(z^2) = \frac{E^{j,2k}_{a,b}(z) + E^{j,2k}_{a,b}(-z)}{2z^j}. \tag{4.5}$$

Two interesting special cases of the assertions (4.4) and (4.5) involving the pseudo-hyperbolic function would occur when we set $a = b = 1$:

$$E^{j,k}_{2,1}(z^2) = \frac{E_{2k}(z;\, j) - E_{2k}(-z;\, j)}{2},$$

and

$$E^{j,k}_{2,1+j}(z^2) = \frac{E_{2k}(z;\, j) + E_{2k}(-z;\, j)}{2z^j}.$$

In (4.4) and (4.5) let $j = 1$ and $k = 0$, to get the known results (see e.g.[10]):

$$E_{2a,b}(z^2) = \frac{E_{a,b}(z) - E_{a,b}(-z)}{2},$$

and

$$E_{2a,b+a}(z^2) = \frac{E_{a,b}(z) + E_{a,b}(-z)}{2z}.$$

Next, by recalling the definitions of Mittag-Leffler functions of six and three parameters (see [10, p.77(34)] Ch. 3] and [26]):

$$E^{\rho,\delta,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\rho)_{nq} z^n}{\Gamma(\alpha n + \beta)(\delta)_{kp}}, \tag{4.6}$$

and

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta) n!}, \tag{4.7}$$

and the operational formula

$$\frac{\partial^m}{\partial z^m} z^n = \frac{\Gamma(n+1)}{\Gamma(n-m+1)} z^{n-m} \ (n \in \mathbb{N}), \tag{4.8}$$

we aim now to derive the following differential relation for $E^{j,k}_{a,b}(z)$.

Proposition 4.2 *If* $m \in \mathbb{N},\ j \geq 1,\ k \geq 0,\ z \in \mathbb{C},\ \Re(a) > 0,\ \Re(b) > 0,$ *then the following results hold:*

$$\frac{\partial^m}{\partial z^m} E^{j,k}_{a,b}(z) = m! E^{m+1,1,j}_{aj,b+ak,j}(z^j), \tag{4.9}$$

$$\frac{\partial^{1-b}}{\partial z^{1-b}} E_{ak}(z;\, aj) = z^{b-1} E^{j,k}_{a,b}(z^a), \tag{4.10}$$

$$(-1)^{\frac{-k}{j}} \frac{\partial^{1-b}}{\partial z^{1-b}} S_{ak}\left((-1)^{\frac{1}{aj}} z;\, aj\right) = z^{b-1} E^{j,k}_{a,b}(z^a). \tag{4.11}$$

Proof Starting from definition (2.1) and making use of formula (4.8), we get

$$\frac{\partial^m}{\partial z^m} E^{j,k}_{a,b}(z) = \sum_{n=\frac{m-k}{j}}^{\infty} \frac{(nj+k)!\, z^{nj+k-m}}{\Gamma(b + a(nj+k))(nj+k-m)!} = \sum_{n=0}^{\infty} \frac{(nj+m)! z^{nj}}{\Gamma(b + a(nj+m))(nj)!}.$$

Now, by using the formula $(\lambda)_m = \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)}$, we lead to the desired result (4.9). Similarly, one can prove the formulas (4.10) and (4.11). ■.

If we set $j = 1$ in (4.9), we get the following differential equation involving the Mittag-Leffler function (4.7):

$$\frac{\partial^m}{\partial z^m} E^{1,k}_{a,b}(z) = m! E^{m+1}_{a,b+am}(z), \tag{4.12}$$

Furthermore, by using definition (2.1) and the operator (4.8) is not difficult to infer the following differentiation recursion formula.

Proposition 4.3 *If* $m \in \mathbb{N}$, $j \geq 1$, $k \geq 0$, $z \in \mathbb{C}$, $\Re(a) > 0$, $\Re(b) > 0$, *then:*

$$\frac{\partial^m}{\partial x^m} \left\{ z^{b-1} E^{j,k}_{a,b}(xz^a) \right\} = z^{b-m-1} E^{j,k}_{a,b-m}(xz^a), \tag{4.13}$$

$$\frac{a}{z^{j-1}} \frac{\partial}{\partial z} E^{j,k}_{a,b}(z) = E^{j,k}_{a,aj+b-1}(z) - (b+ak+1) E^{j,k}_{a,aj+b}(z) + \frac{ak}{z^j} E^{j,k}_{a,b}(z). \tag{4.14}$$

Proof Using the definition (2.1), we write

$$\frac{\partial^m}{\partial z^m} \left\{ z^{b-1} E^{j,k}_{a,b}(xz^a) \right\} = \sum_{n=0}^{\infty} \frac{x^{ni+k} \frac{\partial^m}{\partial z^m} z^{a(nj+k)+b-1}}{\Gamma(b + a(nj+k))}.$$

The result now follows from the general derivative formula (4.8).
Next, we have

$$\begin{aligned}
\frac{\partial}{\partial z} E^{j,k}_{a,b}(z) &= \sum_{n=0}^{\infty} \frac{(nj+k) z^{nj+k-1}}{\Gamma(b + a(nj+k))} \\
&= jz^{-1} \sum_{n=1}^{\infty} n \frac{z^{nj+k}}{\Gamma(b + a(nj+k))} + kz^{-1} E^{j,k}_{a,b}(z) \\
&= jz^{j-1} \sum_{n=0}^{\infty} \left(\frac{anj + aj + ak + b - 1}{aj} - \frac{b + ak - 1}{aj} \right) \frac{z^{nj+k}}{\Gamma(b + a(nj + j + k))} \\
&\quad + kz^{-1} E^{j,k}_{a,b}(z) \\
\frac{\partial}{\partial z} E^{j,k}_{a,b} &= \frac{z^{j-1}}{a} E^{j,k}_{a,aj+b-1}(z) - \frac{z^{j-1}}{a}(b+ak+1) E^{j,k}_{a,b+aj}(z) + kz^{-1} E^{j,k}_{a,b}(z),
\end{aligned}$$

which gives us the formula (4.14). ■.

Applying the formula (2.5) to the relation (4.14), we can obtain a helpful relation in this work's subsequent investigations:

$$az \frac{\partial}{\partial z} E^{j,k}_{a,b}(z) = E^{j,k}_{a,b-1}(z) - (b-1) E^{j,k}_{a,b}(z). \tag{4.15}$$

Further, we make use of (4.13) with $a = \frac{m}{sj}$ and $x = 1$, to derive differentiation recursion involving Fractional values of the parameter a.

Proposition 4.4 *If* $m, s, j \in \mathbb{N}$, $m < s$, $j \geq 1$, $k \geq 0$, $z \in \mathbb{C}$, $\Re(a) > 0$, $\Re(b) > 0$, *then:*

$$\frac{\partial^m}{\partial z^m} \left\{ z^{b-1} E^{j,k}_{\frac{m}{sj},b}(z^{\frac{m}{sj}}) \right\} = z^{b-1} \sum_{n=1}^{s} \frac{z^{-\frac{m}{sj}(nj-k)}}{\Gamma(b - \frac{m}{sj}(nj-k))} + z^{b-1} E^{j,k}_{\frac{m}{sj},b}(z^{\frac{m}{sj}}). \tag{4.16}$$

Proof We have

$$\begin{aligned}\frac{\partial^m}{\partial z^m}\left\{z^{b-1}E^{j,k}_{\frac{m}{sj},b}(z^{\frac{m}{sj}})\right\} &= z^{b-1}\sum_{n=0}^{\infty}\frac{z^{\frac{m}{sj}(nj+k)-m}}{\Gamma(b-m+\frac{m}{sj}(nj+k))}\\ &= z^{b-1}\sum_{n=-s}^{\infty}\frac{z^{\frac{m}{sj}(nj+k-sj)}}{\Gamma(b+\frac{m}{sj}(nj+k-sj))}, \text{ (we let } n=n+s)\\ &= z^{b-1}\sum_{n=1}^{s}\frac{z^{-\frac{m}{sj}(nj-k)}}{\Gamma(b-\frac{m}{sj}(nj-k))}+z^{b-1}\sum_{n=0}^{\infty}\frac{z^{\frac{m}{sj}(nj+k)}}{\Gamma(b+\frac{m}{sj}(nj+k))}\\ &= z^{b-1}\sum_{n=1}^{s}\frac{z^{-\frac{m}{sj}(nj-k)}}{\Gamma(b-\frac{m}{sj}(nj-k))}+z^{b-1}E^{j,k}_{\frac{m}{sj},b}(z^{\frac{m}{sj}}),\end{aligned}$$

which is the desired result. ■.

For $s=1$, (4.16) is

$$\frac{\partial^m}{\partial z^m}\left\{z^{b-1}E^{j,k}_{\frac{m}{j},b}(z^{\frac{m}{j}})\right\} = \frac{z^{b-m+\frac{k}{j}-1}}{\Gamma(b-m+\frac{k}{j})}+z^{b-1}E^{j,k}_{\frac{m}{j},b}(z^{\frac{m}{j}}). \tag{4.17}$$

Evidently, for $j=1$ and $k=0$, (4.16) reduces to the known result [34, Equation 20)]

$$\frac{\partial^m}{\partial z^m}\left\{z^{b-1}E_{\frac{m}{s},b}(z^{\frac{m}{s}})\right\} = z^{b-1}\sum_{n=1}^{s}\frac{z^{-\frac{m}{s}n}}{\Gamma(b-\frac{m}{s}n)}+z^{b-1}E_{\frac{m}{s},b}(z^{\frac{m}{s}}).$$

Now, we establish the derivation of $E^{j,k}_{a,b}(z)$ with respect to the parameters a and b.

Proposition 4.5 *If* $j\geq 1, k\geq 0, z\in\mathbb{C}, \Re(a)>0, \Re(b)>0$, *then:*

$$\frac{\partial}{\partial a}E^{j,k}_{a,b}(z) = z\frac{\partial^2}{\partial z\partial b}E^{j,k}_{a,b}(z). \tag{4.18}$$

Proof starting from definition (2.1), partial differentiation yields

$$\begin{aligned}\frac{\partial}{\partial a}E^{j,k}_{a,b}(z) &= \frac{\partial}{\partial a}\sum_{n=0}^{\infty}\frac{z^{nj+k}}{\Gamma(b+a(nj+k)}\\ &= \sum_{n=0}^{\infty}\frac{\partial}{\partial y}\frac{1}{\Gamma(y)}\bigg|_{y=b+a(nj+k))}\frac{\partial y}{\partial a}z^{nj+k}\\ &= \sum_{n=0}^{\infty}\frac{\psi(y)}{\Gamma(y)}\bigg|_{y=b+a(nj+k)}\frac{\partial y}{\partial a}z^{nj+k}\\ &= -\sum_{n=0}^{\infty}\frac{\psi(y)}{\Gamma(y)}\bigg|_{y=b+a(nj+k)}(nj+k)z^{nj+k}\end{aligned}$$

while similarly

$$\frac{\partial}{\partial b}E^{j,k}_{a,b}(z) = -\sum_{n=0}^{\infty}\frac{\psi(y)}{\Gamma(y)}\bigg|_{y=b+a(nj+k)}z^{nj+k}$$

or

$$z\frac{\partial^2}{\partial z\partial b}E_{a,b}^{j,k}(z) = -\sum_{n=0}^{\infty}\frac{\psi(y)}{\Gamma(y)}\bigg|_{y=b+a(nj+k)}(nj+k)z^{nj+k}.$$

Hence, we observe that

$$\frac{\partial}{\partial a}E_{a,b}^{j,k}(z) = z\frac{\partial^2}{\partial z\partial b}E_{a,b}^{j,k}(z).$$

This complete the proof of (4.18). ■.

Proposition 4.6 *For any integer* $m \geq 0$ *and any* y *independent of* a, b *and* ω, $j \geq 1, k \geq 0, z \in \mathbb{C}, \Re(a) > 0, \Re(b) > 0$, *we have*

$$\frac{\partial^m}{\partial a^m}E_{a,b}^{j,k}(ye^{\omega}) = \frac{\partial^{2m}}{\partial\omega^m\partial a^m}E_{a,b}^{j,k}(ye^{\omega}). \tag{4.19}$$

Proof Partial differentiating m-times gives

$$\frac{\partial^m}{\partial a^m}E_{a,b}^{j,k}(z) = \sum_{n=0}^{\infty}\frac{\partial^m}{\partial y^m}\frac{1}{\Gamma(y)}\bigg|_{y=b+a(nj+k)}(nj+k)^m z^{nj+k},$$

which suggest to let $z = ye^{\omega}$ so that

$$\begin{aligned}\frac{\partial^m}{\partial a^m}E_{a,b}^{j,k}(ye^{\omega}) &= \sum_{n=0}^{\infty}\frac{\partial^m}{\partial y^m}\frac{1}{\Gamma(y)}\bigg|_{y=b+a(nj+k)}(nj+k)^m y^{nj+k}e^{\omega(nj+k)}\\ &= \frac{\partial^m}{\partial\omega^m}\sum_{n=0}^{\infty}\frac{\partial^m}{\partial y^m}\frac{1}{\Gamma(y)}\bigg|_{y=b+a(nj+k)}y^{nj+k}e^{\omega(nj+k)}\end{aligned}$$

Similarly, we can show that

$$\frac{\partial^{2m}}{\partial\omega^m\partial b^m}E_{a,b}^{j,k}(ye^{\omega}) = \frac{\partial^m}{\partial\omega^m}\sum_{n=0}^{\infty}\frac{\partial^m}{\partial y^m}\frac{1}{\Gamma(y)}\bigg|_{y=b+a(nj+k)}y^{nj+k}e^{\omega(nj+k)}$$

Hence, the assertion (4.19) is proved. ■.

The logarithmic derivative of function $E_{a,b}^{j,k}(z)$ is investigated in the following proposition.

Proposition 4.7 *For* $j \geq 1, k \geq 0, z \in \mathbb{C}, \Re(a) > 0, \Re(b) > 0$, *we have*

$$\frac{d\log E_{a,b}^{j,k}(z)}{dz} = \frac{1}{az}\left(\frac{E_{a,b-1}^{j,k}(z)}{E_{a,b}^{j,k}(z)} - (b-1)\right) > 0, \tag{4.20}$$

Proof The logarithmic derivative [34]:

$$\frac{d\log E_{a,b}^{j,k}(z)}{dz} = \frac{\frac{d}{dz}\log E_{a,b}^{j,k}(z)}{E_{a,b}^{j,k}(z)}$$

follows directly from (4.15) as

$$\frac{d\log E_{a,b}^{j,k}(z)}{dz} = \frac{1}{az}\left(\frac{E_{a,b-1}^{j,k}(z)}{E_{a,b}^{j,k}(z)} - (b-1)\right),$$

which is the first part of the assertion (4.20). Next, since $b + anj > (b - 1)$, for $n \geq 0$ because $a > 0$, we have for positive real z and $b > 1$

$$E_{a,b}^{j,k}(z) = \sum_{n=0}^{\infty} \frac{z^{nj+k}}{(b-1+a(nj+k))\Gamma(b-1+a(nj+k))}$$

$$E_{a,b}^{j,k}(z) < \frac{1}{(b-1)} \sum_{n=0}^{\infty} \frac{z^{nj+k}}{(b-1+a(nj+k))}$$

$$E_{a,b}^{j,k}(z) < \frac{E_{a,b-1}^{j,k}(z)}{(b-1)}$$

and that

$$(b-1) < \frac{E_{a,b-1}^{j,k}(z)}{E_{a,b}^{j,k}(z)}.$$

So, for $b > 1$, we get

$$\frac{d \log E_{a,b}^{j,k}(z)}{dz} > 0,$$

which is the proof of the second part of the assertion (4.20). ■.

With a little more precision, we establish another inequality for the logarithmic derivative of the function $E_{a,b}^{j,k}$ in the result that follows.

Proposition 4.8 *For $j \geq 1, k \geq 0, z \in \mathbb{C}, \Re(a) > 0, \Re(b) > 0$, with $b - anj > b - 1 + aj$, for $n \geq 0$, we have for positive z and for $b > 1 - aj$:*

$$\frac{d \log E_{a,b}^{j,k}(z)}{dz} > \frac{j}{z}\left(1 - \frac{1}{\Gamma(b)E_{a,b}^{j,k}(z)}\right). \tag{4.21}$$

Proof We have

$$\begin{aligned}
E_{a,b}^{j,k}(z) &= \frac{1}{\Gamma(b+ak)} + \sum_{n=1}^{\infty} \frac{z^{nj+k}}{\Gamma(b-1+a(nj+k))} \\
&< \frac{1}{\Gamma(b+ak)} + \frac{1}{\Gamma(b-1+a(j+k))}\left(E_{a,b-1}^{j,k}(z) - \frac{1}{\Gamma(b+ak-1)}\right) \\
&< \frac{1}{\Gamma(b)} + \frac{1}{\Gamma(b-1+aj)}\left(E_{a,b-1}^{j,k}(z) - \frac{1}{\Gamma(b-1)}\right) \\
&< \frac{1}{\Gamma(b-1+aj)}\left(\frac{aj}{\Gamma(b)} + E_{a,b-1}^{j,k}(z)\right).
\end{aligned}$$

Hence

$$aj\left(1 - \frac{1}{\Gamma(b)E_{a,b}^{j,k}(z)}\right) < \left(\frac{E_{a,b-1}^{j,k}(z)}{E_{a,b}^{j,k}(z)} - (b-1)\right)$$

and (4.20) becomes

$$\frac{d \log E_{a,b}^{j,k}(z)}{dz} > \frac{j}{z}\left(1 - \frac{1}{\Gamma(b)E_{a,b}^{j,k}(z)}\right)$$

and this completes the proof of (4.21). ■.

5 Applications in kinetic equations

This section looks at the solutions to six generalized versions of Saxena and Kalla's fractional kinetic equations (see [29]; see also [30]):

$$\mathcal{N}(\ll) - \mathcal{N}_0 f(\tau) = -\epsilon^{\nu}{}_0D_\tau^{-\nu}\mathcal{N}(\tau) \ \ (\Re(\nu) > 0), \tag{5.1}$$

where $\mathcal{N}(\tau)$ denotes the number density of a given species at time τ, $\mathcal{N}_0 = \mathcal{N}(0)$ is the number density of that species at time $\tau = 0$, ϵ is a constant, $f \in \mathcal{L}(0, \infty)$ and ${}_0D_\tau^{-\nu}$ is the Riemann-Liouville integral operator defined as (see e.g. [8] and [25]):

$${}_0D_\tau^{-\nu} f(\tau) = \frac{1}{\Gamma(\nu)} \int_0^\tau (\tau - s)^{\nu-1} f(s)ds, \Re(\nu) > 0. \tag{5.2}$$

The results are obtained in a compact form containing the arbitrary order Mittag-Leffler-type function $E_{a,b}^{j,k}(z)$. To begin, we demonstrate that the Mittag–Leffler function $E_{a,b}^{j,k}(z)$ arising in the solution of a generalized fractional kinetic equation with an elementary function in the kernel.

Proposition 5.1 *If $\delta > 0, \nu > 0, j \geq 1, k \geq 0$, then the solution of the equation*

$$\mathcal{N}(\ll) - \mathcal{N}_0\ \tau^{\mu+\nu k-1} = -\delta^{\nu j}{}_0D_\tau^{-\nu j}\mathcal{N}(\tau), \tag{5.3}$$

is given by

$$\mathcal{N}(\ll) = \mathcal{N}_0(-1)^{\frac{k}{j}}\ \Gamma(\mu + \nu k)\ \delta^{-\nu k}\ \tau^{\mu-1}\ E_{\nu,\mu}^{j,k}((-1)^{\frac{1}{j}}(\delta\tau)^\nu), \tag{5.4}$$

where $E_{\nu,\mu}^{j,k}(\tau)$ is the arbitrary order Mittag–Leffler-type function (2.1).

Proof The Laplace transform of the Riemann-Liouville fractional integral operator is given by [8]

$$\mathcal{L}\left[{}_0D_\tau^{-\nu} f(\tau) : s\right] = s^{-\nu}F(s), \tag{5.5}$$

where

$$F(s) = \int_0^\infty e^{-s\tau} f(\tau)\ \ \Re(s) > 0. \tag{5.6}$$

Applying the Laplace transform to both sides of (5.3) gives

$$\begin{aligned}
\mathcal{N}(s) &= \mathcal{N}_0 \left(\int_0^\infty e^{-s\tau}\tau^{\mu+\nu k-1}d\tau\right) - \delta^{\nu j}s^{-\nu j}\mathcal{N}(s) \\
&= \mathcal{N}_0 \left(\frac{\Gamma(\mu+\nu k)}{s^{\mu+\nu k}}\right) - \delta^{\nu j}s^{-\nu j}\mathcal{N}(s) \\
&= \mathcal{N}_0 \left(\frac{\Gamma(\mu+\nu k)}{s^{\mu+\nu k}}\right)(1+\delta^{\nu j}s^{-\nu j})^{-1}.
\end{aligned} \tag{5.7}$$

Taking the Laplace inverse of (5.7) and using

$$\mathcal{L}^{-1}\left[s^{-\nu} : \tau\right] = \frac{\tau^{\nu-1}}{\Gamma(\nu)} \ \ (\Re(\nu) > 0), \tag{5.8}$$

it is found that

$$\mathcal{L}^{-1}\{\mathcal{N}(s)\} = \mathcal{N}_0\Gamma(\mu+\nu k)\sum_{r=0}^{\infty}(-1)^r\delta^{\nu rj}\mathcal{L}^{-1}\left\{s^{-(\nu rj+\mu+\nu k)}\right\}$$

$$\mathcal{N}(\tau) = \mathcal{N}_0\Gamma(\nu+\mu)\,\tau^{\mu-1}\sum_{r=0}^{\infty}\frac{(-1)^r\delta^{\nu rj}\tau^{\nu rj+\nu k}}{\Gamma(\mu+\nu(rj+k))}.$$

Now, from (2.1) we get (5.4). ■.

On letting $j=1$ and $k=0$, in Proposition 5.1, we obtain the result given by Saxena, Mathai, and Haubold [30], whereas for $\mu=1$, $j=1$ and $k=0$, Proposition 5.1, gives us the result given by Haubold and Mathai [11].

Proposition 5.2 *If* $\delta>0$, $\nu>0$, $j\geq 1$, $k\geq 0$, *then the solution of the equation*

$$\mathcal{N}(\ll) - \mathcal{N}_0\,(-1)^{\frac{k}{j}}\tau^{\mu-1}E_{\nu,\mu}^{j,k}\left((-1)^{\frac{1}{j}}\delta^\nu\tau^\nu)\right) = -\delta^{\nu j}{}_0D_\tau^{-\nu j}\mathcal{N}(\ll), \tag{5.9}$$

is given by

$$\mathcal{N}(\tau) = \mathcal{N}_0\left(\frac{\tau^{\mu-1}(-1)^{\frac{k}{j}}}{\nu j}\right) \times\left[E_{\nu,\mu-1}^{j,k}\left((-1)^{\frac{1}{j}}(\delta\tau)^\nu\right) + (\nu j-\nu k-\mu+1)E_{\nu,\mu}^{j,k}\left((-1)^{\frac{1}{j}}(\delta\tau)^\nu\right)\right], \tag{5.10}$$

where $E_{\nu,\mu}^{j,k}(\tau)$ *is the Mittag-Leffler function (2.1).*

Proof Using (2.1) and (5.5) and projecting (5.9) to the Laplace transform, we can

$$\begin{aligned}\mathcal{N}(s) &= \mathcal{L}\{N(\tau):s\} = \mathcal{N}_0\,\delta^{\nu k}s^{-(\nu k+\mu)}\sum_{r=0}^{\infty}(-1)^r\delta^{\nu rj}s^{-\nu rj}(1+\delta^{\nu j}s^{-\nu j})^{-1}\\ &= \mathcal{N}_0\,\delta^{\nu k}\,s^{-(\nu k+\mu)}\,(1+\delta^{\nu j}s^{-\nu j})^{-2}\\ &= \mathcal{N}_0\sum_{r=0}^{\infty}\frac{(-1)^r(2)_r\delta^{\nu rj+\nu k}s^{-(\nu rj+\nu k+\mu)}}{r!},\end{aligned}$$

Hence

$$\mathcal{L}^{-1}\{\mathcal{N}(s)\} = \mathcal{N}_0\sum_{r=0}^{\infty}\frac{(-1)^r(2)_r\delta^{\nu rj+\nu k}\mathcal{L}^{-1}\{s^{-(\nu rj+\nu k+\mu)}\}}{r!}$$

$$\mathcal{N}(\tau) = \mathcal{N}_0\tau^{\mu-1}\sum_{r=0}^{\infty}\frac{(-1)^r(r+1)(-\delta\tau)^{\nu rj+\nu k}}{\Gamma(\mu+\nu(rj+k))}$$

$$\mathcal{N}(\tau) = \mathcal{N}_0\tau^{\mu-1}\sum_{r=0}^{\infty}\frac{(-1)^r(\delta\tau)^{\nu rj+\nu k}}{\Gamma(\mu+\nu(rj+k))} \times\frac{1}{\nu j}\left[(\nu rj+\nu k+\mu-1)+(\nu j-\nu k-\mu+1)\right]. \tag{5.11}$$

From (5.11) we lead to the desired result (5.10). ■.

Now, we look at the solution of a generalized fractional kinetic equation with two parameters δ and σ when $delta$ does not equal σ.

Proposition 5.3 *If* $\delta > 0, \sigma > 0, \delta \neq \sigma, \nu > 0, j \geq 1, k \geq 0,$ *then the solution of the equation*

$$\mathcal{N}(\ll) - \mathcal{N}_0\,(-1)^{\frac{k}{j}}\tau^{\mu-1}E^{j,k}_{\nu,\mu}\left((-1)^{\frac{1}{j}}\delta^{\nu}\tau^{\nu}\right) = -\sigma^{\nu j}\,{}_0D_{\tau}^{-\nu j}\mathcal{N}(\ll), \tag{5.12}$$

is given by

$$\mathcal{N}(\ll) = \mathcal{N}_0\left(\frac{(-1)^{\frac{k}{j}}\tau^{\mu-1}\delta^{\nu j}}{\delta^{\nu j}-\sigma^{\nu j}}\right)E^{j,k}_{\nu,\mu}\left((-1)^{\frac{1}{j}}\delta^{\nu}\tau^{\nu}\right). \tag{5.13}$$

Proof Using (2.1) and (5.5) and projecting (5.12) to the Laplace transform, we can

$$\mathcal{N}(s) = \mathcal{L}\{N(\tau) : s\} = \mathcal{N}_0\,\delta^{\nu k}s^{-(\nu k+\mu)}(1+\delta^{\nu j}s^{-\nu j})^{-1}(1+\sigma^{\nu j}s^{-\nu j})^{-1}$$

$$\mathcal{N}(s) = \mathcal{N}_0\,\delta^{\nu k}s^{-(\nu k+\mu)}\sum_{r=0}^{\infty}(-1)^r\delta^{\nu rj}\sum_{l=0}^{\infty}(-1)^l\sigma^{\nu lj}s^{-(\nu rj+\nu lj)}$$

$$\mathcal{N}(s) = \mathcal{N}_0\left(\frac{\delta^{\nu j}}{\delta^{\nu j}-\sigma^{\nu j}}\right)\sum_{r=0}^{\infty}(-1)^r\delta^{\nu rj+\nu k}\,s^{\nu rj+\nu k+\mu}.$$

Hence

$$\mathcal{L}^{-1}\{\mathcal{N}(s)\} = \mathcal{N}_0\left(\frac{\delta^{\nu j}}{\delta^{\nu j}-\sigma^{\nu j}}\right)\sum_{r=0}^{\infty}(-1)^r\delta^{\nu rj+\nu k}\,\mathcal{L}^{-1}\{s^{\nu rj+\nu k+\mu}\}.$$

gives

$$\mathcal{N}(s) = \mathcal{N}_0\left(\frac{\delta^{\nu j}}{\delta^{\nu j}-\sigma^{\nu j}}\right)\sum_{r=0}^{\infty}\frac{(-1)^r\delta^{\nu rj+\nu k}\,\tau^{\nu rj+\nu k+\mu-1}}{\Gamma(\mu+\nu(rj+k))}. \tag{5.14}$$

The assertion (5.13) now follows from (2.1) and (5.14). ■.

In the next application, we show that the trigonometric function $S_k(z; j)$ in the kernel of the generalized fractional kinetic equation leads to the Mittag–Leffler-type function $E^{j,k}_{a,b}(z)$ as a solution of the equation.

Proposition 5.4 *If* $\delta > 0, \sigma > 0, \delta \neq \sigma, \nu > 0, j \geq 1, k \geq 0,$ *then the solution of the equation*

$$\mathcal{N}(\ll) - \mathcal{N}_0\,\tau^{\mu}S_{\nu k}\left(\delta^{\nu}\tau^{\nu}; \nu j\right) = -\sigma^{\nu j}\,{}_0D_{\tau}^{-\nu j}\mathcal{N}(\ll), \tag{5.15}$$

is given by

$$\mathcal{N}(\ll) = \mathcal{N}_{\prime}\left(\frac{(-1)^{\frac{k}{j}}\delta^{\nu j}\tau^{\mu}}{\delta^{\nu j}-\sigma^{\nu j}}\right)E^{j,k}_{\nu,\mu}\left((-1)^{\frac{1}{j}}\delta^{\nu}\tau^{\nu}\right). \tag{5.16}$$

Proof We refer to the proof of proposition 5.3. ■.

Follows this, we solve a generalized fractional kinetic equation involving the Mittag–Leffler function of three parameters $E^{(c)}_{a,b}(z)$ (see [26]).

Proposition 5.5 *If* $\delta > 0, \sigma > 0, \delta \neq \sigma, \nu > 0, j \geq 1, k \geq 0,$ *then the solution of the equation*

$$\mathcal{N}(\ll) - \mathcal{N}_0\, \tau^{\nu k+\mu-1}\, E^{(\lambda)}_{\nu j, \nu k+\mu}\left(-\delta^{\nu j}\tau^{\nu j}\right) = -\sigma^{\nu j}{}_0D_\tau^{-\nu j}\mathcal{N}(\ll), \tag{5.17}$$

is given by

$$\mathcal{N}(\ll) = \mathcal{N}_0 \left(\frac{\sigma^{\nu j} - \delta^{\nu j}}{\sigma^{\nu j}}\right)^{-\lambda} \left(\frac{(-1)^{\frac{k}{j}}}{\tau\, \sigma^{\nu k}}\right) E^{j,k}_{\nu,\mu}\left((-1)^{\frac{1}{j}}\delta^{\nu}\tau^{\nu}\right). \tag{5.18}$$

Proof Using (2.1) and (5.5) and projecting (5.12) to the Laplace transform, we can

$$\mathcal{N}(s) = \mathcal{N}_0 \sum_{r=0}^{\infty} \frac{(-1)^r (\lambda)^r \delta^{\nu r j} s^{-(\nu j r+\nu k+\mu)}}{r!} \left(1 + \sigma^{\nu j} s^{-\nu j}\right)^{-1}$$

$$\mathcal{N}(s) = \mathcal{N}_0 \left(\frac{\sigma^{\nu j} - \delta^{\nu j}}{\sigma^{\nu j}}\right)^{-\lambda} \sum_{l=0}^{\infty} (-1)^l \sigma^{\nu j s} s^{-(\nu j s+\nu k+\mu)}.$$

Hence

$$\mathcal{L}^{-1}\{\mathcal{N}(s)\} = \mathcal{N}_0 \left(\frac{\sigma^{\nu j} - \delta^{\nu j}}{\sigma^{\nu j}}\right)^{-\lambda} \sum_{l=0}^{\infty} (-1)^l \sigma^{\nu j s} \mathcal{L}^{-1}\{s^{-(\nu j s+\nu k+\mu)}\}$$

$$\mathcal{N}(\tau) = \mathcal{N}_0 \left(\frac{\sigma^{\nu j} - \delta^{\nu j}}{\sigma^{\nu j}}\right)^{-\lambda} \sum_{l=0}^{\infty} \frac{(-1)^l \sigma^{\nu j s} \tau^{\nu j s+\nu k+\mu-1}}{\Gamma(\mu + \nu(sj + k))}. \tag{5.19}$$

The assertion (5.18) now follows from (2.1) and (5.19). ■.

Now, we recall the definition of the Wright function in the form[18]

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n + \mu)},\ \lambda > -1, \mu \in \mathbb{C}. \tag{5.20}$$

Finally, we demonstrate that the Mittag–Leffler-type function $E^{j,k}_{a,b}(z)$ arising in the solution of a generalized fractional kinetic equation with the Wright function $W_{\lambda,\mu}(z)$ in the kernel.

Proposition 5.6 *If* $\delta > 0, \sigma > 0, \delta \neq \sigma, \nu > 0, j \geq 1, k \geq 0,$ *then the solution of the equation*

$$\mathcal{N}(\ll) - \mathcal{N}_0\, \tau^{\nu k+\mu-1}\, W_{\nu j, \nu k+\mu}\left(-\delta^{\nu j}\tau^{\nu j}\right) = -\sigma^{\nu j}{}_0D_\tau^{-\nu j}\mathcal{N}(\ll), \tag{5.21}$$

is given by

$$\mathcal{N}(\ll) = \mathcal{N}_0 \left(\frac{(-1)^{\frac{k}{j}}}{\tau\, \sigma^{\nu k}}\right) E^{j,k}_{\nu,\mu}\left((-1)^{\frac{1}{j}}\delta^{\nu}\tau^{\nu}\right). \tag{5.22}$$

Proof We refer to the proof of Proposition 5.5. ■.

6 Conclusions

Based on the Mittag-Leffler fucntion $E_{a,b}(z)$, the pseudo-hyperbolic function $E_k(z;\, j)$ and the pseudo-trigonometric function $S_k(z;\, j)$, we proposed the Mittag-Leffler function $E_{a,b}^{j,k}(z)$ with arbitrary order. The significance of this generalization comes from the fact that the new Mittag-Leffler function satisfies most of the properties of the original functions mentioned above and provides new relations. In this work, we obtained basic properties, expansion relations, integral representations, differentiation with respect to z, differentiation recursion, and logarithmic derivative for the function $E_{a,b}^{j,k}(z)$. In addition It is important to note that the function $E_{a,b}^{j,k}(z)$ is very compatible with fractional calculus, specifically with fractional differential equations. The results establihed in this work are significant from an application standpoint since we demonstrated that the function $E_{a,b}^{j,k}(z)$ arises in the solutions of six general forms of the fractional kinetic equation integral representation. We conclude by pointing out that the Mittag–Leffler functions are crucial in locating analytical solutions to the fractional diffusion equations. For this particular class of fractional differential equations, we anticipate establishing analogous results in a forthcoming publication.

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Data Availability The results data used to support the findings of this study are included within the article.

Declarations

Conflict of interest No potential conflict of interest was reported by the authors.

Appendix

In this section, we summarize the main results obtained in the previous sections in the form of three tables as follows.

Table 1 Integral representations for the Mittag-Leffler-type function $E^{j,k}_{a,b}(z)$

No.	Integral representation	Proposition
1.	$E^{j,k}_{a,b+1}(z^a) = \frac{2}{\Gamma(b)} \int_0^{\frac{\pi}{2}} (\cos\theta)^{2b-1} \sin\theta \; E_{ak}\left(z(\sin\theta)^2;\, ja\right) d\theta$	3.1
2.	$E^{j,k}_{a,b+1}(z) = \frac{2}{(-1)^{\frac{k}{j}}\Gamma(b)} \int_0^{\frac{\pi}{2}} (\cos\theta)^{2b-1} \sin\theta \; \times S_{ak}\left((-1)^{\frac{1}{aj}} z(\sin\theta)^2;\, ja\right) d\theta$	3.1
3.	$E^{j,k}_{a,b+1}(z^a) = \frac{1}{(-1)^{\frac{k}{j}}\Gamma(b)} \int_0^{\infty} (1+u)^{-(b+1)} \; E_{ak}\left(\frac{zu}{1+u};\, ja\right) du$	3.2
4.	$E^{j,k}_{a,b+1}(z^a) = \frac{1}{(-1)^{\frac{k}{j}}\Gamma(b)} \int_0^{\infty} (1+u)^{-(b+1)} \; S_{ak}\left(\frac{-zu}{1+u};\, ja\right) du$	3.2
5.	$E^{j,k}_{a,b+1}(z) = \frac{z^k}{\Gamma(b+ak)} \int_0^1 \zeta^{ak+b-1} \; E_{aj}\left(z^j(1-\zeta)^{aj}\right) d\zeta$	3.3
6.	$E^{j,k}_{a,b}(z) = \frac{z^k}{\Gamma(ak)} \int_0^1 \zeta^{ak-1}(1-\zeta)^{b-1} \; E_{aj,b}\left(z^j(1-\zeta)^{aj}\right) d\zeta$	3.3
7.	$E^{j,k}_{a,b+1}(z^a) = \frac{1}{\Gamma(b)} \int_0^1 (1-\zeta)^{b-1} \; E_{ak}\left(z\zeta^a;\, ja\right) d\zeta$	3.3
8.	$E^{j,k}_{a,b+1}(z^a) = \frac{1}{(-1)^{\frac{k}{j}}\Gamma(b)} \int_0^1 (1-\zeta)^{b-1} \; S_{ak}\left(-z\zeta^a;\, ja\right) d\zeta$	3.3
9.	$z^{b-1}E^{j,k}_{a,b}(z^a) = \frac{1}{\Gamma(ak)} \int_0^z (z+\zeta)^{ak-1}\zeta^{b-1} \; E_{aj,b}\left(\zeta^{aj}\right) d\zeta$	3.4
10.	$E^{j,k}_{a,b}(z^a) = \frac{z^k}{2\pi i} \int_{H_a} \frac{e^t \; t^{-(b+a(j+k))}}{\left(t^{aj}-z^j\right)} dt$	3.5
11.	$\int_0^1 t^{b-1}E^{j,k}_{a,b}(zt^a)\,(1-t)^{c-1}E^{j,k}_{a,c}(\omega(1-t)^a)dt \; = \frac{1}{z^j-\omega^j} \; \times\omega^k z^j E^{j,k}_{a,b+c+ak}(z) - z^k\omega^j E^{j,k}_{a,b+c+ak}(\omega)$	3.6

Table 2 Pure and differential relations for the Mittag-Leffler-type function $E_{a,b}^{j,k}(z)$

No.	Pure \ differential relation	Proposition
1.	$E_{a,b}^{j,k}(-z) = (-1)^k E_{2a,b}^{j,k/2}(z^2) + (-1)^{j+k} z^j E_{2a,b+aj}^{j,k/2}(z^2)$	4.1
2.	$\frac{\partial^m}{\partial z^m} E_{a,b}^{j,k}(z) = m! E_{aj,b+ak,j}^{m+1,1,j}(z^j)$	4.2
3.	$\frac{\partial^{1-b}}{\partial z^{1-b}} E_{ak}(z; aj) = z^{b-1} E_{a,b}^{j,k}(z^a)$	4.2
4.	$(-1)^{\frac{-k}{j}} \frac{\partial^{1-b}}{\partial z^{1-b}} S_{ak}\left((-1)^{\frac{1}{aj}} z; aj\right) = z^{b-1} E_{a,b}^{j,k}(z^a)$	4.2
5.	$\frac{\partial^m}{\partial x^m}\left\{z^{b-1} E_{a,b}^{j,k}(xz^a)\right\} = z^{b-m-1} E_{a,b-m}^{j,k}(xz^a)$	4.3
6.	$\frac{a}{z^{j-1}} \frac{\partial}{\partial z} E_{a,b}^{j,k}(z) = E_{a,aj+b-1}^{j,k}(z) - (b+ak+1) E_{a,aj+b}^{j,k}(z) + \frac{ak}{z^j} E_{a,b}^{j,k}(z)$	4.3
7.	$\frac{\partial^m}{\partial z^m}\left\{z^{b-1} E_{\frac{m}{sj},b}^{j,k}(z^{\frac{m}{sj}})\right\} = z^{b-1} \sum_{n=1}^{s} \frac{z^{-\frac{m}{sj}(nj-k)}}{\Gamma(b-\frac{m}{sj}(nj-k))} + z^{b-1} E_{\frac{m}{sj},b}^{j,k}(z^{\frac{m}{sj}})$	4.4
8.	$\frac{\partial}{\partial a} E_{a,b}^{j,k}(z) = z \frac{\partial^2}{\partial z \partial b} E_{a,b}^{j,k}(z)$	4.5
9.	$\frac{\partial^m}{\partial a^m} E_{a,b}^{j,k}(ye^{\omega}) = \frac{\partial^{2m}}{\partial \omega^m \partial a^m} E_{a,b}^{j,k}(ye^{\omega})$	4.6
10.	$\frac{d \log E_{a,b}^{j,k}(z)}{dz} = \frac{1}{az}\left(\frac{E_{a,b-1}^{j,k}(z)}{E_{a,b}^{j,k}(z)} - (b-1)\right) > 0$	4.7
11.	$\frac{d \log E_{a,b}^{j,k}(z)}{dz} > \frac{j}{z}\left(1 - \frac{1}{\Gamma(b) E_{a,b}^{j,k}(z)}\right)$	4.8

Table 3 Fractional kinetic equations involving the Mittag–Leffler-type function $E_{a,b}^{j,k}(z)$

No.	Fractional kinetic equation	Solution	Proposition
1.	$\mathcal{N}(\ll) - \mathcal{N}_0\, \tau^{\mu+\nu k-1} = -\delta^{\nu j}\,{}_0D_\tau^{-\nu j}\mathcal{N}(\tau)$	$\mathcal{N}(\ll) = \mathcal{N}_0(-1)^{\frac{k}{j}}\, \Gamma(\mu+\nu k)\, \delta^{-\nu k}\, \tau^{\mu-1}$ $\times\, E_{\nu,\mu}^{j,k}((-1)^{\frac{1}{j}}\,(\delta\tau)^{\nu})$	5.1
2.	$\mathcal{N}(\ll) - \mathcal{N}_0\, (-1)^{\frac{k}{j}}\, \tau^{\mu-1} E_{\nu,\mu}^{j,k}\left((-1)^{\frac{1}{j}}\, \delta^{\nu}\tau^{\nu}\right)$ $= -\delta^{\nu j}\,{}_0D_\tau^{-\nu j}\mathcal{N}(\ll)$	$\mathcal{N}(\tau) = \mathcal{N}_0\left(\frac{\tau^{\mu-1}(-1)^{\frac{k}{j}}}{\nu j}\right) \times \left(E_{\nu,\mu-1}^{j,k}\left((-1)^{\frac{1}{j}}\,(\delta\tau)^{\nu}\right)\right.$ $\left.+(\nu j-\nu k-\mu+1)E_{\nu,\mu}^{j,k}\left((-1)^{\frac{1}{j}}\,(\delta\tau)^{\nu}\right)\right)$	5.2
3.	$\mathcal{N}(\ll) - \mathcal{N}_0\, (-1)^{\frac{k}{j}}\, \tau^{\mu-1} E_{\nu,\mu}^{j,k}\left((-1)^{\frac{1}{j}}\, \delta^{\nu}\tau^{\nu}\right) = -\sigma^{\nu j}\,{}_0D_\tau^{-\nu j}\mathcal{N}(\ll)$	$\mathcal{N}(\ll) = \mathcal{N}_0\left(\frac{(-1)^{\frac{k}{j}}\,\tau^{\mu-1}\delta^{\nu j}}{\delta^{\nu j}-\sigma^{\nu j}}\right)\, E_{\nu,\mu}^{j,k}\left((-1)^{\frac{1}{j}}\,\delta^{\nu}\tau^{\nu}\right)$	5.3
4.	$\mathcal{N}(\ll) - \mathcal{N}_0\, \tau^{\mu}\, S_{\nu k}\left(\delta^{\nu}\tau^{\nu};\, \nu j\right) = -\sigma^{\nu j}\,{}_0D_\tau^{-\nu j}\mathcal{N}(\ll)$	$\mathcal{N}(\ll) = \mathcal{N}_{\prime}\left(\frac{(-1)^{\frac{k}{j}}\,\delta^{\nu j}\tau^{\mu}}{\delta^{\nu j}-\sigma^{\nu j}}\right)\, E_{\nu,\mu}^{j,k}\left((-1)^{\frac{1}{j}}\,\delta^{\nu}\tau^{\nu}\right)$	5.4
5.	$\mathcal{N}(\ll) - \mathcal{N}_0\, \tau^{\nu k+\mu-1}\, E_{\nu j,\nu k+\mu}^{(\lambda)}\left(-\delta^{\nu j}\tau^{\nu j}\right) = -\sigma^{\nu j}\,{}_0D_\tau^{-\nu j}\mathcal{N}(\ll)$	$\mathcal{N}(\ll) = \mathcal{N}_0\left(\frac{\sigma^{\nu j}-\delta^{\nu j}}{\sigma^{\nu j}}\right)^{-\lambda}\left(\frac{(-1)^{\frac{k}{j}}}{\tau\,\sigma^{\nu k}}\right)\, E_{\nu,\mu}^{j,k}\left((-1)^{\frac{1}{j}}\,\delta^{\nu}\tau^{\nu}\right)$	5.5
6.	$\mathcal{N}(\ll) - \mathcal{N}_0\, \tau^{\nu k+\mu-1}\, W_{\nu j,\nu k+\mu}\left(-\delta^{\nu j}\tau^{\nu j}\right) = -\sigma^{\nu j}\,{}_0D_\tau^{-\nu j}\mathcal{N}(\ll)$	$\mathcal{N}(\ll) = \mathcal{N}_0\left(\frac{(-1)^{\frac{k}{j}}}{\tau\,\sigma^{\nu k}}\right)\, E_{\nu,\mu}^{j,k}\left((-1)^{\frac{1}{j}}\,\delta^{\nu}\tau^{\nu}\right)$	5.6

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