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Functional analytic properties and regularity of the Möbius-invariant Willmore flow in $\mathbf{R}^n$

Ruben Jakob[1]

Abstract

In this article we continue the author's investigation of the Möbius-invariant Willmore flow moving parametrizations of umbilic-free tori in $\mathbf{R}^n$ and in the n-sphere $\mathbf{S}^n$. In the main theorems of this article we prove basic properties of the evolution operator of the "DeTurck modification" of the Möbius-invariant Willmore flow and of its Fréchet derivative by means of a combination of the author's results about this topic with the theory of bounded $\mathcal{H}_\infty$-calculus for linear elliptic operators due to Amann, Denk, Duong, Hieber, Prüss and Simonett with Amann's and Lunardi's work on semigroups and interpolation theory. Precisely, we prove local real analyticity of the evolution operator $[F \mapsto \mathcal{P}^*(\,\cdot\,, 0, F)]$ of the "DeTurck modification" of the Möbius-invariant Willmore flow in a small open ball in $W^{4-\frac{4}{p},p}(\Sigma, \mathbf{R}^n)$, for any $p \in (3, \infty)$, about any fixed smooth parametrization $F_0 : \Sigma \longrightarrow \mathbf{R}^n$ of a compact and umbilic-free torus in $\mathbf{R}^n$. We prove moreover that the entire maximal flow line $\mathcal{P}^*(\,\cdot\,, 0, F_0)$, starting to move in a smooth and umbilic-free initial immersion F_0, is real analytic for positive times, and that therefore the Fréchet derivative $D_F \mathcal{P}^*(\,\cdot\,, 0, F_0)$ of the evolution operator in F_0 can be uniquely extended to a family of continuous linear operators $G^{F_0}(t_2, t_1)$ in $L^p(\Sigma, \mathbf{R}^n)$, whose ranges are dense in $L^p(\Sigma, \mathbf{R}^n)$, for every fixed pair of times $t_2 \geq t_1$ within the interval of maximal existence $(0, T_{\max}(F_0))$.

Keywords Willmore flow · Functional analytic properties · Regularity · Conformal invariance

Mathematics Subject Classification 53C42 · 35K46 · 35R01 · 47B12 · 58J35

1 Introduction and state of the art

In [13] the author has proved short-time existence and uniqueness of smooth flow lines of the following non-linear, Willmore-type evolution equation:

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$$\partial_t f_t = -\frac{1}{2}\frac{1}{|A^0_{f_t}|^4}\left(\triangle^{\perp}_{f_t}\mathbf{H}_{f_t} + Q(A^0_{f_t})(\mathbf{H}_{f_t})\right) \equiv -\frac{1}{|A^0_{f_t}|^4}\nabla_{L^2}\mathcal{W}(f_t). \tag{1}$$

Here, $\{f_t\}_{t\in\mathbf{R}_+}$ denotes a differentiable family of $W^{4,2}$-immersions mapping some arbitrarily fixed compact smooth torus Σ into $\mathbf{R}^n$ or into $\mathbf{S}^n$, for $n \geq 3$, without any umbilic points. As already pointed out in the author's article [13], the "umbilic-free condition" $|A^0_{f_t}|^2 > 0$ on Σ implies that $\chi(\Sigma) = 0$, where χ denotes the topological Euler-characteristic, which forces the geometric flow (1) to be only well-defined on differentiable families of sufficiently smooth umbilic-free tori, immersed into $\mathbf{R}^n$ or $\mathbf{S}^n$, with $n \geq 3$. In flow Eq. (1), $\mathcal{W}$ denotes the Willmore-functional

$$\mathcal{W}(f) := \int_{\Sigma} K_f^M + \frac{1}{4}\,|\,\mathbf{H}_f\,|^2\,d\mu_f, \tag{2}$$

which can more generally be considered on C^2-immersions $f : \Sigma \longrightarrow M$, mapping any closed smooth Riemannian orientable surface Σ into an arbitrary smooth Riemannian manifold M, where $K_f^M(x)$ denotes the sectional curvature of M w.r.t. the "immersed tangent plane" $Df_x(T_x\Sigma)$ in $T_{f(x)}M$. The most prominent cases are: $K_f \equiv 0$ if $M = \mathbf{R}^n$ and $K_f \equiv 1$ if $M = \mathbf{S}^n$. Regarding the aims of this article, we will only have to consider the simplest case $M = \mathbf{R}^n$, with $n \geq 3$. We endow the torus Σ with the pullback f^*g_{euc} w.r.t. f of the Euclidean metric of $\mathbf{R}^n$, i.e. with coefficients $g_{ij} := \langle\partial_i f, \partial_j f\rangle_{\mathbf{R}^n}$, and we let A_f denote the second fundamental form of the immersion $f : \Sigma \longrightarrow \mathbf{R}^n$, defined on pairs of tangent vector fields X, Y on Σ by:

$$A_f(X, Y) := D_X(D_Y(f)) - P^{\text{Tan}(f)}(D_X(D_Y(f))) \equiv (D_X(D_Y(f)))^{\perp_f}, \tag{3}$$

where $D_X(V)\lfloor_x$ denotes the classical derivative of a vector field $V : \Sigma \longrightarrow \mathbf{R}^n$ in direction of the tangent vector field $X \in \Gamma(T\Sigma)$ at a point $x \in \Sigma$, and where

$$P^{\text{Tan}(f)} : \bigcup_{x\in\Sigma}\{x\}\times\mathbf{R}^n \longrightarrow \bigcup_{x\in\Sigma}\{x\}\times T_{f(x)}(f(\Sigma))$$

denotes the bundle morphism, which projects the entire ambient space $\mathbf{R}^n$ orthogonally into its subspaces $T_{f(x)}(f(\Sigma))$ – the tangent spaces of the immersion f in the points $f(x)$ for every $x \in \Sigma$ – and where $\perp_f$ abbreviates the bundle morphism $\text{Id}_{\mathbf{R}^n} - P^{\text{Tan}(f)}$. Furthermore, A^0_f denotes the tracefree part of A_f, i.e.

$$A^0_f(X, Y) := A_f(X, Y) - \frac{1}{2}g_f(X, Y)\mathbf{H}_f$$

and

$$\mathbf{H}_f := \text{trace}(A_f) \equiv g_f^{ij}A_f(\partial_i, \partial_j)$$

("Einstein's summation convention") denotes the mean curvature vector of the immersion $f : \Sigma \longrightarrow \mathbf{R}^n$. Finally, $Q(A_f)$ respectively $Q(A^0_f)$ operates on vector fields ϕ which are sections of the normal bundle of f, i.e. which "are normal along f", by assigning:

$$Q(A_f)(\phi) := g_f^{ij}g_f^{kl}A_f(\partial_i, \partial_k)\langle A_f(\partial_j, \partial_l), \phi\rangle_{\mathbf{R}^n},$$

which is by definition again a section of the normal bundle of f.

Furthermore, Weiner computed in [33], Sect. 2, that the first variation of the Willmore functional in a smooth immersion $f : \Sigma \longrightarrow \mathbf{R}^n$, $n \geq 3$, in direction of a smooth section Ψ of

the normal bundle of f:

$$\delta\mathcal{W}(f,\Psi) := \partial_t(\mathcal{W}(f+t\,\Psi))\lfloor_{t=0} = \frac{1}{2}\int_\Sigma \langle \triangle_f^\perp \mathbf{H}_f + Q(A_f^0)(\mathbf{H}_f), \Psi \rangle_{\mathbf{R}^n}\, d\mu_f$$

$$=: \int_\Sigma \langle \nabla_{L^2}\mathcal{W}(f), \Psi \rangle_{\mathbf{R}^n}\, d\mu_f. \tag{4}$$

For the reader's convenience we recall the following Proposition from [13]:

Proposition 1 *Any differentiable family $\{f_t\}$ of C^4-immersions $f_t : \Sigma \longrightarrow \mathbf{R}^n$ without umbilic points, i.e. with $\mid A_{f_t}^0 \mid^2 > 0$ on $\Sigma \,\forall\, t \in [0, T)$, solves the flow equation*

$$\partial_t f_t = -\frac{1}{2}\mid A_{f_t}^0 \mid^{-4}\, \left(\triangle_{f_t}^\perp \mathbf{H}_{f_t} + Q(A_{f_t}^0)(\mathbf{H}_{f_t})\right) \equiv -\mid A_{f_t}^0 \mid^{-4}\, \nabla_{L^2}\mathcal{W}(f_t) \tag{5}$$

if and only if its composition $\Phi(f_t)$ with any applicable Möbius-transformation Φ of $\mathbf{R}^n$ solves the same flow equation, thus, if and only if

$$\partial_t(\Phi(f_t)) = -\mid A_{\Phi(f_t)}^0 \mid^{-4}\, \nabla_{L^2}\mathcal{W}(\Phi(f_t))$$

holds $\forall\, t \in [0, T)$ and for every $\Phi \in \mathrm{M\ddot{o}b}(\mathbf{R}^n)$, for which $\Phi(f_t)$ is well-defined on $\Sigma \times [0, T)$.

Obviously, the above proposition suggests to term the geometric flow (5) the "Möbius-invariant Willmore flow" (MIWF). Actually, the author was able to show in the Appendix of [13], that the expression on the right hand side of Eq. (5) is the analytically simplest modification of the classical differential operator $[f \mapsto \nabla_{L^2}\mathcal{W}(f)]$ from line (4), which yields a "conformally invariant flow". Looking at the right hand side of evolution Eq. (1), one can easily see its "degeneracy" and thus imagine, that the corresponding flow produces many singular flow lines and thus several interesting technical challenges. Now, comparing the MIWF (1) to the classical Willmore flow, given by the equation

$$\partial_t f_t = -\frac{1}{2}\left(\triangle_{f_t}^\perp \mathbf{H}_{f_t} + Q(A_{f_t}^0)(\mathbf{H}_{f_t})\right) \equiv -\nabla_{L^2}\mathcal{W}(f_t)$$

the MIWF has the big advantage, that its flow lines can be conformally mapped—by stereographic projection—either from $\mathbf{R}^n$ into $\mathbf{S}^n$ or from $\mathbf{S}^n$ into $\mathbf{R}^n$, depending on the concrete objectives of the investigation. In particular the "3-sphere" is a simply-connected compact Lie-group—diffeomorphic to $SU(2)$—can be interpreted as the set of quaternions of length 1, is fibered by the Hopf-fibration $\pi : \mathbf{S}^3 \longrightarrow \mathbf{S}^2$ and contains the Clifford-torus—an embedded minimal surface in $\mathbf{S}^3$, which is the global minimizer of the Willmore functional (2) among all compact surfaces of genus ≥ 1 immersed into $\mathbf{S}^3$, on account of [24], Theorem A. This particular mathematical situation actually plays a key role in the proof of the first main theorem of the author's preprint [15], yielding a sufficient condition for global existence and full convergence of smooth flow lines of the MIWF in $\mathbf{S}^3$; see here point (2) of the list below. Moreover, even restricted to the rather narrow class of "Hopf-tori" in $\mathbf{S}^3$, the MIWF seems to develop "singularities"—see Definition 2 below—since it reduces via the Hopf-fibration $\pi : \mathbf{S}^3 \longrightarrow \mathbf{S}^2$ to the "degenerate" parabolic flow:

$$\partial_t \gamma_t = -\frac{1}{(\kappa_{\gamma_t}^2 + 1)^2}\left(2\left(\nabla_{\frac{\gamma_t'}{|\gamma_t'|}}^\perp\right)^2(\kappa_{\gamma_t}) + |\kappa_{\gamma_t}|^2\kappa_{\gamma_t} + \kappa_{\gamma_t}\right)$$

$$\equiv -\frac{1}{(\kappa_{\gamma_t}^2 + 1)^2}\, \nabla_{L^2}\mathcal{E}(\gamma_t) \tag{6}$$

for smooth closed, regular curves $\gamma_t : \mathbf{S}^1 \longrightarrow \mathbf{S}^2$, where "$\kappa_{\gamma_t}$" denotes the curvature vectors of the curves γ_t, and "$\nabla_{L^2}\mathcal{E}$" denotes the L^2-gradient of twice the classical elastic energy, i.e. of the functional

$$\mathcal{E}(\gamma) := \int_{\mathbf{S}^1} 1 + |\kappa_\gamma|^2 \, d\mu_\gamma.$$

See Propositions 3.2 and 3.3 in [14] for technical details of this particular "dimension reduction". The author's most recent research in [16] reveals, that this sort of "degeneracy" of the MIWF does actually not improve under the additional energy condition "$\mathcal{W}(F_0) < 8\pi$" on an initial parametrization F_0 of a smooth Hopf-torus $\pi^{-1}(\text{trace}(\gamma_0))$ in $\mathbf{S}^3$. Hence, the MIWF presents us with many challenging questions and puts the mathematical machinery of modern Geometric Analysis to the test. [1] On the other hand, we can also use "$\mathbf{R}^n$" as ambient space for flow lines of the MIWF, and this allows us to apply heavy machinery from both linear and non-linear "Functional Analysis", in order to investigate the evolution operator of the MIWF—as an operator between appropriately chosen Banach spaces—via its linearization, and this is actually the main technical focus of this article and also of the author's paper [13]. So far, the author has proved four basic results about the MIWF:

(1) Short-time existence and uniqueness of smooth flow lines $\{\mathcal{P}(t, 0, F_0)\}_{t \geq 0}$ of the Möbius-invariant Willmore flow (1), which start moving in C^∞-smooth and umbilic-free initial immersions $F_0 : \Sigma \longrightarrow \mathbf{R}^n$, see Theorem 1 in [13] for the precise statement.

(2) Global existence and full $C^k(\Sigma)$-convergence of any flow line $\{\mathcal{P}(t, 0, F_0)\}_{t \geq 0}$ of the MIWF (1) to a smooth parametrization of a Clifford-torus in $\mathbf{S}^3$, provided $\{\mathcal{P}(t, 0, F_0)\}_{t \geq 0}$ starts moving in a C^∞-smooth immersion $F_0 : \Sigma \longrightarrow \mathbf{S}^3$ which is sufficiently close to a fixed smooth parametrization of a Clifford-torus in a $h^{2+\beta}(\Sigma, \mathbf{R}^4)$-norm. Here, "$h^{2+\beta}(\Sigma, \mathbf{R})$" denotes the "little Hölder space" of differentiation order $2 + \beta$. See Theorem 1.1 in [15] for the precise statement.

(3) Stability of the above "full convergence property" of flow lines of the MIWF (1) w.r.t. perturbations of their umbilic-free initial immersions $F : \Sigma \longrightarrow \mathbf{S}^3$ in any $C^{4,\gamma}(\Sigma, \mathbf{R}^4)$-norm. See here Theorem 1.2 in [15].

(4) Global existence and full $C^{k-1,\alpha}(\Sigma)$-convergence of any flow line $\{\mathcal{P}(t, 0, F_0)\}_{t \geq 0}$ of the MIWF (1) into some C^k-local minimizer of the Willmore functional, which starts moving in a C^∞-smooth immersion $f_0 : \Sigma \longrightarrow \mathbf{R}^3$ that is sufficiently close to a fixed C^k-local minimizer of the Willmore functional in a $C^{k,\alpha}(\Sigma, \mathbf{R}^3)$-norm. See Theorem 1.3 in [15] for the precise statement.

In this article, we will combine the theory of bounded $\mathcal{H}_\infty$-calculus for linear elliptic differential operators of higher order due to Amann, Denk, Duong, Hieber, Prüss and Simonett in [2, 5, 10, 11] with Amann's and Lunardi's work on non-autonomously generated "semigroups" and "interpolation theory" in [3, 21, 22], and with modern regularity criteria due to Shao and Simonett [27–29], in order to prove below in Theorems 3, 4 and 5 basic properties of the evolution operator $\{\mathcal{P}^*(t, 0, F_0)\}_{t \geq 0}$ of the "DeTurck modification" (13) of the MIWF (1) in $\mathbf{R}^n$ and of its "linearization". Just as in the author's first paper [13] about the MIWF, a certain adaption of the "DeTurck trick" will also be used in this paper, in order to be able to apply well developed theories dealing with linear respectively quasilinear parabolic differential equations. Moreover we should also mention here, that certain variants of Proposition 2 and

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[1] One should remark here, that the classical Willmore flow in $\mathbf{S}^3$ actually turns out to have only global smooth flow lines, which start moving in smooth parametrizations of arbitrary smooth Hopf-tori in $\mathbf{S}^3$, see Theorem 2.1 in [14].

of Theorem 1 below in Sect. 2 play key rôles in the proof of the above mentioned Theorem 1.3 of the author's preprint [15] about the MIWF; see point (4) in the list above.

Returning to the above mentioned open question about "singularities" of the MIWF, we should also note here, that the statements of the five main theorems of this article hold mutatis mutandis also for the classical Willmore flow in $\mathbf{R}^n$. Hence modulo obvious changes, Theorems 3, 4 and 5 of this article can be formulated for the classical Willmore flow in $\mathbf{R}^n$ and can then be combined with the main theorem of the author's article [14] about flow lines of the classical Willmore flow in $\mathbf{R}^4$, whose initial immersions parametrize smooth Hopf-tori in $\mathbf{S}^3$, particularly with the aim to follow the lines of Andrews' article [6] and to prove, that generically every such flow line of the classical Willmore flow converges fully and smoothly to some smooth parametrization of the Clifford-torus in $\mathbf{S}^3$—modulo some Möbius-transformation of $\mathbf{S}^3$—without imposing any initial Willmore energy condition. Actually, motivated by Andrews' article [6], Theorems 3, 4 and 5—particularly the final part of Theorem 5—have been discovered and proved by the author exactly with the aim, to control—with "relatively high precision"—the deviation of flow lines $\{\mathcal{P}^*(t, 0, F)\}_{t \geq 0}$ in (102) of the "DeTurck-modified" flow Eq. (13) w.r.t. perturbations of their initial immersions F in small open balls $B_\rho^{4,p}(F_0)$ in $W^{4,p}(\Sigma, \mathbf{R}^n)$ at arbitrarily large times $t >> 1$, provided every flow line $\{\mathcal{P}^*(t, 0, F)\}_{t \geq 0}$ starting in $F \in B_\rho^{4,p}(F_0) \cap C^\infty(\Sigma, \mathbf{R}^n)$ with $\rho > 0$ sufficiently small, certainly exists globally, i.e. for all $t \geq 0$. It is quite a remarkable consequence of this article, that the functional analytic statements of Theorems 4 and 5 turn out to hold along flow lines of the "rather degenerate" MIWF, and not only along flow lines of the classical Willmore flow in $\mathbf{R}^n$, provided we choose $p > 3$ in those two theorems.

This paper is organized as follows: In Sect. 2 we firstly introduce some basic notation and prove fundamental properties of the non-linear differential operator (15) corresponding to the "DeTurck modification" (13) of the MIWF (1) and of its linearization in the setting of parabolic L^p-spaces. Moreover, in Sect. 2 we also prove a new short-time existence result for evolution Eq. (13) and real analyticity—both in space and in time—of maximal flow lines to evolution Eq. (13) starting in smooth—but not necessarily real analytic—and umbilic-free initial immersions of Σ into $\mathbf{R}^n$. To this end we follow the strategies of the papers [12, 18, 26–29], i.e. we use the technique by Escher, Prüss, Shao and Simonett of parameter-dependent diffeomorphisms on closed manifolds, maximal regularity in both L^p- and h^β-spaces and the "Implicit Function Theorem" for real analytic, non-linear operators. In Sect. 3 we prove the main results of this paper, whose proofs heavily rely on the local real analyticity of the evolution operator of evolution Eq. (13)—see the first part of Theorem 4—and the above mentioned real analyticity of smooth flow lines of evolution Eq. (13). In particular, this regularity result is a very effective tool, in order to prove the two final statements of this paper, Theorem 5 (ii) and (iii), yielding the fact that the Fréchet derivative of the evolution operator of evolution Eq. (13) in any smooth and umbilic-free immersion F_0 can be interpreted as a 2-parameter family "$G^{F_0}(t, s)$" of linear continuous operators in $L^p(\Sigma, \mathbf{R}^n)$ mapping $W^{4,p}(\Sigma, \mathbf{R}^n)$ into itself and having <u>dense range in</u> $\underline{L^p(\Sigma, \mathbf{R}^n)}$, for any choice of the two parameters $s \leq t$ within the open interval of maximal existence $(0, T_{\max}(F_0))$ of the considered flow line $\mathcal{P}^*(\cdot, 0, F_0)$ starting in F_0 at time $s = 0$, and for any fixed $p > 3$.

2 Preparation for the proofs of Theorems 4 and 5

We follow the definition of Besov- and Sobolev-Slobodeckij-spaces $W^{s,p}(\mathbf{R}^m_+)$ on half spaces $\mathbf{R}^m_+$ in Sections 2.9.1 and 2.9.3 in [31] and define here similarly L^p- and $W^{s,p}$-functions on smooth domains respectively on compact, closed smooth surfaces, having values in $\mathbf{R}^n$.

Definition 1 Let $p \in (1, \infty)$ and $s \geq 2$ be two real numbers, $n \in \mathbf{N}$, and let Ω be a bounded domain in $\mathbf{R}^2$ with smooth boundary and M a compact, closed smooth surface, equipped with an atlas $\mathcal{A} := \{(\Omega_j, \psi_j)\}_{j=1,\dots,N}$ of coordinate patches $\Omega_j \subset M$ and smooth charts $\psi_j : \Omega_j \xrightarrow{\cong} B_1^2(0)$.

(1) We define the space of L^p-functions on the compact surface M by

$$L^p(M, \mathbf{R}^n) := \{ u : M \longrightarrow \mathbf{R}^n \, | \, u \circ \psi_j^{-1} \in L^p(B_1^2(0), \mathbf{R}^n) \text{ for } j = 1, \dots, N \},$$

and we set

$$\| u \|_{L^p(M,\mathbf{R}^n)} := \max_{j=1,\dots,N} \| u \circ \psi_j^{-1} \|_{L^p(B_1^2(0),\mathbf{R}^n)} .$$

(2) We define the Sobolev-Slobodeckij-space $W^{s,p}(\Omega, \mathbf{R}^n)$ on the bounded domain $\Omega \subset \mathbf{R}^2$ by

$$W^{s,p}(\Omega, \mathbf{R}^n) := \{ u \in L^p(\Omega, \mathbf{R}^n) | \, \exists \, g = (g_1, \dots, g_n) \in W^{s,p}(\mathbf{R}^2, \mathbf{R}^n)$$
$$\text{s.t. } u(x) = g(x) \text{ for } \mathcal{H}^2 - \text{almost every } x \in \Omega \}.$$

Here, the Sobolev-Slobodeckij-space "$W^{s,p}(\mathbf{R}^2, \mathbf{R})$" is defined as in Sect. 2.3.1 in [31].

(3) We define the Sobolev-Slobodeckij-space $W^{s,p}(M, \mathbf{R}^n)$ on the compact surface M by

$$W^{s,p}(M, \mathbf{R}^n) := \{ u \in L^p(M, \mathbf{R}^n) \, | \, u \circ \psi_j^{-1} \in W^{s,p}(B_1^2(0), \mathbf{R}^n) \text{ for } j = 1, \dots, N \}$$

and its norm by

$$\| u \|_{W^{s,p}(M,\mathbf{R}^n)} := \max_{j=1,\dots,N} \| u \circ \psi_j^{-1} \|_{W^{s,p}(B_1^2(0),\mathbf{R}^n)}$$

for any $u \in W^{s,p}(M, \mathbf{R}^n)$.

$\square$

Moreover, we will need:

Definition 2 Let Σ be a smooth compact torus and $n \geq 3$ an integer.

(a) We denote by $\mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbf{R}^n)$ the subset of $C^2(\Sigma, \mathbf{R}^n)$ consisting of umbilic-free immersions, i.e.:

$$\mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbf{R}^n) := \{ f \in C^2(\Sigma, \mathbf{R}^n) \, | \, f \text{ is an immersion with } \, | \, A_f^0 \, |^2 > 0 \text{ on } \Sigma \}.$$

(b) A "flow line" of the MIWF (1) in $\mathbf{R}^n$ is a smooth family $\{f_t\}_{t \in [0,T)}$ of smooth immersions of Σ into $\mathbf{R}^n$, such that the resulting smooth function $f : \Sigma \times [0, T) \longrightarrow \mathbf{R}^n$ satisfies Eq. (1) classically on $\Sigma \times [0, T)$, i.e. such that

$$\partial_t f_t(x) = -\frac{1}{2} \frac{1}{|A_{f_t}^0(x)|^4} \left(\triangle_{f_t}^\perp \mathbf{H}_{f_t}(x) + Q(A_{f_t}^0)(\mathbf{H}_{f_t})(x) \right)$$

holds pointwise in every $(x, t) \in \Sigma \times [0, T)$.

(c) We use exactly the same terminology as in part (b) for smooth solutions $\{f_t\}_{t\in[0,T)}$ of the "relaxed MIWF-equation" in $\mathbf{R}^n$, which is:

$$\left(\partial_t f_t(x)\right)^{\perp_{f_t}} = -\frac{1}{2}\frac{1}{|A^0_{f_t}(x)|^4}\left(\triangle^{\perp}_{f_t}\mathbf{H}_{f_t}(x) + Q(A^0_{f_t})(\mathbf{H}_{f_t})(x)\right)$$

for $(x,t)\in\Sigma\times[0,T)$. Here, $\perp_{f_t}$ abbreviates the projection of the velocity vector $\partial_t f_t(x)$ into the normal space of the immersion f_t within $\mathbf{R}^n$, for every fixed $x\in\Sigma$, as in (3).

(d) Let $F_0 : \Sigma \longrightarrow \mathbf{R}^n$ be a smooth and umbilic-free immersion and $\{F_t\}_{t\in[0,T)}$ a flow line of the MIWF starting in F_0. We call $[0,T)$ the "interval of maximal existence" of the MIWF starting in F_0, if either $T=\infty$, or if there holds $T<\infty$ and there is not: an $\varepsilon>0$ and a smooth solution $\{\tilde{F}_t\}_{t\in[0,T+\varepsilon)}$ of the MIWF with $\tilde{F}_t = F_t$ on Σ for $t\in[0,T)$. In both cases the element $T\in\mathbf{R}_+\cup\{\infty\}$ is uniquely determined by the initial immersion F_0, and we call it the "maximal time of existence" of the MIWF starting in F_0, in symbols "$T_{\max}(F_0)$", respectively we call $\{F_t\}_{t\in[0,T_{\max}(F_0))}$ the "maximal solution" of evolution Eq. (1) starting in F_0.

(e) If $T_{\max}(F_0)$ is finite, then we also call $T_{\max}(F_0)$ "the singular time" of the flow line $\{F_t\}$ of the MIWF starting in F_0. In this case we also say, that the respective flow line of the MIWF "forms a singularity" as $t\nearrow T_{\max}(F_0)$.

$\square$

We should note here, that Definition 2, (b)–(e), makes sense on account of "short-time existence and uniqueness" of flow lines of the MIWF, guaranteed by Theorem 1 in [13]. Now, for any fixed immersion $G : \Sigma \longrightarrow \mathbf{R}^n$ of class C^2 and a smooth chart ψ of an arbitrary coordinate patch Σ' of a fixed smooth compact torus Σ, we will denote throughout this article the resulting partial derivatives on Σ' by ∂_i, $i=1,2$, the coefficients $g_{ij} := \langle\partial_i G, \partial_j G\rangle_{\mathbf{R}^n}$ of the first fundamental form of G w.r.t. ψ and the associated Christoffel-symbols $(\Gamma_G)^m_{kl} := g_G^{mj}\langle\partial_{kl}G, \partial_j G\rangle_{\mathbf{R}^n}$ of $(\Sigma', G^*(g_{\text{eu}}))$. Moreover, we define the first (covariant) derivatives by $\nabla^G_i(v) := \nabla^G_{\partial_i}(v) := \partial_i(v)$, $i=1,2$, and the second covariant derivatives by

$$\nabla^G_{kl}(v) \equiv \nabla^G_k\nabla^G_l(v) := \partial_{kl}v - (\Gamma_G)^m_{kl}\,\partial_m v \tag{7}$$

of any function $v\in C^2(\Sigma,\mathbf{R})$. Moreover, for any vector field $V\in C^2(\Sigma,\mathbf{R}^n)$ we define the projections of its first derivatives onto the normal bundle of the immersed torus $G(\Sigma)$ by

$$\nabla^{\perp_G}_i(V) \equiv (\nabla^G_i(V))^{\perp_G} := \nabla^G_i(V) - P^{\text{Tan}(G)}(\nabla^G_i(V))$$

and the "normal second covariant derivatives" of V w.r.t. the immersion G by

$$\nabla^{\perp_G}_k\nabla^{\perp_G}_l(V) := \nabla^{\perp_G}_k(\nabla^{\perp_G}_l(V)) - (\Gamma_G)^m_{kl}\,\nabla^{\perp_G}_m(V).$$

Using these terms, we define the Beltrami-Laplacian w.r.t. G by $\triangle_G(V) := g_G^{kl}\nabla^G_{kl}(V)$, its projection $(\triangle_G V)^{\perp_G} := \left(g_G^{kl}\nabla^G_k\nabla^G_l(V)\right)^{\perp_G}$ onto the normal bundle of the surface $G(\Sigma)$ and the "normal Beltrami-Laplacian" by $\triangle^{\perp_G}_G(V) := g_G^{kl}\nabla^{\perp_G}_k\nabla^{\perp_G}_l(V)$. We shall note here that Eqs. (3) and (7) together imply:

$$(A_G)_{ij} = A_G(\partial_i, \partial_j) = \partial_{ij}G - (\Gamma_G)^m_{ij}\,\partial_m G = \nabla^G_i\nabla^G_j(G), \tag{8}$$

which shows that the second fundamental form A_G is a covariant tensor field of degree 2 and that there holds:

$$\mathbf{H}_G = g_G^{ij}(A_G)_{ij} = g_G^{ij}\nabla^G_i\nabla^G_j(G) = \triangle_G(G) \tag{9}$$

for the mean curvature of the immersion G. Now we recall from [13], that the main problem about Eq. (5) is its non-parabolicity. We have

$$\triangle_f^\perp \mathbf{H}_f + Q(A_f^0)(\mathbf{H}_f) = (\triangle_f \mathbf{H}_f)^{\perp_f} + 2\, Q(A_f)(\mathbf{H}_f) - \frac{1}{2} \mid \mathbf{H}_f \mid^2 \mathbf{H}_f \tag{10}$$

and by (9):

$$(\triangle_f \mathbf{H}_f)^{\perp_f} = g_f^{ij}\, g_f^{kl}\, \nabla_i^f \nabla_j^f \nabla_k^f \nabla_l^f (f)$$
$$-g_f^{ij}\, g_f^{kl}\, \langle \nabla_i^f \nabla_j^f \nabla_k^f \nabla_l^f (f), \partial_m f \rangle\, g_f^{mr}\, \partial_r (f) \tag{11}$$

for any $W^{4,2}$-immersion $f : \Sigma \longrightarrow \mathbf{R}^n$, which shows that the leading term of the right-hand side of (5), i.e. of $[\{f_t\} \mapsto \mid A_{f_t}^0 \mid^{-4} \nabla_{L^2}\mathcal{W}(f_t)]$, is not uniformly elliptic (of fourth order), even if $\mid A_{f_t}^0 \mid^2$ should stay positive on the torus Σ for all times $t \in [0, T]$. In order to overcome this unpleasant obstruction we have applied in [13] the "DeTurck trick" in the following way: We fix some C^∞-smooth immersion F_0 of Σ into $\mathbf{R}^n$ arbitrarily and compute:

$$\nabla_k^f \nabla_l^f (f) = \left(\partial_{kl} f - (\Gamma_{F_0})_{kl}^m \partial_m(f)\right) + C_{kl}^m(f, F_0)\, \partial_m(f)$$
$$= \nabla_k^{F_0} \nabla_l^{F_0} (f) + C_{kl}^m(f, F_0)\, \partial_m(f)$$

for

$$C_{kl}^m(f, F_0) := \left((\Gamma_{F_0})_{kl}^m - (\Gamma_f)_{kl}^m\right) \quad \text{on} \quad \Sigma'.$$

It is important to note here, that the difference $(\Gamma_{F_0})_{kl}^m - (\Gamma_f)_{kl}^m$ is a tensor field of 3rd degree and that therefore the difference

$$g_f^{ij}\, g_f^{kl} \left(\nabla_i^f \nabla_j^f \nabla_k^f \nabla_l^f (f) - \nabla_i^f \nabla_j^f \left(C_{kl}^m(f, F_0)\, \partial_m(f)\right)\right)$$
$$= g_f^{ij}\, g_f^{kl}\, \nabla_i^f \nabla_j^f \nabla_k^{F_0} \nabla_l^{F_0}(f) \tag{12}$$

is a "scalar", i.e. does not depend on the choice of the local chart ψ, and thus yields a globally well-defined differential operator of fourth order, for $W^{4,2}$-immersions $f : \Sigma \longrightarrow \mathbf{R}^n$ again. Moreover, one can easily verify by (7) and by the derivation formulae in (31) below, that $f \mapsto g_f^{ij}\, g_f^{kl}\, \nabla_i^f \nabla_j^f \nabla_k^{F_0} \nabla_l^{F_0}(f)$ is a non-linear operator of fourth order whose leading term is $g_f^{ij}\, g_f^{kl} \nabla_i^{F_0} \nabla_j^{F_0} \nabla_k^{F_0} \nabla_l^{F_0}(f)$, which is locally elliptic in the sense of Section 1 in [23]. Since the highest order term of the above expression $\nabla_i^f \nabla_j^f (C_{kl}^m(f, F_0)\, \partial_m(f))$ is given by $\nabla_i^f \nabla_j^f \left((\Gamma_{F_0})_{kl}^m - (\Gamma_f)_{kl}^m\right) \partial_m(f)$ and since the latter term is a section of the tangent bundle of the immersion f, we are thus naturally led to firstly consider flow lines of the evolution equation

$$\partial_t(f_t) = -\frac{1}{2} \mid A_{f_t}^0 \mid^{-4} \left(2\, \delta\mathcal{W}(f_t) + \mathcal{T}_{F_0}(f_t)\right) =: \mathcal{M}_{F_0}(f_t), \tag{13}$$

for some arbitrarily fixed C^∞−smooth immersion $F_0 : \Sigma \longrightarrow \mathbf{R}^n$, where the symbol $[f \mapsto \mathcal{T}_{F_0}(f)]$ denotes a globally well-defined differential operator of fourth order, which via locally defined coordinates is concretely given by the formula

$$\mathcal{T}_{F_0}(f) := g_f^{ij}\, g_f^{kl}\, g_f^{mr}\, \langle \nabla_i^f \nabla_j^f \nabla_k^f \nabla_l^f (f), \partial_m f \rangle\, \partial_r f$$
$$-g_f^{ij}\, g_f^{kl}\, \nabla_i^f \nabla_j^f \left((\Gamma_{F_0})_{kl}^m - (\Gamma_f)_{kl}^m\right) \partial_m(f), \tag{14}$$

for any $W^{4,2}$-immersion $f : \Sigma \longrightarrow \mathbf{R}^n$. On account of formula (12) the right-hand side "$\mathcal{M}_{F_0}(f_t)$" of Eq. (13) can thus be expressed by:

$$\mathcal{M}_{F_0}(f_t)(x) = -\frac{1}{2} \mid A_{f_t}^0 \mid^{-4} g_{f_t}^{ij} g_{f_t}^{kl} \nabla_i^{f_t} \nabla_j^{f_t} \nabla_k^{F_0} \nabla_l^{F_0}(f_t)(x)$$
$$+ B(x, D_x f_t, D_x^2 f_t, D_x^3 f_t), \tag{15}$$

for $(x, t) \in \Sigma' \times [0, T]$, and this differential operator is locally elliptic in the sense of Section 1 in [23], as well. Here, the symbols $D_x f_t$, $D_x^2 f_t$, $D_x^3 f_t$ abbreviate the matrix-valued functions $(\partial_1 f_t, \partial_2 f_t)$, $(\nabla_{ij}^{F_0} f_t)_{i,j \in \{1,2\}}$ and $(\nabla_{ijk}^{F_0} f_t)_{i,j,k \in \{1,2\}}$, and the n components of the lower order term $B(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t)$ are "scalars", i.e. well-defined functions on $\Sigma' \times [0, T]$ whose values do not depend on the choice of the chart ψ of Σ'. Hence, there has to exist some well-defined function $B : \Sigma \times \mathbf{R}^{2n} \times \mathbf{R}^{4n} \times \mathbf{R}^{8n} \to \mathbf{R}^n$ whose n components are rational functions in their $14n$ real variables, such that Eq. (15) holds "globally" for any pair $(x, t) \in \Sigma \times [0, T]$, and we arrive at the non-linear equation

$$\partial_t(f_t) = -\frac{1}{2} \mid A_{f_t}^0 \mid^{-4} g_{f_t}^{ij} g_{f_t}^{kl} \nabla_i^{f_t} \nabla_j^{f_t} \nabla_k^{F_0} \nabla_l^{F_0}(f_t) + B(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t) \tag{16}$$

of fourth order with quasilinear leading term, called the "DeTurck modification of the MIWF", whose linearization in any fixed family of $W^{4,p}$-immersions $f_t : \Sigma \longrightarrow \mathbf{R}^n$ belonging to the parabolic L^p-space

$$X_{T,p} := W^{1,p}([0, T]; L^p(\Sigma, \mathbf{R}^n)) \cap L^p([0, T]; W^{4,p}(\Sigma, \mathbf{R}^n)), \tag{17}$$

being well-defined for any $p \in (1, \infty)$, will turn out to yield a uniformly parabolic operator from $X_{T,p}$ to $L^p([0, T]; L^p(\Sigma, \mathbf{R}^n))$—see Theorem 1 below—provided we choose here $p > 3$ and provided the immersions f_t satisfy $\mid A_{f_t}^0 \mid^2 > 0$ on Σ, for every $t \in [0, T]$. The mathematical significance of the choice "$p > 3$" lies in embeddings (29), (37) and (66) below, which might fail to be continuous for $p \in (1, 3]$; see here also the footnote addressing embedding (66). It should be emphasized here, that embeddings (29), (30), (37) and (66) play key roles in the proofs of Theorems 1–5 below.

Now we fix some additional smooth immersion $U_0 \in \mathrm{Imm}_{uf}(\Sigma, \mathbf{R}^n) \cap C^\infty(\Sigma, \mathbf{R}^n)$, consider the maximal flow line $\{\mathcal{P}(t, 0, U_0)\}_{t \in [0, T_{\max}(U_0))}$ of the MIWF-Eq. (1) and choose some $0 < T < T_{\max}(U_0)$, where $T_{\max}(U_0)$ has been defined in Definition 2. We recall from the proof of Theorem 1 in [13], that there is a unique smooth family of smooth diffeomorphisms $\phi_t \equiv \phi_t^{U_0, F_0} : \Sigma \longrightarrow \Sigma$ with $\phi_0 = \mathrm{id}_\Sigma$, $t \in [0, T]$, such that the reparametrization $\{\mathcal{P}(t, 0, U_0) \circ \phi_t^{U_0, F_0}\}_{t \in [0, T]}$ satisfies Eq. (13) respectively (16). Now we fix some $p \in (3, \infty)$ and some open neighborhood $W_{U_0, T, p}$ of the above flow line $\{\mathcal{P}(t, 0, U_0) \circ \phi_t^{U_0, F_0}\}_{t \in [0, T]}$ in the space $X_{T,p}$ from line (17), and we shall follow Spener's strategy in [30], to use the fact that there holds for the Banach space of traces of elements in $X_{T,p}$ at the initial time $t = 0$:

$$\gamma_0(X_{T,p}) \equiv \mathrm{trace}(X_{T,p}) = (L^p(\Sigma, \mathbf{R}^n), W^{4,p}(\Sigma, \mathbf{R}^n))_{1 - \frac{1}{p}, p}$$
$$= B_{p,p}^{4 - \frac{4}{p}}(\Sigma, \mathbf{R}^n) = W^{4 - \frac{4}{p}, p}(\Sigma, \mathbf{R}^n), \tag{18}$$

by Corollary 1.14 in [22] respectively Theorem 4.10.2 in Chapter III of [2], combined with Theorem 4.3.3 in [31] and Theorem 1.2.4 in [32] respectively formula (3.5) in [3], which motivates us to consider the non-linear operator

$$\Psi^{F_0, T} : W_{U_0, T, p} \subset X_{T,p} \longrightarrow W^{4 - \frac{4}{p}, p}(\Sigma, \mathbf{R}^n) \times L^p([0, T]; L^p(\Sigma, \mathbf{R}^n)) =: Y_{T,p}$$

defined by:

$$\Psi^{F_0,T}(\{f_t\}_{t\in[0,T]}) := (\gamma_0(\{f_t\}), \{\partial_t(f_t) - \mathcal{M}_{F_0}(f_t)\}_{t\in[0,T]}), \qquad (19)$$

compare here to Section 1.5 in Chapter III of [2]. We shall at first investigate the properties of exactly this operator in the following theorem, Theorem 1, and then in Theorem 2 we shall precisely address unique short-time existence of solutions to evolution Eq. (16) in the space $X_{T,p}$. The most important techniques in these two theorems are the explicit use of the quasi-linearity of the operator $[f \mapsto \mathcal{M}_{F_0}(f)]$ in lines (13) and (15) in combination with Theorem 2.1 in [18], and the "linearization" of the quasi-linear operator (19) in appropriate points of its open domain $W_{U_0,T,p} \subset X_{T,p}$, yielding linear parabolic differential operators and thus the applicability of classical "Linear Functional Analysis"—in accordance with the classical approach to the Ricci-flow; see e.g. Chapter 5 in [7].

Theorem 1 *Let Σ be a smooth compact torus, $F_0 \in C^\infty_{\mathrm{Imm}}(\Sigma, \mathbf{R}^n)$ a smooth immersion, $U_0 \in \mathrm{Imm}_{uf}(\Sigma, \mathbf{R}^n) \cap C^\infty(\Sigma, \mathbf{R}^n)$ a smooth and umbilic-free initial immersion of Σ into $\mathbf{R}^n$, and let $T \in (0, T_{\max}(U_0))$ and $p \in (3, \infty)$ be arbitrarily chosen, where $T_{\max}(U_0)$ has been defined in Definition 2. Then there is a sufficiently small open neighbourhood $W_{U_0,T,p}$ in the Banach space $X_{T,p}$ about the smooth solution $\{\mathcal{P}(t, 0, U_0) \circ \phi_t^{U_0,F_0}\}_{t\in[0,T]}$ of the modified MIWF-Eq. (16), such that the following statements hold:*

(1) *For any family of immersions $\{f_t\} \in W_{U_0,T,p}$ there holds the identity*

$$\mathcal{M}_{F_0}(\{f_t\}) = K^{F_0}(\{f_t\}).(\{f_t\}) \quad \text{in } L^p([0,T]; L^p(\Sigma, \mathbf{R}^n)), \qquad (20)$$

   where K^{F_0} is the non-linear operator from line (25), here applied to $f_t \in \mathrm{Imm}_{uf}(\Sigma, \mathbf{R}^n)$ for each $t \in [0, T]$.

(2) *The map $\Psi^{F_0,T} : W_{U_0,T,p} \longrightarrow Y_{T,p}$, defined in line (19), is of class C^ω on the open subset $W_{U_0,T,p}$ of the Banach space $X_{T,p}$ from line (17).*

(3) *In any fixed element $\{f_t\} \in W_{U_0,T,p}$ the Fréchet derivative of the second component of $\Psi^{F_0,T}$ is a linear, uniformly parabolic operator of order 4, whose leading operator acts on each component of $f = \{f_t\}$ separately:*

$$(D(\Psi^{F_0,T})_2(f)).(\eta) \equiv \partial_t(\eta) - D(\mathcal{M}_{F_0})(f).(\eta)$$

$$= \partial_t(\eta) + \frac{1}{2} \mid A^0_{f_t} \mid^{-4} g^{ij}_{f_t} g^{kl}_{f_t} \nabla^{F_0}_{ijkl}(\eta) +$$

$$+ B^{ijk}_3 \cdot \nabla^{F_0}_{ijk}(\eta) + B^{ij}_2 \cdot \nabla^{F_0}_{ij}(\eta) + B^i_1 \cdot \nabla^{F_0}_i(\eta) \qquad (21)$$

   on $\Sigma \times [0, T]$, for any element $\eta = \{\eta_t\}$ of the tangent space $T_f W_{U_0,T,p} = X_{T,p}$. Here, the coefficients $\mid A^0_{f_t} \mid^{-4} g^{ij}_{f_t} g^{kl}_{f_t}$ of the leading order term are of class $C^{0,\alpha}([0,T]; C^{0,\alpha}(\Sigma, \mathbf{R}))$, B^{ij}_2 and B^i_1 are the coefficients of $\mathrm{Mat}_{n,n}(\mathbf{R})$-valued, contravariant tensor fields of degrees 2 and 1, which depend on x, $D_x f_t$, $D^2_x f_t$, $D^3_x f_t$ and on $D^4_x f_t$ and are of class
   $L^p([0,T]; L^p(\Sigma, \mathrm{Mat}_{n,n}(\mathbf{R})))$. They depend on the spatial derivatives of third order $\{\nabla^{F_0}_{ijk} f_t\}$ and of fourth order $\{\nabla^{F_0}_{ijkl} f_t\}$ of $\{f_t\}$ affine linearly. Finally B^{ijk}_3 are the coefficients of a $\mathrm{Mat}_{n,n}(\mathbf{R})$-valued, contravariant tensor field of degree 3, which depends on x, $D_x f_t$ and $D^2_x f_t$ only and is of class $C^{0,\alpha}([0,T]; C^{0,\alpha}(\Sigma, \mathrm{Mat}_{n,n}(\mathbf{R})))$, for $\alpha(p) \in (0, 1)$ as in (66) below.

(4) *The Fréchet derivative of $\Psi^{F_0,T}$ yields a topological isomorphism*

$$D\Psi^{F_0,T}(f) : T_f W_{U_0,T,p} = X_{T,p} \xrightarrow{\cong} Y_{T,p}$$

in any fixed family of immersions $f \equiv \{f_t\} \in C^4(\Sigma \times [0, T], \mathbf{R}^n) \cap W_{U_0, T, p}$.

$\square$

Theorem 2 *Let* Σ *be a smooth compact torus,* $p \in (3, \infty)$, *and let* $F_0 \in C^\infty_{\mathrm{Imm}}(\Sigma, \mathbf{R}^n)$ *and* $U_0 \in \mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbf{R}^n) \cap W^{4 - \frac{4}{p}, p}(\Sigma, \mathbf{R}^n)$ *be fixed immersions.*

(1) *There are sufficiently small numbers* $T = T(U_0) > 0$ *and* $\varepsilon = \varepsilon(U_0) > 0$, *such that for every initial immersion* $u_0 \in W^{4 - \frac{4}{p}, p}(\Sigma, \mathbf{R}^n)$ *satisfying*

$$\| U_0 - u_0 \|_{W^{4 - \frac{4}{p}, p}(\Sigma, \mathbf{R}^n)} < \varepsilon \tag{22}$$

there is a unique solution $\{u(t, u_0)\}_{t \in [0, T]}$ *of the non-linear Cauchy problem*

$$\partial_t f_t = \mathcal{M}_{F_0}(f_t) \quad \text{on} \quad \Sigma \times [0, T] \quad \text{with} \quad f_0 = u_0 \quad \text{on} \quad \Sigma, \tag{23}$$

from lines (13) and (16), i.e. of the problem $\Psi^{F_0, T}(\{f_t\}) = (u_0, 0)$, *within the Banach space* $X_{T, p}$.

(2) *Let* $T = T(U_0) > 0$ *and* $\varepsilon = \varepsilon(U_0) > 0$ *be as in the first part of the theorem. Then for every initial immersion* $u_0 \in W^{4 - \frac{4}{p}, p}(\Sigma, \mathbf{R}^n)$ *satisfying condition (22) the unique short-time solution* $\{u(t, u_0)\}_{t \in [0, T]}$ *of Cauchy problem (23) possesses a unique extension* $\{\mathcal{P}^*(t, 0, u_0)\}_{t \in [0, t^+(u_0))}$ *to a maximal, half-sided open interval of existence* $[0, t^+(u_0))$, *which means that* $t^+(u_0) > T$, *that for every* $T^* \in [T, t^+(u_0))$ *the restriction* $\{\mathcal{P}^*(t, 0, u_0)\}_{t \in [0, T^*]}$ *of* $\{\mathcal{P}^*(\cdot, 0, u_0)\}$ *to* $\Sigma \times [0, T^*]$ *is the unique solution of Cauchy problem (23) in* $X_{T^*, p}$, *and that* $t^+(u_0)$ *is maximal in* $\mathbf{R}_+ \cup \infty$ *with this property.*

$\square$

Moreover, for only smooth and umbilic-free initial immersions U_0 we will prove, that the maximal smooth solution of Cauchy problem (23) becomes even real analytic for positive times, i.e. immediately after its initial time. This is the decisive regularity result of this paper. In its proof we will combine slight modifications of several results of Theorem 1 respectively techniques of its proof with the modern method in [12], similarly to the procedure in Sections 3–6 of [28] or Sections 3–4 in [27]. Earlier respectively simpler methods, as e.g. in [8] or in [19], apply in different mathematical situations and cannot yield the statement of Theorem 3 (ii) below.

Theorem 3 *Suppose that* Σ *is a smooth compact torus, that* F_0 *is a fixed* C^∞-*smooth immersion of* Σ *into* $\mathbf{R}^n$, *that* $U_0 \in \mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbf{R}^n) \cap C^\infty(\Sigma, \mathbf{R}^n)$ *is an arbitrary smooth and umbilic-free initial immersion, and that* $p \in (3, \infty)$.

(1) *The maximal solution* $\{\mathcal{P}^*(t, 0, U_0)\}_{t \in [0, t^+(U_0))}$ *of the quasilinear parabolic Cauchy problem (23) - here with* $u_0 = U_0$ - *from the second part of Theorem 2 is of class* $C^\infty(\Sigma \times [0, t^+(U_0)), \mathbf{R}^n)$, *and it is therefore a maximal flow line of evolution Eq. (13) in the sense of Definition 2, (d), with* $T_{\max}(U_0) = t^+(U_0)$.

(2) *Suppose that the smooth torus* Σ *is additionally endowed with a complex structure, and that only the immersion* F_0 *is additionally of class* $C^\omega(\Sigma, \mathbf{R}^n)$. *Then the maximal flow line* $\{\mathcal{P}^*(t, 0, U_0)\}_{t \in [0, T_{\max}(U_0))}$ *of evolution Eq. (13) from the first part of this theorem is additionally of class* $C^\omega(\Sigma \times (0, T_{\max}(U_0)), \mathbf{R}^n)$.

$\square$

We firstly prepare the proofs of Theorems 1–3 by means of the following elementary lemma.

Lemma 1 (1) *For every $p \in (1, \infty)$, the differential operator $[f \mapsto \mathcal{M}_{F_0}(f)]$ from line (13) is a well-defined map from $\mathrm{Imm}_{uf}(\Sigma, \mathbf{R}^n) \cap W^{4,p}(\Sigma, \mathbf{R}^n)$ to $L^p(\Sigma, \mathbf{R}^n)$, and it has a quasilinear structure, which means here precisely that there holds:*

$$\mathcal{M}_{F_0}(f) = K^{F_0}(f).(f), \quad \text{for every } f \in \mathrm{Imm}_{uf}(\Sigma, \mathbf{R}^n) \cap W^{4,p}(\Sigma, \mathbf{R}^n), \quad (24)$$

where K^{F_0} is a non-linear operator

$$K^{F_0} : \mathrm{Imm}_{uf}(\Sigma, \mathbf{R}^n) \longrightarrow \mathcal{L}(W^{4,p}(\Sigma, \mathbf{R}^n), L^p(\Sigma, \mathbf{R}^n)), \quad (25)$$

whose restriction to $\mathrm{Imm}_{uf}(\Sigma, \mathbf{R}^n) \cap W^{4-\frac{4}{p},p}(\Sigma, \mathbf{R}^n)$, endowed with the $W^{4-\frac{4}{p},p}$-norm, is locally Lipschitz continuous, if here additionally $p > 3$.

(2) *For every fixed $h \in \mathrm{Imm}_{uf}(\Sigma, \mathbf{R}^n)$ and every $p \in (1, \infty)$, the value $-K^{F_0}(h)$: $W^{4,p}(\Sigma, \mathbf{R}^n) \longrightarrow L^p(\Sigma, \mathbf{R}^n)$ of operator (25) is of maximal L^p-regularity on $[0, T]$ w.r.t. the pair $(W^{4,p}(\Sigma, \mathbf{R}^n), L^p(\Sigma, \mathbf{R}^n))$, i.e. $-K^{F_0}(h)$ is of class $\mathcal{MR}_p([0, T]; W^{4,p}(\Sigma, \mathbf{R}^n), L^p(\Sigma, \mathbf{R}^n))$ in the notation of [4] or [18].*

Proof (1) First of all we would like to refine the rough, non-linear formulation (15) of the differential operator $\mathcal{M}_{F_0}$ appearing in line (13). To this end, we recall from lines (10)–(14) that

$$
\begin{aligned}
\mathcal{M}_{F_0}(f) = & -\frac{1}{2}\frac{1}{|A_f^0|^4}\Big(g_f^{ij}\, g_f^{kl}\, \nabla_i^f \nabla_j^f \nabla_k^f \nabla_l^f (f) \\
& -g_f^{ij} g_f^{kl}\, \nabla_i^f \nabla_j^f \left((\Gamma_{F_0})_{kl}^m - (\Gamma_f)_{kl}^m \right) \partial_m(f) \\
& +2\, Q(A_f)(\mathbf{H}_f) - \frac{1}{2}\, |\, \mathbf{H}_f\, |^2\, \mathbf{H}_f \Big) \\
= & -\frac{1}{2}\frac{1}{|A_f^0|^4}\Big(g_f^{ij}\, g_f^{kl}\, \nabla_i^f \nabla_j^f \nabla_k^{F_0} \nabla_l^{F_0}(f) \\
& +g_f^{ij} g_f^{kl}\, \left((\Gamma_{F_0})_{kl}^m - (\Gamma_f)_{kl}^m \right) \nabla_i^f \nabla_j^f (\partial_m(f)) \\
& + g_f^{ij} g_f^{kl}\, \nabla_i^f \left((\Gamma_{F_0})_{kl}^m - (\Gamma_f)_{kl}^m \right) \nabla_j^f (\partial_m(f)) \\
& +2\, Q(A_f)(\mathbf{H}_f) - \frac{1}{2}\, |\, \mathbf{H}_f\, |^2\, \mathbf{H}_f \Big), \quad (26)
\end{aligned}
$$

for any fixed umbilic-free immersion $f \in W^{4,p}(\Sigma, \mathbf{R}^n)$ with $p \in (1, \infty)$. Using now formulae (7)–(9) and the first two lines of the general derivation formulae (31) below for covariant tensor fields, one can deduce—similarly to the technical reasoning in the proof of the second part of Theorem 1—the following more systematic formulation of the right hand side in (26):

$$
\begin{aligned}
\mathcal{M}_{F_0}(f)(x) = & -\frac{1}{2}\frac{1}{|A_f^0|^4(x)}\, g_f^{ij}\, g_f^{kl}\, \nabla_i^f \nabla_j^f \nabla_k^{F_0} \nabla_l^{F_0}(f)(x) \\
& +\mathcal{N}^{F_0}(x, D_x f(x), D_x^2 f(x)) \cdot D_x^3 f(x) \\
& +\mathcal{C}^{F_0}(x, D_x f(x), D_x^2 f(x)) \cdot D_x^2 f(x) \\
& +\mathcal{D}^{F_0}(x, D_x f(x), D_x^2 f(x)) \cdot D_x f(x) \quad (27)
\end{aligned}
$$

for $x \in \Sigma$ and for every fixed umbilic-free immersion $f \in W^{4,p}(\Sigma, \mathbf{R}^n)$, where the functions $\mathcal{N}^{F_0} : \Sigma \times \mathbf{R}^{2n} \times \mathbf{R}^{4n} \longrightarrow \mathrm{Mat}_{n,8n}(\mathbf{R})$, $\mathcal{C}^{F_0} : \Sigma \times \mathbf{R}^{2n} \times \mathbf{R}^{4n} \longrightarrow \mathrm{Mat}_{n,4n}(\mathbf{R})$ and $\mathcal{D}^{F_0} : \Sigma \times \mathbf{R}^{2n} \times \mathbf{R}^{4n} \longrightarrow \mathrm{Mat}_{n,2n}(\mathbf{R})$ have the same algebraic structures as the function M^{F_0} in formula (39) below. In particular, the right-hand side of formula (27)

depends only affine linearly on the covariant derivatives of f of fourth and third order. Hence, formula (27) shows especially, that the non-linear differential operator $\mathcal{M}_{F_0}$ maps the intersection $\mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbf{R}^n) \cap W^{4,p}(\Sigma, \mathbf{R}^n)$ into $L^p(\Sigma, \mathbf{R}^n)$.

Now we are able to guess the desired non-linear map K^{F_0} appearing in assertions (24) and (25) to be:

$$
\begin{aligned}
(K^{F_0}(h).\eta)(x) := &-\frac{1}{2}\frac{1}{|A_h^0|^4} g_h^{ij} g_h^{kl} \nabla_i^h \nabla_j^h \nabla_k^{F_0} \nabla_l^{F_0}(\eta)(x) \\
&+ \mathcal{N}^{F_0}(x, D_x h(x), D_x^2 h(x)) \cdot D_x^3 \eta(x) \\
&+ \mathcal{C}^{F_0}(x, D_x h(x), D_x^2 h(x)) \cdot D_x^2 \eta(x) \\
&+ \mathcal{D}^{F_0}(x, D_x h(x), D_x^2 h(x)) \cdot D_x \eta(x),
\end{aligned}
\tag{28}
$$

for $x \in \Sigma$, and for any fixed umbilic-free immersion h and $\eta \in W^{4,p}(\Sigma, \mathbf{R}^n)$, and formula (27) shows us indeed, that the choice of the non-linear map K^{F_0} in (28) yields assertion (24) for every umbilic-free immersion $f \in W^{4,p}(\Sigma, \mathbf{R}^n)$. Now, by formula (28) the linear operator $K^{F_0}(h)$ has continuous coefficients and thus belongs to $\mathcal{L}(W^{4,p}(\Sigma, \mathbf{R}^n), L^p(\Sigma, \mathbf{R}^n))$, for every $h \in \mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbf{R}^n)$ and for every $p > 1$. If we assume additionally that $p > 3$, then we can use the fractional Sobolev embedding

$$
W^{4-\frac{4}{p},p}(\Sigma, \mathbf{R}^n) \hookrightarrow C^{2,\alpha}(\Sigma, \mathbf{R}^n),
\tag{29}
$$

for $0 < \alpha < 1$ with $\alpha \leq 2 - \frac{6}{p}$, and the classical mean value theorem, in order to infer again from formula (28), that the map K^{F_0} in (25) is locally Lipschitz continuous on $\mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbf{R}^n) \cap W^{4-\frac{4}{p},p}(\Sigma, \mathbf{R}^n)$.

(2) From the first part of the lemma we know, that for any fixed $h \in \mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbf{R}^n)$ we obtain a bounded, linear differential operator $-K^{F_0}(h) : W^{4,p}(\Sigma, \mathbf{R}^n) \longrightarrow L^p(\Sigma, \mathbf{R}^n)$ with continuous coefficients. Moreover, again formula (28) shows, that the operator $-K^{F_0}(h)$ is uniformly elliptic in the sense of Proposition 2 below. Hence, Proposition 2 guarantees us, that $\partial_t - K^{F_0}(h)$ maps the Banach space X_T^0—defined below in (71)—isomorphically onto $L^p([0, T], L^p(\Sigma, \mathbf{R}^n))$, for any fixed $p \in (1, \infty)$, thus that $-K^{F_0}(h)$ is of class $\mathcal{MR}_p([0, T]; W^{4,p}(\Sigma, \mathbf{R}^n), L^p(\Sigma, \mathbf{R}^n))$ for every $p \in (1, \infty)$, using the notation of [4] or [18].

$\square$

Proof of Theorem 1 (1) On account of the embedding

$$
X_{T,p} \hookrightarrow C^0([0, T], W^{4-\frac{4}{p},p}(\Sigma, \mathbf{R}^n))
\tag{30}
$$

with T-independent embedding constant—see here Proposition 1.4.2 and Theorem 4.10.2 of Chapter III in [2], Corollary 1.14 in [22] and formula (18) above—and on account of embedding (29), we can derive identity (20) immediately from identity (24), for any family of immersions $\{f_t\} \in W_{U_0,T,p}$, provided the neighbourhood $W_{U_0,T,p}$ about the flow line $\{\mathcal{P}(t, 0, U_0) \circ \phi_t^{U_0,F_0}\}$ in $X_{T,p}$ has been chosen sufficiently small.

(2) The first component of the operator $\Psi^{F_0,T}$ from line (19) is linear and continuous on account of Theorem 4.10.2 in Chapter III of [2], thus it is of class C^ω. In order to prove that also the second component of the operator $\Psi^{F_0,T}$ is of class C^ω, we shall firstly prove its C^1-regularity, following the strategy of the proof of Theorem 2 in [13], because this preparation considerably simplifies the proof of its real analyticity and also the explicit computation of the Fréchet derivative of the operator $\Psi^{F_0,T}$ in any fixed element $\{f_t\}$ of a sufficiently small neighbourhood $W_{U_0,T,p}$ about the flow line $\{\mathcal{P}(t, 0, U_0) \circ \phi_t^{U_0,F_0}\}$ in

$X_{T,p}$.

α) At first we aim at a slight improvement of formula (27) above. To this end, we fix an arbitrary smooth chart $\psi : O \xrightarrow{\cong} \Sigma'$ of an arbitrary coordinate patch Σ' of Σ, which yields partial derivatives $\partial_m, m = 1, 2$, on Σ'. For any C^2-immersion $G : \Sigma \longrightarrow \mathbf{R}^n$ the choice of ψ yields the coefficients $g_{ij} := \langle \partial_i G, \partial_j G \rangle_{\mathbf{R}^n}$ of the first fundamental form of G w.r.t. ψ and the associated Christoffel-symbols $(\Gamma_G)_{kl}^m := g^{mj} \langle \partial_{kl} G, \partial_j G \rangle_{\mathbf{R}^n}$ of $(\Sigma', G^*(g_{\text{eu}}))$. On account of (7) and by the general derivation formulae

$$\nabla_i^G(\omega_k) = \partial_i(\omega_k) - (\Gamma_G)_{ik}^m \omega_m$$
$$\nabla_i^G(\lambda_{jk}) = \partial_i(\lambda_{jk}) - (\Gamma_G)_{ij}^m \lambda_{mk} - (\Gamma_G)_{ik}^m \lambda_{jm}$$
$$\nabla_i^G(\zeta_{jkl}) = \partial_i(\zeta_{jkl}) - (\Gamma_G)_{ij}^m \zeta_{mkl} - (\Gamma_G)_{ik}^m \zeta_{jml} - (\Gamma_G)_{il}^m \zeta_{jkm} \tag{31}$$

on Σ' for the coefficients ω_k, λ_{jk} and ζ_{jkl} (w.r.t. to the chart ψ) of covariant C^1-tensor fields ω, λ and ζ of degrees 1, 2 and 3, one can verify that for any $W^{4,p}$-immersion $f : \Sigma \longrightarrow \mathbf{R}^n$, for any fixed C^∞-immersion $G : \Sigma \longrightarrow \mathbf{R}^n$ and for fixed $i, j, k, l \in \{1, 2\}$ there is a unique rational function $P_{(ijkl)}^G \in C^\infty(\Sigma')[v_1, \ldots, v_{2n}, w_1, \ldots, w_{4n}, y_1, \ldots, y_{8n}]$ (with n real components) in $14n$ real variables, whose coefficients are rational expressions involving the partial derivatives $\partial_i(G^1), \ldots, \partial_i(G^n), \partial_{ij}(G^1), \ldots, \partial_{ij}(G^n), \partial_{ijk}(G^1), \ldots, \partial_{ijk}(G^n)$ of the components of G up to third order, such that

$$\nabla_i^f \nabla_j^f \nabla_k^G \nabla_l^G(f) = \nabla_i^G \nabla_j^G \nabla_k^G \nabla_l^G(f)$$
$$+ P_{(ijkl)}^G(\nabla_1^G(f^1), \ldots, \nabla_2^G(f^n), \nabla_{11}^G(f^1), \ldots, \nabla_{22}^G(f^n), \nabla_{111}^G(f^1), \ldots, \nabla_{222}^G(f^n)) \tag{32}$$

holds on Σ'. We note here that the terms

$$P_{(ijkl)}^G(\nabla_1^G(f^1), \ldots, \nabla_2^G(f^n), \nabla_{11}^G(f^1), \ldots, \nabla_{22}^G(f^n), \nabla_{111}^G(f^1), \ldots, \nabla_{222}^G(f^n))$$

must be the coefficients of a covariant tensor field of fourth degree on Σ', and moreover we note, that this expression is linear w.r.t. the derivatives $\nabla_{rsm}^G(f^1), \ldots, \nabla_{rsm}^G(f^n)$ of third order of the components of f, i.e. we have

$$P_{(ijkl)}^G(\nabla_1^G(f^1), \ldots, \nabla_2^G(f^n), \nabla_{11}^G(f^1), \ldots, \nabla_{22}^G(f^n), \nabla_{111}^G(f^1), \ldots, \nabla_{222}^G(f^n))$$
$$= M_{(ijkl)}^G(\nabla_1^G(f^1), \ldots, \nabla_2^G(f^n), \nabla_{11}^G(f^1), \ldots, \nabla_{22}^G(f^n)) \cdot$$
$$\cdot \left(\nabla_{111}^G(f^1), \nabla_{112}^G(f^1), \ldots, \nabla_{221}^G(f^n), \nabla_{222}^G(f^n) \right)^T$$
$$+ \tilde{P}_{(ijkl)}^G(\nabla_1^G(f^1), \ldots, \nabla_2^G(f^n), \nabla_{11}^G(f^1), \ldots, \nabla_{22}^G(f^n)) \tag{33}$$

where the expression

$$M_{(ijkl)}^G(\nabla_1^G(f^1), \ldots, \nabla_2^G(f^n), \nabla_{11}^G(f^1), \ldots, \nabla_{22}^G(f^n))$$

is a $\text{Mat}_{n,8n}(\mathbf{R})$-valued function, whose entries are rational expressions involving the partial derivatives $\partial_i(G^1), \ldots, \partial_i(G^n), \partial_{ij}(G^1), \ldots, \partial_{ij}(G^n), \partial_{ijk}(G^1), \ldots, \partial_{ijk}(G^n)$ of G up to third order and the derivatives $\nabla_1^G(f^1), \ldots, \nabla_2^G(f^n), \nabla_{11}^G(f^1), \ldots, \nabla_{22}^G(f^n)$ of f up to only second order. Compare here also with the statement of Lemma 3.4 in [30]. Moreover, since there holds

$$| A_f^0 |^2 = g_f^{ik} g_f^{jl} \langle (A_f^0)_{ij}, (A_f^0)_{kl} \rangle_{\mathbf{R}^n} \tag{34}$$

and $(A_f)_{ij} \equiv A_f(\partial_i, \partial_j) = \partial_{ij} f - (\Gamma_f)^m_{ij} \partial_m f$ for every umbilic-free $W^{4,p}$-immersion f by formulae (7) and (8), one can easily verify by means of formula (56) below, that there is a unique rational function $Q^G \in C^\infty(\Sigma')[v_1, \ldots, v_{2n}, w_1, \ldots, w_{4n}]$ in $6n$ real variables, whose coefficients are rational expressions involving the partial derivatives $\partial_i(G^1), \ldots, \partial_i(G^n), \partial_{ij}(G^1), \ldots, \partial_{ij}(G^n)$ of the components of G up to second order, such that

$$| A^0_f |^4 = Q^G(\nabla^G_1(f^1), \nabla^G_1(f^2), \ldots, \nabla^G_2(f^n), \nabla^G_{11}(f^1), \nabla^G_{11}(f^2), \ldots, \nabla^G_{22}(f^n)) \quad (35)$$

holds on Σ', which is a "scalar", i.e. which does not depend on the choice of the chart ψ of Σ'. Combining (32) and (35) and recalling that every immersion $\mathcal{P}(t, 0, U_0) \circ \phi^{U_0, F_0}_t$ of the considered flow line of Eq. (16) does not have any umbilic points on Σ, we obtain the existence of a unique function $R^{F_0} \in C^\infty(\Sigma')[v_1, \ldots, v_{2n}, w_1, \ldots, w_{4n}, y_1, \ldots, y_{8n}]$, whose components are rational functions in their $14n$ real variables, whose coefficients are rational functions of the partial derivatives of the components of F_0 up to third order and which is linear w.r.t. the last $8n$ variables $y_1, \ldots, y_{8n}$ on account of formula (33), such that

$$\begin{aligned}
&\frac{1}{2} | A^0_{f_t} |^{-4} \, g^{ij}_{f_t} g^{kl}_{f_t} \, \nabla^{f_t}_i \nabla^{f_t}_j \nabla^{F_0}_k \nabla^{F_0}_l (f_t) \\
&= \frac{1}{2} | A^0_{f_t} |^{-4} \, g^{ij}_{f_t} g^{kl}_{f_t} \, \nabla^{F_0}_i \nabla^{F_0}_j \nabla^{F_0}_k \nabla^{F_0}_l (f_t) \\
&\quad + R^{F_0}(\nabla^{F_0}_1(f^1_t), \ldots, \nabla^{F_0}_2(f^n_t), \nabla^{F_0}_{11}(f^1_t), \ldots, \nabla^{F_0}_{22}(f^n_t), \nabla^{F_0}_{111}(f^1_t), \ldots, \nabla^{F_0}_{222}(f^n_t))
\end{aligned} \quad (36)$$

holds on $\Sigma' \times [0, T]$ for any family of immersions $\{f_t\}$ in a sufficiently small neighbourhood $W_{U_0, T, p}$ of the fixed flow line $\{\mathcal{P}(t, 0, U_0) \circ \phi^{U_0, F_0}_t\}_{t \in [0, T]}$ in the space $X_{T, p}$ from line (17). At this point we have already silently used the composition of embedding (30) with embedding (29), yielding the continuous embedding

$$X_{T, p} \hookrightarrow C^0([0, T]; C^{2, \alpha}(\Sigma, \mathbf{R}^n)) \quad (37)$$

for any $\alpha \in (0, 1)$ with $\alpha \leq 2 - \frac{6}{p}$, provided $p \in (3, \infty)$. We should note here, that the n components of

$$R^{F_0}(\nabla^{F_0}_1(f^1_t), \ldots, \nabla^{F_0}_2(f^n_t), \nabla^{F_0}_{11}(f^1_t), \ldots, \nabla^{F_0}_{22}(f^n_t), \nabla^{F_0}_{111}(f^1_t), \ldots, \nabla^{F_0}_{222}(f^n_t))$$

are "scalars" as well, i.e. they do not depend on the choice of the chart ψ of Σ'. On account of formulae (27) and (19), we infer from (36) the following representation of the second component of the operator $\Psi^{F_0, T}$:

$$\begin{aligned}
&\partial_t f_t(x) - \mathcal{M}_{F_0}(f_t)(x) \\
&= \partial_t f_t(x) + \frac{1}{2} | A^0_{f_t}(x) |^{-4} \, g^{ij}_{f_t} g^{kl}_{f_t} \nabla^{F_0}_i \nabla^{F_0}_j \nabla^{F_0}_k \nabla^{F_0}_l (f_t)(x) \\
&\quad + R^{F_0}(\nabla^{F_0}_1(f^1_t), \ldots, \nabla^{F_0}_2(f^n_t), \ldots, \nabla^{F_0}_{111}(f^1_t), \nabla^{F_0}_{111}(f^2_t), \ldots, \nabla^{F_0}_{222}(f^n_t))(x) \\
&\quad - \mathcal{N}^{F_0}(x, D_x f_t(x), D^2_x f_t(x)) \cdot D^3_x f_t(x) \\
&\quad - \mathcal{C}^{F_0}(x, D_x f_t(x), D^2_x f_t(x)) \cdot D^2_x f_t(x) \\
&\quad - \mathcal{D}^{F_0}(x, D_x f_t(x), D^2_x f_t(x)) \cdot D_x f_t(x) \\
&=: \partial_t f_t(x) + \frac{1}{2} | A^0_{f_t}(x) |^{-4} \, g^{ij}_{f_t} g^{kl}_{f_t} \nabla^{F_0}_i \nabla^{F_0}_j \nabla^{F_0}_k \nabla^{F_0}_l (f_t)(x) \\
&\quad + \mathcal{F}^{F_0}(x, D_x f_t(x), D^2_x f_t(x), D^3_x f_t(x))
\end{aligned} \quad (38)$$

for any family of immersions $\{f_t\} \in W_{U_0,T,p}$ and for $(x,t) \in \Sigma' \times [0,T]$, if the neighbourhood $W_{U_0,T,p}$ of the flow line $\{\mathcal{P}(t,0,U_0) \circ \phi_t^{U_0,F_0}\}_{t\in[0,T]}$ in $X_{T,p}$ is chosen sufficiently small. Here and in the sequel, the symbols $D_x f, D_x^2 f, D_x^3 f, \ldots$ abbreviate the matrix-valued functions $(\partial_1 f, \partial_2 f)$, $(\nabla_{ij}^{F_0} f)_{i,j\in\{1,2\}}$, $(\nabla_{ijk}^{F_0} f)_{i,j,k\in\{1,2\}}$, ... On account of the arbitrariness of the choice of the coordinate patch Σ' and its chart ψ and by the compactness of Σ Eq. (38) gives rise to a unique and well-defined function $\mathcal{F}^{F_0} : \Sigma \times \mathbf{R}^{2n} \times \mathbf{R}^{4n} \times \mathbf{R}^{8n} \longrightarrow \mathbf{R}^n$ whose n components are rational functions in their $14n$ real variables, such that (38) holds on $\Sigma \times [0,T]$, and not only on $\Sigma' \times [0,T]$. It is important to note, that by (27) and (36) the function $f \mapsto \mathcal{F}^{F_0}(\cdot, D_x f, D_x^2 f, D_x^3 f)$ in (38) has the same algebraic structure as the function $f \mapsto R^{F_0}(\nabla_1^{F_0}(f^1), \ldots, \nabla_{222}^{F_0}(f^n))$, in particular $f \mapsto \mathcal{F}^{F_0}(\cdot, D_x f, D_x^2 f, D_x^3 f)$ is affine linear w.r.t. the derivatives $(\nabla_{ijk}^{F_0} f)_{i,j,k\in\{1,2\}}$ of third order, similarly to the illustration in formula (33) on Σ'. Hence, there are functions $M^{F_0} : \Sigma \times \mathbf{R}^{2n} \times \mathbf{R}^{4n} \longrightarrow \mathrm{Mat}_{n,8n}(\mathbf{R})$,

$$[(x, v_1, \ldots, v_{2n}, w_1, \ldots, w_{4n}) \mapsto M^{F_0}(x, v_1, \ldots, v_{2n}, w_1, \ldots, w_{4n})],$$

and $\tilde{\mathcal{F}}^{F^0} : \Sigma \times \mathbf{R}^{2n} \times \mathbf{R}^{4n} \longrightarrow \mathbf{R}^n$ depending smoothly on x and rationally on the $6n$ variables v_i and w_j, such that

$$\begin{aligned}
&\mathcal{F}^{F_0}(x, v_1, \ldots, v_{2n}, w_1, \ldots, w_{4n}, y_1, \ldots, y_{8n}) \\
&= M^{F_0}(x, v_1, \ldots, v_{2n}, w_1, \ldots, w_{4n}) \cdot (y_1, y_2, \ldots, y_{8n-1}, y_{8n})^T \\
&+ \tilde{\mathcal{F}}^{F_0}(x, v_1, \ldots, v_{2n}, w_1, \ldots, w_{4n}).
\end{aligned} \tag{39}$$

Now, since the entire flow line $\{\mathcal{P}(t,0,U_0) \circ \phi_t^{U_0,F_0}\}$, starting in the fixed umbilic-free initial immersion U_0, solves Eq. (16) within the space $C^\infty(\Sigma \times [0,T], \mathbf{R}^n)$ and thus consists of umbilic-free immersions, all summands in line (38) remain uniformly bounded on $\Sigma \times [0,T]$. Hence, there are open connected neighbourhoods $\mathcal{O}_t$ of the graphs

$$\begin{aligned}
G_t &:= \{(x, D_x(\mathcal{P}(t,0,U_0) \circ \phi_t^{U_0,F_0})(x), D_x^2(\mathcal{P}(t,0,U_0) \circ \phi_t^{U_0,F_0})(x)) \,|\, x \in \Sigma\} \\
&\subset \Sigma \times \mathbf{R}^{6n}
\end{aligned}$$

for $t \in [0,T]$, such that M^{F_0} and $\tilde{\mathcal{F}}^{F_0}$ are C^∞-smooth on the open subset $\bigcup_{t\in[0,T]} \mathcal{O}_t$ of $\Sigma \times \mathbf{R}^{6n}$. Now, on account of the compactness of Σ and of $[0,T]$ there is some $\delta > 0$, such that there holds:

$$\mathrm{dist}_{\Sigma \times \mathbf{R}^{6n}}(G_t, \partial \mathcal{O}_t) > 2\delta \qquad \forall\, t \in [0,T]. \tag{40}$$

Similarly to the proof of Theorem 2 in [13] and in view of inequality (40) and of embedding (37) we define an open neighbourhood $\mathcal{V}_\delta$ of the function

$$[(x,t) \mapsto (D_x(\mathcal{P}(t,0,U_0) \circ \phi_t^{U_0,F_0})(x), D_x^2(\mathcal{P}(t,0,U_0) \circ \phi_t^{U_0,F_0})(x))]$$

within the space $C^0([0,T], C^{0,\alpha}(\Sigma, \mathbf{R}^{6n}))$, with the exponent $\alpha = \alpha(p) \in (0,1)$ from line (37), by setting

$$\begin{aligned}
\bar{h} \in \mathcal{V}_\delta :&\Longleftrightarrow \| (D_x(\mathcal{P}(t,0,U_0) \circ \phi_t^{U_0,F_0}), D_x^2(\mathcal{P}(t,0,U_0) \circ \phi_t^{U_0,F_0})) - \bar{h}_t \|_{L^\infty(\Sigma, \mathbf{R}^{6n})} \\
&< \delta \ \forall\, t \in [0,T].
\end{aligned}$$

It is an immediate consequence of this definition and of inequality (40), that there holds:

$$\bar{h} \in \mathcal{V}_\delta \Longrightarrow (x, \bar{h}_t(x)) \in \mathcal{O}_t \subset \Sigma \times \mathbf{R}^{6n} \quad \forall (x,t) \in \Sigma \times [0,T]. \tag{41}$$

Now, in order to show that the non-linear operator

$$W_{U_0,T,p} \ni [(x,t) \mapsto f_t(x)] \mapsto [(x,t) \mapsto \mathcal{F}^{F_0}(x, D_x f_t(x), D_x^2 f_t(x), D_x^3 f_t(x))] \in$$
$$\in L^p([0,T], L^p(\Sigma, \mathbf{R}^n))$$

is of class C^1 on a sufficiently small open neighborhood $W_{U_0,T,p}$ of the flow line $\{\mathcal{P}(t, 0, U_0) \circ \phi_t^{U_0, F_0}\}$ in $X_{T,p}$, we shall proceed similarly to the proof of Theorem 2 in [13] and consider here the Banach space

$$B := C^0([0,T]; C^{0,\alpha}(\Sigma, \mathbf{R}^{6n})) \times L^p([0,T]; W^{1,p}(\Sigma, \mathbf{R}^{8n})), \tag{42}$$

equipped with the canonical product norm

$$\| h \|_B := \| \bar{h} \|_{C^0([0,T];C^{0,\alpha}(\Sigma,\mathbf{R}^{6n}))} + \| h^* \|_{L^p([0,T];W^{1,p}(\Sigma,\mathbf{R}^{8n}))}$$

for $h = (\bar{h}, h^*) \in B$. We firstly derive from Eq. (39), that the classical derivative of the function $\mathcal{F}^{F_0}$ in any point $(v, w, y) \in \mathbf{R}^{2n} \times \mathbf{R}^{4n} \times \mathbf{R}^{8n}$, in which $\mathcal{F}^{F_0}$ is smooth, and in direction of any fixed vector $(\mathbf{v}, \mathbf{w}, \mathbf{y}) \in \mathbf{R}^{2n} \times \mathbf{R}^{4n} \times \mathbf{R}^{8n}$ reads:

$$D_{(v,w,y)}\mathcal{F}^{F_0}(x, (v, w, y)) \cdot (\mathbf{v}, \mathbf{w}, \mathbf{y})$$
$$= D_{(v,w)}(M^{F_0}(x, v, w) \cdot y) \cdot (\mathbf{v}, \mathbf{w}) + M^{F_0}(x, v, w) \cdot \mathbf{y}$$
$$+ D_{(v,w)}\tilde{\mathcal{F}}^{F_0}(x, v, w) \cdot (\mathbf{v}, \mathbf{w}). \tag{43}$$

Now we fix some $h \in B$, whose first component $\bar{h} \in C^0([0,T]; C^{0,\alpha}(\Sigma, \mathbf{R}^{6n}))$ is contained in $\mathcal{V}_\delta$, and we use statement (41), Eq. (43) and the mean value theorem, in order to compute for any $\eta \in B$ whose first component $\bar{\eta} \in C^0([0,T]; C^{0,\alpha}(\Sigma, \mathbf{R}^{6n}))$ satisfies $\| \bar{\eta} \|_{C^0([0,T];C^{0,\alpha}(\Sigma,\mathbf{R}^{6n}))} \leq \epsilon$:

$$|M^{F_0}(x, \bar{h}_t(x) + \bar{\eta}_t(x)) \cdot \eta_t^*(x) - M^{F_0}(x, \bar{h}_t(x)) \cdot \eta_t^*(x)|$$
$$\leq C(\Sigma, F_0, \bar{h}, \epsilon) |\bar{\eta}_t(x)| |\eta_t^*(x)| \tag{44}$$

$\forall (x, t) \in \Sigma \times [0, T]$ and also:

$$|D_{(v,w)}(M^{F_0}(x, \bar{h}_t(x) + \bar{\eta}_t(x)) \cdot h_t^*(x)) - D_{(v,w)}(M^{F_0}(x, \bar{h}_t(x)) \cdot h_t^*(x))|$$
$$\leq C(\Sigma, F_0, \bar{h}, \epsilon) |\bar{\eta}_t(x)| |h_t^*(x)| \tag{45}$$

$\forall (x, t) \in \Sigma \times [0, T]$, provided $\epsilon = \epsilon(F_0, \bar{h}) > 0$ is sufficiently small. Now we obtain by means of an integration of estimates (44) and (45) the estimates

$$\| M^{F_0}(\cdot, \bar{h} + \bar{\eta}) \cdot \eta^* - M^{F_0}(\cdot, \bar{h}) \cdot \eta^* \|_{L^p(\Sigma \times [0,T], \mathbf{R}^n)}$$
$$\leq C(\Sigma, F_0, \bar{h}, \epsilon) \| \eta^* \|_{L^p(\Sigma \times [0,T], \mathbf{R}^{8n})} \| \bar{\eta} \|_{L^\infty(\Sigma \times [0,T], \mathbf{R}^{6n})} \tag{46}$$

and

$$\| D_{(v,w)}(M^{F_0}(\cdot, \bar{h} + \bar{\eta}) \cdot h^*) - D_{(v,w)}(M^{F_0}(\cdot, \bar{h}) \cdot h^*) \|_{L^p(\Sigma \times [0,T], \mathbf{R}^{6n^2})}$$
$$\leq C(\Sigma, F_0, \bar{h}, \epsilon) \| h^* \|_{L^p(\Sigma \times [0,T], \mathbf{R}^{8n})} \| \bar{\eta} \|_{L^\infty(\Sigma \times [0,T], \mathbf{R}^{6n})}, \tag{47}$$

for any $\eta \in B$ whose first component $\bar{\eta}$ satisfies $\| \bar{\eta} \|_{C^0([0,T];C^{0,\alpha}(\Sigma,\mathbf{R}^{6n}))} \leq \epsilon$. Moreover, we infer from the fact that $\bar{h} \in \mathcal{V}_\delta$ and from the definition of $\mathcal{V}_\delta$:

$$\| D_{(v,w)}(M^{F_0}(\cdot, \bar{h} + \bar{\eta}) \cdot \eta^*) \|_{L^p(\Sigma \times [0,T], \mathbf{R}^{6n^2})}$$
$$\leq C(\Sigma, F_0, \bar{h}, \epsilon) \| \eta^* \|_{L^p(\Sigma \times [0,T], \mathbf{R}^{8n})} . \tag{48}$$

Now we derive from the linearity of the map $[(v, w, y) \mapsto D_{(v,w)}(M^{F_0}(x, v, w) \cdot y)]$ w.r.t. y the identity

$$
\begin{aligned}
&D_{(v,w)}(M^{F_0}(x, \bar{h} + s\,\bar{\eta}) \cdot (h^* + s\eta^*)) - D_{(v,w)}(M^{F_0}(x, \bar{h}) \cdot h^*) \\
&= D_{(v,w)}(M^{F_0}(x, \bar{h} + s\,\bar{\eta}) \cdot h^*) - D_{(v,w)}(M^{F_0}(x, \bar{h}) \cdot h^*) \\
&\quad + s\, D_{(v,w)}(M^{F_0}(x, \bar{h} + s\,\bar{\eta}) \cdot \eta^*),
\end{aligned}
\tag{49}
$$

and we infer from Eq. (43), from estimates (46)–(48), from Eq. (49) and from the mean value theorem that the operator

$$
\mathcal{F}_\sharp^{F_0} : \mathcal{V}_\delta \times L^p([0, T]; W^{1,p}(\Sigma, \mathbf{R}^{8n})) \subset B \longrightarrow L^p([0, T]; L^p(\Sigma, \mathbf{R}^n)) \tag{50}
$$

mapping $h = (\bar{h}, h^*) \mapsto \mathcal{F}^{F_0}(\cdot, \bar{h}, h^*)$, satisfies:

$$
\| \mathcal{F}_\sharp^{F_0}(h + \eta) - \mathcal{F}_\sharp^{F_0}(h) - D_{(v,w,y)}\mathcal{F}^{F_0}(\cdot, h) \cdot \eta \|_{L^p([0,T];L^p(\Sigma,\mathbf{R}^n))}
$$

$$
= \| \int_0^1 \left(D_{(v,w,y)}\mathcal{F}^{F_0}(\cdot, h + s\,\eta) - D_{(v,w,y)}\mathcal{F}^{F_0}(\cdot, h) \right) ds \cdot \eta \|_{L^p([0,T];L^p(\Sigma,\mathbf{R}^n))}
$$

$$
\leq \| \int_0^1 \left[D_{(v,w)}(M^{F_0}(\cdot, \bar{h} + s\,\bar{\eta}) \cdot h^*) - D_{(v,w)}(M^{F_0}(\cdot, \bar{h}) \cdot h^*) \right.
$$

$$
\left. + s\, D_{(v,w)}(M^{F_0}(\cdot, \bar{h} + s\,\bar{\eta}) \cdot \eta^*) \right] ds \cdot \bar{\eta} \|_{L^p([0,T];L^p(\Sigma,\mathbf{R}^n))}
$$

$$
+ \| \int_0^1 \left[M^{F_0}(\cdot, \bar{h} + s\bar{\eta}) - M^{F_0}(\cdot, \bar{h}) \right] ds \cdot \eta^* \, ds \|_{L^p([0,T];L^p(\Sigma,\mathbf{R}^n))}
$$

$$
+ \| \int_0^1 \left[D_{(v,w)}\tilde{\mathcal{F}}^{F_0}(\cdot, \bar{h} + s\,\bar{\eta}) - D_{(v,w)}\tilde{\mathcal{F}}^{F_0}(\cdot, \bar{h}) \right] ds \cdot \bar{\eta} \|_{L^p([0,T];L^p(\Sigma,\mathbf{R}^n))}
$$

$$
\leq C(\Sigma, F_0, \bar{h}, \epsilon) \left[\| h^* \|_{L^p([0,T];L^p(\Sigma,\mathbf{R}^{8n}))} \| \bar{\eta} \|^2_{L^\infty(\Sigma \times [0,T], \mathbf{R}^{6n})} \right.
$$

$$
+ \| \bar{\eta} \|_{L^\infty(\Sigma \times [0,T], \mathbf{R}^{6n})} \| \eta^* \|_{L^p([0,T];L^p(\Sigma, \mathbf{R}^{8n}))}
$$

$$
+ \| \eta^* \|_{L^p([0,T];L^p(\Sigma, \mathbf{R}^{8n}))} \| \bar{\eta} \|_{L^\infty(\Sigma \times [0,T], \mathbf{R}^{6n})}
$$

$$
\left. + \| \bar{\eta} \|_{L^p([0,T];L^p(\Sigma,\mathbf{R}^{6n}))} \| \bar{\eta} \|_{L^\infty(\Sigma \times [0,T], \mathbf{R}^{6n})} \right]
$$

$$
\leq \hat{C}(\Sigma, T, F_0, h, \epsilon, p) \| \eta \|^2_B,
$$

for any $\eta \in B$ whose first component $\bar{\eta} \in C^0([0, T]; C^{0,\alpha}(\Sigma, \mathbf{R}^{6n}))$ satisfies $\| \bar{\eta} \|_{C^0([0,T];C^{0,\alpha}(\Sigma,\mathbf{R}^{6n}))} \leq \epsilon$. This estimate proves that the operator $\mathcal{F}_\sharp^{F_0}$ in line (50) is Fréchet differentiable in any point $h \in \mathcal{V}_\delta \times L^p([0, T]; W^{1,p}(\Sigma, \mathbf{R}^{8n}))$ and that the Fréchet derivative of $\mathcal{F}_\sharp^{F_0}$ in any such $h \in B$ is concretely given by the formula:

$$
\begin{aligned}
D\mathcal{F}_\sharp^{F_0}(h).\eta &= D_{(v,w,y)}\mathcal{F}^{F_0}(\cdot, h) \cdot \eta \\
&= D_{(v,w)}(M^{F_0}(\cdot, \bar{h}) \cdot h^*) \cdot \bar{\eta} + M^{F_0}(\cdot, \bar{h}) \cdot \eta^* + D_{(v,w)}\tilde{\mathcal{F}}^{F_0}(\cdot, \bar{h}) \cdot \bar{\eta}, \tag{51}
\end{aligned}
$$

where we have used Eq. (43) in the second equality. Moreover, one can easily deduce from formulae (44), (45) and (51), that the Fréchet derivative $D\mathcal{F}_\sharp^{F_0}(h) : B \longrightarrow L^p([0, T]; L^p(\Sigma, \mathbf{R}^n))$ depends continuously on $h \in B$. Furthermore, since we had taken the exponent $\alpha = \alpha(p) \in (0, 1)$ from embedding (37), we can now use this embedding, in order to see that the linear map

$$
X_{T,p} \ni [(x, t) \mapsto f_t(x)] \overset{\mathcal{L}}{\mapsto} [(x, t) \mapsto (D_x f_t(x), D_x^2 f_t(x), D_x^3 f_t(x))] \in B
$$

is continuous and thus maps a sufficiently small open neighborhood $W_{U_0,T,p}$ of the flow line $\{\mathcal{P}(t,0,U_0)\circ\phi_t^{U_0,F_0}\}_{t\in[0,T]}$ in $X_{T,p}$ into the product $\mathcal{V}_\delta\times L^p([0,T];W^{1,p}(\Sigma,\mathbf{R}^{8n}))$. Hence, the composition

$$W_{U_0,T,p}\ni[(x,t)\mapsto f_t(x)]\mapsto(\mathcal{F}_\sharp^{F_0}\circ\mathcal{L})([(x,t)\mapsto f_t(x)])\equiv$$
$$\equiv[(x,t)\mapsto\mathcal{F}^{F_0}(x,D_xf_t(x),D_x^2f_t(x),D_x^3f_t(x))]\in L^p([0,T];L^p(\Sigma,\mathbf{R}^n))$$

has just turned out to be a non-linear C^1-operator, and we infer from formula (51) that

$$D\mathcal{F}^{F_0}(\,\cdot\,,D_xf_t,D_x^2f_t,D_x^3f_t).(D_x\eta_t,D_x^2\eta_t,D_x^3\eta_t)$$
$$= D_{(v,w)}(M^{F_0}(\,\cdot\,,D_xf_t,D_x^2f_t)\cdot D_x^3f_t)\cdot(D_x\eta_t,D_x^2\eta_t)$$
$$+M^{F_0}(\,\cdot\,,D_xf_t,D_x^2f_t)\cdot D_x^3\eta_t+D_{(v,w)}\tilde{\mathcal{F}}^{F_0}(\,\cdot\,,D_xf_t,D_x^2f_t)\cdot(D_x\eta_t,D_x^2\eta_t),\quad(52)$$

for every $\{f_t\}_{t\in[0,T]}\in W_{U_0,T,p}$ and $\{\eta_t\}\in T_fW_{U_0,T,p}=X_{T,p}$. Repeating the above reasoning starting in line (42)—using now Eqs. (34) and (35) instead of Eq. (39) and again embedding (37)—and adjusting the definition of the auxiliary Banach space B in (42) appropriately, one can also show that the operator

$$X_{T,p}\supset W_{U_0,T,p}\ni[(x,t)\mapsto f_t(x)]\mapsto[(x,t)\mapsto\mathcal{A}(x,D_xf_t(x),D_x^2f_t(x))]$$
$$:=[(x,t)\mapsto\frac{1}{2}|A_{f_t}^0|^{-4}(x)]\in C^0(\Sigma\times[0,T],\mathbf{R})\quad(53)$$

and also the operator

$$X_{T,p}\supset W_{U_0,T,p}\ni[(x,t)\mapsto f_t(x)]\mapsto[(x,t)\mapsto Q(D_xf_t(x),D_x^4f_t(x))]$$
$$:=[(x,t)\mapsto g_{f_t}^{ij}(x)\,g_{f_t}^{kl}(x)\,\nabla_i^{F_0}\nabla_j^{F_0}\nabla_k^{F_0}\nabla_l^{F_0}(f_t)(x)]\in L^p([0,T];L^p(\Sigma,\mathbf{R}^n))\quad(54)$$

are of class C^1 on a sufficiently small open neighborhood $W_{U_0,T,p}$ of the flow line $\{\mathcal{P}(t,0,U_0)\circ\phi_t^{U_0,F_0}\}_{t\in[0,T]}$ in $X_{T,p}$. Hence, both the first and the second component of the operator $\Psi^{F_0,T}:W_{F_0,T,p}\longrightarrow Y_{T,p}$ have turned out to be of class C^1. We should mention here already for later use, that the above function

$$Q:\mathbf{R}^{2n}\times\mathbf{R}^{16n}\longrightarrow\mathbf{R}^n,\qquad(v,p)\mapsto Q(v,p),\quad(55)$$

in (54) is rational in its first $2n$ variables $v_1,\ldots,v_{2n}$ and linear in its last $16n$ variables $p_1,\ldots p_{16n}$. This follows immediately from the definition in (54) and from the identity

$$(g_f)^{ij}\equiv((g_f)_{ij})^{-1}=\frac{1}{\det(g_f)}((g_f)_{ij})^\sharp\quad(56)$$

by Cramer's rule, for any C^1-immersion $f:\Sigma\longrightarrow\mathbf{R}^n$. See here also [13], p. 1165.

β) Now we go on proving higher regularity of the operator $\Psi^{F_0,T}:W_{U_0,T,p}\longrightarrow Y_{T,p}$, following the lines of the proof of Proposition 3.5 in Appendix A.1 of [30]. To this end, we first recall the special algebraic structure of the function

$$X_{T,p}\supset W_{U_0,T,p}\ni[(x,t)\mapsto f_t(x)]$$
$$\mapsto[(x,t)\mapsto\mathcal{F}^{F_0}(x,D_xf_t(x),D_x^2f_t(x),D_x^3f_t(x))]\in L^p([0,T];L^p(\Sigma,\mathbf{R}^n))\quad(57)$$

in (38), which is expressed by the affine linear structure of the function $\mathcal{F}^{F_0}(x,v_1,\ldots,v_{2n},w_1,\ldots,w_{4n},y_1,\ldots,y_{8n})$ in formula (39) w.r.t. the variable $y\in$

$\mathbf{R}^{8n}$, representing the third covariant derivatives $\nabla_{111}^{F_0}(f_t), \ldots, \nabla_{222}^{F_0}(f_t)$ in Eqs. (16) and (38). Now, the function $M^{F_0} : \Sigma \times \mathbf{R}^{2n} \times \mathbf{R}^{4n} \longrightarrow \mathrm{Mat}_{n,8n}(\mathbf{R})$, i.e.

$$[(x, v_1, \ldots, v_{2n}, w_1, \ldots, w_{4n}) \mapsto M^{F_0}(x, v_1, \ldots, v_{2n}, w_1, \ldots, w_{4n})], \qquad (58)$$

and the function $\tilde{\mathcal{F}}^{F^0} : \Sigma \times \mathbf{R}^{2n} \times \mathbf{R}^{4n} \longrightarrow \mathbf{R}^n$ depend rationally on the $6n$ variables v_i and w_j, representing the derivatives $\nabla_1^{F_0}(f_t), \nabla_2^{F_0}(f_t), \nabla_{11}^{F_0}(f_t), \ldots, \nabla_{22}^{F_0}(f_t)$ in Eqs. (16) and (38). We recall here that the embedding (37) guarantees, that we have for the covariant derivatives up to second order:

$$\nabla_1^{F_0}(f_t), \nabla_2^{F_0}(f_t), \nabla_{11}^{F_0}(f_t), \ldots, \nabla_{22}^{F_0}(f_t) \in C^0(\Sigma \times [0, T], \mathbf{R}^n), \qquad (59)$$

whereas there holds trivially:

$$\nabla_{111}^{F_0}(f_t), \ldots, \nabla_{222}^{F_0}(f_t), \nabla_{1111}^{F_0}(f_t), \ldots, \nabla_{2222}^{F_0}(f_t) \in L^p([0, T]; L^p(\Sigma, \mathbf{R}^n)) \quad (60)$$

for any $\{f_t\} \in W_{U_0, T, p}$. Therefore, we may proceed as in the proof of Proposition 3.5 in [30], making use of the fact that by Proposition 2.5 in [28] the bilinear maps

$$C^0(\Sigma \times [0, T], \mathbf{R}) \bigotimes C^0(\Sigma \times [0, T], \mathbf{R}) \longrightarrow C^0(\Sigma \times [0, T], \mathbf{R}),$$
$$u_1 \otimes u_2 \mapsto [(x, t) \mapsto u_1(x, t) \cdot u_2(x, t)] \qquad (61)$$

and

$$C^0(\Sigma \times [0, T], \mathrm{Mat}_{n,m}(\mathbf{R})) \bigotimes L^p([0, T]; L^p(\Sigma, \mathbf{R}^m)) \longrightarrow L^p([0, T]; L^p(\Sigma, \mathbf{R}^n)),$$
$$(M, Y) \mapsto [(x, t) \mapsto M(x, t) \cdot Y(x, t)] \qquad (62)$$

are continuous and thus real analytic, for any fixed pair of integers $m, n \in \mathbf{N}$, and that—similarly to Proposition 6.4 in [29]—the non-linear operator

$$\{u \in C^0(\Sigma \times [0, T], \mathbf{R}) | \inf_{\Sigma \times [0, T]} |u| > \delta\} \longrightarrow C^0(\Sigma \times [0, T], \mathbf{R}), \quad u \mapsto \frac{1}{u}, \quad (63)$$

is real analytic, for any fixed $\delta > 0$. Now, due to formula (39) the operator in line (57) can be expressed as the sum of the two following operators:
a) The composition of the linear continuous operator

$$W_{U_0, T, p} \ni [(x, t) \mapsto f_t(x)] \mapsto [(x, t) \mapsto (x, D_x f_t(x), D_x^2 f_t(x))]$$
$$\in C^0(\Sigma \times [0, T], \Sigma \times \mathbf{R}^{6n})$$

with the function $[(x, v, w) \mapsto M^{F_0}(x, v_1, \ldots, v_{2n}, w_1, \ldots, w_{4n})]$, matrix-multiplied by the $\mathbf{R}^{8n}$-valued function $[(x, t) \mapsto D_x^3 f_t(x) \in L^p([0, T]; L^p(\Sigma, \mathbf{R}^{8n}))$ of all covariant derivatives of third order of $\{f_t\} \in W_{U_0, T, p}$,
b) the composition of the above linear function $[(x, t) \mapsto f_t(x)] \mapsto [(x, t) \mapsto (x, D_x f_t(x), D_x^2 f_t(x))]$ with the function

$$[(x, v, w) \mapsto \tilde{\mathcal{F}}^{F_0}(x, v_1, \ldots, v_{2n}, w_1, \ldots, w_{4n})]. \qquad (64)$$

Now, taking into account that the functions M^{F_0} and $\tilde{\mathcal{F}}^{F_0}$ in (58) and (64) are rational expressions in the variables $v_1, \ldots, v_{2n}, w_1, \ldots, w_{4n}$ and in the partial derivatives of the smooth immersion F_0 up to third order, and recalling the regularity of the covariant derivatives of any fixed $\{f_t\} \in W_{U_0, T, p}$ up to second order respectively up to fourth order in (59) and (60), we can successively apply statements (61)–(63), in order to argue as in the proof of Proposition 3.5 in [30], that the non-linear operators composed above in (a) and (b) are real analytic maps from $W_{U_0, T, p}$ to $L^p([0, T]; L^p(\Sigma, \mathbf{R}^n))$, implying that the non-linear

operator in line (57) is actually real analytic from $W_{U_0, T, p}$ to $L^p([0, T], L^p(\Sigma, \mathbf{R}^n))$, as well. Moreover, we can see that the operator $\{f_t\} \mapsto \{Q(D_x f_t, D_x^4 f_t)\}$ in formula (54) is the composition of the linear continuous operator

$$
W_{U_0, T, p} \ni [(x, t) \mapsto f_t(x)] \mapsto [(x, t) \mapsto (D_x f_t(x), D_x^4 f_t(x))]
$$
$$
\in C^0([0, T]; C^{1,\alpha}(\Sigma, \mathbf{R}^{2n})) \times L^p([0, T]; L^p(\Sigma, \mathbf{R}^{16n}))
$$

with the function $Q : \mathbf{R}^{2n} \times \mathbf{R}^{16n} \to \mathbf{R}$ from line (55), which is rational in the variables $v_1, \ldots, v_{2n}$ and linear in the variables $p_1, \ldots, p_{16n}$. We may therefore argue again by means of statements (61)–(63), that the non-linear operator in line (54) is actually real analytic. Similarly, one can show by means of formulae (8), (9), (34), (61) and (63) that the non-linear operator

$$
W_{U_0, T, p} \ni [(x, t) \mapsto f_t(x)] \mapsto [(x, t) \mapsto \mathcal{A}(x, D_x f_t(x), D_x^2 f_t(x))]
$$
$$
\equiv [(x, t) \mapsto |A_{f_t}^0(x)|^{-4}] \in C^0(\Sigma \times [0, T], \mathbf{R})
$$

from line (53) is real analytic. Hence, combining these three results we can finally conclude by means of formula (38)—and again using statement (62) - that the "DeTurck-modification"

$$
\mathcal{M}_{F_0} : W_{U_0, T, p} \longrightarrow L^p([0, T]; L^p(\Sigma, \mathbf{R}^n))
$$

of the original differential operator $[\{f_t\} \mapsto \{|A_{f_t}^0|^{-4} \nabla_{L^2} \mathcal{W}(f_t)\}]$ in formulae (13) and (15) is a real analytic operator from $W_{U_0, T, p}$ to $L^p([0, T]; L^p(\Sigma, \mathbf{R}^n))$. Since the operator $\partial_t : X_{T, p} \longrightarrow L^p([0, T]; L^p(\Sigma, \mathbf{R}^n))$ is linear and continuous, we have thus proved the assertion of the third part of the theorem.

(3) Just as in the proof of the second part of Theorem 2 in [13], one can infer from a combination of the result of the third part of this theorem with formulae (38) and (52)–(54) and with the chain- and product rule for non-linear C^1-operators, that the Fréchet derivative of the second component of $\Psi^{U_0, T}$ in any fixed $\{f_t\} \in W_{U_0, T, p}$ is given by the formula:

$$
(D(\Psi^{F_0, T})_2(\{f_t\})).(\{\eta_t\}) \equiv \partial_t \eta_t - D(\mathcal{M}_{F_0})(\{f_t\}).(\{\eta_t\})
$$
$$
= \partial_t \eta_t + \frac{1}{2} |A_{f_t}^0|^{-4} \left(D_v Q(D_x f_t, D_x^4 f_t) \cdot D_x(\eta_t) \right.
$$
$$
\left. + g_{f_t}^{ij} g_{f_t}^{kl} \nabla_i^{F_0} \nabla_j^{F_0} \nabla_k^{F_0} \nabla_l^{F_0}(\eta_t) \right)
$$
$$
+ D_{(v,w)} \mathcal{A}(\cdot, D_x f_t, D_x^2 f_t) \cdot \left(D_x \eta_t, D_x^2 \eta_t \right) g_{f_t}^{ij} g_{f_t}^{kl} \nabla_i^{F_0} \nabla_j^{F_0} \nabla_k^{F_0} \nabla_l^{F_0}(f_t)
$$
$$
+ D_{(v,w,y)} \mathcal{F}^{F_0}(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t) \cdot \left(D_x \eta_t, D_x^2 \eta_t, D_x^3 \eta_t \right)
$$
$$
= \partial_t \eta_t + \frac{1}{2} |A_{f_t}^0|^{-4} g_{f_t}^{ij} g_{f_t}^{kl} \nabla_i^{F_0} \nabla_j^{F_0} \nabla_k^{F_0} \nabla_l^{F_0}(\eta_t)
$$
$$
+ B_3^{ijk}(\cdot, D_x f_t, D_x^2 f_t) \cdot \nabla_{ijk}^{F_0}(\eta_t)
$$
$$
+ B_2^{ij}(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t, D_x^4 f_t) \cdot \nabla_{ij}^{F_0}(\eta_t)
$$
$$
+ B_1^i(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t, D_x^4 f_t) \cdot \nabla_i^{F_0}(\eta_t) \tag{65}
$$

on $\Sigma \times [0, T]$, for $\eta = \{\eta_t\} \in T_f W_{U_0, T, p} = X_{T, p}$, where $B_3^{ijk}(\cdot, D_x f_t, D_x^2 f_t)$, $B_2^{ij}(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t, D_x^4 f_t)$, $B_1^i(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t, D_x^4 f_t)$ are coefficients of $\mathrm{Mat}_{n,n}(\mathbf{R})$-valued, contravariant tensor fields of degrees 3, 2 and 1. It is important to note here, that a quick inspection of formula (54) shows, that the gradient $D_v Q(v, p)$

of the function Q in line (55) w.r.t. v depends linearly on the variables $p_1, \ldots, p_{16n}$, just as the function $[(v, p) \mapsto Q(v, p)]$ itself does. See here again [13], p. 1165. Combining this information with formulae (52), (53) and (65), we see that the coefficients $B_1^i(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t, D_x^4 f_t)$ and $B_2^{ij}(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t, D_x^4 f_t)$ depend affine linearly on the derivatives of third order $\nabla_{ijk}^{F_0} f_t$ and of fourth order $\nabla_{ijkl}^{F_0} f_t$ of any fixed element $\{f_t\} \in W_{U_0,T,p}$. Now, since the derivatives of third and fourth order $\nabla_{ijk}^{F_0} f_t$ and $\nabla_{ijkl}^{F_0} f_t$ are of class $L^p([0, T]; L^p(\Sigma, \mathbf{R}^n))$ by (60), and since all the remaining factors respectively summands in $B_2^{ij}(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t, D_x^4 f_t)$ and $B_1^i(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t, D_x^4 f_t)$ are of class $C^0([0, T]; C^{0,\alpha}(\Sigma))$ on account of formulae (39), (52) and (65) and on account of embedding (37), these two types of coefficients are of class $L^p([0, T]; L^p(\Sigma, \mathrm{Mat}_{n,n}(\mathbf{R})))$, provided the open neighborhood $W_{U_0,T,p}$ of the fixed flow line $\{\mathcal{P}(t, 0, U_0) \circ \phi_t^{U_0,F_0}\}_{t \in [0,T]}$ in $X_{T,p}$ is sufficiently small. Compare here also with the proof of Lemma 3.4 in [30]. Moreover, using the fact that the map $M^{F_0} : \Sigma \times \mathbf{R}^{2n} \times \mathbf{R}^{4n} \longrightarrow \mathrm{Mat}_{n,8n}(\mathbf{R})$ in (58) depends smoothly on x and rationally on the $6n$ variables v_i and w_j, we conclude that the coefficients $B_3^{ijk}(\cdot, D_x f_t, D_x^2 f_t)$ are of class $C^{0,\alpha}([0, T], C^{0,\alpha}(\Sigma, \mathrm{Mat}_{n,n}(\mathbf{R})))$, for some appropriate $\alpha = \alpha(p) \in (0, 1)$, on account of formulae (52) and (65), and on account of the continuous embedding

$$X_{T,p} \hookrightarrow C^{0,\alpha}([0, T]; C^{2,\alpha}(\Sigma, \mathbf{R}^n)), \tag{66}$$

for some small $\alpha \in (0, 1)$ only depending on p, provided $p \in (3, \infty)$. [2] Using again embedding (66), one can see immediately in formula (65), that the leading term

$$\{\eta_t\} \mapsto \left\{ \partial_t \eta_t + \frac{1}{2} \mid A_{f_t}^0 \mid^{-4} g_{f_t}^{ij} g_{f_t}^{kl} \nabla_i^{F_0} \nabla_j^{F_0} \nabla_k^{F_0} \nabla_l^{F_0} (\eta_t) \right\} \tag{67}$$

of the second component of the Fréchet derivative $D\Psi^{F_0,T}(\{f_t\})$ is a uniformly parabolic linear differential operator of fourth order in the sense of Proposition 2 below, mapping $X_{T,p}$ into $L^p([0, T]; L^p(\Sigma, \mathbf{R}^n))$, for any fixed $\{f_t\} \in W_{U_0,T,p}$, whose coefficients are of class $C^{0,\alpha}([0, T], C^{0,\alpha}(\Sigma, \mathbf{R}))$, and it obviously acts on each component of $\{\eta_t\} \in X_{T,p}$ separately. This proves all assertions of the fourth part of the theorem.

(4) First of all, the coefficients of the leading term (67) of the second component of $D\Psi^{F_0,T}(\{f_t\})$ are of class $C^2(\Sigma \times [0, T], \mathbf{R})$, if $\{f_t\} \in C^4(\Sigma \times [0, T], \mathbf{R}^n) \cap W_{U_0,T,p}$. Hence, they satisfy the regularity requirements of Proposition 2 below. Moreover, the leading operator in (67) is obviously uniformly elliptic and diagonal on $\Sigma \times [0, T]$ in the sense of Proposition 2 below for $g := g_{F_0}$, since the functions $f_t : \Sigma \longrightarrow \mathbf{R}^n$ have to stay umbilic-free immersions for every $t \in [0, T]$, if the open neighborhood $W_{U_0,T,p}$ of the flow line $\{\mathcal{P}(t, 0, U_0) \circ \phi_t^{U_0,F_0}\}_{t \in [0,T]}$ in X_T is sufficiently small, taking here embedding (66) again into account. Moreover, the remaining coefficients $B_3^{ijk}(\cdot, D_x f_t, D_x^2 f_t)$, $B_2^{ij}(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t, D_x^4 f_t)$ and $B_1^i(\cdot, D_x f_t, D_x^2 f_t, D_x^3 f_t, D_x^4 f_t)$ of the second component of $D\Psi^{F_0,T}(\{f_t\})$ in (65) are of class $C^0(\Sigma \times [0, T], \mathrm{Mat}_{n,n}(\mathbf{R}))$, and thus their restrictions to $\Omega_\kappa \times [0, T]$, for any fixed coordinate patch $\Omega_\kappa \subset \Sigma$ of a finite atlas of Σ as in Section 10 of [5], satisfy the conditions of Proposition 2 below. Hence, we

---

[2] Embedding (66) obviously improves embedding (37)—at least if T remains fixed—and it can be proved exactly as the statement of the third part of Lemma 3.3 in [30] by means of identification of fractional Sobolev spaces with appropriate Besov spaces and identification of certain Besov spaces with appropriate Hölder-Zygmund spaces, taken from [3, 32], furthermore by means of continuous embeddings between appropriate Besov spaces in [3] and by means of appropriate interpolation theorems, taken from [3, 22].

infer from this proposition, that both the linear operator

$$\partial_t - D(\mathcal{M}_{F_0})(\{f_t\}) : X_T^0 := \{\{u_t\} \in X_{T,p} \,|\, u_0 \equiv 0\} \xrightarrow{\cong} L^p([0,T]; L^p(\Sigma, \mathbf{R}^n))$$

and the Fréchet derivative $D\Psi^{F_0,T}(\{f_t\}) : X_{T,p} \xrightarrow{\cong} Y_{T,p}$ of the non-linear operator $\Psi^{F_0,T}$ in (19) are isomorphisms, in any $\{f_t\} \in C^4(\Sigma \times [0,T], \mathbf{R}^n) \cap W_{U_0,T,p}$. $\qquad\square$

In order to justify the proof of the fourth part of Theorem 1, we combine results due to Amann, Duong, Hieber and Simonett in [2, 4, 5, 11], yielding the following general statement.

Proposition 2 *Let Σ be a smooth, compact, orientable surface without boundary, $g \in \Gamma(T^{0,2}\Sigma)$ a C^∞-smooth Riemannian metric on Σ, ∇^g the corresponding Levi-Civita-connection on Σ, $n \in \mathbf{N}$, $p \in (1, \infty)$, and $T > 0$ arbitrarily fixed, and let $\psi : O \xrightarrow{\cong} \Sigma'$ be a smooth chart of an arbitrary coordinate patch Σ' of Σ, yielding partial derivatives ∂_m, $m = 1, 2$, and the coefficients $g_{ij} := g(\partial_i, \partial_j)$ of the metric g, restricted to Σ'. Moreover, let*

$$\partial_t + L : X_{T,p} \longrightarrow L^p([0,T]; L^p(\Sigma, \mathbf{R}^n))$$

be a linear differential operator of order 4, whose leading operator of fourth order is diagonal, i.e. acts on each component of f separately:

$$(\partial_t + L)(f)(x,t) := \partial_t(f_t)(x) + A_4^{ijkl}(x,t) \, \nabla_{ijkl}^g(f_t)(x)$$
$$+ A_3^{ijk}(x,t) \, \nabla_{ijk}^g(f_t)(x) + A_2^{ij}(x,t) \, \nabla_{ij}^g(f_t)(x)$$
$$+ A_1^i(x,t) \, \nabla_i^g(f_t)(x) + A_0(x,t) \, (f_t)(x), \tag{68}$$

in every pair $(x,t) \in \Sigma' \times [0,T]$, and L has to satisfy the following "structural hypotheses":

(1) *A_4^{ijkl}, A_3^{ijk}, A_2^{ij}, A_1^i, A_0 are the coefficients of contravariant and continuous tensor fields A_4, A_3, A_2, A_1, A_0 on $\Sigma \times [0,T]$ of degrees $4, 3, \ldots, 0$. Moreover, the tensor A_4 has to be the square $E \otimes E$ of a contravariant real-valued symmetric tensor field E of order 2, i.e. $A_4^{ijkl}(x,t) = E^{ij}(x,t) E^{kl}(x,t)$ with $E^{ij}(x,t) = E^{ji}(x,t) \in \mathbf{R}$, for every $(x,t) \in \Sigma' \times [0,T]$.*

(2) *The tensor field E is required to be positive definite on $\Sigma \times [0,T]$, i.e. there has to be a number $\Lambda \geq 1$ such that there holds:*

$$E^{ij}(x,t) \xi_i \xi_j \geq \Lambda^{-1/2} g^{ij}(x) \xi_i \xi_j \tag{69}$$

for any vector $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$ and for any pair $(x,t) \in \Sigma' \times [0,T]$.

(3) *There holds $\sup_{\Sigma \times [0,T]} |A_r| \leq \Lambda$, for $r = 0, 1, 2, 3, 4$.*

Then the following three statements hold:

(1) *For every $\theta \in (0, \pi/2)$ there are numbers $\omega = \omega(\Sigma, \Lambda, p, \theta, n) > 0$ and $M = M(\Sigma, \Lambda, p, \theta, n) \geq 1$, such that for each fixed $t^* \in [0,T]$ the elliptic operator*

$$\omega \, \mathrm{Id}_{L^p(\Sigma, \mathbf{R}^n)} + L_{t^*} := \omega \, \mathrm{Id}_{L^p(\Sigma, \mathbf{R}^n)} + \left(A_4^{ijkl}(\cdot, t^*) \, \nabla_{ijkl}^g + A_3^{ijk}(\cdot, t^*) \, \nabla_{ijk}^g \right.$$
$$\left. + A_2^{ij}(\cdot, t^*) \, \nabla_{ij}^g + A_1^i(\cdot, t^*) \, \nabla_i^g + A_0(\cdot, t^*) \right) \tag{70}$$

in $L^p(\Sigma, \mathbf{R}^n)$ maps $W^{4,p}(\Sigma, \mathbf{R}^n)$ isomorphically onto $L^p(\Sigma, \mathbf{R}^n)$ and is of type $(M, \pi - \theta)$ in the sense of formula (1.1) in [5] respectively of formula (4.7.8) in Chapter III of [2].

(2) *The operator*

$$(\gamma_0, \partial_t + L) : X_{T,p} \xrightarrow{\cong} Y_{T,p} \equiv W^{4-\frac{4}{p},p}(\Sigma, \mathbf{R}^n) \times L^p([0, T], L^p(\Sigma, \mathbf{R}^n))$$

is a topological isomorphism, where $\gamma_0 : X_{T,p} \longrightarrow W^{4-\frac{4}{p},p}(\Sigma, \mathbf{R}^n)$ *is the trace operator at time* $t = 0$, *mapping* $\{u_t\} \mapsto u_0$; *see also formula (18).*

(3) *The parabolic linear operator*

$$\partial_t + L : X_T^0 := \{\{u_t\} \in X_{T,p} \mid \gamma_0(\{u_t\}) \equiv 0\} \xrightarrow{\cong} L^p([0, T], L^p(\Sigma, \mathbf{R}^n)) \quad (71)$$

is a topological isomorphism.

Proof We fix an arbitrary uniformly parabolic operator L as in (68) satisfying the structural hypotheses (1), (2) and (3) of this proposition, and we choose a finite atlas $\{(\Omega_\kappa, \psi_\kappa)\}_{\kappa=1,\dots,K}$ of smooth charts for Σ as in Section 10 of [5]. Since the coefficients of L are assumed to be continuous, we can consider—instead of the original operator $\partial_t + L$ in (68)—for every fixed $t^* \in [0, T]$ the linear, uniformly elliptic differential operator

$$\eta \mapsto L_{t^*}(\eta) := \big(A_4^{ijkl}(\cdot, t^*) \nabla_{ijkl}^g + A_3^{ijk}(\cdot, t^*) \nabla_{ijk}^g + $$
$$+ A_2^{ij}(\cdot, t^*) \nabla_{ij}^g + A_1^i(\cdot, t^*) \nabla_i^g + A_0(\cdot, t^*) \big)(\eta) \quad (72)$$

in $L^p(\Sigma, \mathbf{R}^n)$ with domain $W^{4,p}(\Sigma, \mathbf{R}^n)$. Now, by the second assumption of this proposition the highest order term

$$W^{4,p}(\Sigma, \mathbf{R}^n) \ni \eta \mapsto \mathcal{A}_{t^*}(\eta) := E^{ij}(\cdot, t^*) E^{kl}(\cdot, t^*) \nabla_{ijkl}^g(\eta) \in L^p(\Sigma, \mathbf{R}^n) \quad (73)$$

of the linear operator L_{t^*} in line (72) is uniformly elliptic and diagonal, in every fixed $t^* \in [0, T]$. Hence, the spectrum of its "symbol", which is given by the family of linear maps

$$(\mathcal{A}_{t^*})_\sharp(x, \xi) := \mathrm{Id}_{\mathbf{C}^n} \cdot \Big(\sum_{i,j,k,l} E^{ij}(x, t^*) E^{kl}(x, t^*) \xi_i \xi_j \xi_k \xi_l \Big) : \mathbf{C}^n \longrightarrow \mathbf{C}^n, \quad (74)$$

for $x \in \Omega_\kappa$, for every $\xi \in \mathbf{R}^2$ with $|\xi| = 1$ and $\kappa \in \{1, \dots, K\}$, is precisely given by:

$$\sigma((\mathcal{A}_{t^*})_\sharp(x, \xi)) = \Big\{ \sum_{i,j,k,l} E^{ij}(x, t^*) E^{kl}(x, t^*) \xi_i \xi_j \xi_k \xi_l \Big\}$$

for every $x \in \Omega_\kappa$, for $\xi \in \mathbf{R}^2$ with $|\xi| = 1$ and for $\kappa \in \{1, \dots, K\}$, and these are real positive numbers, being uniformly bounded from below in terms of Λ^{-1}, on account of the required uniform positive definiteness of the tensor field E on $\Sigma \times [0, T]$ by assumption (69). In particular, the "principal symbol" of the operator L_{t^*} in line (72) is "parameter elliptic" in every fixed point $x \in \Sigma$ with "angle of ellipticity"$= 0$ in the sense of Section 10 in [5] respectively Section 7 in [11], for every fixed $t^* \in [0, T]$; see here also Section 5.1 in [10]. Since the coefficients of the linear operators $\{L_t\}_{t \in [0,T]}$ are also assumed to be uniformly bounded in $L^\infty(\Sigma)$ in terms of Λ, we can thus infer from Theorem 10.1 in [5] and from Theorem 7.1 in [11], that for any fixed $\theta \in (0, \pi/2)$ there are numbers $\omega = \omega(\Sigma, \Lambda, p, \theta, n) > 0$ and $M = M(\Sigma, \Lambda, p, \theta, n) \geq 1$, such that

(1) the linear operator $\omega \, \mathrm{Id}_{L^p(\Sigma, \mathbf{R}^n)} + L_{t^*}$ in $L^p(\Sigma, \mathbf{R}^n)$ yields an isomorphism

$$\omega \, \mathrm{Id}_{L^p(\Sigma, \mathbf{R}^n)} + L_{t^*} : W^{4,p}(\Sigma, \mathbf{R}^n) \xrightarrow{\cong} L^p(\Sigma, \mathbf{R}^n),$$

(2) the linear operator $\omega \operatorname{Id}_{L^p(\Sigma, \mathbf{R}^n)} + L_{t^*}$ is of positive type $(M, \pi - \theta)$, which means precisely that the inverse

$$\left((\omega + \lambda) \operatorname{Id}_{L^p(\Sigma, \mathbf{R}^n)} + L_{t^*}\right)^{-1} : L^p(\Sigma, \mathbf{R}^n) \longrightarrow L^p(\Sigma, \mathbf{R}^n)$$

exists and is bounded for every complex number λ, which is contained in the sector

$$S_\vartheta := \{\, \lambda \in \mathbf{C}^* \mid |\arg(\lambda)| \leq \vartheta \,\} \cup \{0\}$$

with apex in 0 and with half opening angle $\vartheta := \pi - \theta \in \left(\frac{\pi}{2}, \pi\right)$, and that there holds the estimate

$$\left\| \left((\omega + \lambda) \operatorname{Id}_{L^p(\Sigma, \mathbf{R}^n)} + L_{t^*}\right)^{-1} \right\|_{\mathcal{L}(L^p(\Sigma, \mathbf{R}^n), L^p(\Sigma, \mathbf{R}^n))} \leq \frac{M}{1 + |\lambda|} \tag{75}$$

for every $\lambda \in S_\vartheta$, and for every fixed $t^* \in [0, T]$,

(3) the linear operator $\omega \operatorname{Id}_{L^p(\Sigma, \mathbf{R}^n)} + L_{t^*}$ admits a bounded $\mathcal{H}_\infty$-calculus in $L^p(\Sigma, \mathbf{R}^n)$ with angle θ, which means precisely that for any bounded holomorphic function $f : \mathring{S}_\theta \longrightarrow \mathbf{C}$ there holds

$$\| f(\omega \operatorname{Id}_{L^p(\Sigma, \mathbf{R}^n)} + L_{t^*}) \|_{\mathcal{L}(L^p(\Sigma, \mathbf{R}^n), L^p(\Sigma, \mathbf{R}^n))} \leq M \ \| f \|_{L^\infty(\mathring{S}_\theta)},$$

in the sense of Sections 2 and 7 in [11], again for some sufficiently large constant $M = M(\Sigma, \Lambda, p, \theta, n) > 0$. By formula (2.4) in [5] respectively Corollary 7.2 in [11] the third conclusion implies particularly, that

(4) the operator $\omega \operatorname{Id}_{L^p(\Sigma, \mathbf{R}^n)} + L_{t^*}$ has bounded imaginary powers in $L^p(\Sigma, \mathbf{R}^n)$ with angle θ, i.e. that there holds for every fixed $t^* \in [0, T]$:

$$\| (\omega \operatorname{Id}_{L^p(\Sigma, \mathbf{R}^n)} + L_{t^*})^{i s} \|_{\mathcal{L}(L^p(\Sigma, \mathbf{R}^n), L^p(\Sigma, \mathbf{R}^n))} \leq M \, e^{\theta s},$$

for every $s \in \mathbf{R}$, and for every fixed $\theta \in (0, \pi/2)$.

On account of our assumption, that the coefficients $A_4^{ijkl}, \ldots, A_0$ of the family of linear operators $L = \{L_t\}_{t \in [0, T]}$ are continuous on $\Sigma \times [0, T]$ there holds trivially

$$[t \mapsto L_t] \in C^0([0, T], \mathcal{L}(W^{4,p}(\Sigma, \mathbf{R}^n), L^p(\Sigma, \mathbf{R}^n))).$$

Hence, combining the above four statements (1)–(4) with the facts that $L^p(\Sigma, \mathbf{R}^n)$ is a UMD-space by Theorem 4.5.2 in Chapter III of [2], and that the trace operator γ_0 maps $X_{T,p}$ continuously onto $W^{4-\frac{4}{p}, p}(\Sigma, \mathbf{R}^n)$ by formula (18), we can deduce the second and the third assertion of this proposition from a combination of Theorem 4.10.8 in Chapter III of [2] with Theorem 7.1 in [4]. $\qquad \square$

Proof of Theorem 2 (1) Formulae (20) respectively (24) motivate us, to consider the quasi-linear parabolic Cauchy problem

$$\partial_t f_t = K^{F_0}(f_t).(f_t) \quad \text{on} \quad \Sigma \times [0, T], \quad \text{with} \quad f_0 = u_0 \quad \text{on} \quad \Sigma \tag{76}$$

in the space $X_{T,p}$, instead of Cauchy problem (23). Now, on account of embedding (29) we can choose some small open neighbourhood V_{U_0} of the umbilic-free immersion U_0 in $W^{4-\frac{4}{p}, p}(\Sigma, \mathbf{R}^n)$ which is still contained in $\operatorname{Imm}_{\mathrm{uf}}(\Sigma, \mathbf{R}^n)$, and we can infer from Lemma 1, that the non-linear operator K^{F_0} in formula (76) is of class $C^{0,1}(V_{U_0}, \mathcal{L}(W^{4,p}(\Sigma, \mathbf{R}^n), L^p(\Sigma, \mathbf{R}^n)))$, and that especially the linear differential operator $-K^{F_0}(U_0)$ is of maximal L^p-regularity on $[0, T]$ w.r.t. $(W^{4,p}(\Sigma, \mathbf{R}^n), L^p(\Sigma, \mathbf{R}^n))$. We may therefore apply Theorem 2.1 - here with interpolation parameter $\mu = 1$—and

Remark 2.3 in [18] to Cauchy problem (76), and we obtain from this theorem some small positive time $T = T(U_0) > 0$ and an $\varepsilon = \varepsilon(U_0) > 0$, such that for any immersion $u_0 \in V_{U_0}$ with

$$\| u_0 - U_0 \|_{W^{4-\frac{4}{p},p}(\Sigma,\mathbf{R}^n)} < \varepsilon \tag{77}$$

the quasilinear parabolic Cauchy problem (76) has exactly one solution $[t \mapsto u(t, u_0)]$ in $X_{T(U_0),p}$. Now, on account of the evolution equation in (76) and on account of embedding (37) the unique short-time solution $[t \mapsto u(t, u_0)]$ of Cauchy problem (76) in $X_{T(U_0),p}$ has to consist of umbilic-free immersions only. Hence, formula (24) tells us, that the function $[t \mapsto u(t, u_0)]$ also solves the original Cauchy problem (23) in $X_{T(U_0),p}$, for every initial immersion $u_0 \in V_{U_0}$ satisfying condition (77). Now vice versa, any short-time solution of the original Cauchy problem (23) in $X_{T^*(u_0),p}$, for some sufficiently small $T^*(u_0) > 0$, has to consist of only umbilic-free immersions—for the same reason as above—and therefore any such short-time solution is a short-time solution of Cauchy problem (76) in $X_{T^*(u_0),p}$ as well, for any fixed initial immersion $u_0 \in V_{U_0}$ satisfying condition (77). Therefore, the above unique short-time solution $[t \mapsto u(t, u_0)]$ of Cauchy problem (76) in $X_{T(U_0),p}$ also solves Cauchy problem (23) uniquely in $X_{T(U_0),p}$, for every initial immersion $u_0 \in V_{U_0}$ satisfying condition (77) with $\varepsilon = \varepsilon(U_0) > 0$ sufficiently small, which proves the assertion.

(2) The proof of the second part of the theorem now follows immediately from the result of the first part of this theorem, combined with Lemma 1 of this article and Corollary 2.2 and Remark 2.3 of [18].

$\square$

Now we finally arrive at the proof of Theorem 3:

(1) Theorem 1 in [13] yields the existence of a C^∞-smooth short-time solution $\{\mathcal{P}(t, 0, U_0)\}_{t \in [0,\tau]}$ of the MIWF-Eq. (1) starting in U_0 for some $\tau > 0$, and its proof also yields a smooth family of smooth diffeomorphisms $\phi_t^{U_0,F_0} : \Sigma \longrightarrow \Sigma$ with $\phi_0^{U_0,F_0} = \mathrm{id}_\Sigma$, for $t \in [0, \tau]$, such that the composition $\{\mathcal{P}(t, 0, U_0) \circ \phi_t^{U_0,F_0}\}_{t \in [0,\tau]}$ is a C^∞-smooth solution of the modified evolution Eq. (13), i.e. it solves Cauchy problem (23) with $u_0 = U_0$ in the class $C^\infty(\Sigma \times [0, \tau], \mathbf{R}^n)$. This short-time solution can be continued to a unique maximal flow line of evolution Eq. (13) starting in U_0, in the sense of Definition 2, (d). On the other hand this maximal flow line also solves problem (23) with $u_0 = U_0$ in the space $X_{T^*,p}$, for any $T^* \in (0, T_{\max}(U_0))$, and thus coincides with the unique maximal solution $\{\mathcal{P}^*(t, 0, U_0)\}_{t \in [0, t^+(U_0))}$ of Cauchy problem (23) from the second part of Theorem 2 restricted to $\Sigma \times [0, T^*]$, which implies especially that $T_{\max}(U_0) \leq t^+(U_0)$. Now we show that actually $T_{\max}(U_0) = t^+(U_0)$ holds. To this end, we choose some $T \in (0, t^+(U_0))$, and we recall that the unique solution $\{\mathcal{P}^*(t, 0, U_0)\}_{t \in [0,T]}$ of problem (23) in $X_{T,p}$ has to be of class $C^{0,\alpha}([0, T]; C^{2,\alpha}(\Sigma, \mathbf{R}^n))$, on account of embedding (66), for some small $\alpha = \alpha(p) \in (0, 1)$. Therefore Lemma 1 guarantees us, that evolution Eq. (13) has the quasilinear structure appearing in formulae (24) or (76) along its solution $\{\mathcal{P}^*(t, 0, U_0)\}_{t \in [0,T]}$, and this fact inspires us, to consider the linear operator

$$L^{F_0,U_0} := \partial_t - K^{F_0}(\mathcal{P}^*(\cdot, 0, U_0)) :$$
$$C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma \times [0, T], \mathbf{R}^n) \longrightarrow C^{\alpha,\frac{\alpha}{4}}(\Sigma \times [0, T], \mathbf{R}^n), \tag{78}$$

which is uniformly parabolic and has coefficients of class $C^{\alpha, \frac{\alpha}{4}}(\Sigma \times [0, T], \mathbf{R}^n)$, thus satisfies all conditions of Corollary 3 in [13] and yields therefore an isomorphism

$$L^{F_0, U_0} : \{\{\eta_t\} \in C^{4+\alpha, 1+\frac{\alpha}{4}}(\Sigma \times [0, T], \mathbf{R}^n) \,|\, \eta_0 = U_0\} \overset{\cong}{\longrightarrow} C^{\alpha, \frac{\alpha}{4}}(\Sigma \times [0, T], \mathbf{R}^n)$$

between affine linear spaces, again with the exponent α from embedding (66). Moreover, there holds the linear parabolic differential Eq.

$$\begin{aligned} L^{F_0, U_0}(\mathcal{P}^*(t, 0, U_0)) &= \partial_t(\mathcal{P}^*(t, 0, U_0)) - K^{F_0}(\mathcal{P}^*(t, 0, U_0)).(\mathcal{P}^*(t, 0, U_0)) \\ &= 0 \quad \text{for a.e. } t \in [0, T] \end{aligned} \tag{79}$$

on account of Eqs. (23), (24) and (68), and also $\mathcal{P}^*(0, 0, U_0) = U_0$. Hence, since obviously $C^{4+\alpha, 1+\frac{\alpha}{4}}(\Sigma \times [0, T], \mathbf{R}^n)$ is a Banach subspace of $X_{T,p}$, we can therefore conclude as in the proof of Theorem 3 in [13], that the family of immersions $\{\mathcal{P}^*(t, 0, U_0)\}_{t \in [0, T]}$ solves Cauchy problem (23) not only in $X_{T,p}$ but actually in the smaller Banach space $C^{4+\alpha, 1+\frac{\alpha}{4}}(\Sigma \times [0, T], \mathbf{R}^n)$. Now we can combine this new information again with the linear parabolic Eq. (79) and with the smoothness of U_0, in order to infer by means of Proposition 3 in [13] higher regularity of class $C^{4+k+\alpha, 1+\frac{k+\alpha}{4}}(\Sigma \times [0, T], \mathbf{R}^n)$, inductively for every $k \in \mathbf{N}_0$, just as in the proof of Theorem 3 in [13]. Hence, the unique solution $\{\mathcal{P}^*(t, 0, U_0)\}_{t \in [0, T]}$ of Cauchy problem (23) in $X_{T,p}$ has turned out to be of class $C^{\infty}(\Sigma \times [0, T], \mathbf{R}^n)$, for every $T \in (0, t^+(U_0))$, proving that the maximal solution $\{\mathcal{P}^*(t, 0, U_0)\}_{t \in [0, t^+(U_0))}$ of problem (23) from the second part of Theorem 2 is a maximal flow line of evolution Eq. (13) in the sense of Definition 2, (d), and that also $T_{\max}(U_0) \geq t^+(U_0)$ has to hold.

(2) Now we consider the unique maximal solution $\{\mathcal{P}^*(t, 0, U_0)\}_{t \in [0, T_{\max}(U_0))}$ of problem (23) from the first part of this theorem—starting in a smooth and umbilic-free immersion U_0 - and we assume here additionally that Σ is a complex compact torus and that the immersion F_0 is not only smooth but real analytic on Σ w.r.t. the prescribed complex structure. On account of the complicated, quasilinear structure of the differential operator $[f \mapsto \mathcal{M}_{F_0}(f)]$ in formula (27) respectively (38), we cannot directly apply Simonett's and Shao's regularity theory for linear parabolic equations on uniformly regular Riemannian manifolds [28, 29] to smooth solutions of our Eq. (23), in order to prove their real analyticity, locally in space and time. But we still succeed here as Shao did in [27] or in Section 4 of [28] - discussing the real analyticity of the Willmore flow respectively of the Ricci-DeTurck flow—localizing the quasilinear differential operator in (24) respectively (76) in terms of a particular partition of unity, then adopting the argument of the proof of Proposition 3.7 in [28] via Theorem 4.2 in [12] for each localized differential operator, and then conclude by means of Proposition 2.5 and Lemma 3.8 in [28] and Proposition 6.4 in [29]. Moreover, we will have to work here in slightly different function spaces, namely in little Hölder spaces $h^s(\Sigma, \mathbf{R}^n)$ instead of L^p-spaces, thus exactly following Shao's work in [27] or Sections 4–6 of his paper [28]. Our argument therefore clearly deviates from the proof of Theorem 3.14 in Appendix A.3 in [30], discussing the real analyticity of short-time solutions of the elastic energy flow in $\mathbf{R}^{m+1}$. Henceforth, we will work with a finite conformal atlas $\mathcal{A} := \{(O_j, \varphi_j)\}_{j=1,\ldots,N}$ of Σ with $\varphi_j : O_j \overset{\cong}{\longrightarrow} B_1^2(0)$, $j = 1, \ldots, N$, with invers maps $\psi_j := (\varphi_j)^{-1}$, and we shall consider some arbitrary point $p \in \Sigma$ and some coordinate patch (O_{j_p}, φ_{j_p}) of $\mathcal{A}$ containing p. As explained in Section 3.1 in [28], we can add a conformal chart $\varphi_\iota : O_\iota \overset{\cong}{\longrightarrow} B_1^2(0)$ to the atlas $\mathcal{A}$, satisfying $\varphi_\iota(p) = 0$, resulting in another conformal atlas $\tilde{\mathcal{A}} = \mathcal{A} \cup (O_\iota, \varphi_\iota)$. Furthermore, we introduce as in Section 3.1 of [28] or in Section 4 of [29] open balls $B_i := B_{i\varepsilon_0}^2(0)$, with $i = 1, \ldots, 5$

and $0 < \varepsilon_0 < \frac{1}{5}$, cut-off functions $\chi \in C_c^\infty(B_2, [0, 1])$ and $\varsigma \in C_c^\infty(B_5, [0, 1])$ with $\chi \equiv 1$ on B_1 and $\varsigma \equiv 1$ on B_4, the truncated shift

$$\theta_\mu(y) := y + \mu\,\chi(y), \quad \text{for} \quad y \in B_1^2(0), \tag{80}$$

of points y in $B_1^2(0)$, with $\mu \in B_r^2(0)$ for some sufficiently small $r \in (0, \varepsilon_0)$—see here Section 2 in [12]—and the induced truncated shift about p on Σ:

$$\Theta_\mu(x) := \begin{cases} (\varphi_\iota)^{-1}(\theta_\mu(\varphi_\iota(x)) : x \in O_\iota \\ \qquad\qquad\qquad\quad x : x \notin O_\iota. \end{cases}$$

By Proposition 2.4 in [12]—see also Lemma 3.1 in [28]—the truncated shift θ_μ induces topological automorphisms of $h^s(U, \mathbf{R}^n)$ for any open subset $U \subset \mathbf{R}^2$ containing the closed ball $\overline{B_3} = B_{3\varepsilon_0}^2(0)$ and for any Hölder exponent $s \in \mathbf{R}_+ \setminus \mathbf{N}_0$, simply in terms of composition:

$$u \mapsto (\theta_\mu)^*(u) = u \circ \theta_\mu$$

with invers

$$u \mapsto (\theta^\mu)_*(u) = u \circ (\theta_\mu)^{-1},$$

for functions $u : U \longrightarrow \mathbf{R}^n$. Similarly, for functions $u : \Sigma \longrightarrow \mathbf{R}^n$ the truncated shift Θ_μ induces a transformation

$$u \mapsto \Theta_\mu^*(u) := \varphi_\iota^* \theta_\mu^* \psi_\iota^*(\varsigma u) + (1 - \varsigma)\,u,$$

where we identified ς with its pullback $\varphi_\iota^* \varsigma$ from $B_1^2(0)$ onto $O_\iota \subset \Sigma$, and this transformation is again an automorphism of $h^s(\Sigma, \mathbf{R}^n)$ for any Hölder exponent $s \in \mathbf{R}_+ \setminus \mathbf{N}_0$, on account of Proposition 3.3 in [28]. Now, as in Section 3.4 in [28] we choose a time $T \in (0, T_{\max}(U_0))$, another time $t_0 \in (0, T)$, a small open interval $B_{3\varepsilon_0}(t_0) \subset (0, T)$ about t_0 and another bump function $\xi \in C_c^\infty((0, T), \mathbf{R})$ with

$$\text{supp}(\xi) \subset (t_0 - 2\varepsilon_0, t_0 + 2\varepsilon_0) \quad \text{and} \quad \xi \equiv 1 \quad \text{on} \quad [t_0 - \varepsilon_0, t_0 + \varepsilon_0],$$

yielding the time-dependent time-translations

$$\varrho_\lambda(t) := t + \lambda\,\xi(t), \quad \text{for} \ t \in [0, T], \tag{81}$$

for fixed parameters $\lambda \in \mathbf{R}$. Furthermore, following the procedure in Section 3.4 of [28], we consider the time-dependent shift $\theta_{\xi(t)\cdot\mu}(y) = y + \chi(y)\,\xi(t)\,\mu$, inducing

$$\tilde{T}_\mu(u)(y, t) := (\theta_{\xi(t)\cdot\mu})^*(u)(y, t), \quad \text{for} \ (y, t) \in U \times [0, T],$$

for functions $u : U \times [0, T] \longrightarrow \mathbf{R}^n$, and analogously

$$T_\mu(u)(x, t) := (\Theta_{\xi(t)\cdot\mu})^*(u)(x, t), \quad \text{for} \ (x, t) \in \Sigma \times [0, T], \tag{82}$$

for functions $u : \Sigma \times [0, T] \longrightarrow \mathbf{R}^n$ and for $|\mu| < r$, with r as in (80). Finally, together with the time-shift in (81) we consider the local modifications

$$\begin{aligned} u_{\lambda,\mu}(x, t) &:= (\Phi_{\lambda,\mu}^* u)(x, t) := (T_\mu \circ \varrho_\lambda^*)(u)(x, t) \\ &= u(x + \xi(t)\,\chi(x)\,\mu, t + \xi(t)\,\lambda), \ \text{for} \ (x, t) \in \Sigma \times [0, T], \end{aligned} \tag{83}$$

of any given function $u : \Sigma \times [0, T] \longrightarrow \mathbf{R}^n$, with $|\lambda|^2 + |\mu|^2 < r^2$. By Proposition 3.10 in [28] and by the open mapping theorem the local modifications in (83) induce topological automorphisms of the Banach space

$$\mathbf{E}_1 := C^0([0, T]; h^{4+\beta}(\Sigma, \mathbf{R}^n)) \cap C^1([0, T]; h^\beta(\Sigma, \mathbf{R}^n)) \tag{84}$$

for some arbitrary $\beta \in (0, 1)$, provided r in (80) and (83) is chosen sufficiently small. We note here for later use, that there holds

$$u_{\lambda,\mu}(x, 0) = u(x + \xi(0) \chi(x) \mu, \xi(0) \lambda) = u(x, 0), \quad \text{for} \quad x \in \Sigma,$$

for any fixed function $u : \Sigma \times [0, T] \longrightarrow \mathbf{R}^n$. Moreover, Proposition 3.10 in [28] yields a certain operator

$$[(\lambda, \mu) \mapsto B_{(\lambda,\mu)}] \in C^\omega(B_r^3(0), C^0([0, T], \mathcal{L}(h^{4+\beta}(\Sigma, \mathbf{R}^n), h^\beta(\Sigma, \mathbf{R}^n)))) \tag{85}$$

with $B_{(\lambda,0)} \equiv 0$ such that

$$\partial_t u_{\lambda,\mu}(x, t) = (1 + \xi'(t) \lambda) (\Phi_{\lambda,\mu})^*(\partial_t u_t)(x) + B_{(\lambda,\mu)}(u_{\lambda,\mu})(x, t) \tag{86}$$

holds $\forall (x, t) \in \Sigma \times [0, T]$ and for any fixed family of functions $\{u_t\} \in \mathbf{E}_1$. Now we apply formulae (83) and (86) directly to the flow line $\{\mathcal{P}^*(t, 0, U_0)\}_{t \in [0,T]}$—which we shall henceforth abbreviate by $\{u_t^*\}_{t \in [0,T]}$—and we shall use the important fact, that the right hand side of evolution Eq. (13) is given by the time-independent, quasilinear operator in formula (27) respectively (28) along smooth flow lines, on account of the first part of Theorem 1. We see therefore, that the function $u_{\lambda,\mu}^* \equiv (\Phi_{\lambda,\mu})^*(u^*)$ satisfies $u_{\lambda,\mu}^*(\cdot, 0) = u^*(\cdot, 0) = U_0$ on Σ and the equation

$$\begin{aligned}
\partial_t u_{\lambda,\mu}^*(x, t) &= (1 + \xi'(t) \lambda) (\Phi_{\lambda,\mu})^*(\partial_t u_t^*)(x) + B_{(\lambda,\mu)}(u_{\lambda,\mu}^*)(x, t) \\
&= (1 + \xi'(t) \lambda) T_\mu \circ \varrho_\lambda^*(K^{F_0}(u_t^*).(u_t^*))(x) + B_{(\lambda,\mu)}(u_{\lambda,\mu}^*)(x, t) \\
&= (1 + \xi'(t) \lambda) T_\mu \big(K^{F_0}(\varrho_\lambda^* u_t^*).(\varrho_\lambda^* u_t^*)\big)(x) + B_{(\lambda,\mu)}(u_{\lambda,\mu}^*)(x, t) \\
&= (1 + \xi'(t) \lambda) T_\mu \big(K^{F_0}((T_\mu)^{-1}(u_{\lambda,\mu}^*)).((T_\mu)^{-1}(u_{\lambda,\mu}^*))\big)(x, t) + B_{(\lambda,\mu)}(u_{\lambda,\mu}^*)(x, t) \\
&= (1 + \xi'(t) \lambda) T_\mu \circ \mathcal{M}_{F_0} \circ (T_\mu)^{-1}(u_{\lambda,\mu}^*)(x, t) + B_{(\lambda,\mu)}(u_{\lambda,\mu}^*)(x, t)
\end{aligned} \tag{87}$$

for $(x, t) \in \Sigma \times [0, T]$, where we have used the time-independence of the operator K^{F_0} from formula (28) in the third line. Following now the notation of Section 4 in [27] or Sections 4–6 in [28], we shall abbreviate $E_0 := h^\beta(\Sigma, \mathbf{R}^n)$ and $E_1 := h^{4+\beta}(\Sigma, \mathbf{R}^n)$, and moreover

$$\mathbf{E}_0 := C^0([0, T]; h^\beta(\Sigma, \mathbf{R}^n)),$$

for the same $\beta \in (0, 1)$ as in the definition of the space $\mathbf{E}_1$ in (84). Hence, in this proof we shall exchange the parabolic L^p-spaces $L^p([0, T]; L^p(\Sigma, \mathbf{R}^n))$ and $X_{T,p}$ from (17) by the smaller function spaces $\mathbf{E}_0$ and $\mathbf{E}_1$, in order to argue correctly throughout this proof. Since the considered flow line $\{u_t^*\}_{t \in [0,T]}$ of Eq. (13) is smooth and thus contained in the Banach space $\mathbf{E}_1$, we can choose a small open neighborhood $V_{U_0,T,p}$ about $\{u_t^*\}_{t \in [0,T]}$ in $\mathbf{E}_1$, and we shall introduce the non-linear operator $\Psi : V_{U_0,T,p} \times B_r^3(0) \longrightarrow E_1 \times \mathbf{E}_0$, sending the triple $(\{f_t\}, (\lambda, \mu))$ to the pair

$$\begin{pmatrix} \gamma_0(\{f_t\}) - U_0 \\ \partial_t f_t - (1 + \xi'(t) \lambda) T_\mu \circ \mathcal{M}_{F_0} \circ (T_\mu)^{-1}(\{f_t\}) - B_{(\lambda,\mu)}(\{f_t\}) \end{pmatrix}.$$

We should note here for later use, that statement (87) is equivalent to:

$$\Psi(u^*_{\lambda,\mu}, (\lambda, \mu)) = 0 \quad \text{in} \quad E_1 \times \mathbf{E}_0, \tag{88}$$

for every pair $(\lambda, \mu) \in B^3_r(0)$. Now we shall prove exactly as in Section 4 of [27], that this operator is real analytic, provided $V_{U_0,T,p}$ and $r > 0$ are chosen sufficiently small. We can firstly infer from (85) and Proposition 2.5 in [28] - exactly as on p. 12 in [27]—that there holds:

$$[(\{f_t\}, (\lambda, \mu)) \mapsto B_{(\lambda,\mu)}(\{f_t\})] \in C^\omega(V_{U_0,T,p} \times B^3_r(0), \mathbf{E}_0). \tag{89}$$

Now we aim to prove, that the technically most challenging operator

$$[(\{f_t\}, \mu) \mapsto (T_\mu \circ \mathcal{M}_{F_0} \circ (T_\mu)^{-1})(\{f_t\})] \tag{90}$$

within the above definition of Ψ is of class $C^\omega(V_{U_0,T,p} \times B^2_r(0), \mathbf{E}_0)$. To this end, we proceed exactly as on pp. 12–13 in [27], making again use of the fact, that the non-linear differential operator $[f \mapsto \mathcal{M}_{F_0}(f)]$ has actually a quasilinear structure $[f \mapsto K^{F_0}(f).(f)]$, which is given by formula (28) respectively (38), i.e. that there holds:

$$\begin{aligned}
\mathcal{M}_{F_0}(f_t)(x) &= (K^{F_0}(f_t).(f_t))(x) \\
&= -\frac{1}{2} \frac{1}{|A^0_{f_t}|^4(x)} g^{ij}_{f_t} g^{kl}_{f_t} \nabla^{F_0}_i \nabla^{F_0}_j \nabla^{F_0}_k \nabla^{F_0}_l (f_t)(x) \\
&\quad + \tilde{\mathcal{N}}^{F_0}(x, D_x f_t(x), D^2_x f_t(x)) \cdot D^3_x f_t(x) \\
&\quad + \tilde{\mathcal{C}}^{F_0}(x, D_x f_t(x), D^2_x f_t(x)) \cdot D^2_x f_t(x) \\
&\quad + \tilde{\mathcal{D}}^{F_0}(x, D_x f_t(x), D^2_x f_t(x)) \cdot D_x f_t(x)
\end{aligned} \tag{91}$$

for $(x, t) \in \Sigma \times [0, T]$ and every $\{f_t\} \in V_{U_0,T,p}$—provided $V_{U_0,T,p}$ is chosen sufficiently small in $\mathbf{E}_1$—where the functions $\tilde{\mathcal{N}}^{F_0}, \tilde{\mathcal{C}}^{F_0}$ and $\tilde{\mathcal{D}}^{F_0}$ in (91) have the same algebraic structure as the functions $\mathcal{N}^{F_0}, \mathcal{C}^{F_0}$ and $\mathcal{D}^{F_0}$ in formulae (28) and (38). In order to correctly deal with the quasilinearity of the decisive factor $[f \mapsto g^{ij}_f g^{kl}_f \nabla^{F_0}_i \nabla^{F_0}_j \nabla^{F_0}_k \nabla^{F_0}_l(f)]$ of the leading differential operator in (91), we have to localize this operator on the particular coordinate patch O_{j_p} containing the point p, which we had chosen above. As explained on p. 68 in [28], we can use a fixed localization system $\{(\pi_j, \zeta_j)\}_{j=1,\ldots,N}$ subordinate to our original conformal atlas $\mathcal{A} = \{(O_j, \varphi_j)\}_{j=1,\ldots,N}$, meeting properties (L1)–(L3) on p. 49 in [28], together with the additional localization function $\zeta_\iota := \varphi^*_\iota(\zeta)$ on the auxiliary coordinate patch O_ι—similarly to the original bump functions $\zeta_j := \varphi^*_j(\zeta)$, which are pullbacks w.r.t. the original charts φ_j of a fixed bump function $\zeta \in C^\infty_c(B^2_1(0), [0, 1])$ appearing in condition (L2) on p. 49 in [28]—in order to define a new, more suitable partition of unity subordinate to our original atlas $\mathcal{A}$ by:

$$\tilde{\pi}_j := \pi^2_j - \tilde{\pi}_{j_p} \pi^2_j, \quad 1 \leq j \leq N, \quad \text{with} \quad \tilde{\pi}_{j_p} := \zeta_{j_p} + \zeta_\iota - \zeta_{j_p} \zeta_\iota.$$

The particular bump function $\pi := \sqrt{\tilde{\pi}_{j_p}}$ has support in the original chart $(O_{j_p}, \varphi_{j_p}) \in \mathcal{A}$ containing the chosen point p, and its construction guarantees that $\psi_\iota(B_4) \subset [\pi = 1]$, provided $\varepsilon_0 > 0$ is chosen appropriately. Now first of all, we obtain the operator identity

$$K^{F_0}(h) = \pi^2 \cdot K^{F_0}(h) + \sum_{j \neq j_p} \tilde{\pi}_j \cdot K^{F_0}(h) \quad \text{in} \quad \mathcal{L}(E_1, E_0), \tag{92}$$

for every fixed $h \in \mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbf{R}^n)$. According to formula (31) we have for any immersion $h \in E_1$:

$$\pi(x) \cdot \left(g_h^{ij}(x)\, g_h^{kl}(x)\, \nabla_i^{F_0} \nabla_j^{F_0} \nabla_k^{F_0} \nabla_l^{F_0}(h)(x) \right)$$

$$= \pi(x) \cdot \left(g_h^{ij}(x)\, g_h^{kl}(x)\, \partial_{ijkl} h(x) + D_3^{ijk}(x, \partial_x h(x), \partial_x^2 h(x)) \cdot \partial_{ijk} h(x) \right.$$

$$\left. + \tilde{D}(x, \partial_x h(x), \partial_x^2 h(x)) \right), \tag{93}$$

for $x \in O_{j_p}$, where $\partial_x h(x) := (\partial_1 h, \partial_2 h)(x)$ and $\partial_x^2 h(x) := (\partial_{11}^2 h, \ldots, \partial_{22}^2 h)(x)$, and where D_3^{ijk} and $\tilde{D}$ are $\mathrm{Mat}_{n,n}(\mathbf{R})$- respectively $\mathbf{R}^n$-valued functions on O_{j_p}, whose components are rational functions of the partial dervatives of h up to second order—as precisely indicated in (93)—and additionally of the partial derivatives of F_0 up to fourth order, which are here real analytic. Using the fixed chart $\psi_{j_p} = (\varphi_{j_p})^{-1} : B_1^2(0) \xrightarrow{\cong} O_{j_p}$, the partial derivatives in (93) correspond to partial derivatives on $B_1^2(0)$, and therefore the differential operator on the right-hand side of (93) is a well-defined quasilinear differential operator on O_{j_p}, which extends smoothly to the zero-operator on entire Σ. Similarly to p. 67 in [28] and to p. 13 in [27] we infer from formulae (31) and (56), that the right-hand side in (93) can be written as a quotient, whose nominator is a sum of finitely many products of locally defined, linear differential operators up to order $o = 4$, i.e. of

$$\left[v \mapsto c_\alpha^o\, \partial^\alpha(v) \right], \quad \text{for functions } v \in C^4(O_{j_p}, \mathbf{R}^n), \tag{94}$$

and whose denominator is exactly given by:

$$\left[v \mapsto \left(\det\left[(\langle \partial_i(v), \partial_j(v)\rangle)_{ij} \right] \right)^2 \right], \quad \text{for functions } v \in C^1(O_{j_p}, \mathbf{R}^n), \tag{95}$$

here applied to the restriction of the immersion $h \in E_1$ to the coordinate patch $O_{j_p} \subset \Sigma$, where $\alpha = (\alpha_1, \alpha_2) \in (\mathbf{N}_0)^2$ with $1 \le o := |\alpha| = \alpha_1 + \alpha_2 \le 4$, and where especially $c_\alpha^4(x) = \pi(x)$ for $|\alpha| = 4$. In view of the proof of Proposition 3.7 in [28], we shall interpret here the operators in (94) as differential operators mapping $h_{\mathrm{cp}}^{4+\beta}(\Sigma, \mathbf{R}^n)$ into $h_{\mathrm{cp}}^\beta(\Sigma, \mathbf{R}^n)$, adopting here the non-standard notation

$$h_{\mathrm{cp}}^s(\Sigma, \mathbf{R}^n) := \{ u \in h^s(\Sigma, \mathbf{R}^n) \,|\, \mathrm{supp}(u) \subseteq \psi_\iota(\overline{B_5}) \}, \quad \text{for } s \in \mathbf{R}_+ \setminus \mathbf{N}_0,$$

from Section 3.1 in [28], and we should note here, that the above coefficients c_α^o, with $o = |\alpha| \in \{1, 2, 3\}$, are smooth real-valued functions on the entire patch O_{j_p} and even real analytic functions on any open subset $U_{j_p} \subset O_{j_p}$ satisfying $U_{j_p} \subset [\pi = 1]$, because they are composed of certain partial derivatives of the components of the real analytic immersion F_0, and they might also contain the smooth bump function π from (93) as a factor. As mentioned at the beginning of Section 3.2 in [28], we should choose here the open set U_{j_p} in such a way, that $\psi_\iota(\overline{B_3}) \subset U_{j_p} \subset \psi_\iota(B_4)$ holds, in view of the fact that $\psi_\iota(B_4)$ is contained in $[\pi = 1]$, by construction. Hence, exactly as in the proof of Proposition 3.7 in [28] we can conclude via Theorem 4.2 in [12] combined with Lemma 3.1 in [28], that the maps

$$\left[\mu \mapsto \Theta_\mu \circ \left(c_\alpha^o\, \partial^\alpha \right) \circ (\Theta_\mu)^{-1} \right] \tag{96}$$

are of class $C^\omega(B_r^2(0), \mathcal{L}(h_{\mathrm{cp}}^{4+\beta}(\Sigma, \mathbf{R}^n), h_{\mathrm{cp}}^\beta(\Sigma, \mathbf{R}^n)))$, for each order $o = |\alpha| \in \{1, 2, 3, 4\}$, provided $0 < r \ll \varepsilon_0$. Now we may apply Lemma 3.8 in [28] respec-

tively Lemma 5.1 in [12] to statement (96)—here with the basic Banach space $\mathcal{L}(h_{\mathrm{cp}}^{4+\beta}(\Sigma, \mathbf{R}^n), h_{\mathrm{cp}}^{\beta}(\Sigma, \mathbf{R}^n))$—and we infer, that the maps

$$\left[\mu \mapsto T_\mu \circ \left(c_\alpha^o \, \partial^\alpha \right) \circ (T_\mu)^{-1} \right] \tag{97}$$

are of class $C^\omega(B_r^2(0), C^0([0, T], \mathcal{L}(h_{\mathrm{cp}}^{4+\beta}(\Sigma, \mathbf{R}^n), h_{\mathrm{cp}}^{\beta}(\Sigma, \mathbf{R}^n))))$, for each order $o = |\alpha| \in \{1, 2, 3, 4\}$, provided $r > 0$ is sufficiently small. See here the definition in (82) and also formula (4.8) in [28]. Since the "cut-off" differential operator in (93) extends trivially to entire Σ and is a rational expression of linear differential operators as pointed out in (94) and (95), we can finally combine the result in (97) via Proposition 6.4 in [29] and Proposition 2.5 in [28], in order to conclude that the map

$$\left[(\{f_t\}, \mu) \mapsto T_\mu \circ \left(\pi \cdot \left(g_{T_\mu^{-1}(\{f_t\})}^{ij} \, g_{T_\mu^{-1}(\{f_t\})}^{kl} \, \nabla_i^{F_0} \nabla_j^{F_0} \nabla_k^{F_0} \nabla_l^{F_0} (T_\mu^{-1}(\{f_t\})) \right) \right) \right] \tag{98}$$

is of class $C^\omega(V_{U_0,T,p} \times B_r^2(0), \mathbf{E}_0)$, where we should keep in mind the fact, that the derivatives of fourth and third order in (93) contribute only affine linearly to (98). On account of formulae (7)–(9) and (34) exactly the same method can be employed, in order to prove that the remaining factor $\frac{1}{2|A_{f_t}^0|^4}$ in (91) gives rise to a map

$$\left[(\{f_t\}, \mu) \mapsto T_\mu \circ \left(\pi \cdot \frac{1}{2\,|A_{(T_\mu)^{-1}(\{f_t\})}^0|^4} \right) \right] \tag{99}$$

of class $C^\omega(V_{U_0,T,p} \times B_r^2(0), C^0([0, T]; h^\beta(\Sigma, \mathbf{R})))$. Now, combining the results in (98) and (99) again via Proposition 2.5 in [28], we infer that the entire leading operator on the right hand side of (91) yields a map

$$\left[(\{f_t\}, \mu) \mapsto T_\mu \circ \left(\pi^2 \cdot \frac{1}{2} \frac{1}{|A_{(T_\mu)^{-1}(\{f_t\})}^0|^4} \, g_{(T_\mu)^{-1}(\{f_t\})}^{ij} \, g_{(T_\mu)^{-1}(\{f_t\})}^{kl} \right.\right.$$
$$\left.\left. \cdot \nabla_i^{F_0} \nabla_j^{F_0} \nabla_k^{F_0} \nabla_l^{F_0} ((T_\mu)^{-1}(\{f_t\})) \right) \right] \tag{100}$$

of regularity class $C^\omega(V_{U_0,T,p} \times B_r^2(0), \mathbf{E}_0)$. Since the remaining three terms in (91) of order < 4 can be handled in exactly the same way, the above method shows that the map

$$\left[(\{f_t\}, \mu) \mapsto \left(T_\mu \circ \left(\pi^2 \cdot \mathcal{M}_{F_0} \right) \circ (T_\mu)^{-1} \right)(\{f_t\}) \right]$$

is of regularity class $C^\omega(V_{U_0,T,p} \times B_r^2(0), \mathbf{E}_0)$. Since each remaining map $[(\{f_t\}, \mu) \mapsto (T_\mu \circ (\tilde{\pi}_j \cdot \mathcal{M}_{F_0}) \circ (T_\mu)^{-1})(\{f_t\})]$, for $j \neq j_p$, according to our decomposition in (92), is trivially of regularity class $C^\omega(V_{U_0,T,p} \times B_r^2(0), \mathbf{E}_0)$ as well, we sum them up and finally infer from $\sum_{j=1}^N \tilde{\pi}_j = 1$ on Σ the correctness of our claim in (90). Together with statement (89) we conclude again via Proposition 2.5 in [28], that the map Ψ is of class $C^\omega(V_{U_0,T,p} \times B_r^3(0), \mathbf{E}_0)$ indeed, provided $V_{U_0,T,p}$ and $r > 0$ are chosen sufficiently small.

Furthermore, one can prove exactly as in the second and third part of Theorem 1, that the quasilinear differential operator

$$\partial_t - \mathcal{M}_{F_0} : V_{U_0,T,p} \subset \mathbf{E}_1 \longrightarrow \mathbf{E}_0$$

is real analytic—here w.r.t. the norms of the Banach spaces $\mathbf{E}_1$ and $\mathbf{E}_0$, differing from the corresponding L^p-spaces—and that its Fréchet-derivative

$$\partial_t - D\mathcal{M}_{F_0}(\{f_t\}) : T_{\{f_t\}} V_{U_0,T,p} = \mathbf{E}_1 \longrightarrow \mathbf{E}_0$$

in any fixed $\{f_t\} \in V_{U_0,T,p}$ is a linear, uniformly parabolic operator of fourth order, concretely given by formula (21) again. Moreover, since the considered flow line $\{u_t^*\}_{t\in[0,T]} \equiv \{u(t, U_0)\}_{t\in[0,T]}$ is C^∞-smooth, one can argue as in the proof of Proposition 2—but now via Theorems 3.3 and 3.7 in [29], here applied simply with interpolation parameter $\gamma = 1$—that for each fixed time $s \in [0, T]$ the linear operator

$$\left(\gamma_0, \partial_t - D\mathcal{M}_{F_0}(u_s^*)\right) : \mathbf{E}_1 \xrightarrow{\cong} E_1 \times \mathbf{E}_0$$

is a topological isomorphism. Now, again on account of formula (21) combined with the smoothness of $\{u_t^*\}_{t\in[0,T]}$, we can infer from Lemma 2.8 (a) in [9]—here again with interpolation parameter $\mu = 1$—that

$$\left(\gamma_0, \partial_t - D\mathcal{M}_{F_0}(\{u_t^*\})\right) : \mathbf{E}_1 \xrightarrow{\cong} E_1 \times \mathbf{E}_0 \tag{101}$$

is an isomorphism as well, similarly to the statement of the fourth part of Theorem 1. On the other hand, the Fréchet-derivative of the real analytic map $\Psi : V_{U_0,T,p} \times B_r^3(0) \longrightarrow E_1 \times \mathbf{E}_0$ w.r.t. its first argument in the special point $(\{f_t\}, (\mu, \lambda)) = (\{u_t^*\}, (0, 0))$ is exactly the linear, isomorphic operator in (101), due to $B_{(0,0)} \equiv 0$. Since we also know that $\Psi(u_{\lambda,\mu}^*, (\lambda, \mu)) = 0$ for all $(\lambda, \mu) \in B_r^3(0)$ by (88), the Implicit Function Theorem for real analytic operators [[34], Theorem 4.B and Corollary 4.23] yields some small open, non-empty ball $B_{\epsilon_0}^3(0) \subset B_r^3(0)$ and a unique real analytic map $h : B_{\epsilon_0}^3(0) \longrightarrow V_{U_0,T,p}$, such that $h(\lambda, \mu) = u_{\lambda,\mu}^*$ for $(\lambda, \mu) \in B_{\epsilon_0}^3(0)$, implying especially that $(\lambda, \mu) \mapsto u_{\lambda,\mu}^*$ is a real analytic function from $B_{\epsilon_0}^3(0)$ into $\mathbf{E}_1$. Hence, Theorem 1.1 in [28] respectively Theorem 4.1 in [29] finally yield the asserted real analyticity of the flow line $\{u_t^*\}_{t\in[0,T]}$ about both any point $p \in \Sigma$ and any time $t_0 \in (0, T)$, and for every $T \in (0, T_{\max}(U_0))$.

$\square$

3 Main theorems 4 and 5 and their proofs

In this section we will prove the main results of this article. We shall start with a further investigation of the evolution operator of the "DeTurck modification" (13) of the MIWF (1) and of its linearization. For any fixed immersion $F_0 \in \mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbf{R}^n) \cap C^\infty(\Sigma, \mathbf{R}^n)$ and for any fixed $p \in (3, \infty)$ we will denote by $B_\rho^{4-\frac{4}{p},p}(F_0)$ the open ball of radius $\rho > 0$ about F_0 in the space of traces $W^{4-\frac{4}{p},p}(\Sigma, \mathbf{R}^n)$ from line (18), and we will abbreviate X_T for the basic Banach space $X_{T,p}$ from line (17), and similarly $Y_T := Y_{T,p}$.

Theorem 4 *Suppose that Σ is a smooth compact torus and that F_0 is a smooth and umbilic-free immersion, i.e. of class $C^\infty(\Sigma, \mathbf{R}^n) \cap \mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbf{R}^n)$. Furthermore, let $T \in (0, T_{\max}(F_0))$ and $p \in (3, \infty)$ be arbitrarily chosen, where $T_{\max}(F_0)$ has been defined in Definition 2. Then the following statements hold:*

(1) *There is some $\rho = \rho(\Sigma, F_0, T, p) > 0$, such that for every $F \in W^{4-\frac{4}{p},p}(\Sigma, \mathbf{R}^n)$ with $\| F - F_0 \|_{W^{4-\frac{4}{p},p}(\Sigma,\mathbf{R}^n)} < \rho$ the existence interval $[0, t^+(F))$ of the maximal solution $\{\mathcal{P}^*(\cdot, 0, F)\}$ of Cauchy problem (23) from the second part of Theorem 2 contains the prescribed interval $[0, T]$, and the resulting evolution operator*

$$\mathcal{P}^*(\cdot, 0, \cdot) : B_\rho^{4-\frac{4}{p},p}(F_0) \subset W^{4-\frac{4}{p},p}(\Sigma, \mathbf{R}^n) \longrightarrow X_T, \tag{102}$$

mapping any element F of the open ball $B_\rho^{4-\frac{4}{p},p}(F_0)$ about F_0 in $W^{4-\frac{4}{p},p}(\Sigma, \mathbf{R}^n)$ to the unique solution $\{\mathcal{P}^*(t, 0, F)\}_{t\in[0,T]}$ of Cauchy problem (23) in X_T, is of class C^ω.

(2) *For any fixed $\xi \in W^{4,p}(\Sigma, \mathbf{R}^n)$ the function $[(x,t) \mapsto (D_F\mathcal{P}^*(t, 0, F_0).\xi)(x)]$ arising from the Fréchet derivative*

$$D_F\mathcal{P}^*(\,\cdot\,, 0, F_0) : W^{4-\frac{4}{p},p}(\Sigma, \mathbf{R}^n) \longrightarrow X_T \tag{103}$$

of the evolution operator $[F \mapsto \mathcal{P}^*(\,\cdot\,, 0, F)]$ in line (102) in the immersion F_0 is the unique solution $\{z_t\}$ of the linear uniformly parabolic initial value problem (104) in $C^1([0, T]; L^p(\Sigma, \mathbf{R}^n)) \cap C^0([0, T]; W^{4,p}(\Sigma, \mathbf{R}^n))$, which starts moving at time $t = 0$ in the given function $\xi \in W^{4,p}(\Sigma, \mathbf{R}^n)$: [3]

$$\begin{aligned}
\partial_t u_t(x) &= (D(\mathcal{M}_{F_0})(\mathcal{P}^*(t, 0, F_0)).u_t)(x) \\
&\equiv -\frac{1}{2} \mid A^0_{\mathcal{P}^*(t,0,F_0)}(x) \mid^{-4} g^{ij}_{\mathcal{P}^*(t,0,F_0)} g^{kl}_{\mathcal{P}^*(t,0,F_0)} \nabla^{F_0}_i \nabla^{F_0}_j \nabla^{F_0}_k \nabla^{F_0}_l(u_t)(x) \\
&\quad - B_3^{ijk}(x, D_x\mathcal{P}^*(t, 0, F_0), D_x^2\mathcal{P}^*(t, 0, F_0)) \cdot \nabla^{F_0}_{ijk}(u_t)(x) \\
&\quad - B_2^{ij}(x, D_x\mathcal{P}^*(t, 0, F_0), D_x^2\mathcal{P}^*(t, 0, F_0), D_x^3\mathcal{P}^*(t, 0, F_0), D_x^4\mathcal{P}^*(t, 0, F_0)) \cdot \nabla^{F_0}_{ij}(u_t)(x) \\
&\quad - B_1^{i}(x, D_x\mathcal{P}^*(t, 0, F_0), D_x^2\mathcal{P}^*(t, 0, F_0), D_x^3\mathcal{P}^*(t, 0, F_0), D_x^4\mathcal{P}^*(t, 0, F_0)) \cdot \nabla^{F_0}_{i}(u_t)(x) \\
&\text{for } (x, t) \in \Sigma \times [0, T] \quad \text{and with } u_0 = \xi \text{ on } \Sigma,
\end{aligned} \tag{104}$$

where

$$\begin{aligned}
&B_3^{ijk}(\,\cdot\,, D_x\mathcal{P}^*(t, 0, F_0), D_x^2\mathcal{P}^*(t, 0, F_0)), \\
&B_2^{ij}(\,\cdot\,, D_x\mathcal{P}^*(t, 0, F_0), D_x^2\mathcal{P}^*(t, 0, F_0), D_x^3\mathcal{P}^*(t, 0, F_0), D_x^4\mathcal{P}^*(t, 0, F_0)), \\
&B_1^{i}(\,\cdot\,, D_x\mathcal{P}^*(t, 0, F_0), D_x^2\mathcal{P}^*(t, 0, F_0), D_x^3\mathcal{P}^*(t, 0, F_0), D_x^4\mathcal{P}^*(t, 0, F_0))
\end{aligned}$$

are coefficients of $\mathrm{Mat}_{3,3}(\mathbf{R})$-valued, contravariant tensor fields of degrees $3, 2$ and 1, whose basic properties have been proved in Theorem 1 (iii).

(3) *The Fréchet derivative in line (103) of the evolution operator $[F \mapsto \mathcal{P}^*(\,\cdot\,, 0, F)]$ in the umbilic-free immersion F_0 has the particular property, to yield for every fixed time $t \in [0, T]$ a linear continuous operator*

$$D_F\mathcal{P}^*(t, 0, F_0) : W^{4,p}(\Sigma, \mathbf{R}^n) \longrightarrow W^{4,p}(\Sigma, \mathbf{R}^n), \tag{105}$$

and this linear operator has a unique continuous linear extension

$$D_F\mathcal{P}^*(t, 0, F_0) : L^p(\Sigma, \mathbf{R}^n) \longrightarrow L^p(\Sigma, \mathbf{R}^n) \tag{106}$$

for every fixed time $t \in [0, T]$, whose range is contained in $W^{4,p}(\Sigma, \mathbf{R}^n)$, if $t \in (0, T]$.

(4) *The differential Eq. (104) generates a parabolic fundamental solution*

$$G^{F_0}(t, s) : L^p(\Sigma, \mathbf{R}^n) \longrightarrow L^p(\Sigma, \mathbf{R}^n), \quad \forall s \leq t \in [0, T],$$

with range in $W^{4,p}(\Sigma, \mathbf{R}^n)$ for $t > s$, which extends the family of linear operators $[(t, s) \mapsto D_F\mathcal{P}^*(t, s, \mathcal{P}^*(s, 0, F_0))]$ continuously and linearly from the space $W^{4,p}(\Sigma, \mathbf{R}^n)$ to the space $L^p(\Sigma, \mathbf{R}^n)$, i.e. we have

$$G^{F_0}(t, s) = D_F\mathcal{P}^*(t, s, \mathcal{P}^*(s, 0, F_0)) : W^{4,p}(\Sigma, \mathbf{R}^n) \longrightarrow W^{4,p}(\Sigma, \mathbf{R}^n), \tag{107}$$

---

[3] According to Definition 4.1.1 in [21] this means exactly, that the function $[(x, t) \mapsto (D_F\mathcal{P}^*(t, 0, F_0).\xi)(x)]$ is the <u>unique strict solution</u> of initial value problem (104) w.r.t. the Banach spaces $X := L^p(\Sigma, \mathbf{R}^n)$ and $D := W^{4,p}(\Sigma, \mathbf{R}^n)$.

for every pair $s \leq t \in [0, T]$. In particular, this uniquely extended family of linear operators in $\mathcal{L}(L^p(\Sigma, \mathbf{R}^n), L^p(\Sigma, \mathbf{R}^n))$—again denoted by $[(t, s) \mapsto D_F \mathcal{P}^*(t, s, \mathcal{P}^*(s, 0, F_0))]$—has the semigroup property:

$$D_F \mathcal{P}^*(t, s, \mathcal{P}^*(s, 0, F_0)) \circ D_F \mathcal{P}^*(s, r, \mathcal{P}^*(r, 0, F_0)) = D_F \mathcal{P}^*(t, r, \mathcal{P}^*(r, 0, F_0)) \quad (108)$$

in $L^p(\Sigma, \mathbf{R}^n)$, for any $0 \leq r \leq s \leq t \leq T$.

(5) For every fixed $t \in (0, T]$ and $\xi^* \in (L^p(\Sigma, \mathbf{R}^n))^*$ the function

$$y_s := \left(D_F \mathcal{P}^*(t, s, \mathcal{P}^*(s, 0, F_0)) \right)^* (\xi^*), \quad \text{for } s \in [0, t],$$

is of class

$$\{y_s\} \in C^1([0, t); (L^p(\Sigma, \mathbf{R}^n))^*) \cap C^0([0, t); D(A_0^*)) \cap C^0([0, t]; (L^p(\Sigma, \mathbf{R}^n))^*), \quad (109)$$

it is the unique classical solution[4] of the adjoint linear <u>terminal value problem</u>:

$$\partial_s y_s = -\left(D(\mathcal{M}_{F_0})(\mathcal{P}^*(s, 0, F_0)) \right)^* (y_s) \quad \text{in } (L^p(\Sigma, \mathbf{R}^n))^*, \ \forall s \in [0, t),$$
$$y_t = \xi^* \quad \text{in } (L^p(\Sigma, \mathbf{R}^n))^*, \quad (110)$$

and its time inverse $\{\tilde{y}_s\} := \{y_{t-s}\}$ is the unique mild solution [5] of the initial value problem:

$$\partial_s \tilde{y}_s = \left(D(\mathcal{M}_{F_0})(\mathcal{P}^*(t - s, 0, F_0)) \right)^* (\tilde{y}_s) \quad \text{in } (L^p(\Sigma, \mathbf{R}^n))^*, \ \forall s \in (0, t],$$
$$\tilde{y}_0 = \xi^* \quad \text{in } (L^p(\Sigma, \mathbf{R}^n))^*. \quad (111)$$

Proof (1) On account of the first part of Theorem 3 there is a unique, maximal flow line $\{\mathcal{P}^*(t, 0, F_0)\}_{t \in [0, T_{\max}(F_0))}$ of evolution Eq. (16), in the sense of Definition 2, (d), because the immersion F_0 was supposed to be smooth and umbilic-free. Furthermore we know from Theorem 1, that for any fixed $T \in (0, T_{\max}(F_0))$ there is some open neighborhood $W_{F_0, T, p}$ about the smooth solution $\{\mathcal{P}^*(t, 0, F_0)\}_{t \in [0, T]}$ of Eq. (13) within the space X_T, such that the operator $\Psi^{F_0, T}$ in line (19) is a C^ω-map from $W_{F_0, T, p}$ to Y_T, whose Fréchet derivative in the particular element $\{\mathcal{P}^*(t, 0, F_0)\}_{t \in [0, T]} \in W_{F_0, T, p} \cap C^\infty(\Sigma \times [0, T], \mathbf{R}^n)$ yields a topological isomorphism between X_T and Y_T. Noting also that there holds

$$\Psi^{F_0, T}(\{\mathcal{P}^*(t, 0, F_0)\}_{t \in [0, T]}) = (F_0, 0) \in Y_T,$$

by definition of the operator $\Psi^{F_0, T}$ in (19) and since $\{\mathcal{P}^*(t, 0, F_0)\}_{t \in [0, T]}$ solves Eq. (13), we infer from the Inverse Mapping Theorem for non-linear C^ω-operators [[34], Theorem 4.B and Corollary 4.23], that there is some small open ball $B_\rho((F_0, 0)) \subset Y_T$, with $\rho = \rho(\Sigma, F_0, T, p) > 0$, and an appropriate further open neighborhood $W^*_{F_0, T, p} \subset W_{F_0, T, p}$ of $\{\mathcal{P}^*(t, 0, F_0)\}_{t \in [0, T]}$ in X_T, such that

$$\Psi^{F_0, T} : W^*_{F_0, T, p} \xrightarrow{\cong} B_\rho((F_0, 0))$$

is a C^ω-diffeomorphism. Hence, by definition of the map $\Psi^{F_0, T}$ the restriction of the inverse map $(\Psi^{F_0, T})^{-1}$ to the product $B_\rho^{4 - \frac{4}{p}, p}(F_0) \times \{0\} \subset B_\rho((F_0, 0))$ yields exactly

---

[4] See here Definition 4.1.1 in [21].

[5] See here Definition 4.1.4 in [21].

the unique maximal solutions $\{\mathcal{P}^*(t, 0, F)\}_{t \in [0,T]}$ of the parabolic Cauchy problems in (23) restricted to $\Sigma \times [0, T]$ from the second part of Theorem 2, i.e.:

$$X_T \ni \{\mathcal{P}^*(t, 0, F)\}_{t \in [0,T]} = (\Psi^{F_0, T})^{-1}((F, 0))$$

$$\forall \, F \in B_\rho^{4 - \frac{4}{p}, p}(F_0) \subset W^{4 - \frac{4}{p}, p}(\Sigma, \mathbf{R}^n), \tag{112}$$

and this map consequently has to be of class C^ω as a non-linear operator from $W^{4 - \frac{4}{p}, p}(\Sigma, \mathbf{R}^n)$ to X_T.

(2) Now we show that the Gateaux derivative of the evolution operator $[F \mapsto \mathcal{P}^*(\,\cdot\,, 0, F)]$ of Eq. (13) in the fixed umbilic-free immersion F_0 has to coincide with the <u>unique strict solution</u> of the linear parabolic initial value problem (104); compare here to the footnote attached to Eq. (104) regarding this terminology. To this end, we firstly fix some function $\xi \in W^{4, p}(\Sigma, \mathbf{R}^n) \setminus \{0\}$ arbitrarily, in order to define the auxiliary function

$$y(\,\cdot\,, h) := \mathcal{P}^*(\,\cdot\,, 0, F_0 + h\,\xi) \in X_T,$$

for sufficiently small $|h| \geq 0$. We can immediately see that there holds $y(0, h) = \mathcal{P}^*(0, 0, F_0 + h\,\xi) = F_0 + h\,\xi$. Recalling statement (102) of the first part of this theorem we set $h_*(T, \xi) := \frac{\rho}{\|\xi\|_{W^{4, p}(\Sigma, \mathbf{R}^n)}}$ for the same $\rho > 0$ as in statement (102), in order to guarantee that the initial immersions $y(0, h)$ satisfy $\| \, y(0, h) - F_0 \, \|_{W^{4, p}(\Sigma)} < \rho$ for every $|h| < h_*(T, \xi)$. Furthermore, we have that

$$\frac{y(0, h) - y(0, 0)}{h} = \frac{F_0 + h\,\xi - F_0}{h} = \xi \quad \text{for } h \neq 0. \tag{113}$$

On account of the second part of Theorem 1 there is an open neighborhood $W_{F_0, T, p}$ of the flow line $\{y(t, 0)\}_{t \in [0,T]} = \{\mathcal{P}^*(t, 0, F_0)\}_{t \in [0,T]}$ in the space X_T, such that the operator

$$\mathcal{M}_{F_0} : W_{F_0, T, p} \xrightarrow{\; C^\omega \;} L^p([0, T]; L^p(\Sigma, \mathbf{R}^n)), \; \{f_t\}_{t \in [0,T]} \mapsto \{\mathcal{M}_{F_0}(f_t)\}_{t \in [0,T]} \tag{114}$$

is real analytic, and on account of the first part of this theorem the evolution operator $\mathcal{P}^*(\,\cdot\,, 0, \,\cdot\,)$ in line (102) maps the open $W^{4 - \frac{4}{p}, p}$-ball $B_\rho^{4 - \frac{4}{p}, p}(F_0)$ about F_0 into the open neighborhood $W_{F_0, T, p}$ of $\{y(t, 0)\}_{t \in [0,T]}$ real analytically, thus in particular locally Lipschitz continuously. Hence, we conclude that for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$, such that

$$\| \, y(\,\cdot\,, h) - y(\,\cdot\,, 0) \, \|_{X_T} \equiv \| \, \mathcal{P}^*(\,\cdot\,, 0, F_0 + h\,\xi) - \mathcal{P}^*(\,\cdot\,, 0, F_0) \, \|_{X_T} < \varepsilon, \tag{115}$$

for any $|h| < \delta(\varepsilon)$. Therefore, we can choose some positive $h^* < h_*(T, \xi) = \frac{\rho}{\|\xi\|_{W^{4, p}(\Sigma, \mathbf{R}^n)}}$ that small, such that every convexly combined function $[(s, x) \mapsto (\sigma \, y(s, h) + (1 - \sigma) \, y(s, 0))(x)]$ is contained in the open neighborhood $W_{F_0, T, p}$ of the flow line $\{y(t, 0)\}_{t \in [0,T]}$ in X_T, for every fixed $\sigma \in [0, 1]$, provided we have $0 \leq |h| < h^*$. Now, since we know from the second part of Theorem 2 respectively from the first part of this theorem, that for every fixed $F \in B_\rho^{4 - \frac{4}{p}, p}(F_0)$ the solution $\{\mathcal{P}^*(t, 0, F)\}_{t \in [0,T]}$ of flow Eq. (13) is an element of the space $X_T = W^{1, p}([0, T]; L^p(\Sigma, \mathbf{R}^n)) \cap L^p([0, T]; W^{4, p}(\Sigma, \mathbf{R}^n))$, the differential Eq. (13), i.e.

$$\partial_t(\mathcal{P}^*(\,\cdot\,, 0, F)) = \mathcal{M}_{F_0}(\mathcal{P}^*(\,\cdot\,, 0, F)) \quad \text{in} \quad L^p([0, T]; L^p(\Sigma, \mathbf{R}^n)),$$

and embedding (30) imply the integral equation

$$\mathcal{P}^*(t, 0, F) - F = \int_0^t \mathcal{M}_{F_0}(\mathcal{P}^*(s, 0, F)) \, ds \quad \text{in} \quad L^p(\Sigma, \mathbf{R}^n) \tag{116}$$

for every $t \in [0, T]$ and for every fixed $F \in B_\rho^{4-\frac{4}{p}, p}(F_0)$. Moreover, we recall here that for any $\eta \in X_T$ the function $(D(\mathcal{M}_{F_0})(\sigma \, y(\,\cdot\,, h) + (1 - \sigma) \, y(\,\cdot\,, 0)))(\eta)$ is not only an abstract element of the Banach space $L^p([0, T]; L^p(\Sigma, \mathbf{R}^n))$, but it can be "concretely computed" in terms of the right hand side of formula (21), at least in L^1-almost every $s \in [0, T]$. Recalling now statements (113) and (114) and the fact that we have $\{\sigma \, y(s, h) + (1 - \sigma) \, y(s, 0))\}_{s \in [0, T]} \in W_{F_0, T, p}$, for every fixed $\sigma \in [0, 1]$ and every $0 \le |h| < h^*$, we can combine Eq. (116) with the generalized mean value theorem, in order to infer for the difference quotients

$$z_h(\,\cdot\,) := \frac{1}{h} \left(y(\,\cdot\,, h) - y(\,\cdot\,, 0) \right) \in X_T, \quad \text{being defined for } 0 < |h| < h^*,$$

the crucial equation

$$
\begin{aligned}
z_h(t) &= \frac{y(0, h) - y(0, 0)}{h} + \int_0^t \frac{1}{h} \Big(\mathcal{M}_{F_0}(y(s, h)) - \mathcal{M}_{F_0}(y(s, 0)) \Big) \, ds \\
&= \xi + \int_0^t \int_0^1 D(\mathcal{M}_{F_0}) \big(\sigma \, y(s, h) + (1 - \sigma) \, y(s, 0) \big). \left(\frac{y(s, h) - y(s, 0)}{h} \right) d\sigma \, ds \\
&\equiv \xi + \int_0^t \left(\int_0^1 D(\mathcal{M}_{F_0}) \big(\sigma \, y(s, h) + (1 - \sigma) \, y(s, 0) \big) \, d\sigma \right) \left(\frac{y(s, h) - y(s, 0)}{h} \right) ds \\
&\equiv \xi + \int_0^t N_{F_0, h\xi}(s).(z_h(s)) \, ds \quad \text{on } \Sigma,
\end{aligned}
\tag{117}
$$

for every $t \in [0, T]$ and for $0 < |h| < h^*$. Here, we have used the abbreviation

$$N_{F_0, h\xi}(s) := \int_0^1 D(\mathcal{M}_{F_0}) \big(\sigma \, y(s, h) + (1 - \sigma) \, y(s, 0) \big) \, d\sigma,$$

for $0 < |h| < h^*$, which again has to be interpreted pointwise in L^1-almost every $s \in [0, T]$ by means of the right hand side of formula (21). Now, using again statement (115) and the fact from line (114), that $\mathcal{M}_{F_0} : W_{F_0, T, p} \longrightarrow L^p([0, T]; L^p(\Sigma, \mathbf{R}^n))$ is an operator of class C^ω, we infer that for every $\varepsilon > 0$ there is some $\tilde{\delta}(\varepsilon) > 0$, such that

$$
\begin{aligned}
&\| N_{F_0, h\xi}(\,\cdot\,) - D(\mathcal{M}_{F_0})(y(\,\cdot\,, 0)) \|_{\mathcal{L}(X_T, L^p([0, T], L^p(\Sigma, \mathbf{R}^n)))} \\
&\le \int_0^1 \| D(\mathcal{M}_{F_0}) \big(\sigma \, y(\,\cdot\,, h) + (1 - \sigma) \, y(\,\cdot\,, 0) \big) \\
&\quad - D(\mathcal{M}_{F_0})(y(\,\cdot\,, 0)) \|_{\mathcal{L}(X_T, L^p([0, T], L^p(\Sigma, \mathbf{R}^n)))} \, d\sigma < \varepsilon
\end{aligned}
\tag{118}
$$

holds, provided $0 < |h| < \tilde{\delta}(\varepsilon)$. Now, again using the result of the first part of this theorem, i.e. statement (102), we can immediately infer from the definition of the family $\{z_h\}$ and from the chain rule, that $\{z_h\}$ converges in X_T to the Fréchet derivative $D_F \mathcal{P}^*(\,\cdot\,, 0, F_0).(\xi)$ applied to ξ respectively to the Gateaux derivative $D_\xi \mathcal{P}^*(\,\cdot\,, 0, F_0)$ w.r.t. ξ of the evolution operator $[F \mapsto \{\mathcal{P}^*(t, 0, F)\}_{t \in [0, T]}]$ as $h \to 0$:

$$
\begin{aligned}
\lim_{h \to 0} z_h &\equiv \lim_{h \to 0} \frac{y(s, h) - y(s, 0)}{h} \\
&= \lim_{h \to 0} \left(\left[(x, t) \mapsto \frac{\mathcal{P}^*(t, 0, F_0 + h\,\xi)(x) - \mathcal{P}^*(t, 0, F_0)(x)}{h} \right] \right) \\
&\equiv [(x, t) \mapsto D_\xi \mathcal{P}^*(t, 0, F_0)(x)] =: z^* \quad \text{in } X_T.
\end{aligned}
\tag{119}
$$

We note here that it is an immediate consequence of the above construction of the function z^* and of embedding (30), that there holds

$$[(x, t) \mapsto D_\xi \mathcal{P}^*(t, 0, F_0)(x)] \equiv z^* \in C^0([0, T], W^{4-\frac{4}{p}, p}(\Sigma, \mathbf{R}^n)). \qquad (120)$$

Inserting the convergences (118) and (119) into formula (117), we infer the following integral equation from the triangle inequality for "Bochner's integral" and from Hölder's inequality, when letting tend $h \searrow 0$:

$$
\begin{aligned}
z_t^* &= \xi + \int_0^t D(\mathcal{M}_{F_0})(y(s, 0)).(z_s^*) \, ds \\
&\equiv \xi + \int_0^t D(\mathcal{M}_{F_0})(\mathcal{P}^*(s, 0, F_0)).(z_s^*) \, ds \quad \text{in} \quad L^p(\Sigma, \mathbf{R}^n)
\end{aligned}
\qquad (121)
$$

for every $t \in [0, T]$, taking here also (120) into account. Since we know already from convergence (119), that the limit function z^* is an element of X_T, and since we know from the second part of Theorem 1, that the Fréchet derivative $D(\mathcal{M}_{F_0})(\mathcal{P}^*(\cdot, 0, F_0))$ maps the Banach space X_T continuously into $L^p([0, T]; L^p(\Sigma, \mathbf{R}^n))$, we may differentiate Eq. (121) w.r.t. t and obtain together with computation (65) an equivalent reformulation of Eq. (121), stating that the function $[(x, t) \mapsto z_t^*(x)] \equiv [(x, t) \mapsto D_\xi \mathcal{P}^*(t, 0, F_0)(x)] \in X_T$ from line (119) solves the following linear parabolic differential system of equations:

$$
\begin{aligned}
\partial_t(z_t^*) &= D(\mathcal{M}_{F_0})(\mathcal{P}^*(t, 0, F_0)).(z_t^*) \\
&= -\frac{1}{2} \mid A_{\mathcal{P}^*(t,0,F_0)}^0 \mid^{-4} g_{\mathcal{P}^*(t,0,F_0)}^{ij} g_{\mathcal{P}^*(t,0,F_0)}^{kl} \nabla_i^{F_0} \nabla_j^{F_0} \nabla_k^{F_0} \nabla_l^{F_0}(z_t^*) \\
&\quad - B_3^{ijk}(\cdot, D_x \mathcal{P}^*(t, 0, F_0), D_x^2 \mathcal{P}^*(t, 0, F_0)) \cdot \nabla_{ijk}^{F_0}(z_t^*) \\
&\quad - B_2^{ij}(\cdot, D_x \mathcal{P}^*(t, 0, F_0), D_x^2 \mathcal{P}^*(t, 0, F_0), D_x^3 \mathcal{P}^*(t, 0, F_0), D_x^4 \mathcal{P}^*(t, 0, F_0)) \cdot \nabla_{ij}^{F_0}(z_t^*) \\
&\quad - B_1^i(\cdot, D_x \mathcal{P}^*(t, 0, F_0), D_x^2 \mathcal{P}^*(t, 0, F_0), D_x^3 \mathcal{P}^*(t, 0, F_0), D_x^4 \mathcal{P}^*(t, 0, F_0)) \cdot \nabla_i^{F_0}(z_t^*)
\end{aligned}
\qquad (122)
$$

in $L^p([0, T]; L^p(\Sigma, \mathbf{R}^n))$, with $z_0^* = \xi$ on Σ, if $\xi \in W^{4,p}(\Sigma, \mathbf{R}^n)$, just as asserted in line (104). Since we know that $\{\mathcal{P}^*(t, 0, F_0)\}_{t \in [0,T]} \in C^\infty(\Sigma \times [0, T], \mathbf{R}^n)$, we infer from the fourth part of Theorem 1, that the Fréchet derivative $D(\Psi^{F_0,T})(\mathcal{P}^*(\cdot, 0, F_0))$ is a topological isomorphism between X_T and Y_T, which implies that the function $[(x, t) \mapsto z_t^*(x)] \equiv [(x, t) \mapsto D_\xi \mathcal{P}^*(t, 0, F_0)(x)]$ in line (119) has to be the unique solution of initial value problem (122) within the space X_T. Furthermore, we know already from the proof of the fourth part of Theorem 1, that the linear operator "$\partial_t - D(\mathcal{M}_{F_0})(\mathcal{P}^*(\cdot, 0, F_0))$" satisfies all requirements of Proposition 2. Hence, Proposition 2 guarantees us, that the linear differential operators

$$
\begin{aligned}
\omega \, \mathrm{Id}_{L^p(\Sigma, \mathbf{R}^n)} - A_t &:= \omega \, \mathrm{Id}_{L^p(\Sigma, \mathbf{R}^n)} - D(\mathcal{M}_{F_0})(\mathcal{P}^*(t, 0, F_0)) : \\
W^{4,p}(\Sigma, \mathbf{R}^n) &\longrightarrow L^p(\Sigma, \mathbf{R}^n)
\end{aligned}
\qquad (123)
$$

are of type (M, ϑ) for every fixed $t \in [0, T]$, where the angle $\vartheta \in (\pi/2, \pi)$ is arbitrarily fixed, and where $M = M(\Sigma, F_0, T, p, \vartheta) \geq 1$ and $\omega = \omega(\Sigma, F_0, T, p, \vartheta) > 0$ are sufficiently large real numbers. In particular, the differential operators $A_t \equiv D(\mathcal{M}_{F_0})(\mathcal{P}^*(t, 0, F_0))$ in line (123) are "sectorial" in $L^p(\Sigma, \mathbf{R}^n)$—in the sense of Definition 2.0.1 in [21]—on account of estimate (75), uniformly for every $t \in [0, T]$. Since the operators A_t also depend smoothly on t, the family of linear operators $\{A_t\}$ satisfies all principal hypotheses of Section 6.1 in [21] for the couple of Banach spaces

$X := L^p(\Sigma, \mathbf{R}^n)$ and $D := W^{4,p}(\Sigma, \mathbf{R}^n)$, adopting here the notation of Chapter 6 in [21]. Hence, we may apply the results of Sections 6.1 and 6.2 of [21], and we thus obtain a "parabolic fundamental solution" respectively an "evolution operator" $\{G(t, s)\} \equiv \{G^{F_0}(t, s)\} \subset \mathcal{L}(X, X)$, for $0 \leq s \leq t \leq T$, which is characterized by the following 5 properties:

(1) There exists some constant $C(\Sigma, F_0, T, p) > 0$, such that

$$\| G(t, s).(\xi) \|_{L^p(\Sigma, \mathbf{R}^n)} \leq C(\Sigma, F_0, T, p) \| \xi \|_{L^p(\Sigma, \mathbf{R}^n)}, \tag{124}$$

for every $\xi \in L^p(\Sigma, \mathbf{R}^n)$ and for every $0 \leq s \leq t \leq T$. In particular, the operator $G(t, s)$ yields a linear continuous map

$$G(t, s) : L^p(\Sigma, \mathbf{R}^n) \longrightarrow L^p(\Sigma, \mathbf{R}^n), \tag{125}$$

for $0 \leq s \leq t \leq T$; and it has the additional property that $\text{range}(G(t, s)) \subset W^{4,p}(\Sigma, \mathbf{R}^n)$, if $s < t$.

(2) For every $0 \leq r \leq s \leq t \leq T$ there holds the semigroup property:

$$G(t, s) \circ G(s, r) = G(t, r) \tag{126}$$

and also $G(s, s) = \text{Id}$ on $L^p(\Sigma, \mathbf{R}^n)$, $\forall s \in [0, T]$.

(3) For every fixed $s \in [0, T)$ and for every $\xi \in L^p(\Sigma, \mathbf{R}^n)$ the function $[t \mapsto G(t, s).(\xi)]$ is continuous on $[s, T]$ and continuously differentiable in $(s, T]$, and the derivative $\partial_t G(t, s).(\xi)$ satisfies the linear differential equation

$$\partial_t G(t, s).(\xi) = (A_t \circ G(t, s)).(\xi), \qquad \forall t \in (s, T]. \tag{127}$$

(4) For every fixed $t \in (0, T]$ and every $\xi \in W^{4,p}(\Sigma, \mathbf{R}^n)$ the function $[s \mapsto G(t, s).(\xi)]$ is continuous on $[0, t]$ and continuously differentiable in $[0, t)$, and the derivative $\partial_s G(t, s).(\xi)$ satisfies the linear differential equation

$$\partial_s G(t, s).(\xi) = -(G(t, s) \circ A_s).(\xi), \qquad \forall s \in [0, t). \tag{128}$$

(5) There is some constant $C = C(\Sigma, F_0, T, p) > 0$ such that there holds the estimate

$$\| G(t, s).(\xi) \|_{W^{4,p}(\Sigma, \mathbf{R}^n)} \leq \frac{C}{t - s} \| \xi \|_{L^p(\Sigma, \mathbf{R}^n)}, \tag{129}$$

for every $\xi \in L^p(\Sigma, \mathbf{R}^n)$, for $0 \leq s < t \leq T$.

It is important to mention here, that the above "parabolic fundamental solution" $G^{F_0}(t, s)$ is actually uniquely determined by the above 5 properties according to Section 3 in [1] respectively Theorem 4.4.1 of Chapter II in [2]. Moreover, we can infer from the above properties of the differential operators $A_t := D(\mathcal{M}_{F_0})(\mathcal{P}^*(t, 0, F_0))$ and from Corollary 6.1.9 and Proposition 6.2.2 in [21], that the above "parabolic fundamental solution" $[(x, t) \mapsto (G^{F_0}(t, 0).\xi)(x)]$ is the unique strict solution of the initial value problem

$$\partial_t(z_t^*) = D(\mathcal{M}_{F_0})(\mathcal{P}^*(t, 0, F_0)).(z_t^*) \equiv (A_t).(z_t^*), \quad \text{for } t \in [0, T],$$
$$z_0 = \xi \in W^{4,p}(\Sigma, \mathbf{R}^n) \tag{130}$$

w.r.t. the Banach spaces $X := L^p(\Sigma, \mathbf{R}^n)$ and $D := W^{4,p}(\Sigma, \mathbf{R}^n)$, i.e. that the mild solution $[(x, t) \mapsto (G^{F_0}(t, 0).\xi)(x)]$ of problem (130) satisfies:

$$[(x, t) \mapsto (G^{F_0}(t, 0).\xi)(x)] \in C^1([0, T]; L^p(\Sigma, \mathbf{R}^n)) \cap C^0([0, T]; W^{4,p}(\Sigma, \mathbf{R}^n)) \tag{131}$$

and that there is no further such solution of initial value problem (130). Now, since the Banach space in (131) is contained in the space X_T, and since the function $[(x, t) \mapsto z_t^*(x)] \equiv [(x, t) \mapsto D_\xi \mathcal{P}^*(t, 0, F_0)(x))]$ solves the same initial value problem <u>uniquely</u> in X_T, we can conclude the identity

$$D_\xi \mathcal{P}^*(t, 0, F_0) = G^{F_0}(t, 0).\xi \quad \text{on} \quad \Sigma, \quad \forall t \in [0, T], \tag{132}$$

for every $\xi \in W^{4,p}(\Sigma, \mathbf{R}^n)$, i.e. that the Gateaux derivative $[(x, t) \mapsto D_\xi \mathcal{P}^*(t, 0, F_0)(x)]$ in direction of any fixed function $\xi \in W^{4,p}(\Sigma, \mathbf{R}^n)$ is given by the application of the unique parabolic fundamental solution $G^{F_0}(t, 0)$ to any initial function $\xi \in W^{4,p}(\Sigma, \mathbf{R}^n)$, which is generated by the smooth family of uniformly elliptic differential operators $\{D(\mathcal{M}_{F_0})(\mathcal{P}^*(t, 0, F_0))\}_{t \in [0,T]}$ in (122) and (123). Combining statements (131) and (132) we verify the asserted uniqueness of the solution $[(x, t) \mapsto D_\xi \mathcal{P}^*(t, 0, F_0)(x)]$ of Eq. (104) and its asserted regularity: $[(x, t) \mapsto D_\xi \mathcal{P}^*(t, 0, F_0)(x)] \in C^1([0, T]; L^p(\Sigma, \mathbf{R}^n)) \cap C^0([0, T]; W^{4,p}(\Sigma, \mathbf{R}^n))$, improving the previous regularity statement (120) significantly.

(3) Combining the second property of the parabolic fundamental solution $\{G^{F_0}(t, s)\}$ with statements (129) and (132) we see that

$$D_F \mathcal{P}^*(t, 0, F_0) = G^{F_0}(t, 0) : W^{4,p}(\Sigma, \mathbf{R}^n) \longrightarrow W^{4,p}(\Sigma, \mathbf{R}^n) \tag{133}$$

is a continuous operator for every fixed $t \in [0, T]$, and that this linear operator can be uniquely extended by means of statement (125) to a continuous linear operator

$$G^{F_0}(t, 0) : L^p(\Sigma, \mathbf{R}^n) \longrightarrow L^p(\Sigma, \mathbf{R}^n),$$

for every $t \in [0, T]$, whose range is contained in $W^{4,p}(\Sigma, \mathbf{R}^n)$ for every $t \in (0, T]$. We shall term this unique extension $G^{F_0}(t, 0)$ again "$D_F \mathcal{P}^*(t, 0, F_0)$" in the sequel.

(4) We have already shown in the second part of this theorem, that the family of linear operators $A_t := D(\mathcal{M}_{F_0})(\mathcal{P}^*(t, 0, F_0))$, $t \in [0, T]$, generates a unique parabolic fundamental solution $G^{F_0}(t, s)$, and that the function in identity (132) is the unique strict solution to initial value problem (130), for any initial function $\xi \in W^{4,p}(\Sigma, \mathbf{R}^n)$. Since the immersion $\mathcal{P}^*(s, 0, F_0)$ is of class $C^\infty(\Sigma, \mathbf{R}^n)$ for every $s \in [0, T]$ on account of the first part of Theorem 3, we may interchange the immersion F_0 with the immersion $\mathcal{P}^*(s, 0, F_0)$ and the initial time 0 with another initial time $s \in [0, T]$ in the first part of this theorem, and we infer that the evolution operator $[F \mapsto \mathcal{P}^*(\cdot, s, F)]$ is of class C^ω in a small ball $B_\rho^{4-\frac{4}{p},p}(\mathcal{P}^*(s, 0, F_0))$ about $\mathcal{P}^*(s, 0, F_0)$ in $W^{4-\frac{4}{p},p}(\Sigma, \mathbf{R}^n)$. Moreover, we infer from the second part of this theorem that the resulting Fréchet derivative $D_F \mathcal{P}^*(t, s, \mathcal{P}^*(s, 0, F_0))$ of the evolution operator $[F \mapsto \mathcal{P}^*(\cdot, s, F)]$ in the immersion $\mathcal{P}^*(s, 0, F_0)$ yields functions $[(x, t) \mapsto D_F \mathcal{P}^*(t, s, \mathcal{P}^*(s, 0, F_0)).\xi(x)]$, for $\xi \in W^{4,p}(\Sigma, \mathbf{R}^n)$, which are the unique strict solutions of the initial value problem

$$\partial_t(z_t^*) = D(\mathcal{M}_{F_0})(\mathcal{P}^*(t, s, \mathcal{P}^*(s, 0, F_0))).(z_t^*), \quad \text{for } t \in [s, T],$$
$$z_s = \xi \in W^{4,p}(\Sigma, \mathbf{R}^n) \tag{134}$$

w.r.t. the Banach spaces $X := L^p(\Sigma, \mathbf{R}^n)$ and $D := W^{4,p}(\Sigma, \mathbf{R}^n)$. Because of the identity

$$\mathcal{P}^*(t, 0, F_0) = \mathcal{P}^*(t - s, 0, \mathcal{P}^*(s, 0, F_0)) = \mathcal{P}^*(t, s, \mathcal{P}^*(s, 0, F_0)) \quad \forall 0 \leq s \leq t \leq T,$$

we have here:

$$D(\mathcal{M}_{F_0})(\mathcal{P}^*(t, s, \mathcal{P}^*(s, 0, F_0))) = D(\mathcal{M}_{F_0})(\mathcal{P}^*(t, 0, F_0)) \equiv A(t), \tag{135}$$

$\forall \, 0 \leq s \leq t \leq T$. Hence, fixing $s \in [0, T]$ initial value problem (134) becomes the problem

$$\partial_t(z_t^*) = A(t).(z_t^*), \quad \text{for } t \in [s, T], \quad z_s = \xi \in W^{4,p}(\Sigma, \mathbf{R}^n) \qquad (136)$$

whose unique strict solution with values in $W^{4,p}(\Sigma, \mathbf{R}^n)$ is given by the functions $[(x, t) \mapsto G^{F_0}(t, s).\xi(x)]$, since we know from the proof of the second part of this theorem, that our particular parabolic fundamental solution $\{G^{F_0}(t, s)\}$ is uniquely generated by the family of operators $\{A_t\}_{t \in [0,T]}$, so that we can argue here again by means of property (3) of $\{G^{F_0}(t, s)\}$ combined with Corollary 6.1.9 and Proposition 6.2.2 in [21]. Hence, combining this with the fact, that $[(x, t) \mapsto D_F \mathcal{P}^*(t, s, \mathcal{P}^*(s, 0, F_0)).(\xi)(x)]$ is the unique strict solution of initial value problem (136) with values in $W^{4,p}(\Sigma, \mathbf{R}^n)$ as well, we obtain the identity

$$D_F \mathcal{P}^*(t, s, \mathcal{P}^*(s, 0, F_0)).(\xi) = G^{F_0}(t, s).(\xi), \quad \forall \, 0 \leq s \leq t \leq T, \qquad (137)$$

for any $\xi \in W^{4,p}(\Sigma, \mathbf{R}^n)$, which proves - again on account of estimate (129) - that the linear operators $G^{F_0}(t, s) : L^p(\Sigma, \mathbf{R}^n) \longrightarrow L^p(\Sigma, \mathbf{R}^n)$ in (125) are the unique continuous linear extensions of the linear operators

$$D_F \mathcal{P}^*(t, s, \mathcal{P}^*(s, 0, F_0)) : W^{4,p}(\Sigma, \mathbf{R}^n) \longrightarrow W^{4,p}(\Sigma, \mathbf{R}^n),$$

induced by the Fréchet derivative $D_F \mathcal{P}^*(\,\cdot\,, s, \mathcal{P}^*(s, 0, F_0))$, from $W^{4,p}(\Sigma, \mathbf{R}^n)$ to $L^p(\Sigma, \mathbf{R}^n)$, for every $0 \leq s \leq t \leq T$. Finally, combining this insight with formula (126), we verify immediately, that the family of linear operators $[(t, s) \mapsto D_F \mathcal{P}^*(t, s, \mathcal{P}^*(s, 0, F_0))]$ has the asserted semigroup property (108) both on $W^{4,p}(\Sigma, \mathbf{R}^n)$ and on $L^p(\Sigma, \mathbf{R}^n)$.

(5) For ease of notation we introduce the abbreviation

$$D_F \mathcal{P}^*(t_2, t_1, F_0) := D_F \mathcal{P}^*(t_2, t_1, \mathcal{P}^*(t_1, 0, F_0)) \qquad (138)$$

for every pair of times $t_2 \geq t_1$ in $[0, T]$, and we know from the fourth part of this theorem, that these are linear operators mapping $W^{4,p}(\Sigma, \mathbf{R}^n)$ into itself—induced by the unique strict solutions of Cauchy problem (134)—which can be uniquely and continuously extended to $L^p(\Sigma, \mathbf{R}^n)$ by the parabolic fundamental solution $\{G^{F_0}(t, s)\}_{s \leq t}$ to Cauchy problem (136), so that we shall simply write in the sequel:

$$D_F \mathcal{P}^*(t_2, t_1, F_0) : L^p(\Sigma, \mathbf{R}^n) \longrightarrow L^p(\Sigma, \mathbf{R}^n) \qquad (139)$$

for $t_2 \geq t_1$ in $[0, T]$. We shall note here, that the notation introduced in line (138) is a slight abuse of our original notation, but this abbreviation will not cause any confusion in the sequel, because from the very beginning the flow line $\mathcal{P}^*(\,\cdot\,, 0, F_0)$ of Eq. (13) was supposed to start moving in the immersion F_0 at time $t = 0$, and not at any later time $t_1 > 0$. Now, since the linear operators in line (139) are well-defined in entire $L^p(\Sigma, \mathbf{R}^n)$ and continuous w.r.t. the L^p-norm on both sides of (139), they are especially closed linear operators in $L^p(\Sigma, \mathbf{R}^n)$, for every pair $t_2 \geq t_1$ in $[0, T]$. Theorem 5.29 in Chapter III of [17] therefore guarantees, that every operator in line (139) has a uniquely determined maximal, densely defined and closed linear adjoint operator in $(L^p(\Sigma, \mathbf{R}^n))^*$:

$$(D_F \mathcal{P}^*(t_2, t_1, F_0))^* : (L^p(\Sigma, \mathbf{R}^n))^* \longrightarrow (L^p(\Sigma, \mathbf{R}^n))^* \cong L^q(\Sigma, \mathbf{R}^n), \qquad (140)$$

for $q = \frac{p}{p-1}$. Moreover we know, that the linear operators A_t from statements (123) and (135) depend smoothly on the time $t \in [0, T]$, that they are uniformly sectorial in the Banach space $L^p(\Sigma, \mathbf{R}^n)$ on account of statement (123) and that the domain of A_t,

which is $W^{4,p}(\Sigma, \mathbf{R}^n)$, is densely contained in $L^p(\Sigma, \mathbf{R}^n)$. In particular, the operators A_t are closed linear operators in $L^p(\Sigma, \mathbf{R}^n)$. Therefore, again Theorem 5.29 in Chapter III of [17] guarantees, that A_t has a uniquely determined maximal, densely defined and closed linear adjoint operator

$$(A_t)^* : D((A_t)^*) \subset (L^p(\Sigma, \mathbf{R}^n))^* \longrightarrow (L^p(\Sigma, \mathbf{R}^n))^*, \tag{141}$$

for every $t \in [0, T]$, and furthermore the adjoint operators in (141) constitute a smooth family of uniformly sectorial linear operators in $(L^p(\Sigma, \mathbf{R}^n))^* \cong L^q(\Sigma, \mathbf{R}^n)$ on account of Proposition 1.2.3 in Chapter I of [2].
Now we use the adjoint operators in (140), in order to define the family of functions $\{y_s\} \in C^0([0, t], (L^p(\Sigma, \mathbf{R}^n))^*)$ by

$$y_s := \left(D_F \mathcal{P}^*(t, s, F_0)\right)^*(\xi^*), \quad \text{for } s \in [0, t]. \tag{142}$$

It is important to note here, that the fourth property of the parabolic fundamental solution $[(t, s) \mapsto G^{F_0}(t, s)]$ together with Eq. (137) actually guarantees, that the above defined family of functions $\{y_s\}$ is of class $C^0([0, t]; (L^p(\Sigma, \mathbf{R}^n))^*)$. Since the family of operators $\{D_F \mathcal{P}^*(t_2, t_1, F_0)\}_{t_1 \leq t_2}$ in line (139) has the semigroup property (108), the family of functions $\{y_s\}$ evolves according to the evolution of the "adjoint Fréchet derivatives" of the evolution operator $\mathcal{P}^*(\cdot, \cdot, F_0)$ of the modification (13) of the MIWF:

$$\left(D_F \mathcal{P}^*(\tilde{s}, s, F_0)\right)^*(y_{\tilde{s}}) = \left(D_F \mathcal{P}^*(\tilde{s}, s, F_0)\right)^* \circ \left(D_F \mathcal{P}^*(t, \tilde{s}, F_0)\right)^*(\xi^*)$$
$$= \left(D_F \mathcal{P}^*(t, \tilde{s}, F_0) \circ D_F \mathcal{P}^*(\tilde{s}, s, F_0)\right)^*(\xi^*) =$$
$$= \left(D_F \mathcal{P}^*(t, s, F_0)\right)^*(\xi^*) = y_s \quad \text{for } 0 \leq s \leq \tilde{s} < t. \tag{143}$$

Since we know from the fourth part of this theorem—formula (107)—that the family of Fréchet derivatives $\{D_F \mathcal{P}^*(\tilde{s}, s, F_0)\}_{s \leq \tilde{s}}$ is the unique "parabolic fundamental solution" $\{G(\tilde{s}, s)\}_{s \leq \tilde{s}}$ for the linear system (104), we can apply here formula (128), in order to conclude that for every fixed $y \in (L^p(\Sigma, \mathbf{R}^n))^*$ the function $s \mapsto \left(D_F \mathcal{P}^*(\tilde{s}, s, F_0)\right)^*(y)$ is continuously differentiable on every interval $[0, \tilde{s})$ with $\tilde{s} < t$ and that it satisfies the linear differential equation

$$\partial_s \left(\left(D_F \mathcal{P}^*(\tilde{s}, s, F_0)\right)^*(y)\right) = -\left(D_F \mathcal{P}^*(\tilde{s}, s, F_0) \circ A_s\right)^*(y), \quad \forall s \in [0, \tilde{s}). \tag{144}$$

We can therefore take the derivative of the family of functions $\{y_s\}$ w.r.t. s in Eq. (143), and we infer from a combination of Eq. (144), here applied to the function $y := y_{\tilde{s}} \in (L^p(\Sigma, \mathbf{R}^n))^*$, again with Eq. (143) and with statement (141), that the family $\{y_s\}$ is of class $C^1([0, t); (L^p(\Sigma, \mathbf{R}^n))^*) \cap C^0([0, t]; (L^p(\Sigma, \mathbf{R}^n))^*)$ and that $\{y_s\}$ is a solution of the following linear differential equation:

$$\partial_s y_s = \partial_s \left(\left(D_F \mathcal{P}^*(\tilde{s}, s, F_0)\right)^*(y_{\tilde{s}})\right) = -\left(A_s\right)^*(y_s), \quad \forall s \in [0, \tilde{s}), \tag{145}$$

for every $\tilde{s} < t$, proving that $\{y_s\}$ is actually a mild solution of Eq. (110) on $[0, t)$. We note, that by Eq. (145) y_s is not only contained in $(L^p(\Sigma, \mathbf{R}^n))^*$, but even in the smaller space $D(A_{\tilde{s}}^*) \cong D(A_0^*)$ for every $s \in [0, t)$. Now we consider the functions

$$\tilde{y}_s := y_{t-s} = \left(D_F \mathcal{P}^*(t, t-s, F_0)\right)^*(\xi^*), \quad \text{for } s \in [0, t]. \tag{146}$$

Obviously, we know already that $\{\tilde{y}_s\}$ is of class $C^1((0, t]; (L^p(\Sigma, \mathbf{R}^n))^*) \cap C^0$ $([0, t]; (L^p(\Sigma, \mathbf{R}^n))^*)$, and we infer from (145) that it solves the initial value problem:

$$\partial_s \tilde{y}_s = \left(A_{t-s}\right)^* (\tilde{y}_s) \quad \forall s \in (0, t], \quad \text{and} \quad \tilde{y}_0 = \xi^*, \tag{147}$$

i.e. that $\{\tilde{y}_s\}$ is the unique mild solution of this initial value problem, as asserted in (111), taking also its definition in (146) into account, and moreover that $\tilde{y}_s \in D((A_{t-s})^*) \cong D(A_0^*)$ for every $s \in (0, t]$. We know already that the adjoint linear operators in (141) constitute a C^∞-smooth family of densely defined and uniformly sectorial linear operators in $(L^p(\Sigma, \mathbf{R}^n))^* \cong L^q(\Sigma, \mathbf{R}^n)$. Hence, the family of linear operators $\{(A_{t-s})^*\}_{s \in [0,t]}$ in (147) satisfies all requirements of Section 6.1 in [21], and we may thus apply Lemma 6.2.1 in [21] to the mild solution $\{\tilde{y}_s\}$ of problem (147), in order to obtain even better regularity of $\{\tilde{y}_s\}$, namely:

$$\{\tilde{y}_s\} \in C^1((0, t]; (L^p(\Sigma, \mathbf{R}^n))^*) \cap C^0((0, t]; D(A_0^*)) \cap C^0([0, t]; (L^p(\Sigma, \mathbf{R}^n))^*). \tag{148}$$

This means exactly that $\{\tilde{y}_s\}$ is a classical solution of Cauchy problem (147), in the sense of Definition 4.1.1 in [21]. We can therefore improve the above regularity statement about the original family of functions $\{y_s\}$, namely:

$$\{y_s\} \in C^1([0, t); (L^p(\Sigma, \mathbf{R}^n))^*) \cap C^0([0, t); D(A_0^*)) \cap C^0([0, t]; (L^p(\Sigma, \mathbf{R}^n))^*),$$

which means that $\{y_s\}$ is a classical solution of the adjoint terminal value problem:

$$\partial_s y_s = -\left(A_s\right)^*(y_s) \quad \forall s \in [0, t), \quad \text{with} \quad y_t = \xi^*,$$

as asserted in (110). Finally, if $\{x_s\}$ is any given classical solution of this terminal value problem, then its time inverse $\{x_{t-s}\}$ is a classical solution of initial value problem (147). Since we also know that $\overline{D((A_{t-s})^*)} = (L^p(\Sigma, \mathbf{R}^n))^*$ for every $s \in [0, t]$, that $\xi^* \in (L^p(\Sigma, \mathbf{R}^n))^*$ and that the family of linear operators $\{(A_{t-s})^*\}_{s \in [0,t]}$ in (147) satisfies all requirements of Section 6.1 in [21], we may infer here from Corollary 6.2.4 in [21], that the family $\{x_{t-s}\}$ actually has to be the unique mild solution of initial value problem (147), and that therefore $x_{t-s} = \tilde{y}_s$ for every $s \in [0, t]$. Hence, together with (146) we arrive at: $x_s = \tilde{y}_{t-s} = y_s$, for $s \in [0, t]$, proving that such $\{x_s\}$ has to be the concretely given family of functions in (142), which has shown the "uniqueness part" of the assertion in (110).

□

Combining Theorems 3 and 4, we can now prove the final result of this article. The application of the strong regularity result of Theorem 3 (ii) in the proof of the second part of Theorem 5 below cannot be substituted by Theorem 1.1 of [20], whose requirements are not satisfied in our mathematical situation. Exactly this technical difficulty motivated the author to prove Theorem 3 (ii).

Theorem 5 *Suppose that Σ is a smooth compact torus, being endowed with a complex structure, and suppose that $F_0 \in \mathrm{Imm}_{\mathrm{uf}}(\Sigma, \mathbf{R}^n) \cap C^\omega(\Sigma, \mathbf{R}^n)$ is a real analytic and umbilic-free immersion. Furthermore, let $T \in (0, T_{\max}(F_0))$ and $p \in (3, \infty)$ be arbitrarily chosen, where $T_{\max}(F_0)$ has been defined in Definition 2. Then the following statements hold:*

(1) *The Gateaux derivative $[(x, t) \mapsto (D_F \mathcal{P}^*(t, 0, F_0).\xi)(x)]$ of the evolution operator in line (102) is real analytic in time $t \in (0, T)$, for every fixed $\xi \in W^{4,p}(\Sigma, \mathbf{R}^n)$, and the unique classical solution $[(x, s) \mapsto (D_F \mathcal{P}^*(T, s, F_0))^*(\xi^*)(x)]$ of the adjoint*

linear terminal value problem ([110]) is real analytic in time $t \in (0, T)$, for every fixed $\xi^* \in (L^p(\Sigma, \mathbf{R}^n))^*$.

(2) *For any fixed $t_0 \in (0, T)$ we denote $F_1 := \mathcal{P}^*(t_0, 0, F_0)$. Then the family of linear differential operators $\{A_t\}_{t \in [t_0, T]} \equiv \{D(\mathcal{M}_{F_0})(\mathcal{P}^*(t, t_0, F_1))\}_{t \in [t_0, T]}$ generating the linear evolution Eq. ([104]) on $\Sigma \times [t_0, T]$ in the second part of Theorem [4], and also its "L^p-adjoint family" of linear operators $\{-(A_t)^*\}_{t \in [t_0, T]}$—along the flow line $[t \mapsto \mathcal{P}^*(t, t_0, F_1)]$ of the modified MIWF ([13])—have the "backwards-uniqueness-property" in the sense of Section 1 in [20]. This means here precisely the following two statements:*

a) *For any function $\xi \in W^{4,p}(\Sigma, \mathbf{R}^n)$ the function $[(x, t) \mapsto D_F \mathcal{P}^*(t, t_0, F_1).\xi(x)]$ from the second part of Theorem [4] vanishes identically on $\Sigma \times [t_0, T]$, if there should hold: $D_F \mathcal{P}^*(t, t_0, F_1).\xi \equiv 0$ on Σ at time $t = T$, i.e. if $[(x, t) \mapsto D_F \mathcal{P}^*(t, t_0, F_1).\xi(x)]$ solves the linear terminal value problem:*

$$\partial_t z_t = D(\mathcal{M}_{F_0})(\mathcal{P}^*(t, t_0, F_1))(z_t), \quad \text{for } t \in [t_0, T], \quad \text{with } z_T \equiv 0 \text{ on } \Sigma. \quad (149)$$

b) *For every $t \in (t_0, T]$ and every $\xi^* \in (L^p(\Sigma, \mathbf{R}^n))^*$ the unique classical solution*

$$y_s := \left(D_F \mathcal{P}^*(t, s, \mathcal{P}^*(s, t_0, F_1)) \right)^*(\xi^*), \quad \text{for } s \in [t_0, t],$$

of the adjoint linear terminal value problem:

$$\partial_s y_s = -\left(D(\mathcal{M}_{F_0})(\mathcal{P}^*(s, t_0, F_1)) \right)^*(y_s) \text{ in } (L^p(\Sigma, \mathbf{R}^n))^*, \quad \text{for } s \in [t_0, t),$$
$$\text{with} \quad y_t = \xi^* \text{ in } (L^p(\Sigma, \mathbf{R}^n))^* \quad (150)$$

from the fifth part of Theorem [4] vanishes identically on $\Sigma \times [t_0, t]$, if it also satisfies $y_{t_0} = 0$ in $(L^p(\Sigma, \mathbf{R}^n))^*$ at time $s = t_0$.

(3) *For any fixed $t_0 \in (0, T)$ and every $t \in [t_0, T]$ the unique linear continuous extension*

$$G^{F_0}(t, t_0) : L^p(\Sigma, \mathbf{R}^n) \longrightarrow L^p(\Sigma, \mathbf{R}^n)$$

of the Fréchet derivative $D_F \mathcal{P}^*(t, t_0, F_1)$ from line ([107]) has dense range in $L^p(\Sigma, \mathbf{R}^n)$, where we have set again $F_1 := \mathcal{P}^*(t_0, 0, F_0)$.

Proof (1) From Theorem [3] (ii) and from the definition of the differential operator $\mathcal{M}_{F_0}$ in formula ([13]) we can immediately deduce the fact, that all coefficients on the right hand side of Eq. ([104]) respectively ([122]) have to be real analytic functions on $\Sigma \times (0, T)$. Since we also know from the proof of the second part of Theorem [4], that the linear operators $A_t = D(\mathcal{M}_{F_0})(\mathcal{P}^*(t, 0, F_0))$ from line ([123]) are sectorial in $L^p(\Sigma, \mathbf{R}^n)$, uniformly for every $t \in [0, T]$, we may apply Theorems 8.1.1 and 8.3.9 in [21] to the linear parabolic system ([104]) and infer that its unique strict solution $[(x, t) \mapsto (D_F \mathcal{P}^*(t, 0, F_0).\xi)(x)]$, starting in some $\xi \in W^{4,p}(\Sigma, \mathbf{R}^n)$ at time $t = 0$—which is provided by the second part of Theorem [4]—is real analytic in $t \in (0, T)$, with values in $W^{4,p}(\Sigma, \mathbf{R}^n)$. Similarly, we can argue that the unique classical solution

$$y_s := [(x, s) \mapsto (D_F \mathcal{P}^*(T, s, \mathcal{P}^*(s, 0, F_0)))^*(\xi^*)(x)]$$
$$\equiv [(x, s) \mapsto (G^{F_0}(T, s))^*(\xi^*)(x)]$$

of the adjoint linear terminal value problem ([110]) is real analytic in $s \in (0, T)$, for every fixed $\xi^* \in (L^p(\Sigma, \mathbf{R}^n))^*$. For, we know from the proof of the fifth part of Theorem [4], that the linear operators $A(t)^*$ from ([141]) satisfy all requirements of Section 6.1 in [21], and additionally that they depend real analytically on $t \in (0, T)$, just as the

operators $A(t)$ do here. We can therefore argue again by means of Theorems 8.1.1 and 8.3.9 in [21], that the unique classical solution $\{\tilde{y}_s\} = \{y_{T-s}\}$ of Eq. (111), starting in $\tilde{y}_0 = \xi^* \in (L^p(\Sigma, \mathbf{R}^n))^*$ and of the regularity stated in line (148), is real analytic in $s \in (0, T)$, with values in $D(A_0^*)$. This implies immediately that also the unique classical solution $\{y_s\} \equiv (D_F \mathcal{P}^*(T, s, \mathcal{P}^*(s, 0, F_0)))^*(\xi^*) = \{\tilde{y}_{T-s}\}$ of problem (110) is real analytic in $s \in (0, T)$, with values in $D(A_0^*)$, for any $\xi^* \in (L^p(\Sigma, \mathbf{R}^n))^*$.

(2) a) Suppose that the unique strict solution $[(x, t) \mapsto D_F \mathcal{P}^*(t, t_0, F_1).\xi(x)]$ of the linear system (122) would also satisfy $z_T^* \equiv 0$ on Σ, at some fixed time $T \in (0, T_{\max}(F_0))$ and for some $\xi \in W^{4,p}(\Sigma, \mathbf{R}^n)$. Since the flow line $\{\mathcal{P}^*(\cdot, 0, F_0)\}$ of the modified MIWF (13) exists smoothly in the time interval $(0, T_{\max}(F_0))$, and since the statements of Theorem 3 (ii) and of the first part of this theorem hold in every fixed time interval $(0, T)$, for $T \in (0, T_{\max}(F_0))$, we may vary here our choice of starting and final time, namely setting $t_0' := T$ and $T' := T + \varepsilon$, for any sufficiently small $\varepsilon > 0$. Using the uniqueness of the strict solution to the initial value problem (104) w.r.t. the spaces $X := L^p(\Sigma, \mathbf{R}^n)$ and $D := W^{4,p}(\Sigma, \mathbf{R}^n)$ by the second part of Theorem 4, the unique short-time extension, again denoted by $\{z_t^*\}$, of the solution $\{z_t^*\} := [(x, t) \mapsto D_F \mathcal{P}^*(t, t_0, F_1).\xi(x)]$ of Eq. (104) from the interval $[t_0, T]$ to the larger interval $[t_0, T + \varepsilon]$ is given by the unique short-time solution of the initial value problem (104), which starts at time $t = T$ in the function $z_T^* \in W^{4,p}(\Sigma, \mathbf{R}^n)$. Since this short-time solution is explicitly given by the function

$$[(t, x) \mapsto D_F \mathcal{P}^*(t, T, \mathcal{P}^*(T, t_0, F_1)).(z_T^*)(x)] \quad \text{for} \quad (x, t) \in \Sigma \times [T, T + \varepsilon]$$

on account of Eq. (137), we conclude from $z_T^* \equiv 0$ on Σ that $z_t^* \equiv 0$ on $\Sigma \times [T, T + \epsilon]$. Hence, on account of the real analytic dependence of the function $[t \mapsto z_t^*]$ on the time $t \in [t_0, T + \epsilon)$ by the first part of this theorem, we can therefore conclude that $z_t^* \equiv 0$ on $\Sigma \times [t_0, T + \epsilon]$. This proves already the first assertion of the second part of the theorem.

b) Now we consider the unique classical solution of the adjoint linear terminal value problem (150), provided by the fifth part of Theorem 4. The unique classical extension $\{z_s^*\}$ of the mild and classical solution

$$[(x, s) \mapsto (D_F \mathcal{P}^*(t, s, \mathcal{P}^*(s, t_0, F_1)))^*(\xi^*)(x)]$$
$$\equiv [(x, s) \mapsto (G^{F_0}(t, s))^*(\xi^*)(x)] \quad \text{for} \quad (x, s) \in \Sigma \times [t_0, t]$$

of the terminal value problem (110) from the interval $[t_0, t]$ to the larger interval $(0, t]$ is given by the unique classical solution of the adjoint terminal value problem (110), which stops at time $t = t_0$ in the function $z_{t_0}^* \in (L^p(\Sigma, \mathbf{R}^n))^*$. Since this classical solution is explicitly given by the function

$$[(x, s) \mapsto (D_F \mathcal{P}^*(t_0, s, \mathcal{P}^*(s, 0, F_0)))^*(z_{t_0}^*)(x)]$$
$$\equiv [(x, s) \mapsto (G^{F_0}(t_0, s))^*(z_{t_0}^*)(x)] \quad \text{for} \quad (x, s) \in \Sigma \times (0, t_0]$$

on account of the fifth part of Theorem 4, we conclude from the assumption "$z_{t_0}^* \equiv 0$ on Σ", that $z_s^* \equiv 0$ on $\Sigma \times (0, t_0]$. Hence, on account of the real analytic dependence of the function $[s \mapsto z_s^*]$ on the time $s \in (0, t)$ by the first part of this theorem, we can therefore conclude that the original solution $\{z_s^*\}_{s \in [t_0, t]}$ of the adjoint linear terminal value problem (150) has to vanish identically on $\Sigma \times [t_0, t]$, which proves also the second assertion of the second part of this theorem.

(3) We fix some $t \in (t_0, T]$ arbitrarily, and we consider some arbitrary function $\eta \in (L^p(\Sigma, \mathbf{R}^n))^* \cong L^q(\Sigma, \mathbf{R}^n)$, $q = \frac{p}{p-1}$, which satisfies:

$$\int_{\Sigma} \langle \eta, D_F \mathcal{P}^*(t, t_0, F_1).(\xi) \rangle_{\mathbf{R}^n} \, d\mu_{\mathcal{P}^*(t,t_0,F_1)} = 0 \quad \forall \xi \in W^{4,p}(\Sigma, \mathbf{R}^n). \tag{151}$$

We try to prove in the sequel, that any such function η has to vanish on Σ. To this end, we recall from the fourth part of Theorem 4, that the family of solutions of the homogeneous, linear system (104) respectively (122) constitutes a parabolic fundamental solution $\{G^{F_0}(t, s)\}_{s \leq t}$ in $L^p(\Sigma, \mathbf{R}^n)$, which coincides with the family of Fréchet derivatives $\{D_F \mathcal{P}^*(t, s, \mathcal{P}^*(s, 0, F_0))\}_{s \leq t}$ on $W^{4,p}(\Sigma, \mathbf{R}^n)$, and we recall from the proof of the fifth part of Theorem 4, that for every pair of times $t_2 \geq t_1$ in $[t_0, T]$ the operators

$$D_F \mathcal{P}^*(t_2, t_1, F_1) := D_F \mathcal{P}^*(t_2, t_1, \mathcal{P}^*(t_1, t_0, F_1)) \tag{152}$$

from line (138)—up to exchanging $F_1 := \mathcal{P}^*(t_0, 0, F_0)$ with F_0—can be considered as continuous linear operators

$$D_F \mathcal{P}^*(t_2, t_1, F_1) : L^p(\Sigma, \mathbf{R}^n) \longrightarrow L^p(\Sigma, \mathbf{R}^n), \tag{153}$$

which have uniquely determined maximal, densely defined and closed linear adjoint operators in $(L^p(\Sigma, \mathbf{R}^n))^*$:

$$(D_F \mathcal{P}^*(t_2, t_1, F_1))^* : (L^p(\Sigma, \mathbf{R}^n))^* \longrightarrow (L^p(\Sigma, \mathbf{R}^n))^* \cong L^q(\Sigma, \mathbf{R}^n), \tag{154}$$

see statement (140). We use the adjoint operators in line (154) and the test function η in line (151), in order to define the family of functions $\{y_s\} \in C^0([t_0, t]; (L^p(\Sigma, \mathbf{R}^n))^*)$ by

$$y_s := \left(D_F \mathcal{P}^*(t, s, F_1) \right)^*(\eta), \quad \text{for } s \in [t_0, t].$$

We can immediately infer from the fifth part of Theorem 4, that we have here

$$\{y_s\} \in C^1([t_0, t]; (L^p(\Sigma, \mathbf{R}^n))^*) \cap C^0([t_0, t); D(A_0^*)) \cap C^0([t_0, t]; (L^p(\Sigma, \mathbf{R}^n))^*) \tag{155}$$

and that $\{y_s\}$ is the unique classical solution of the linear equation

$$\partial_s y_s = -\left(A_s \right)^*(y_s), \quad \forall s \in [t_0, t), \tag{156}$$

satisfying the terminal condition $y_t = \eta$ in $(L^p(\Sigma, \mathbf{R}^n))^*$, which is just the adjoint linear terminal value problem (150), with ξ^* replaced by η. Moreover, on account of the definition of the family $\{y_s\}$ and by assumption (151) we have:

$$\int_{\Sigma} \langle y_{t_0}, \xi \rangle \, d\mu_{F_1} = \int_{\Sigma} \langle \eta, D_F \mathcal{P}^*(t, t_0, F_1).(\xi) \rangle \, d\mu_{\mathcal{P}^*(t,t_0,F_1)} = 0$$
$$\forall \xi \in W^{4,p}(\Sigma, \mathbf{R}^n). \tag{157}$$

Since we have $y_{t_0} \in (L^p(\Sigma, \mathbf{R}^n))^* \cong L^q(\Sigma, \mathbf{R}^n)$ by construction, Eq. (157) implies that $y_{t_0} = 0$ in $(L^p(\Sigma, \mathbf{R}^n))^*$. Combining this insight with the fact that $\{y_s\}$ is a classical solution of the adjoint linear terminal value problem (150) of regularity class in line (155), we can apply the second uniqueness result of the second part of this theorem, guaranteeing that there holds $y_s \equiv 0$ on Σ for every $s \in [t_0, t]$. Combining this again with the definition

of the family of functions $\{y_s\}$, and with the fact that $\eta \in (L^p(\Sigma, \mathbf{R}^n))^* \cong L^q(\Sigma, \mathbf{R}^n)$, we can now conclude that

$$0 \equiv \int_{\Sigma} \langle y_t, \xi \rangle_{\mathbf{R}^n} \, d\mu_{\mathcal{P}^*(t,t_0,F_1)} = \int_{\Sigma} \langle \eta, D_F\mathcal{P}^*(t,t,F_1).(\xi) \rangle_{\mathbf{R}^n} \, d\mu_{\mathcal{P}^*(t,t_0,F_1)}$$

$$= \int_{\Sigma} \langle \eta, \xi \rangle_{\mathbf{R}^n} \, d\mu_{\mathcal{P}^*(t,t_0,F_1)} \tag{158}$$

for an arbitrary function $\xi \in W^{4,p}(\Sigma, \mathbf{R}^n)$. Hence, there has to hold $\eta = 0$ in $L^q(\Sigma, \mathbf{R}^n)$ by (158). Since this conclusion holds for every $\eta \in L^q(\Sigma, \mathbf{R}^n)$ satisfying Eq. (151), we have proved that the linear operator

$$G^{F_0}(t,t_0) \equiv D_F\mathcal{P}^*(t,t_0,F_1) : L^p(\Sigma, \mathbf{R}^n) \longrightarrow W^{4,p}(\Sigma, \mathbf{R}^n),$$

appearing in (151), must have dense range in $L^p(\Sigma, \mathbf{R}^n)$, which proves the assertion of the last part of the theorem in the considered case of a fixed time $t \in (t_0, T]$. In the special case "$t = t_0$" this assertion is trivially true.

$\square$

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