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ORIGINAL PAPER

Cauchy problem for the ES-BGK model with the correct Prandtl number

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Abstract
In this paper, we establish the existence of weak solutions to the ellipsoidal BGK model (ES-BGK model) of the Boltzmann equation with the correct Prandtl number, which corresponds to the case when the Knudsen parameter is $-1/2$.

Keywords BGK model · Ellipsoidal BGK model · Boltzmann equation · Kinetic theory of gases · Cauchy problem · Correct Prandtl number

Mathematics Subject Classification 82C40 · 35Q20 · 76P05 · 35F25

1 Introduction

This paper studies the global in time existence of weak solutions to the Cauchy problem of the ES-BGK model:

$$\begin{aligned} \partial_t f + v\cdot\nabla_x f &= A_\nu(\mathcal{M}_\nu(f) - f),\\ f(x,v,0) &= f_0(x,v), \end{aligned} \tag{1.1}$$

in the critical case ($\nu = -1/2$). The particle distribution function $f(x, v, t)$ is the number density of the molecules on the position $x \in \mathbb{R}^3$, with the velocity $v \in \mathbb{R}^3$ at time $t \geq 0$. The Knudsen parameter ν is chosen in the range $-1/2 \leq \nu < 1$, and $A_\nu = 1/(1-\nu)$. The non-isotropic Gaussian $\mathcal{M}_\nu(f)$ parametrized by ν is defined by

$$\mathcal{M}_\nu(f) = \frac{\rho}{\sqrt{\det(2\pi\mathcal{T}_\nu)}}\exp\left(-\frac{1}{2}(v-U)^\top\mathcal{T}_\nu^{-1}(v-U)\right). \tag{1.2}$$

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Here the local density ρ, momentum U, temperature T and stress tensor Θ are defined through the following relations:

$$\begin{aligned} \rho(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)dv, \\ \rho(x,t)U(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)v dv, \\ 3\rho(x,t)T(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)|v-U(x,t)|^2 dv, \\ \rho(x,t)\Theta(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)\big(v-U(x,t)\big)\otimes\big(v-U(x,t)\big)dv. \end{aligned} \tag{1.3}$$

Note that elements of Θ are given by $(1 \le i, j \le 3)$

$$\rho(x,t)\Theta_{ij}(x,t) = \int_{\mathbb{R}^3} f(x,v,t)\big(v-U_i(x,t)\big)\big(v-U_j(x,t)\big)dv.$$

The temperature tensor $\mathcal{T}_\nu$ is given as a linear combination of the temperature and the stress tensor:

$$\begin{aligned} \mathcal{T}_\nu &= (1-\nu)T Id + \nu\Theta \\ &= \begin{pmatrix} (1-\nu)T+\nu\Theta_{11} & \nu\Theta_{12} & \nu\Theta_{13} \\ \nu\Theta_{21} & (1-\nu)T+\nu\Theta_{22} & \nu\Theta_{23} \\ \nu\Theta_{31} & \nu\Theta_{32} & (1-\nu)T+\nu\Theta_{33} \end{pmatrix}, \end{aligned}$$

where Id is the 3×3 identity matrix. We note that on (x,t) where $\rho = 0$, $\mathcal{M}_\nu(f)$ is defined to be zero. The range of ν is restricted to $1/2 \le \nu < 1$ since it is the minimum condition that guarantees the non-negative definiteness of the temperature tensor at least at the formal level [2]. We also mention that the horizontal cross-section of the non-isotropic Gaussian $\mathcal{M}_\nu(f)$ is an ellipsoid, whereas the horizontal-cross section of the usual Maxwellian is a sphere. This is why the model is called the ellipsoidal BGK model.

A direct computation shows that the ellipsoidal Gaussian satisfies

$$\int_{\mathbb{R}^3} \{\mathcal{M}_\nu(f)(x,v,t) - f(x,v,t)\} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0,$$

which leads to the conservation laws of mass, momentum and energy:

$$\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3} f(x,v,t)\begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dxdv = 0.$$

The celebrated H-theorem was verified by Andries et al [2]:

$$\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3} f \ln f dvdx \le 0. \tag{1.4}$$

The Boltzmann equation is the fundamental model for the description of gases at the mesoscopic level. In practice, the BGK model [4] is widely used in place of the Boltzmann equation due to its reliable performance in numerical simulations at much lower computational costs. But the compressible Navier-Stokes limit of the original BGK model shows that the Prandtl number—The ratio between the heat conductivity and the viscosity—is not computed correctly. Holway managed this problem by introducing a free parameter $\nu \in [-1/2, 1)$ and generalizing the local Maxwellian into a non-isotropic Gaussian [13]. When $\nu = 0$, (1.1) reduces to the original BGK model [4] and $\nu = -1/2$ is the choice that yields the correct Prantl number. The ES-BGK model, however, was not employed popularly in the community since the H-theorem was not known. The H-theorem was verified later in [2], and the model got popularized [1, 10–12, 15, 18, 23]. To motivate the current work, we briefly review the results that are directly relevant to this work. Brull et al. derived ES-BGK model systematically using an entropy minimization argument [5]. The entropy production estimate for ES-BGK model was obtained in [22] for the non-critical case $-1/2 < \nu < 1$ and in [14] for the critical case $\nu = -1/2$. The weak solutions and the unique mild solution in the non-critical case, were established in [16], and [8, 19, 20] respectively. The existence of classical solutions near-equilibrium was studied in [21] for $-1/2 \le \nu < 1$. The results on the stationary solution for the ES-BGK in a bounded interval can be found in [3] for the non-critical case and in [6] for the critical case.

All in all, the existence of the ES-BGK model in the non-critical case has been rather thoroughly studied, while many problems remain open for the critical case. One of the main reasons is that, in the non-critical case ($-1/2 < \nu < 1$), the temperature tensor enjoys the following equivalence type estimate [6, 19, 21]:

$$\min\{1-\nu, 1+2\nu\}T\,Id \le \mathcal{T}_\nu \le \max\{1-\nu, 1+2\nu\}T\,Id.$$

Therefore, many estimates of the temperature tensor can be reduced to similar estimates of the local temperature. In the critical case $\nu = -1/2$, however, such estimate breaks down, and the temperature tensor has to be treated with more care. Especially, the existence of weak solutions for (1.1) in the critical case ($\nu = -1/2$) has not been addressed, which is the main purpose of this work. In this regard, our main result is as follows:

Theorem 1.1 *Let $\nu = -1/2$. Suppose that $f_0(x, v) \ge 0$ satisfies*

$$\int_{\mathbb{R}^6} (1 + |v|^2 + |x|^2 + |\ln f_0|) f_0 \, dx dv < \infty.$$

Then, for any final time T^f there exists a non-negative weak solution $f(x, v, t) \in L^1([0, T^f], \mathbb{R}^3 \times \mathbb{R}^3)$ to (1.1):

$$-\int_{\mathbb{R}^3\times\mathbb{R}^3} f_0\phi(0)\,dxdv - \int_0^{T^f}\int_{\mathbb{R}^3\times\mathbb{R}^3} f(\partial_t\phi + v\cdot\nabla_x\phi)\,dxdvdt$$
$$= A_\nu \int_0^{T^f}\int_{\mathbb{R}^3\times\mathbb{R}^3} \big(\mathcal{M}_\nu(f) - f\big)\phi\,dxdvdt$$

for every $\phi \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)$ with $\phi(x, v, T^f) = 0$. Moreover, f satisfies

$$\int_0^{T^f}\int_{\mathbb{R}^6} (1 + |v|^2 + |x|^2 + |\ln f|) f \, dxdvdt < \infty,$$

the conservation laws:

$$\int_{\mathbb{R}^6} f(t) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dxdv = \int_{\mathbb{R}^6} f_0 \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dxdv,$$

and the entropy dissipation ($t_2 \ge t_1 \ge 0$):

$$\int_{\mathbb{R}^6} f(t_2)\ln f(t_2)\,dxdv \le \int_{\mathbb{R}^6} f(t_1)\ln f(t_1)\,dxdv.$$

2 Proof of Theorem 1.1

2.1 Approximate problem

For $n = 1, 2, \cdots$, we set up our approximate problem of (1.1) by

$$\begin{aligned} \partial_t f_n + v \cdot \nabla_x f_n &= A_{-1/2+1/n}\big(\mathcal{M}_{-1/2+1/n}(f_n) - f_n\big), \\ f_n(x, v, 0) &= f_{0,n}(x, v), \end{aligned} \tag{2.1}$$

where f_0^n is the regularized initial data:

$$f_{0,n}(x, v) = f_0(x, v) + \frac{1}{n} m(x, v),$$

with $m(x, v)$ is defined by ($q > 5$)

$$m(x, v) = e^{-|v|^2}(1 + |x|^2)^{-q/2},$$

and $\mathcal{M}_{-1/2+1/n}(f_n)$ corresponds to the non-isotropic Gaussian defined in (1.2) with $\nu = -1/2 + 1/n$:

$$\mathcal{M}_{-1/2+1/n}(f_n) = \frac{\rho_n}{\sqrt{\det(2\pi \mathcal{T}_{-1/2+1/n,n})}} \exp\left(-\frac{1}{2}(v - U_n)^\top \mathcal{T}^{-1}_{-1/2+1/n,n}(v - U_n)\right),$$

where ρ_n, U_n, T_n and Θ_n are the macroscopic fields constructed from the particle distribution function f_n through the relation (1.3), and $\mathcal{T}_{-1/2+1/n,n}$ is the temperature tensor constructed from f_n in the case $\nu = -1/2 + 1/n$:

$$\begin{aligned} \mathcal{T}_{-1/2+1/n,n} &= \left(1 - \left(\frac{1}{2} - \frac{1}{n}\right)\right) T_n Id + \left(\frac{1}{2} - \frac{1}{n}\right)\Theta_n \\ &= \left(\frac{1}{2} + \frac{1}{n}\right) T_n Id + \left(\frac{1}{2} - \frac{1}{n}\right)\Theta_n. \end{aligned}$$

We note that the approximate equation (2.1) corresponds to the ES-BGK model with non-critical Prandtl parameter ($-1/2 < \nu < 1$), whose existence theory is considered in [16]:

Proposition 2.1 *Let T^f be any final time. For each $n = 1, 2, 3, \cdots$, there exists a global weak solution $f_n(x, v, t) \geq 0$ to (2.1):*

$$\begin{aligned} &- \int_{\mathbb{R}^3\times\mathbb{R}^3} f_{0,n}\phi(0)dxdv - \int_0^{T^f}\int_{\mathbb{R}^3\times\mathbb{R}^3} f_n(\partial_t\phi + v\cdot\nabla_x\phi)dxdvdt \\ &\quad = A_{-1/2+1/n}\int_0^{T^f}\int_{\mathbb{R}^3\times\mathbb{R}^3}\big(\mathcal{M}_{-1/2+1/n}(f_n) - f_n\big)\phi dxdvdt \end{aligned}$$

for every $\phi \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)$ with $\phi(x, v, T^f) = 0$. Moreover

1. *f_n satisfies*

$$\int_0^{T^f}\int_{\mathbb{R}^6}(1 + |v|^2 + |x|^2 + |\ln f_n|) f_n dxdvdt < C,$$

for some $C > 0$ independent of n.

2. *The conservation laws hold:*

$$\int_{\mathbb{R}^6} f_n(t) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dxdv = \int_{\mathbb{R}^6} f_{0,n} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dxdv.$$

3. f_n *satisfies the entropy dissipation:*

$$\int_{\mathbb{R}^6} f_n(t_2) \ln f_n(t_2) dxdv \le \int_{\mathbb{R}^6} f_n(t_1) \ln f_n(t_1) dxdv. \quad (t_2 \ge t_1)$$

4. *For any compact set* $K_x \subseteq \mathbb{R}^3_x$, f_n *satisfies the following moment estimate:*

$$\int_0^{T^f} \int_{K_x} \int_{\mathbb{R}^3} |v|^3 f_n(x, v, t) dvdxdt \le C_{K_x}.$$

5. $\mathcal{T}_{-1/2+1/n,n}$ *is strictly positive definite:*

$$\kappa^\top \mathcal{T}_{-1/2+1/n,n}(x,t)\kappa \ge C_{T^f, f_{0,n},n}(1+|x|^2)^{-q/2} > 0, \textit{ for any } \kappa \in \mathbb{S}^2.$$

Remark 2.1 (1) The 3rd moment is established by Perthame in [17]. (2) The strictly positive definiteness in (5) holds due to the fact that the regularized initial data $f_{0,n}$ has a strict lower bound. See Theorem 2.1. in [16].

The following estimate is also crucially used for the weak L^1 compactness of $\mathcal{M}_{-1/2+1/n}$.

2.2 Weak compactness of f_n and $\mathcal{M}_{-1/2+1/n}(f_n)$

We deduce from Proposition 2.1 and Dunford-Pettis theorem [7, 9] that there exists $f \in L^1$ such that f_n, $f_n v$ converge to f, fv weakly $L^1(\mathbb{R}^3 \times \mathbb{R}^3 \times [0, T^f])$. This, combined with the velocity averaging lemma gives

$$\begin{aligned}
\rho_n &= \int_{\mathbb{R}^3} f_n dv \rightharpoonup \int_{\mathbb{R}^3} f dv = \rho \quad \text{in} \quad L^1([0, T^f], \mathbb{R}^3_x), \\
\rho_n U_n &= \int_{\mathbb{R}^3} f_n v dv \rightharpoonup \int_{\mathbb{R}^3} f v dv = \rho U \quad \text{in} \quad L^1([0, T^f], \mathbb{R}^3).
\end{aligned}$$

Similarly, but this time combined with Proposition 2.1 (4), it can be shown that

$$\int_{\mathbb{R}^3} f_n v_i v_j dv \rightharpoonup \int_{\mathbb{R}^3} f v_i v_j dv$$

in $L^1([0, T^f], K_x \times \mathbb{R}^3)$, so that

$$\begin{aligned}
&\rho_n \mathcal{T}_{-1/2+1/n,n} + \rho_n \left\{ \Big(\frac{1}{2} - \frac{1}{3n}\Big)|U_n|^2 Id + \Big(-\frac{1}{2} + \frac{1}{n}\Big)\rho_n U_n \otimes U_n \right\} \\
&\quad = \int_{\mathbb{R}^3} f_n \left\{ \Big(\frac{1}{2} - \frac{1}{3n}\Big)|v|^2 Id + \Big(-\frac{1}{2} + \frac{1}{n}\Big) v \otimes v \right\} dv \\
&\quad \rightharpoonup \int_{\mathbb{R}^3} f \left\{ \frac{1}{2}|v|^2 Id - \frac{1}{2} v \otimes v \right\} dv \\
&\quad = \rho \mathcal{T}_{-1/2} + \rho \left\{ \frac{1}{2}|U|^2 Id - \frac{1}{2}\rho U \otimes U \right\},
\end{aligned}$$

in $L^1([0,T^f], K_x\times\mathbb{R}^3)$. Therefore, we have almost everywhere convergence of macroscopic fields on a set where ρ does not vanish:

$$\begin{aligned} \rho_n &\to \rho \qquad \text{a.e on } \mathbb{R}^3\times[0,T^f],\\ U_n &\to U \qquad \text{a.e on } \mathbb{E},\\ \mathcal{T}_{-1/2+1/n,n} &\to \mathcal{T}_{-1/2} \qquad \text{a.e on } \mathbb{E}, \end{aligned} \tag{2.2}$$

where $\mathbb{E}$ is defined by

$$\mathbb{E}=\{(x,t)\in\mathbb{R}^3\times(0,T^f)\mid\rho(x,t)\neq 0\}. \tag{2.3}$$

On the other hand, the weak compactness of $\mathcal{M}_{-1/2+1/n}(f_n)$ in $L^1((0,T^f)\times\mathbb{R}^3\times\mathbb{R}^3)$ follows from the following inequality established in Lemma 2.3 of [16] with a $C>0$ independent of n:

$$\int_0^{T^f}\int_{\mathbb{R}^6}(1+|v|^2+|x|^2+|\ln\mathcal{M}_{-1/2+1/n}(f_n)|)\mathcal{M}_{-1/2+1/n}(f_n)dxdvdt<C.$$

Therefore, we can find $M\in L^1([0,T^f],\mathbb{R}^3\times\mathbb{R}^3)$ such that $\mathcal{M}_{-1/2+1/n}$ converges weakly in L_1 to M as $n\to\infty$.

2.3 Conclusion of the proof

It remains to check that

$$M=\mathcal{M}_{-1/2}(\rho,U,\mathcal{T}_{-1/2}).$$

For this, we define

$$\mathbb{A}=\left\{(x,t)\in\mathbb{R}^3_x\times[0,T^f]|\ k^\top\mathcal{T}_{-1/2}k\neq 0 \text{ for all non zero } k\in\mathbb{R}^3\right\}$$

and consider (Recall that $\mathbb{E}$ is defined in (2.3).)

$$\begin{aligned} &\int_0^{T^f}\int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v}\mathcal{M}_{-1/2+1/n}(f_n)\phi dvdxdt\\ &\quad=\int_{\mathbb{A}\cap\mathbb{E}}\int_{\mathbb{R}^3_v}\mathcal{M}_{-1/2+1/n}(f_n)\phi dvdxdt+\int_{\mathbb{A}\cap\mathbb{E}^c}\int_{\mathbb{R}^3_v}\mathcal{M}_{-1/2+1/n}(f_n)\phi dvdxdt\\ &\qquad+\int_{\mathbb{A}^c\cap\mathbb{E}^c}\int_{\mathbb{R}^3_v}\mathcal{M}_{-1/2+1/n}(f_n)\phi dvdxdt+\int_{\mathbb{A}^c\cap\mathbb{E}}\int_{\mathbb{R}^3_v}\mathcal{M}_{-1/2+1/n}(f_n)\phi dvdxdt\\ &\quad:=I_1+I_2+I_3+I_4. \end{aligned}$$

Below, we consider each integrals separately to show that M coincides with $\mathcal{M}_{-1/2}$ on each subset of $\mathbb{R}^3\times\mathbb{R}^3$.

• I_1: Since $\rho\neq 0$, we find from (2.2) that $\mathcal{M}_{-1/2+1/n}(\rho_n,U_n,\mathcal{T}_{-1/2+1/n,n})$ converges almost everywhere to $\mathcal{M}_{-1/2}(\rho,U,\mathcal{T}_{-1/2})$. Therefore, using Fatou's Lemma, we get

$$\int_{\mathbb{A}\cap\mathbb{E}}\int_{\mathbb{R}^3_v}\mathcal{M}_{-1/2}(\rho,U,\mathcal{T}_{-1/2})\phi dvdxdt\leq\lim_{n\to\infty}\int_{\mathbb{A}\cap\mathbb{E}}\int_{\mathbb{R}^3_v}\mathcal{M}_{-1/2+1/n}(f_n)\phi dvdxdt.$$

But we have from the definition of M that

$$\int_{\mathbb{A}\cap\mathbb{E}}\int_{\mathbb{R}^3_v}\mathcal{M}_{-1/2+1/n}(f_n)\phi dvdxdt=\int_{\mathbb{A}\cap\mathbb{E}}\int_{\mathbb{R}^3_v}M\phi dvdxdt.$$

This yields

$$\int_{\mathbb{A}\cap\mathbb{E}}\int_{\mathbb{R}^3_v}\mathcal{M}_{-1/2+1/n}(\rho,U,\mathcal{T}_{-1/2+1/n})\phi dvdxdt \le \int_{\mathbb{A}\cap\mathbb{E}}\int_{\mathbb{R}^3_v} M\phi dvdxdt. \tag{2.4}$$

To show the reverse inequality, we choose $\phi = 1$ and observe from the definition of M that

$$\int_{\mathbb{A}\cap\mathbb{E}}\int_{\mathbb{R}^3_v} Mdvdxdt = \lim_{n\to\infty}\int_{\mathbb{A}\cap\mathbb{E}}\int_{\mathbb{R}^3_v}\mathcal{M}_{-1/2+1/n}(f_n)dvdxdt.$$

Since $\mathcal{T}_{-1/2+1/n}$ is strictly positive definite by Proposition 2.1 (5), we can take the change of variable:

$$X = \mathcal{T}^{-1/2}_{-1/2+1/n}(v-U)$$

to compute

$$\lim_{n\to\infty}\int_{\mathbb{A}\cap\mathbb{E}}\int_{\mathbb{R}^3_v}\mathcal{M}_{-1/2+1/n}(f_n)dvdxdt = \lim_{n\to\infty}\int_{\mathbb{A}\cap\mathbb{E}}\int_{\mathbb{R}^3_v}\rho_n dvdxdt = \int_{\mathbb{A}\cap\mathbb{E}}\int_{\mathbb{R}^3_v}\rho dvdxdt.$$

The last line comes from (2.2). Now, since $\mathcal{T}_{-1/2}(x,t)$ is also strictly positive definite on $\mathbb{A}$. We can take the change of variable:

$$Y = \mathcal{T}^{-1/2}_{-1/2}(v-U),$$

to get

$$\int_{\mathbb{A}\cap\mathbb{E}}\rho dxdt = \int_{\mathbb{A}\cap\mathbb{E}}\int_{\mathbb{R}^3_v}\mathcal{M}_{-\frac{1}{2}}(\rho,U,\mathcal{T}_{-1/2})dvdxdt.$$

In summary, we have on $\mathbb{A}\cap\mathbb{E}$

$$\int_{\mathbb{A}\cap\mathbb{E}}\int_{\mathbb{R}^3_v} Mdvdxdt = \int_{\mathbb{A}\cap\mathbb{E}}\int_{\mathbb{R}^3_v}\mathcal{M}_{-\frac{1}{2}}(\rho,U,\mathcal{T}_{-1/2})dvdxdt. \tag{2.5}$$

From (2.4) and (2.5), we conclude that

$$M = \mathcal{M}_{-\frac{1}{2}}(\rho,U,\mathcal{T}_{-1/2})$$

almost everywhere on $\mathbb{A}\cap\mathbb{E}$.

• I_2: Using the same argument of case I_1, we find

$$\begin{aligned}\int_{\mathbb{A}\cap\mathbb{E}^c}\int_{\mathbb{R}^3_v} Mdvdxdt &= \lim_{n\to\infty}\int_{\mathbb{A}\cap\mathbb{E}^c}\int_{\mathbb{R}^3_v}\mathcal{M}_{-1/2+1/n}(f_n)dvdxdt = \lim_{n\to\infty}\int_{\mathbb{A}\cap\mathbb{E}^c}\rho_n dxdt\\ &= \int_{\mathbb{A}\cap\mathbb{E}^c}\rho dxdt = 0.\end{aligned}$$

Therefore, $M = 0 = \mathcal{M}_{-1/2}(\rho,U,\mathcal{T}_{-1/2})$.

• I_3 : Since $\rho = 0$, we have $\mathcal{M}_{-1/2}(f) = 0$ by definition. Therefore, by the Fatou's lemma and the fact that $\mathcal{M}_\nu(f_n)$ converges in weak L^1 to M, we have

$$\begin{aligned}0 &= \int_{\mathbb{A}^c\cap\mathbb{E}^c}\int_{\mathbb{R}^3_v}\mathcal{M}_{-1/2}(f)\phi dvdxdt\\ &\le \lim_{n\to\infty}\int_{\mathbb{A}^c\cap\mathbb{E}^c}\int_{\mathbb{R}^3_v}\mathcal{M}_{-1/2+1/n}(f_n)\phi dvdxdt\\ &= \int_{\mathbb{A}^c\cap\mathbb{E}^c}\int_{\mathbb{R}^3_v} M\phi dvdxdt.\end{aligned}$$

On the other hand, fixing ϕ to 1 and proceeding as in the previous case, we get

$$\int_{\mathbb{A}^c\cap\mathbb{E}^c}\int_{\mathbb{R}^3_v} M dv dx dt = \lim_{n\to\infty}\int_{\mathbb{A}^c\cap\mathbb{E}^c}\int_{\mathbb{R}^3_v} \mathcal{M}_{-1/2+1/n}(f_n) dv dx dt = \lim_{n\to\infty}\int_{\mathbb{A}^c\cap\mathbb{E}^c} \rho_n dx dt$$
$$= \int_{\mathbb{A}^c\cap\mathbb{E}^c} \rho dx dt = 0.$$

- I_4: $(x,t)\in\mathbb{A}^c$ means that there exists a non-zero vector $k(x,t)\in\mathbb{R}^3$ such that

$$k^\top(x,t)\mathcal{T}_{-1/2}(x,t)k(x,t) = 0.$$

We can find through an explicit computation using

$$Y^\top(X\otimes X)Y = \{X\cdot Y\}^2 \quad (X, Y\in\mathbb{R}^3).$$

Since $(x,t)\in\mathbb{A}$, the statement $k^\top\mathcal{T}_{-1/2}k = 0$, is equivalent to $k^\top\left\{\rho\mathcal{T}_{-1/2}\right\}k = 0$. But

$$\begin{aligned} k^\top\left\{\rho_{-1/2}\right\}\mathcal{T}_{-1/2}k &= k^\top\rho\left(\frac{3}{2}T_{-1/2} - \frac{1}{2}\Theta_{-1/2}\right)k \\ &= k^\top\left\{\frac{1}{2}\int_{\mathbb{R}^3_v} f|v-U|^2 dv\right\}k - k^\top\left\{\frac{1}{2}\int_{\mathbb{R}^3_v} f(v-U)\otimes(v-U)dv\right\}k \\ &= \frac{1}{2}\int_{\mathbb{R}^3_v} f|v-U|^2|k|^2 dv - \frac{1}{2}\int_{\mathbb{R}^3_v} f\{(v-U)\cdot k\}^2 dv \\ &= \frac{1}{2}\int_{\mathbb{R}^3_v} f\left\{|v-U|^2|k|^2 - \{(v-U)\cdot k\}^2\right\}dv. \end{aligned}$$

Recalling

$$|v-U|^2|k|^2 - \{(v-U)\cdot k\}^2 \geq 0.$$

One finds that

$$f(t,x,v)\left\{|v-U|^2|k|^2 - \{(v-U)\cdot k\}^2\right\} = 0$$

on $(t,x,v)\in\mathbb{A}^c\cap\mathbb{E}\times\mathbb{R}^3_v$. If f is identically zero on the set, we are done. If not, there exists a measurable set B of strictly positive measure such that

$$f(t,x,v) > 0 \text{ on } B.$$

Therefore,

$$\left\{|v-U|^2|k|^2 - \{(v-U)\cdot k\}^2\right\} = 0$$

on B, which is possible only when $v - U(x,t)$ and $k(x,t)$ are parallel on B. Combining the conclusion This is contradiction since k does not depend on v. From this, we conclude that

$$f(t,x,v) = 0 \quad \text{for } (x,t)\in\mathbb{E} \text{ and } \forall v\in\mathbb{R}^3.$$

Therefore, we have desired result from the same argument as in the case of I_2.

Combining the arguments above, we conclude that $M = \mathcal{M}_{-1/2}$ on $\mathbb{R}^3\times\mathbb{R}^3$, and the proof of main theorem is completed.

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Declarations

Conflicts of interest The authors state that there is no conflict of interest.

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