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ORIGINAL PAPER

On a sharp weighted Sobolev inequality on the upper half-space and its applications

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Abstract
In this paper, we establish a sharp weighted Sobolev inequality on the upper half-space. We also discourse existence and nonexistence of minimizer . As an application, we study a quasilinear problem on the upper half-space.

Keywords Weighted Sobolev inequality · Trace inequality · Upper half-space · Minimizers · Neumann problem

Mathematics Subject Classification 35J85 · 47J20 · 46E35

1 Introduction and main results

1.1 Overview

In the past decades, the inequalities of Sobolev type have played a fundamental role both from the theoretical point of view in calculus of variations as well as from the point of view of applications in the development of many branches of mathematics and physics. In [16], G. Talenti proved the following Gagliardo–Nirenberg–Sobolev inequality: for $p \in (1, n)$, there exists a constant $C_0(n, p) > 0$ such that

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$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{\frac{p}{p^*}} \leq C_0 \int_{\mathbb{R}^n} |\nabla u|^p dx, \quad \text{for all} \;\; u \in C_0^1(\mathbb{R}^n), \tag{1.1}$$

where $p^* := \frac{np}{n-p}$ denotes the critical Sobolev exponent and the extremals have the form

$$u(x) = \left(a + b|x|^{\frac{p}{p-1}}\right)^{\frac{p-n}{p}}, \;\; a, \;\; b > 0.$$

In studying boundary value problems in the domain Ω instead of the entire space $\mathbb{R}^n$, the so-called Sobolev trace inequalities are particularly of importance. For the Euclidean upper half-space

$$\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > 0\},$$

the Sobolev trace embedding states that there exists a constant $C_1(n, p) > 0$ such that

$$\left(\int_{\mathbb{R}^{n-1}} |u|^{p_*} dx'\right)^{\frac{p}{p_*}} \leq C_1 \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx\right), \quad \text{for all} \;\; u \in C_0^1(\mathbb{R}^n), \tag{1.2}$$

where $p_* := \frac{p(n-1)}{n-p}$ denotes the critical Sobolev trace exponent. In [9], some sharp constants related to (1.2) with $p = 2$ have been established by J. Escobar and the extremals are given by the form of

$$u(x', x_n) = \left(\frac{\varepsilon}{(\varepsilon + x_n)^2 + |x' - x_0'|^2}\right)^{(n-2)/2}, \quad \varepsilon > 0$$

Moreover, the author conjectured that for $p \in (1, n)$, the Sobolev trace inequality (1.2) admits the similar extremals to the case $p = 2$. This conjecture was shown to be true by B. Nazaret in [15] via a mass transportation method.

1.2 Main purpose and motivation

The main purpose of this present paper is two-folds. First, we aim to obtain some optimal conditions on $q > 0$ and α to the validity of the weighted anisotropic Sobolev inequality

$$\left(\int_{\mathbb{R}^n_+} \frac{|u|^q}{(1 + x_n)^\alpha} dx\right)^{\frac{p}{q}} \leq B_0 \int_{\mathbb{R}^n_+} |\nabla u|^p dx, \quad \text{for all} \;\; u \in C_0^\infty(\mathbb{R}^n), \tag{1.3}$$

for some constant $B_0(n, q, p, \alpha) > 0$. It is worthwhile to mention that an important feature of this inequality is the fact that the anisotropic weight function $w(x', x_n) = (1+x_n)^{-\alpha}$ does not belong to any Lebesgue space $L^p(\mathbb{R}^n_+)$. Second, we want to provide some conditions on existence and nonexistence of minimizes to best constants of this inequality.

Remark 1.1 Notice that, the inequality (1.3) holds true for any $\alpha \geq 0$ and $q = p^*$. Indeed, for any $u \in C_0^\infty(\mathbb{R}^n)$, defining

$$\tilde{u}(x', x_n) := \begin{cases} u(x', x_n), & \text{if} \;\; x_n \geq 0 \\ -3u(x', -x_n) + 4u(x', -x_n/2), & \text{if} \;\; x_n < 0, \end{cases}$$

and performing a straightforward computation we see that $\tilde{u} \in C_0^1(\mathbb{R}^n)$ and there exists a constant $\tilde{C} = \tilde{C}(n, p) > 0$ such that

$$\int_{\mathbb{R}^n} |\nabla \tilde{u}|^p dx \leq \tilde{C} \int_{\mathbb{R}_+^n} |\nabla u|^p dx.$$

This, combined with (1.1), applied to $\tilde{u}$, yields that

$$\left(\int_{\mathbb{R}_+^n} |u|^{p^*} dx \right)^{\frac{p}{p^*}} \leq K_0 \left(\int_{\mathbb{R}_+^n} |\nabla u|^p dx \right), \quad \text{for all } u \in C_0^\infty(\mathbb{R}^n), \tag{1.4}$$

for some $K_0(n, p) > 0$, which clearly implies that (1.3) holds for any $\alpha \geq 0$ and $q = p^*$.

Remark 1.2 The inequality (1.3) is false on the region

$$\mathcal{R}_2 := \{(q, \alpha) : q > p^* \text{ and } \alpha \in \mathbb{R}\},$$

which is enough to assume $\alpha > 0$. To see this, let $u_0 \in C_0^\infty(\mathbb{R}^n) \setminus \{0\}$ and define $u_\lambda(x) = \lambda^{(n-p)/p} u_0(\lambda x)$ for $x \in \mathbb{R}^n$ and $\lambda > 0$. A straightforward calculation shows that there exists $C_1 > 0$ independent of λ such that

$$\int_{\mathbb{R}_+^n} |\nabla u_\lambda|^p dx = C_1$$

and making a change of variables, for any $\lambda \geq 1$ we get

$$\int_{\mathbb{R}_+^n} \frac{|u_\lambda|^q}{(1 + x_n)^\alpha} dx = \lambda^{\frac{(n-p)q}{p} - n} \int_{\mathbb{R}_+^n} \frac{|u_0(y)|^q}{(1 + \frac{y_n}{\lambda})^\alpha} dy \geq \lambda^{\frac{(n-p)q}{p} - n} \int_{\mathbb{R}_+^n} \frac{|u_0(y)|^q}{(1 + y_n)^\alpha} dy.$$

Assume by contradiction that the inequality holds. Then, in particular, for some $C_2 > 0$ we have

$$0 < B_0^{-1} \leq \frac{\int_{\mathbb{R}_+^n} |\nabla u_\lambda|^p dx}{\left(\int_{\mathbb{R}_+^n} \frac{|u_\lambda|^q}{(1+x_n)^\alpha} dx \right)^{\frac{p}{q}}} \leq \frac{C_1}{C_2 \lambda^{n-p-\frac{np}{q}}} \to 0,$$

if $q > p^*$ and $\lambda \to +\infty$ we reach a contradiction and hence the inequality is false for all $q > p^*$ and $\alpha > 0$.

Remark 1.3 The inequality (1.3) also is false on the region

$$\mathcal{R}_1 := \{(q, \alpha) : q < p_* \text{ and } \alpha \in \mathbb{R}\}.$$

To see this, is sufficient to assume $\alpha > 1$. Consider a function $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi(x) = 1$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| \geq 2$. Let us define $\phi_t(x) = \phi(x/t)$ for $x \in \mathbb{R}^n$ and $t > 0$. A straightforward calculation shows that there are constants $C_1, C_2 > 0$ independent of t such that

$$\int_{\mathbb{R}_+^n} |\nabla \phi_t|^p dx = C_1 t^{n-p}$$

and

$$\begin{aligned} \int_{\mathbb{R}_+^n} \frac{|\phi_t|^q}{(1 + x_n)^\alpha} dx &\geq \int_0^{\frac{t}{\sqrt{2}}} \int_{|x'| \leq t/\sqrt{2}} \frac{1}{(1 + x_n)^\alpha} dx' dx_n \\ &= \left[\frac{1}{(\alpha - 1)} - \frac{1}{(\alpha - 1)(1 + t/\sqrt{2})^{\alpha - 1}} \right] C_2 t^{n-1}, \quad \text{for all } \alpha > 1. \end{aligned}$$

Hence, if the inequality (1.3) holds we have

$$0 < B_0^{-1} \leq \frac{\int_{\mathbb{R}^n_+} |\nabla \phi_t|^p dx}{\left(\int_{\mathbb{R}^n_+} \frac{|\phi_t|^q}{(1+x_n)^\alpha} dx \right)^{\frac{p}{q}}} \leq C_4 t^{n-p-\frac{p(n-1)}{q}},$$

for t large and any $\alpha > 1$. In particular, if $q < p_*$ we obtain a contradiction by letting $t \to +\infty$.

We also mention that Hardy–Sobolev type inequalities on the upper-half space appear in many papers, see for instance, [7, 10, 11, 14, 17] and references therein. In [7] Chen–Li proved the inequality

$$\left(\int_{\mathbb{R}^n_+} \frac{|u|^{2^*(\alpha)}}{x_n^\alpha} dx \right)^{2/2^*(\alpha)} \leq C \int_{\mathbb{R}^n_+} |\nabla u|^2 dx, \quad \text{for all } \ u \in C_0^\infty(\mathbb{R}^n_+),$$

where $2^*(\alpha) = 2(n-\alpha)/(n-2)$ for all $\alpha \in (0, 2]$. In fact, in all these papers the inequalities are derived for functions $u \in C_0^\infty(\mathbb{R}^n_+)$.

Finally, we observe that an inspection in Remark 1.2 shows that inequality (1.3) does not hold for $p = n$ and $\alpha > 1$. Indeed, in this case there are constants $C_1 > 0$ and $C_2 > 0$ such that

$$\int_{\mathbb{R}^n_+} |\nabla \phi_t|^n dx = C_1 \quad \text{and} \quad \int_{\mathbb{R}^n_+} \frac{|\phi_t|^q}{(1+x_n)^\alpha} dx \geq C_2 t^{n-1},$$

which yields that

$$\frac{\int_{\mathbb{R}^n_+} |\nabla \phi_t|^n dx}{\left(\int_{\mathbb{R}^n_+} \frac{|\phi_t|^q}{(1+x_n)^\alpha} dx \right)^{\frac{n}{q}}} \to 0, \qquad \text{as} \quad t \to \infty.$$

For a related inequality when $q = n$, we refer the reader to [1].

1.3 Main results

Motivated by the mentioned papers and the previous Remarks, a natural question is

whether inequality (1.3) holds for $q < p^*$?

A complete answer is given below concerning this specifically issue. To this purpose, we will borrow some idea from [6,Proposition 3.5] and [11,Thorem 1], see also [8] for $p = 2$. In this context, our first main result reads as follows.

Theorem 1.1 *Let $1 < p < n$ and $\alpha > 1$. Then there exists $C = C(n, \alpha, p) > 0$ such that*

$$\left(\int_{\mathbb{R}^n_+} \frac{|u|^{p_*}}{(1+x_n)^\alpha} dx \right)^{\frac{p}{p_*}} \leq C \int_{\mathbb{R}^n_+} |\nabla u|^p dx, \quad \forall\, u \in C_0^\infty(\mathbb{R}^n). \tag{1.5}$$

Furthermore, the power $\alpha > 1$ is optimal in the sense that this inequality is false for any $\alpha \leq 1$.

Let $1 < p < n$ and $\alpha > 1$. Since $(1 + x_n)^{\alpha} \geq 1$ for $x_n \geq 0$, by inequality (1.4) we see that

$$\left(\int_{\mathbb{R}^n_+} \frac{|u|^{p^*}}{(1 + x_n)^{\alpha}} dx \right)^{\frac{p}{p^*}} \leq K_0 \int_{\mathbb{R}^n_+} |\nabla u|^p dx, \quad \text{for all } u \in C_0^{\infty}(\mathbb{R}^n).$$

Thus, by interpolating this inequality with (1.5), for all $q \in [p_*, p^*]$ it holds that

$$\left(\int_{\mathbb{R}^n_+} \frac{|u|^{q}}{(1 + x_n)^{\alpha}} dx \right)^{\frac{p}{q}} \leq C \int_{\mathbb{R}^n_+} |\nabla u|^p dx, \quad \text{for all } \ u \in C_0^{\infty}(\mathbb{R}^n), \tag{1.6}$$

for some constant $C = C(n, \alpha, p, q) > 0$. In fact, we have an improvement of (1.6). Precisely, defining the function

$$\alpha(q) := n - \frac{(n-p)q}{p} \in [0, 1], \quad q \in [p_*, p^*],$$

we see that $\alpha(p_*) = 1$, $\alpha(p^*) = 0$ and the following result holds:

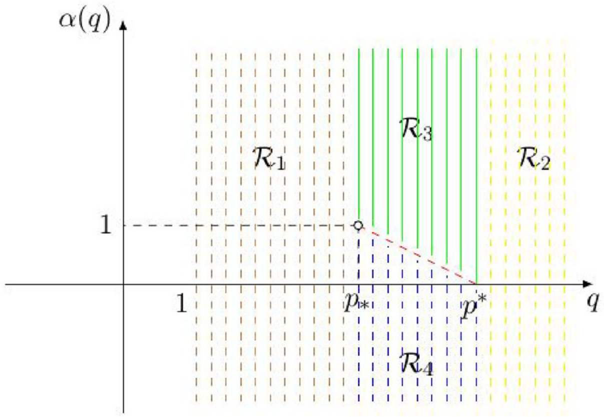
Corollary 1.1 *Let $1 < p < n$. Then the inequality (1.6) holds true on the region*

$$\mathcal{R}_3 := \left\{(q, \alpha) : p_* \leq q \leq p^* \text{ and } \alpha > \alpha(q)\right\} \cup \{(p^*, \alpha(p^*))\}$$

and is false on the region

$$\mathcal{R}_4 := \{(p_*, \alpha(p_*))\} \cup \{(q, \alpha) : p_* \leq q \leq p^* \text{ and } -\infty < \alpha < \alpha(q)\}.$$

For a better comprehension we present a graphic that corresponds the regions obtained in the previous results.



As another consequence of Theorem 1.1, we derive an inequality with exponential weight that was proved and used in [6] to study the asymptotic profile of ground state of a Henon equation with Neumann boundary conditions when $p = 2$. In fact, for any $\tau > 0$ and $\alpha > 1$, there exists $C_0 = C_0(\tau, \alpha) > 0$ such that

$$\exp(\tau x_n) \geq C_0(1 + x_n)^{\alpha}, \quad \text{for all} \quad x_n \geq 0.$$

Thus, as a consequence of inequality 1.6, we have

Corollary 1.2 *Let* $1 < p < n$ *and* $\tau > 0$*. Then, for any* $q \in [p_*, p^*]$ *there exists* $C_1 = C_1(\tau, q, p)$ *such that*

$$\left(\int_{\mathbb{R}^n_+} \frac{|u|^q}{\exp(\tau x_n)} dx\right)^{\frac{p}{q}} \leq C_1 \int_{\mathbb{R}^n_+} |\nabla u|^p dx, \quad \textit{for all} \quad u \in C^1_0(\mathbb{R}^n).$$

Furthermore, the condition $q \geq p_*$ *is necessary.*

Next, denote by $C^\infty_\delta(\mathbb{R}^n_+)$ the space of $C^\infty_0(\mathbb{R}^n)-$functions restricted to $\mathbb{R}^n_+$. For $1 < p < n$ we define the Sobolev space E as the completion of $C^\infty_\delta(\mathbb{R}^n_+)$ with respect to the norm

$$\|u\|_E := \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

In view of Corollary 1.1, we have the following embedding result.

Corollary 1.3 *Let* $1 < p < n$*. Then the weighted Sobolev embedding*

$$E \hookrightarrow L^q\left(\mathbb{R}^n_+, \frac{1}{(1+x_n)^\alpha}\right) \tag{1.7}$$

is continuous, for all pair $(q, \alpha) \in \mathcal{R}_3$.

This paper is organized as follows. In Sect. 2, we give the proofs of Theorem 1.1 and Corollary 1.1. Section 3 is devoted to existence and nonexistence of minimizers for the best constant in inequality (1.5). In Sect. 4, as an application of inequality (1.5), we investigate the existence of solutions to a quasilinear elliptic equation in the upper half-space with Neumann boundary condition. In Sect. 5, some open questions are given.

2 Proofs of Theorem 1.1 and Corollary 1.1

Proof of Theorem 1.1 Let $v \in C^\infty_0(\mathbb{R}^n)$ and $\sigma \in \mathbb{R}$ with $\sigma \neq -1$. Integrating by parts one has

$$\begin{aligned}(\sigma+1)\int_{\mathbb{R}^n_+}(1+x_n)^\sigma |v| dx &= \int_{\mathbb{R}^n_+} \partial_{x_n}((1+x_n)^{\sigma+1})|v| dx \\ &= -\int_{\mathbb{R}^n_+}(1+x_n)^{\sigma+1}(|v|)_{x_n} dx - \int_{\mathbb{R}^{n-1}} |v| dx',\end{aligned}$$

where we used that the unit outward normal on the boundary $\mathbb{R}^{n-1}$ is $\nu = (0', -1)$. Thus,

$$|\sigma+1|\int_{\mathbb{R}^n_+}(1+x_n)^\sigma |v| dx \leq \int_{\mathbb{R}^n_+}(1+x_n)^{\sigma+1}|\nabla v| dx + \int_{\mathbb{R}^{n-1}} |v| dx'.$$

Applying this inequality with $v = |u|^q$ for $u \in C^\infty_0(\mathbb{R}^n)$, $q > 1$ and using that $\sigma + 1 < 0$ we obtain

$$|\sigma+1|\int_{\mathbb{R}^n_+}(1+x_n)^\sigma |u|^q dx \leq q\int_{\mathbb{R}^n_+} |u|^{q-1}|\nabla u| dx + \int_{\mathbb{R}^{n-1}} |u|^q dx'. \tag{2.1}$$

Now, choosing $q = p_*$ and using the trace inequality (1.2) we obtain $C_1(n, p) > 0$ such that

$$\int_{\mathbb{R}^{n-1}} |u|^{p_*} dx' \le \left(C_1(n, p) \int_{\mathbb{R}^n_+} |\nabla u|^p dx \right)^{\frac{n-1}{n-p}}. \tag{2.2}$$

On the other hand, by using the Hölder inequality and (1.4) it follows that

$$\begin{aligned} \int_{\mathbb{R}^n_+} |u|^{p_*-1} |\nabla u| dx &\le \left(\int_{\mathbb{R}^n_+} |u|^{p^*} dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx \right)^{\frac{1}{p}} \\ &\le K_0^{\frac{n(p-1)}{p(n-p)}} \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx \right)^{\frac{n-1}{n-p}}. \end{aligned} \tag{2.3}$$

Combining inequalities (2.1), (2.2) and (2.3) we conclude that

$$|\sigma + 1| \int_{\mathbb{R}^n_+} \frac{|u|^{p_*}}{(1 + x_n)^{-\sigma}} dx \le (p_* K_0^{\frac{n(p-1)}{p(n-p)}} + C_1^{\frac{n-1}{n-p}}) \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx \right)^{\frac{n-1}{n-p}}.$$

Thus, choosing $\alpha = -\sigma > 1$, we obtain

$$\left(\int_{\mathbb{R}^n_+} \frac{|u|^{p_*}}{(1 + x_n)^{\alpha}} dx \right)^{\frac{1}{p_*}} \le \left(\frac{p_* K_0^{\frac{n(p-1)}{p(n-p)}} + C_1^{\frac{n-1}{n-p}}}{|-\alpha + 1|} \right)^{\frac{1}{p_*}} \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx \right)^{\frac{1}{p}},$$

which is the desired inequality.

Next we shall prove that (1.5) is false for any $\alpha \le 1$. To this end, it is enough to consider $\alpha = 1$. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi(x) = 1$ for $|x| \le 1$ and $\phi(x) = 0$ for $|x| \ge 2$. For any $t > 0$, we define $\phi_t(x) = \phi(x/t)$ for all $x \in \mathbb{R}^n$. A straightforward calculation shows that there are constants $C_1, C_2 > 0$ independent of t such that

$$\int_{\mathbb{R}^n_+} |\nabla \phi_t|^p dx = C_1 t^{n-p}$$

and using that $|(x', x_n)| \le t$ if $x_n \le t/\sqrt{2}$ and $|x'| \le t/\sqrt{2}$, we conclude that

$$\int_{\mathbb{R}^n_+} \frac{|\phi_t|^{p_*}}{1 + x_n} dx \ge \int_0^{\frac{t}{\sqrt{2}}} \int_{|x'| \le \frac{t}{\sqrt{2}}} \frac{1}{1 + x_n} dx' dx_n \quad = \log \left(\frac{t}{\sqrt{2}} + 1 \right) C_2 t^{n-1}.$$

Now, assume by contradiction that (1.5) holds true. Then, we have

$$0 < C_0 \le \frac{\int_{\mathbb{R}^n_+} |\nabla \phi_t|^p dx}{\left(\int_{\mathbb{R}^n_+} \frac{|\phi_t|^{p_*}}{1 + x_n} dx \right)^{\frac{p}{p_*}}} \le \frac{C_1}{\left[C_2 \log \left(\frac{t}{\sqrt{2}} + 1 \right) \right]^{\frac{p}{p_*}}},$$

which is a contradiction since the right-hand side of the last inequality goes to zero as $t \to \infty$ and this completes the proof of Theorem 1.1. □

Proof of Corollary 1.1 If $q = p_*$ this result is exactly Theorem 1.1. When $q = p^*$, inequality (1.6) was proved in Remark 1.1. For each $q \in (p_*, p^*)$ it holds that

$$q = p_*\theta + p^*(1-\theta), \quad \text{with} \quad \theta = \frac{p^* - q}{p^* - p_*} \in (0, 1).$$

From Hölder's inequality, ones has that

$$\int_{\mathbb{R}^n_+} \frac{|u|^q}{(1+x_n)^\alpha} dx \le \left(\int_{\mathbb{R}^n_+} \frac{|u|^{p_*}}{(1+x_n)^{\frac{\alpha}{\theta}}} dx \right)^\theta \left(\int_{\mathbb{R}^n_+} |u|^{p^*} dx \right)^{1-\theta}. \tag{2.4}$$

Now take into account that $\alpha/\theta > \alpha(q)/\theta = 1$, Theorem 1.1 and inequality (1.4) yield hat

$$\begin{aligned} \int_{\mathbb{R}^n_+} \frac{|u|^q}{(1+x_n)^\alpha} dx &\le \left(C_0 \int_{\mathbb{R}^n_+} |\nabla u|^p dx \right)^{\frac{\theta p_*}{p}} \left(K_0 \int_{\mathbb{R}^n_+} |\nabla u|^p dx \right)^{\frac{(1-\theta)p^*}{p}} \\ &= C_1 \left(\int_{\mathbb{R}^n_+} |\nabla u|^p dx \right)^{\frac{q}{p}}, \end{aligned} \tag{2.5}$$

which is the desired inequality. Next we will prove that inequality (1.6) is false for all $0 < \alpha < \alpha(q)$ with $q \in (p_*, p^*)$. For this purpose, let $u \in C_0^\infty(\mathbb{R}^n) \setminus \{0\}$ and define $u_\lambda(x) := u(\lambda x)$ for $x \in \mathbb{R}^n_+$ and $\lambda \in (0, 1]$. It is easy to see that there are constants $C_1, C_2 > 0$, independent of λ, such that

$$\int_{\mathbb{R}^n_+} |\nabla u_\lambda|^p dx = C_1 \lambda^{p-n},$$

and using that $0 < \lambda \le 1$, after a change of variables we obtain

$$\int_{\mathbb{R}^n_+} \frac{|u_\lambda|^q}{(1+x_n)^\alpha} dx \ge \int_{\mathbb{R}^n_+} \frac{|u_\lambda|^q}{\left(\frac{1}{\lambda} + x_n\right)^\alpha} dx = C_2 \lambda^{-(n-\alpha)}.$$

Now, assume by contradiction that (1.6) holds true. Then, we have

$$0 < C_0 \le \frac{\int_{\mathbb{R}^n_+} |\nabla u_\lambda|^p dx}{\left(\int_{\mathbb{R}^n_+} \frac{|u_\lambda|^q}{(1+x_n)^\alpha} dx \right)^{\frac{p}{q}}} \le \left(\frac{C_1}{C_2^{\frac{p}{q}}} \right) \lambda^{p-n+\frac{p(n-\alpha)}{q}}.$$

If $0 < \alpha < \alpha(q)$ we obtain a contradiction by taking $\lambda \to 0$. To finish, we will prove that inequality (1.6) is false when $q = p^*$ and $\alpha < \alpha(p^*)$. For this purpose, let $u \in C_0^\infty(\mathbb{R}^n) \setminus \{0\}$ and define $u_\lambda(x) := u(\lambda x)$ for $x \in \mathbb{R}^n_+$ and $\lambda > 0$. It is easy to check that there exists a constant $C_1, C_2 > 0$, independent of λ, such that

$$\int_{\mathbb{R}^n_+} |\nabla u_\lambda|^p dx = C_1 \lambda^{p-n},$$

and for all $\alpha \le 0$ we have

$$\int_{\mathbb{R}^n_+} \frac{|u_\lambda|^q}{(1+x_n)^\alpha} = \int_{\mathbb{R}^n_+} |u_\lambda|^q (1+x_n)^{-\alpha} \ge \int_{\mathbb{R}^n_+} |u_\lambda|^q (x_n)^{-\alpha} = C_2 \lambda^{\alpha-n}.$$

Hence, if the inequality holds we have

$$0 < C_0 \le \frac{\int_{\mathbb{R}^n_+} |\nabla u_\lambda|^p dx}{\left(\int_{\mathbb{R}^n_+} \frac{|u_\lambda|^q}{(1+x_n)^\alpha} dx \right)^{\frac{p}{q}}} \le \left(\frac{C_1}{C_2^{\frac{p}{q}}} \right) \lambda^{p-n+\frac{p(n-\alpha)}{q}}.$$

In particular, if $\alpha < \alpha(p^*)$ and $q = p^*$ we obtain a contradiction by taking the limit as $\lambda \to 0$, and this finishes the proof. □

3 Existence and nonexistence of minimizers

In this section, we analyze existence and nonexistence of minimizers for the best constant in inequality (1.6). Precisely, we consider the variational problem

$$l(q,\alpha) := \inf_{\{u\in E\setminus\{0\}\}} \frac{\int_{\mathbb{R}^n_+} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^n_+} \frac{|u|^q}{(1+x_n)^\alpha} dx\right)^{\frac{p}{q}}}.$$

Notice that $l(q,\alpha)$ is positive if and only if (1.6) holds and further $C = 1/l(q,\alpha)$.

For $q = p_*$, we have the following nonexistence result of minimizers.

Theorem 3.1 *Assume $1 < p < n$ and $\alpha = 2$. Then $l(q,\alpha)$ has no minimizer for $q = p_*$.*

To prove Theorem 3.1 we requires two technical lemmas. For every $\varepsilon > 0$, the functions

$$u_\varepsilon(x',x_n) = \left(\frac{n-p}{p-1}\right)^{\frac{n-p}{p}} \left(\frac{\varepsilon^{\frac{2}{p}}}{(\varepsilon+x_n)^2+|x'|^2}\right)^{\frac{n-p}{2(p-1)}}, \quad (x',x_n)\in\mathbb{R}^n_+$$

are minimizers of the best constant in the trace inequality (1.2), i.e.,

$$\frac{1}{C_1} = \frac{\int_{\mathbb{R}^n_+} |\nabla u_\varepsilon|^p}{\left(\int_{\mathbb{R}^{n-1}} |u_\varepsilon|^{p_*}\right)^{\frac{p}{p_*}}},$$

and solves the quasilinear elliptic problem

$$\begin{cases} -div(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \mathbb{R}^n_+, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial x_n} = |u|^{p_*-2}u & \text{on } \mathbb{R}^{n-1}. \end{cases} \tag{3.1}$$

Moreover, if we define

$$m := I(u_\varepsilon),$$

where $I : E \to \mathbb{R}$ is the functional energy associated to problem (3.1)

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^n_+} |\nabla u|^p dx - \frac{1}{p_*}\int_{\mathbb{R}^{n-1}} |u|^{p_*} dx',$$

by using a simple calculation we see that

$$m = \frac{p-1}{p(n-1)} \frac{\left(\int_{\mathbb{R}^n_+} |\nabla u_\varepsilon|^p\right)^{\frac{n-1}{p-1}}}{\left(\int_{\mathbb{R}^{n-1}} |u_\varepsilon|^{p_*}\right)^{\frac{n-p}{p-1}}} = \frac{p-1}{p(n-1)} C_1^{-\frac{n-1}{p-1}}.$$

Furthermore, the following characterization holds true

$$m = \inf_{\{u\in C_0^\infty(\mathbb{R}^n)\setminus\{0\}\}} \max_{t>0} I(tu).$$

The next result allows us to compare the energy of minimizers of the best constants C_1 and C in the inequalities (1.2) and (1.5), respectively.

Lemma 3.1 *Assume $1 < p < n$ and $\alpha \geq 2$. Let $w \in E$ be a least energy solution of*

$$\begin{cases} -div(|\nabla w|^{p-2}\nabla w) = \frac{|w|^{p_*-2}w}{(1+x_n)^{\alpha}}, & in \quad \mathbb{R}^n_+, \\ \frac{\partial w}{\partial x_n} = 0, & on \quad \mathbb{R}^{n-1}. \end{cases} \tag{3.2}$$

Then the following estimate holds:

$$J(w) := \frac{1}{p}\int_{\mathbb{R}^n_+} |\nabla w|^p dx - \frac{1}{p_*}\int_{\mathbb{R}^n_+} \frac{|w|^{p_*}}{(1+x_n)^{\alpha}} dx > m.$$

Proof Suppose by contradiction that there exists a solution w of (3.2) with

$$J(w) \leq m.$$

Let w^* be the Steiner symmetrization of $|w|$ with respect to $x' \in \mathbb{R}^{n-1}$. Then, $w^* \in E$ and by [5,Proposition 3.1], we see that for any $t > 0$, $J(tw^*) \leq J(tw)$ and hence

$$\max_{t>0} J(tw^*) \leq \max_{t>0} J(tw) = J(w) \leq m.$$

Using integration by parts we see that

$$\begin{aligned} \int_{\mathbb{R}^n_+} \frac{|w^*|^{p_*}}{(1+x_n)^{\alpha}} dx &= \frac{1}{1-\alpha}\int_{\mathbb{R}^n_+} |w^*|^{p_*} \frac{\partial}{\partial x_n}(1+x_n)^{-\alpha+1} dx \\ &= \frac{1}{1-\alpha}\left(-p_*\int_{\mathbb{R}^n_+} \frac{|w^*|^{p_*-1}}{(1+x_n)^{\alpha-1}} \frac{\partial w^*}{\partial x_n} dx - \int_{\mathbb{R}^{n-1}} |w^*|^{p_*} dx'\right), \end{aligned}$$

which implies that

$$\begin{aligned} J(tw^*) &= \frac{t^p}{p}\int_{\mathbb{R}^n_+} |\nabla w^*|^p dx - \frac{t^{p_*}}{p_*(\alpha-1)}\left(p_*\int_{\mathbb{R}^n_+} \frac{|w^*|^{p_*-1}}{(1+x_n)^{\alpha-1}} \frac{\partial w^*}{\partial x_n} dx + \int_{\mathbb{R}^{n-1}} |w^*|^{p_*} dx'\right) \\ &\geq I(tw^*) - \frac{t^{p^*}}{\alpha-1}\int_{\mathbb{R}^n_+} \frac{|w^*|^{p_*-1}}{(1+x_n)^{\alpha-1}} \frac{\partial w^*}{\partial x_n} dx, \end{aligned}$$

where in the last inequality we used that $\alpha \geq 2$. Since $\max_{t>0} I(tw^*) \geq m$ and $\frac{\partial w^*}{\partial x_n} \leq 0$, we conclude that $\int_{\mathbb{R}^n_+} \frac{|w^*|^{p_*-1}}{(1+x_n)^{\alpha-1}} \frac{\partial w^*}{\partial x_n} dx = 0$, which implies that $\frac{\partial w^*}{\partial x_n} = 0$ on $\mathbb{R}^n_+$ and $\max_{t>0} I(tw^*) = m$. It follows that for some $t^* > 0$ such that t^*w^* is a least energy solution to (3.1). Then

$$|\nabla w^*|^{p-2}\frac{\partial w^*}{\partial x_n} = |w^*|^{p_*-2}w^* \quad \text{on } \mathbb{R}^{n-1}$$

and $w^* = 0$ on $\mathbb{R}^{n-1}$. Due to $\frac{\partial w^*}{\partial x_n} = 0$ on $\mathbb{R}^n_+$, $w^* \equiv 0$ on $\mathbb{R}^n_+$, which contradicts the fact that $m > 0$. □

Another ingredient in the proof of Theorem 3.1 is the following estimate.

Lemma 3.2 *If $\alpha \leq 2$, then*

$$\lim_{\varepsilon\to\infty} \max_{t>0} J(tu_{\varepsilon}) \leq m. \tag{3.3}$$

Proof By using a straightforward computation, one has

$$\max_{t>0} J(tu_\varepsilon) = \frac{p-1}{p(n-1)} \frac{\left(\int_{\mathbb{R}^n_+} |\nabla u_\varepsilon|^p\right)^{\frac{n-1}{p-1}}}{\left(\int_{\mathbb{R}^n_+} \frac{|u_\varepsilon|^{p_*}}{(1+x_n)^\alpha}\right)^{\frac{n-p}{p-1}}}.$$

On the other hand, we have

$$\begin{aligned}\frac{\left(\int_{\mathbb{R}^n_+} |\nabla u_\varepsilon|^p\right)^{\frac{n-1}{p-1}}}{\left(\int_{\mathbb{R}^n_+} \frac{|u_\varepsilon|^{p_*}}{(1+x_n)^\alpha}\right)^{\frac{n-p}{p-1}}} &= \frac{\left(\int_{\mathbb{R}^n_+} |\nabla u_\varepsilon|^p\right)^{\frac{n-1}{p-1}}}{\left(\int_{\mathbb{R}^{n-1}} |u_\varepsilon|^{p_*}\right)^{\frac{n-p}{p-1}}} \left(\frac{\int_{\mathbb{R}^{n-1}} |u_\varepsilon|^{p^*}}{\int_{\mathbb{R}^n_+} \frac{|u_\varepsilon|^{p_*}}{(1+x_n)^\alpha}}\right)^{\frac{n-p}{p-1}} \\ &= \frac{p(n-1)m}{p-1} \left(\frac{\int_{\mathbb{R}^{n-1}} |u_\varepsilon|^{p^*}}{\int_{\mathbb{R}^n_+} \frac{|u_\varepsilon|^{p_*}}{(1+x_n)^\alpha}}\right)^{\frac{n-p}{p-1}}.\end{aligned}$$

Therefore, to conclude the proof, it is enough to prove that

$$\lim_{\varepsilon\to\infty} \frac{\int_{\mathbb{R}^{n-1}} |u_\varepsilon|^{p_*}}{\int_{\mathbb{R}^n_+} \frac{|u_\varepsilon|^{p_*}}{(1+x_n)^\alpha}} \leq 1.$$

To this purpose, we observe that by using polar coordinates we obtain

$$\begin{aligned}\int_{\mathbb{R}^n_+} \frac{|u_\varepsilon|^{p_*}}{(1+x_n)^\alpha} &= \left(\frac{n-p}{p-1}\right)^{n-1} \int_{\mathbb{R}^n_+} \left(\frac{\varepsilon^{\frac{2}{p}}}{(\varepsilon+x_n)^2+|x'|^2}\right)^{\frac{p(n-1)}{2(p-1)}} \frac{1}{(1+x_n)^\alpha} \\ &= \omega_{n-2} \left(\frac{n-p}{p-1}\right)^{n-1} \int_0^\infty \int_0^\infty \left(\frac{\varepsilon^{\frac{2}{p}}}{(\varepsilon+t)^2+r^2}\right)^{\frac{p(n-1)}{2(p-1)}} \frac{r^{n-2}}{(1+t)^\alpha} dt dr\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} |u_\varepsilon|^{p_*} &= \omega_{n-2} \left(\frac{n-p}{p-1}\right)^{n-1} \int_0^\infty \left(\frac{\varepsilon^{\frac{2}{p}}}{\varepsilon^2+r^2}\right)^{\frac{p(n-1)}{2(p-1)}} r^{n-2} dr \\ &= \omega_{n-2} \left(\frac{n-p}{p-1}\right)^{n-1} (\alpha-1) \int_0^\infty \left(\frac{\varepsilon^{\frac{2}{p}}}{\varepsilon^2+r^2}\right)^{\frac{p(n-1)}{2(p-1)}} r^{n-2} dr \int_0^\infty \frac{1}{(1+t)^\alpha} dt \\ &= \omega_{n-2} \left(\frac{n-p}{p-1}\right)^{n-1} (\alpha-1) \int_0^\infty \int_0^\infty \left(\frac{\varepsilon^{\frac{2}{p}}}{\varepsilon^2+r^2}\right)^{\frac{p(n-1)}{2(p-1)}} \frac{r^{n-2}}{(1+t)^\alpha} dt dr.\end{aligned}$$

Then we have that

$$\frac{\int_{\mathbb{R}^{n-1}} |u_\varepsilon|^{p_*}}{\int_{\mathbb{R}^n_+} \frac{|u_\varepsilon|^{p_*}}{(1+x_n)^\alpha}} = \frac{(\alpha-1) \int_0^\infty \int_0^\infty \left(\frac{1}{\varepsilon^2+r^2}\right)^{\frac{p(n-1)}{2(p-1)}} \frac{r^{n-2}}{(1+t)^\alpha} dt dr}{\int_0^\infty \int_0^\infty \left(\frac{1}{(\varepsilon+t)^2+r^2}\right)^{\frac{p(n-1)}{2(p-1)}} \frac{r^{n-2}}{(1+t)^\alpha} dt dr}$$

$$= \frac{(\alpha-1)\int_0^\infty\int_0^\infty \left(\frac{1}{1+r^2}\right)^{\frac{p(n-1)}{2(p-1)}} \frac{r^{n-2}}{(1+t)^\alpha} dtdr}{\int_0^\infty\int_0^\infty \left(\frac{\varepsilon^{\frac{2}{p}}}{(\varepsilon+t)^2+r^2}\right)^{\frac{p(n-1)}{2(p-1)}} \frac{r^{n-2}}{(1+t)^\alpha} dtdr}$$

$$= \frac{(\alpha-1)\int_0^\infty\int_0^\infty \left(\frac{1}{1+r^2}\right)^{\frac{p(n-1)}{2(p-1)}} \frac{r^{n-2}}{(1+t)^\alpha} dtdr}{\int_0^\infty\int_0^\infty \left(\frac{\varepsilon}{\varepsilon+t}\right)^{\frac{n-1}{p-1}} \left(\frac{1}{1+r^2}\right)^{\frac{p(n-1)}{2(p-1)}} \frac{r^{n-2}}{(1+t)^\alpha} dtdr}$$

Applying the Lebesgue Dominate Convergence Theorem, we obtain

$$\lim_{\varepsilon\to\infty} \frac{\int_{\mathbb{R}^{n-1}} |u_\varepsilon|^{p_*}}{\int_{\mathbb{R}^n_+} \frac{|u_\varepsilon|^{p_*}}{(1+x_n)^\alpha}} = \alpha - 1 \le 1.$$

proving the claim and this concludes the proof. □

Proof of Theorem 3.1 Assume by contradiction that $l(p_*, \alpha)$ has a minimizer u_0, which is a least energy solution of problem (3.2), then

$$m = \lim_{\varepsilon\to\infty} \max_{t>0} J(tu_\varepsilon) \ge \inf_{\{\varphi\in E\setminus\{0\}\}} \max_{t>0} J(t\varphi) = J(u_0) > m,$$

which is a contradiction and this completes the proof. □

When $q \in (p_*, p^*)$, we have the following result of existence on minimizers.

Theorem 3.2 *Assume* $1 < p < n$ *and* $\alpha > 1$*. Then,* $l(q, \alpha)$ *has a minimizer for every* $q \in (p_*, p^*)$.

Proof The proof is similar to [6]. For the sake of completeness, we give the details. By the Steiner symmetrization, we see that

$$l(q,\alpha) = \inf_{\{u\in\mathcal{D}^{1,p}_r(\mathbb{R}^n_+)\setminus\{0\}\}} \frac{\int_{\mathbb{R}^n_+} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^n_+} a(x)|u|^q\right)^{\frac{p}{q}}},\ a(x) = (1+x_n)^{-\alpha},$$

where

$$\mathcal{D}^{1,p}_r(\mathbb{R}^n_+) := \left\{u \in \mathcal{D}^{1,p}(\mathbb{R}^n_+) : u(x', x_n) = u(|x'|, x_n),\ x' \in \mathbb{R}^{n-1},\ x_n > 0\right\}.$$

Let $\{\phi_l\} \subset \mathcal{D}^{1,p}_r(\mathbb{R}^n_+) \cap C_0^\infty(\mathbb{R}^n)$ be a minimizing sequence of $l(q, \alpha)$. We may assume that $supp(\phi_l) \subset B(0, R_l)$ with $R_1 < R_2 < \cdots < R_l < \cdots$ and $\lim_{l\to\infty} R_l = \infty$. We define

$$\mathcal{D}^{1,p}_+(R_l) \equiv \mathcal{D}^{1,p}(\mathbb{R}^n_+) \cap H^1_0(B(0, R_l)).$$

We may take a better minimizing sequence $\{u_l\}$ of $l(q, \alpha)$ such that u_l is a minimizer of

$$l(\alpha, q, B(0, R_l)) = \inf_{\{u\in\mathcal{D}^{1,p}_+(R_l)\setminus\{0\}\}} \frac{\int_{\mathbb{R}^n_+\cap B(0,R_l)} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^n_+\cap B(0,R_l)} a(x)|u|^q\right)^{\frac{p}{q}}}.$$

we may assume that

$$\int_{\mathbb{R}^n_+\cap B(0,R_l)} |\nabla u_l|^p dx = \int_{\mathbb{R}^n_+\cap B(0,R_l)} a(x)|u_l|^q dx.$$

Then, we see that

$$\begin{cases} -div(|\nabla u_l|^{p-2}\nabla u_l) = \frac{u_l^q}{(1+x_n)^\alpha}, \text{ in } & \mathbb{R}^n_+ \cap B(0,R_l),\\ u_l = 0, \quad \text{on} & \mathbb{R}^n_+ \cap \partial B(0,R_l),\\ \frac{\partial u_l}{\partial x_n} = 0, \quad \text{on} & \mathbb{R}^{n-1} \cap B(0,R_l). \end{cases}$$

By the moving plane method, we see that $u_l \in \mathcal{D}^{1,p}_r(\mathbb{R}^n_+)$ and $u_l(0) > u_l(x)$ for any $x \neq 0$. Then, applying the standard blow up argument, we see that $\{\|u_l\|_{L^\infty}\}$ is bounded. Since

$$\int_{\mathbb{R}^n_+\cap B(0,R_l)} |\nabla u_l|^p dx \le \|u_l\|_{L^\infty}^{q-p_*} \int_{\mathbb{R}^n_+\cap B(0,R_l)} a(x)|u_l|^{p_*} dx,$$

by Proposition 2.3 in [6], we deduce that $\{\|u_l\|_{L^\infty}\}$ is bounded away from 0. Taking a subsequence, if necessary, we may assume that

$$\begin{cases} u_l \rightharpoonup u_0 \quad \text{weakly in } E,\\ u_l \to u_0 \quad \text{strongly in } L^p(\mathbb{R}^n_+, a),\\ u_l(x) \to u_0(x) \quad \text{for a.e. } x \in \mathbb{R}^n_+. \end{cases}$$

u_l converges weakly to some u_0 in $\mathcal{D}^{1,p}_r(\mathbb{R}^n_+)$ and $u_l \to u_0$ in $C^2(\mathbb{R}^n_+ \cap B(0,R))$ for each $R > 0$.

Furthermore, we see that the limit $u_0 \in \mathcal{D}^{1,p}_r(\mathbb{R}^n_+)$ is a solution of

$$\begin{cases} -div(|\nabla u_0|^{p-2}\nabla u_0) = \frac{u_0^q}{(1+x_n)^\alpha}, \quad u > 0, \quad \text{in} & \mathbb{R}^n_+,\\ |\frac{\partial u_0}{\partial \nu}|^{p-2}\nabla u_0 \cdot \nu = 0, \quad \text{on} & \mathbb{R}^{n-1}. \end{cases} \tag{3.4}$$

From the standard regularity theory, we see that $\{\|u_l\|_{C^1(R^+_n\cap B(0,R_l))}\}$ is bounded. Since $\{\int_{\mathbb{R}^n_+} a(x)|u_l|^q\}_l$ is bounded, we deduce that $\lim_{|x'|\to\infty} u_l(x',0) = 0$ uniformly for $l \ge 1$. For each $x' \in \mathbb{R}^{n-1} \cap B(0,R_l)$, we get

$$\begin{aligned} (u_l(x',0))^q &= -q\int_0^{\sqrt{(R_l)^2-|y'|^2}} u_l^q(x',x_n)\frac{\partial u_l(x',x_n)}{\partial x_n}dx_n\\ &\le \frac{q}{2}\int_0^{\sqrt{(R_l)^2-|y'|^2}} (u_l(x',0))^{q-p_*}(u_l(x',x_n))^{pp_*}dx_n\\ &\quad + \frac{p}{2}\int_0^{\sqrt{(R_l)^2-|y'|^2}} (u_l(x',0))^{q-p_*}|\nabla u_l|^p dx_n. \end{aligned} \tag{3.5}$$

Thus, we see that for some $C > 0$, independent of $l \ge 1$ and $D > 0$,

$$\begin{aligned} \int_{\mathbb{R}^{n-1}\setminus B(0,D)} (u_l(x',x_n))^q dx' &\le \int_{\mathbb{R}^{n-1}\setminus B(0,D)} (u_l(x',0))^q dx'\\ &\le C\max_{|x'|=D}(u_l(x',0))^{q-p_*}\int_{\mathbb{R}^n_+\cap B(0,R_l)} (|\nabla u_l|^p + (u_l)^{p^*})dx \end{aligned} \tag{3.6}$$

By Theorem 1.1, we have that

$$
\begin{aligned}
\int_{\mathbb{R}^n_+\setminus B(0,D)} a(x)u_l^q dx &\le \int_{\{(x',x_n)\in\mathbb{R}^n_+:|x'|\ge D/\sqrt{2}\}} a(x)u_l^q dx \\
&\quad + \int_{\{(x',x_n)\in\mathbb{R}^n_+:x_n\ge D/\sqrt{2}\}} a(x)u_l^q dx \\
&\le C \max_{|x'|=D/\sqrt{2}} (u_l(x',0))^{q-p_*} \int_{\mathbb{R}^n_+\cap B(0,R_l)} (|\nabla u_l|^p + u_l^{p^*})dx \\
&\quad + C\frac{1}{(1+D/2\sqrt{2})^\alpha} \int_{\mathbb{R}^n_+\cap B(0,R_l)} \frac{u_l^q}{(1+x_n/2)^\alpha} dx \\
&\le C \max_{|x'|=D/\sqrt{2}} (u_l(x',0))^{q-p_*} \int_{\mathbb{R}^n_+\cap B(0,R_l)} (|\nabla u_l|^p + u_l^{p^*})dx \\
&\quad + C_1\frac{1}{(1+D/2\sqrt{2})^\alpha} \int_{\mathbb{R}^n_+\cap B(0,R_l)} |\nabla u_l|^p dx.
\end{aligned} \tag{3.7}
$$

This means that

$$
\lim_{D\to\infty} \int_{\mathbb{R}^n_+\setminus B(0,D)} a(x)u_l^q dx = 0 \tag{3.8}
$$

uniformly for $l \ge 1$, which implies that

$$
\lim_{l\to\infty} \int_{\mathbb{R}^n_+} a(x)u_l^q dx = \int_{\mathbb{R}^n_+} a(x)u_0^q dx. \tag{3.9}
$$

Since

$$
\liminf_{l\to\infty} \int_{\mathbb{R}^n_+} |\nabla u_l|^p dx \ge \int_{\mathbb{R}^n_+} |\nabla u_0|^p dx,
$$

we see that u_0 is a minimizer of $l(\alpha, q)$ and this completes the proof. □

4 Application

In this section, with the purpose to illustrate an application of inequality (1.5), motivated by the works [12, 13], where the authors have consider quasilinear elliptic problem with Robin boundary conditions on the upper-half space, we investigate the existence of solutions to the following quasilinear elliptic equation with Neumann boundary condition and involving anisotropic weight:

$$
\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda b(x)|u|^{q-1}u + |u|^{p^*-2}u & \text{in } \mathbb{R}^n_+, \\ \quad |\nabla u|^{p-2}\nabla u\cdot\nu = 0 & \text{on } \mathbb{R}^{n-1}, \end{cases}, \tag{$\mathcal{P}_\ge$}
$$

where $1 < q < p < n$, ν denotes the unit outward normal on the boundary, $\lambda > 0$ is a parameter and the weight function $b : \mathbb{R}^n_+ \to \mathbb{R}$ is a positive continuous function satisfying the assumption

$$
\int_{\mathbb{R}^n_+} \frac{b^{\frac{p*}{p*-q}}}{a^{\frac{q}{p*-q}}} dx < \infty, \tag{4.1}
$$

where $a(x) = (1 + x_n)^{-\alpha}$ for $\alpha > 1$.

Remark 4.1 We quote that assumption (4.1) is inspired by the one in the paper [2] and we can check that the function $b(x) = 1/(1 + x_n)^{\frac{\alpha q}{p_*}}(1 + |x|)^{\theta}$ with $\theta > n(p_* - q)/q$ satisfies the assumption (4.1). In fact, if $\theta > n(p_* - q)/q$ one has

$$\int_{\mathbb{R}^n_+} \frac{b^{\frac{p_*}{p_*-q}}}{a^{\frac{q}{p_*-q}}} dx \leq \int_{\mathbb{R}^n_+} \frac{1}{(1 + x_n)^{\frac{\alpha q}{p_*-q}} (1 + |x|)^{\frac{\theta q}{p_*-q}}} (1 + x_n)^{\frac{\alpha q}{p_*-q}} dx = \int_{\mathbb{R}^n_+} \frac{1}{(1 + |x|)^{\frac{\theta q}{p_*-q}}} dx < \infty.$$

By carrying out a direct minimization argument similar in the spirit as those in [3, 4] we are able to prove the following result.

Theorem 4.1 *Let $1 < q < p < n$ and assume that (4.1) holds. Then there exists $\lambda^* > 0$ such that problem ($\mathcal{P}_{\geq}$) possesses at least a nonzero weak solution for all $\lambda \in (0, \lambda^*)$.*

Here, by a weak solution of problem ($\mathcal{P}_{\geq}$), we mean a nontrivial function $u \in E$ verifying

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \lambda \int_{\mathbb{R}^n_+} b|u|^{q-2} u\varphi dx + \int_{\mathbb{R}^n_+} |u|^{p_*-2} u\varphi dx, \quad \forall \varphi \in E.$$

To prove Theorem 4.1 we shall need the following result.

Lemma 4.1 *Assume that (4.1) holds. Then the weighted Sobolev embedding $E \hookrightarrow L^q\left(\mathbb{R}^n_+, b\right)$ is continuous and compact.*

Proof Notice that by the Hölder inequality

$$\int_{\mathbb{R}^n_+} b|u|^q dx = \int_{\mathbb{R}^n_+} \frac{b}{a^{\frac{q}{p_*}}} (a^{\frac{1}{p_*}} |u|)^q dx \leq \left(\int_{\mathbb{R}^n_+} \frac{b^{\frac{p_*}{p_*-q}}}{a^{\frac{q}{p_*-q}}} dx \right) \left(\int_{\mathbb{R}^n_+} a|u|^{p_*} dx \right)^{\frac{q}{p_*}}.$$

Thus, by Theorem 1.1 and hypothesis (4.1) we conclude that the embedding $E \hookrightarrow L^q\left(\mathbb{R}^n_+, b\right)$ is continuous. We claim that, up to a subsequence, $u_k \to 0$ strongly in $L^q(\mathbb{R}^n_+, b)$ whenever $u_k \rightharpoonup 0$ weakly in E. Indeed, for $R > 0$ to be chosen later on, we can write

$$\int_{\mathbb{R}^n_+} b|u_k|^q dx = \int_{B_R^+} b|u_k|^q dx + \int_{\mathbb{R}^n_+ \setminus B_R^+} b|u_k|^q dx, \tag{4.2}$$

where $B_R^+ = \mathbb{R}^n_+ \cap B_R$. Since the restriction operator $u \mapsto u_{|_{B_R^+}}$ is continuous from E into $E(B_R^+) := \left\{ v_{|_{B_R^+}} : v \in E \right\}$ and the embedding $E(B_R^+) \hookrightarrow L^q(B_R^+, b)$ is compact, for any $\varepsilon > 0$ there exists $k_1 \in \mathbb{N}$ such that for any $1 < q \leq p_*$

$$\int_{B_R^+} b|u_k|^q dx < \frac{\varepsilon}{2}, \quad \forall k \geq k_1. \tag{4.3}$$

On the other hand, by using the Hölder inequality and the fact that (u_k) is bonded in E, we can invoke assumption (4.1) to choose $R > 0$ large enough such that

$$\int_{\mathbb{R}^n_+ \setminus B_R^+} b|u_k|^q dx \leq \left(\int_{\mathbb{R}^n_+ \setminus B_R^+} \frac{b^{\frac{p_*}{p_*-q}}}{a^{\frac{q}{p_*-q}}} dx \right)^{\frac{p_*-q}{p_*}} \left(\int_{\mathbb{R}^n_+ \setminus B_R^+} a|u_k|^{p_*} dx \right)^{\frac{p_*}{q}} \leq \frac{\varepsilon}{2}.$$

This combined with (4.2) and (4.3) imply the desired convergence. □

In view of Lemma 4.1 the energy functional associated to problem ($\mathcal{P}_{\geq}$), $I_\lambda : E \to \mathbb{R}$ given by

$$I_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^n_+} |\nabla u|^p dx - \frac{\lambda}{q}\int_{\mathbb{R}^n_+} b|u|^q dx - \frac{1}{p^*}\int_{\mathbb{R}^n_+} |u|^{p^*} dx,$$

is well defined and of C^1 class. Furthermore, standard arguments show that critical points of I_λ are weak solutions of problem ($\mathcal{P}_{\geq}$) and reciprocally.

Lemma 4.2 *Let $1 < q < p < n$ and assume (4.1). Then there exists $\lambda^* > 0$ such that for all $0 < \lambda < \lambda^*$ the following statements hold:*

(i) *there are $\gamma, \rho > 0$ such that $I_\lambda(u) \geq \gamma$ if $\|u\|_E = \rho$;*
(ii) *$I_\lambda(tu_0) < 0$ for any $u_0 \in E \setminus \{0\}$ and $t > 0$ small enough.*

Proof It follows from Lemma 4.1 and inequality (1.4) that

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{p}\|u\|^p - \frac{\lambda C_1}{q}\|u\|^q - \frac{C_2}{p^*}\|u\|^{p^*} \\ &\geq \left(\frac{1}{p}\rho^{p-q} - \frac{\lambda C}{q} - \frac{C}{p}\rho^{p^*-q}\right)\rho^q, \end{aligned}$$

where $\rho = \|u\| > 0$. Since $q < p < p^*$, one can choose $\rho > 0$ such that $\frac{1}{p}\rho^{p-q} - \frac{C}{p}\rho^{p^*-q} > \frac{1}{2p}\rho^{p-q}$. Thus, for $0 < \lambda < \frac{1}{2pC}\rho^{p-q}$, there exists $\gamma_\lambda > 0$ in order that

$$I_\lambda(u) \geq \gamma_\lambda \quad \forall\, u \in E, \text{ with } \|u\|_E = \rho$$

where $\gamma_\varepsilon = \left(\frac{1}{2p}\rho^{p-q} - \frac{\lambda C}{q}\right)\rho^{p-q}$, and assertion (i) is proved.

Since $q < p < p^*$, for any $u_0 \in E \setminus \{0\}$ one has

$$I_\lambda(tu_0) = t^q\left[\frac{t^{p-q}}{p}\int_{\mathbb{R}^n_+} |\nabla u_0|^p dx - \frac{\lambda}{q}\int_{\mathbb{R}^n_+} b|u_0|^q dx - \frac{t^{p^*-q}}{p^*}\int_{\mathbb{R}^n_+} |u_0|^{p^*} dx\right] < 0,$$

for $t > 0$ small enough and this proves assertion (ii). □

Proof of Theorem 4.1 By Lemma 4.2 we see that

$$-\infty < C_\lambda := \inf_{\overline{B}_\rho} I_\lambda(u) < 0.$$

By applying the Ekeland variational principle we get a sequence $(u_n) \subset E$ such that

$$I'_\lambda(u_n) \to C_\lambda \quad \text{and} \quad \|I'_\lambda(u_n)\|_{E^*} \to 0. \tag{4.4}$$

where E^* denotes the dual space of E. Since (u_n) is bounded, up to a subsequence, we may assume that $u_n \rightharpoonup u$ weakly in E. By using standard arguments we see that u is a weak solution. We claim that u is nontrivial. In fact, defining $l := \lim_{n\to\infty} \|\nabla u_n\| \geq 0$, from (4.4) and the compact embedding $E \hookrightarrow L^q(\mathbb{R}^n_+, b)$ we obtain

$$l = \lim_{n\to\infty}\int_{\mathbb{R}^n_+} |\nabla u_n|^p = \lim_{n\to\infty}\int_{\mathbb{R}^n_+} |u_n|^{p^*}.$$

Thus, using again (4.4) we conclude that

$$0 \leq (\frac{1}{p} - \frac{1}{p^*})l = C_\lambda < 0,$$

which is a contradiction and this completes the proof. □

5 Final comments

In this section we raise some questions related our results that have been of interest by many authors in different contexts.

Question 1 The question of optimal constants and attainability has been the subject of many papers, see for instance [9, 15, 16] and references therein. In this context, it is very important to investigate the optimality and attainability of the constant $C(n, \alpha, p)$ in Theorem 1.1. □

Question 2 In view of Theorem 1.1 and Corollary 1.2 it is natural to ask if the weight function $(1+x_n)^{-\alpha}$ is optimal in the sense that, if $w : \mathbb{R}^n_+ \to \mathbb{R}$ verifies the inequality in Theorem 1.1 then there are constants $\alpha > 1$ and $c_3 > 0$ such that

$$0 \leq w(x) \leq \frac{c_3}{(1+x_n)^{\alpha}}, \quad \text{a.e. in} \quad \mathbb{R}^n_+.$$

□

Question 3 Is inequality (1.6) true or false for $\alpha = \alpha(q)$ with $q \in (p_*, p^*)$? □

Data Availibility Statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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