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ORIGINAL PAPER

Regularity properties of the cubic biharmonic Schrödinger equation on the half line

Engin Başakoğlu[1]

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Abstract
In this paper we study the regularity properties of the cubic biharmonic Schrödinger equation posed on the right half line. We prove local well-posedness and obtain a smoothing result in the low-regularity spaces on the half line. In particular we prove that the nonlinear part of the solution on the half line is smoother than the initial data obtaining a full derivative gain in certain cases. Moreover, in the defocusing case, we establish global well-posedness and global smoothing in the higher order regularity spaces by making use of the global-wellposedness result of Özsarı and Yolcu (Commun Pure Appl Phys 18(6):3285–3316, 2019) in the energy space. Also this paper improves the well-posedness result of Özsarı and Yolcu (Commun Pure Appl Phys 18(6):3285–3316, 2019) in the case of cubic nonlinearity.

Keywords Initial boundary value problem · Local wellposedness · Global wellposedness · Smoothing

Mathematics Subject Classification 35G30 · 35Q55

1 Introduction

This paper aims to study the initial boundary value problem (IBVP) for the cubic biharmonic nonlinear Schrödinger equations (biharmonic NLS) on the half line

$$\begin{cases} iu_t + \partial_x^4 u + \mu|u|^2 u = 0, \quad x \in \mathbb{R}^+, t \in \mathbb{R}^+, \\ u(0,t) = h_1(t), \quad u_x(0,t) = h_2(t), \\ u(x,0) = g(x) \end{cases} \tag{1}$$

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✉ Engin Başakoğlu
engin.basakoglu@boun.edu.tr

1 Department of Mathematics, Boğaziçi University, Bebek, 34342 Istanbul, Turkey

where $\mu = \pm 1$ and the data (g, h_1, h_2) are taken in the space $H_x^s(\mathbb{R}^+) \times H_t^{\frac{2s+3}{8}}(\mathbb{R}^+) \times H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)$ with the compatibility conditions $g(0) = h_1(0)$ when $\frac{1}{2} < s \le \frac{3}{2}$, and $g(0) = h_1(0)$, $g'(0) = h_2(0)$ when $\frac{3}{2} < s \le \frac{9}{2}$. These compatibility conditions are necessary since the solutions we are concerned with have continuous L^2 traces for $s > \frac{1}{2}$. For the notion of traces of functions in $H^s(\mathbb{R})$, we assume, for our throughout discussion, that $s \ne n+\frac{1}{2}$ for $n = 0, 1, 2, \ldots$. Note that choosing data triples $(g, h_1, h_2) \in H_x^s(\mathbb{R}^+) \times H_t^{\frac{2s+3}{8}}(\mathbb{R}^+) \times H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)$ is due to the local smoothing inequalities of [32,34]:

$$\left\| \partial_x^k e^{it\partial_x^4} g \right\|_{L_x^\infty H_{t\in(0,T)}^{\frac{2s+3-2k}{8}}} \lesssim \|g\|_{H^s},$$

for $k = 0, 1$ and these inequalities are sharp in the sense that the numbers $\frac{2s+3}{8}$ and $\frac{2s+1}{8}$ cannot be replaced by any bigger number and hence taking such data makes sense. We also verify the appropriateness of the selected spaces in our computations.

Fourth order NLS with power-type nonlinearity

$$iu_t + \Delta u + \lambda \Delta^2 u + |u|^p u = 0, \quad x \in \mathbb{R}^n, t \in \mathbb{R}$$

was introduced by Karpman and Shagalov [30,31] to consider the effect of the small fourth order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Indeed, when $\lambda < 0$, they studied the stability/instability of solutions depending on certain restrictions on the parameters λ, p. When Laplacian is removed, the equation

$$iu_t + \lambda \Delta^2 u + \mu |u|^p u = 0, \quad x \in \mathbb{R}^n, t \in \mathbb{R} \tag{2}$$

is called biharmonic NLS, in addition it is said to be defocusing if $\lambda\mu > 0$, and focusing if $\lambda\mu < 0$. From a physical point of view, as a model equation, biharmonic NLS arises in many context such as deep water wave dynamics [11], vortex filaments [19], solitary waves [30,31]. Furthermore it was used as a model equation in [29], [38] to study the stability of solitons in magnetic materials once the effective quasi particle mass becomes infinite. Fourth order NLS with various nonlinearities have been extensively studied on the well-posed in the periodic and non-periodic settings. As half line problems are relavent to the initial value problems posed in the non-periodic setting, here it is better to review some of those posed on $\mathbb{R}^d$. So we write

$$\begin{cases} iu_t + \kappa \Delta u + \lambda \Delta^2 u + F(u) = 0, \\ u(x, 0) = g(x). \end{cases} \tag{3}$$

The initial value problem (IVP) (3) on $\mathbb{R}^n \times (0, \infty)$ with $\kappa = 0$, $\lambda = 1$ and nonlinearities $F(u) = \partial_x(|u|^{p-1}u)$, $2 \le p \in \mathbb{N}$ have been studied in [25] in terms of well-posedness and scattering of the solution. In particular, it turns out that when $n = 1$ and $p = 3$, the authors obtained the local well-posedness of (3) in the Sobolev spaces $H^s(\mathbb{R})$ for $s \ge 0$. Furthermore this result is almost sharp in the sense that the flow map from $H^s(\mathbb{R})$ to $C(\mathbb{R}, H^s(\mathbb{R}))$ is not C^3. The local and global well-posedness for the IVP (3) on $\mathbb{R} \times \mathbb{R}$ with $\kappa = 0$, $\lambda = -1$ and $F(u) = \pm|u|^2 u$, were established in [36] for data $g \in H^s(\mathbb{R})$ with $s \ge -\frac{1}{2}$, also the equation was shown to be ill-posed below this range ($s < -\frac{1}{2}$), by proving that the flow map is not uniform continuous. In [28], the IVP (3) on $\mathbb{R} \times \mathbb{R}$ with $\kappa = 1$, $\lambda \ne 0$ and the nonlinearity

$$F(u) = -\frac{1}{2}|u|^2 u + c_1|u|^4 u + c_2(\partial_x u)^2 \bar{u} + c_3|\partial_x u|^2 + c_4 u^2 \partial_x^2 \bar{u} + c_5|u|^2 \partial_x^2 u \tag{4}$$

(with certain restrictions on the constants) was proved to be locally well-posed in $H^s(\mathbb{R})$, $s \geq \frac{1}{2}$ by the restricted norm method. For higher dimensions, Pausader [35] showed that the Eq. (3) with $\kappa = 0$, $\lambda = 1$ and $F(u) = |u|^2 u$ is globally well-posed for $n \leq 8$, and ill-posed for $n \geq 9$. For the other well-posedness results related to Eq. (3) see for instance [1,10,20–24]. Initial boundary value problems for the fourth order NLS have been recently started to be addressed. In the case of the half line, Hu etal [27] obtained a solution of some form of the Eq. (3) in the IBVP setting (with a similar nonlinearity as in (4)) after reformulating the problem as a Riemann-Hilbert problem. Özsarı-Yolcu [34] studied the IBVP of the Eq. (2) with $\lambda = 1$, $\mu \in \mathbb{C}$ and the inhomogeneous Dirichlet-Neumann boundary data on the half line where they make use of the unified transform method in obtaining the solution. By making some assumptions on the relation of s and p, the authors obtained the local well-posedness in $H^s(\mathbb{R}^+)$ for $s \in (\frac{1}{2}, \frac{9}{2})$, $s \neq \frac{3}{2}$, and $s \in [0, \frac{1}{2})$ separately. Moreover, for the defocussing problem they established the global well-posedness in the energy space $H^2(\mathbb{R}^+)$. It is remarkable to note that [34] is the first treatment of the fourth order Schrödinger equations on a half line subject to the inhomogeneous boundary conditions. Lastly, more recently Filho–Cavalcante–Gallego [18] addressed the IBVP of the cubic biharmonic NLS (2) when $\lambda = -1$ with the same set of initial-boundary data as in [34]. The authors proved the local well-posedness in $H^s(\mathbb{R}^+)$ for $0 \leq s < \frac{1}{2}$ by the Fourier restriction norm method and using the Duhamel boundary forcing operator for the corresponding linear equation.

In this paper we continue the program initiated in [16] that establishes the regularity properties of cubic NLS on a half line using the tools available in the case of the full line. Biharmonic cubic NLS is higher order dispersive PDE version of cubic NLS, so as expected, we obtain well-posedness in a less regular space by adapting the estimates of [16]. We will use Laplace transform methods proposed by Bona–Sun–Zhang [2] to divide the problem into a linear IBVP on the half line and nonlinear IVP on the full line after extending the data into $\mathbb{R}$. By this method we can write the explicit solution for a linear IBVP and then using it, we set up an equivalent integral equation on $\mathbb{R} \times \mathbb{R}$ for the full solution. We then examine the integral equation with the $X^{s,b}$ method, see [3,4]. To state our theorems we begin with a definition.

Definition 1.1 We say that the biharmonic NLS Eq. (1) is locally well-posed in $H^s(\mathbb{R}^+)$ if for any data $(g, h_1, h_2) \in H_x^s(\mathbb{R}^+) \times H_t^{\frac{2s+3}{8}}(\mathbb{R}^+) \times H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)$ with the additional compatibility conditions discussed above, the integral equation (8) has a unique solution in

$$X^{s,b}(\mathbb{R} \times [0,T]) \cap C_t^0 H_x^s([0,T] \times \mathbb{R}) \cap C_x^0 H_t^{\frac{2s+3}{8}}(\mathbb{R} \times [0,T])$$

for some $b < \frac{1}{2}$ and sufficiently small $T = T(\|g\|_{H^s(\mathbb{R}^+)}, \|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)}, \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)})$. Furthermore, if u_1 and u_2 are two such solutions coming from different extensions g_{e1} and g_{e2}, then their restriction to $\mathbb{R}^+ \times [0,T]$ are the same. In addition, if $g_n \to g$ in $H^s(\mathbb{R}^+)$, $h_{n1} \to h_1$ in $H^{\frac{2s+3}{8}}(\mathbb{R}^+)$ and $h_{n2} \to h_2$ in $H^{\frac{2s+1}{8}}(\mathbb{R}^+)$, then $u_n \to u$ in the space above.

We state our local result below and note that it improves the result for the cubic biharmonic NLS in [34] which establishes the well-posedness for $s \geq 0$. As already mentioned [34] utilizes the uniform transform method of Fokas to obtain the local well-posedness for the biharmonic NLS with power nonlinearities. The method is based on inverse-scattering techniques and used to obtain representation formula for the solution of the linear biharmonic Schrödinger equation. In order to establish the local theory we will need to obtain some essential estimates regarding the linear and nonlinear terms of the integral equation representation for the solution in Sect. 4 below.

Theorem 1.2 *For any* $s \in (-\frac{1}{3}, \frac{9}{2})$, $s \neq \frac{1}{2}, \frac{3}{2}$, *the* Eq. (1) *is locally well-posed in* $H^s(\mathbb{R}^+)$ *with the local existence time* T *satisfying* $T \approx \left(C + \|g\|_{H^s(\mathbb{R}^+)}\right)^{-\frac{8}{2s+3}}$ *where the constant* C *depends on* $\|g\|_{L^2} + \|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)}$.

Next theorem is concerned with the smoothing result of the Eq. (1) that is, it demonstrates that the nonlinear part of the solution is smoother than the initial data. It reads that smoothing vanishes at the upper end point $s = \frac{9}{2}$, nevertheless, the gain of a derivative at the lower end point $s = -\frac{1}{3}$ is still $\frac{1}{3}$. The proof of the smoothing theorem below will be based on the restricted norm method of Bourgain [3,4] and in the sequel, we will denote the operator W_0^t as the linear part of the solution of the Eq. (1).

Theorem 1.3 *Fix* $s \in (-\frac{1}{3}, \frac{9}{2})$, $s \neq \frac{1}{2}, \frac{3}{2}$, $(g, h_1, h_2) \in H_x^s(\mathbb{R}^+) \times H_t^{\frac{2s+3}{8}}(\mathbb{R}^+) \times H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)$ *with the compatibility conditions given for the* Eq. (1). *Then for* $a < \min(1, 2s+1, \frac{9}{2} - s)$ *and* t *in the local existence interval* $[0, T]$, *we have*

$$u(x,t) - W_0^t(g, h_1, h_2) \in C_t^0 H_x^{s+a}([0, T] \times \mathbb{R}^+).$$

The smoothing estimates of this sort were obtained for NLS in certain papers in the periodic, see [7,9,12,14] and non-periodic cases, see [5,33]. The first smoothing result related to the initial boundary value problem is established for cubic NLS, [16]. Also using the same approach as in [16], the papers [8,17] establish the regularity properties of the Boussinesq equation and the Zakharov system on the half line respectively. In order to prove the above theorems we take advantage of the Duhamel formulation by which we run a fixed point argument. With this formulation we express the solution as a superposition of the linear evolutions which incorporate the boundary term and the initial data with the nonlinearity. Also to estimate the terms coming from Duhamel formula, we first solve the corresponding linear problem by taking Laplace transform of the equation in the temporal variable and inverting back by the Mellin transform so that we obtain an explicit formula for the linear evolution after extending the initial data to the whole line. Afterwards the nonlinear part of the formula will be treated by the $X^{s,b}$ method. Note that in the boundary value problems $b < \frac{1}{2}$ is necessary in order to carry out the contraction argument, while $b > \frac{1}{2}$ is required on the full line. As for the uniqueness, the solution we constructed is the unique fixed point of the Duhamel operator (18) by the contraction argument, yet it is not clear if the restriction of the fixed point of (18) to the half line is independent of the different extensions of the initial data. In this regard, the proof of uniqueness in our case proceeds in two steps: one is for the case $s > \frac{1}{2}$ where we exploit the Sobolev embedding and well-known Gronwall's inequality on $\mathbb{R}^+$, and the other is for the low regularity case $-\frac{1}{3} < s < \frac{1}{2}$ where we make use of the uniqueness obtained for $s > \frac{1}{2}$ and the smoothing estimate of Theorem 1.3 to establish the uniqueness in this range, also in contrast to the case $s > \frac{1}{2}$, it is not immediate to exhibit that different extensions produce the same solution. In particular, in order to establish uniqueness down to the local theory threshold $H^{-\frac{1}{3}}(\mathbb{R}^+)$, we require smoothing estimate of Theorem 1.3.

When $\mu = 1$ in (1) (the defocusing case), the following theorem provides bounds for higher order Sobolev norms. This is based on smoothing result obtained in Theorem 1.3 and a priori estimate at the energy level, Lemma 7.1.

Theorem 1.4 *Let* $\mu = 1$ *in the* Eq. (1). *In the case* $s \in [2, \frac{5}{2})$, $g \in H^s(\mathbb{R}^+)$, $h_1 \in H^{\frac{2s+3}{8}}(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$ *and* $h_2 \in H^{\frac{2s+1}{8}}(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$, *the associated local solution is* *global and the smoothing result holds globally. Furthermore, for $2 < s < \frac{5}{2}$ the solution has the growth bound*

$$\|u(t)\|_{H^s(\mathbb{R}^+)} \lesssim \langle T \rangle.$$

Here we note that the Eq. (1) does not satisfy the mass and energy conservations once the boundary data h_1 and h_2 are nonzero. Hence the global well-posedness at the energy level, H^2, for the Eq. (1) is a nontrivial problem in the presence of inhomogeneous boundary conditions, see Theorem 1.3 of [34]. The Lemma 7.1, which is the key to obtaining the growth bound in Theorem 1.4, results from the proof of Theorem 1.3 of [34].

As far as we know this work is the first treatment of the fourth order biharmonic Schrödinger equation subject to the inhomogeneous boundary conditions where well-posed solutions are constructed below the L^2 space.

Now we outline the organization of the paper. In Sect. 2, we define the notion of a solution. To be more precise we reformulate (1) as an integral equation (Duhamel's formula) and set this to be a solution map which we then show is a contraction in a suitable metric space. Thus by using the Duhamel's formula, the solution we constructed is a superposition of a linear and a nonlinear evolutions. We also introduce the space $H^s(\mathbb{R}^+)$ and discuss whenever one can extend the initial and boundary data. In Sect. 3 we illustrate, by an application of the Laplace transform on the half line, how to find the explicit solution formula for the linear problem with zero initial data. In Sect. 4, we state and prove linear and nonlinear a priori estimates. Linear estimates relate to two separate processes one is for a solution to a free fourth order Schrödinger equation and the other is for a solution to IBVP subject to the inhomogeneous boundary data. The estimates for the latter also clarify the regularity level of the boundary data h_1, h_2 and the selection of the spaces they are taken. In the remaining part of the Sect. 4, we prove the multilinear estimates associated to the nonlinear term coming from the integral part of the solution representation. In Sect. 5, we prove Theorem 1.2 by establishing the local well-posedness theory via the contraction argument and argue the dependence of the local existence time to the initial and boundary data. Theorems 1.3 and 1.4 are proved in Sect. 6 and the uniqueness is proved in Sect. 5.1. Lastly Sect. 7 is an appendix that involves some inequalities that will be needed repeatedly in the text.

1.1 Notation

We define the one dimensional Fourier transform as

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx$$

similarly the space time Fourier transform

$$\widehat{f}(\xi, \tau) = \mathcal{F}f(\xi, \tau) = \int_{\mathbb{R}^2} e^{-ix\xi - it\tau} f(x, t)dxdt.$$

Sobolev space $H^s(\mathbb{R})$ is defined via the norm

$$\|g\|_{H^s} = \|g\|_{H^s(\mathbb{R})} = \|\langle \xi \rangle^s \widehat{g}(\xi)\|_{L^2(\mathbb{R})}$$

where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$ (or equivalently $1 + |\xi|$). For $s > -\frac{1}{2}$, Sobolev spaces $H^s(\mathbb{R})$ on the half line are defined as

$$H^s(\mathbb{R}^+) = \left\{ g \in \mathcal{D}(\mathbb{R}^+) \ : \ \exists \widetilde{g} \in H^s(\mathbb{R}) \text{ such that } \widetilde{g}\chi_{(0,\infty)} = g \right\}$$

with the norm

$$\|g\|_{H^s(\mathbb{R}^+)} = \inf\left\{ \|\widetilde{g}\|_{H^s(\mathbb{R})} : \widetilde{g}\chi_{(0,\infty)} = g \right\}.$$

The restriction $s > -\frac{1}{2}$ is necessary because multiplication with the characteristic function $\chi_{(0,\infty)}$ is not well-defined for H^s distributions when $s \leq -\frac{1}{2}$. Moreover we write $W^t g$ for the linear biharmonic Scrödinger propagator

$$W^t g(x,t) = e^{it\Delta^2} g = \int e^{ix\xi + it\xi^4} \widehat{g}(\xi, t) d\xi.$$

For a space time function f, the notation D_0 means the evaluation at the boundary $x = 0$, that is

$$D_0\big(f(x,t)\big) = f(0,t).$$

Throughout we write η for a smooth compactly supported function that is equal to 1 on $[-1, 1]$ and $\operatorname{supp}\eta \subset [-2, 2]$. Also let $\rho \in C^\infty$ be a cut-off function satisfying $\rho = 1$ on $[0, \infty)$ and $\operatorname{supp}\rho \subset [-1, \infty)$. Lastly we use the notation $a \lesssim b$ meaning that $a \leq Cb$ for some absolute constant C, we define $a \gtrsim b$ likewise and write $a \sim b$ for $a \lesssim b \lesssim a$.

2 Notion of a solution

In order to find solutions of (1) we start with constructing the solution of the linear IBVP

$$\begin{cases} iu_t + u_{xxxx} = 0 \\ u(0,t) = h_1(t), \quad u_x(0,t) = h_2(t), \\ u(x,0) = g(x), \end{cases} \tag{5}$$

with the compatibility conditions $g(0) = h_1(0)$ for $\frac{1}{2} < s \leq \frac{3}{2}$ and $g(0) = h_1(0)$, $g'(0) = h_2(0)$ for $\frac{3}{2} < s \leq \frac{9}{2}$. We shall denote the solution of (5) by $W_0^t(g, h_1, h_2)$. This solution can be written as

$$W_0^t(g, h_1, h_2) = W_0^t(0, h_1 - p_1, h_2 - p_2) + W^t g_e$$

where g_e is an extension of g to the full line $\mathbb{R}$ such that $\|g_e\|_{H^s(\mathbb{R})} \lesssim \|g\|_{H^s(\mathbb{R}^+)}$ and the traces $p_1(t) = \eta(t) D_0(W^t g_e)$, $p_2(t) = \eta(t) D_0\big(\partial_x\big[W^t g_e\big]\big)$ are well well-defined and belong to the spaces $H^{\frac{2s+3}{8}}(\mathbb{R}^+)$, $H^{\frac{2s+1}{8}}(\mathbb{R}^+)$ respectively, by Lemma 4.1 below. As a result we decomposed the solution operator as a sum of free biharmonic Scrödinger evolution and the boundary operator corresponding to the zero initial data. Therefore we consider

$$\begin{cases} iu_t + u_{xxxx} = 0, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ u(0,t) = h_1(t), \quad u_x(0,t) = h_2(t), \\ u(x,0) = 0 \end{cases} \tag{6}$$

where $W_0^t(0, h_1, h_2)$ denotes the solution to this problem. By an application of the Laplace transform described in the next section, we obtain explicit representation for $W_0^t(0, h_1, h_2)$.

Lemma 2.1 *Assume that h_1 and h_2 are Schwartz functions. The solution of (6) can explicitly be written in the form*

$$u(x,t) = \frac{-1+i}{\pi}\Big[W_1 h_2 - i W_2 h_1 - W_3 h_1 - W_4 h_2 \Big]$$

$$-\frac{\sqrt{2}i}{\pi}\Big[W_5h_2+\Big(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\Big)W_6h_1+\Big(\frac{-\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\Big)W_7h_1-W_8h_2\Big]$$

where

$$W_1h_2(x,t)=\int_0^\infty e^{i\beta^4t-\beta x}\beta^2\widehat{h}_2(\beta^4)\rho(\beta x)d\beta,$$
$$W_2h_1(x,t)=\int_0^\infty e^{i\beta^4t-\beta x}\beta^3\widehat{h}_1(\beta^4)\rho(\beta x)d\beta,$$
$$W_3h_1(x,t)=\int_0^\infty e^{i\beta^4t+i\beta x}\beta^3\widehat{h}_1(\beta^4)d\beta,$$
$$W_4h_2(x,t)=\int_0^\infty e^{i\beta^4t+i\beta x}\beta^2\widehat{h}_2(\beta^4)d\beta,$$
$$W_5h_2(x,t)=\int_0^\infty e^{-i\beta^4t}e^{[-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}]\beta x}\beta^2\widehat{h}_2(-\beta^4)\rho(\beta x)d\beta,$$
$$W_6h_1(x,t)=\int_0^\infty e^{-i\beta^4t}e^{[-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}]\beta x}\beta^3\widehat{h}_1(-\beta^4)\rho(\beta x)d\beta,$$
$$W_7h_1(x,t)=\int_0^\infty e^{-i\beta^4t}e^{[-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}]\beta x}\beta^3\widehat{h}_1(-\beta^4)\rho(\beta x)d\beta,$$
$$W_8h_2(x,t)=\int_0^\infty e^{-i\beta^4t}e^{[-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}]\beta x}\beta^2\widehat{h}_2(-\beta^4)\rho(\beta x)d\beta.$$

Here by an abuse of notation we take

$$\widehat{h}_j(\xi)=\mathcal{F}\big(\chi_{(0,\infty)}h_j\big)(\xi)=\int_0^\infty e^{-i\xi t}h_j(t)dt. \tag{7}$$

We use this explicit form to obtain bounds on $W_0^t(0,h_1,h_2)$ in Sect. 4 below. Next, by the Duhamel formulation, we consider the integral equation equivalent to (1) on $[0,T]$, $t\le T<1$:

$$u(t)=\eta(t)W^tg_e+\eta(t)\int_0^t W^{t-t'}F(u)dt'+\eta(t)W_0^t\big(0,h_1-p_1-q_1,h_2-p_2-q_2\big)(t), \tag{8}$$

where

$$F(u)=\eta(t/T)|u|^2u,\qquad p_1(t)=\eta(t)D_0(W^tg_e),\qquad p_2(t)=\eta(t)D_0\big(\partial_x\big[W^tg_e\big]\big),$$
$$q_1(t)=\eta(t)D_0\Big(\int_0^t W^{t-t'}F(u)dt'\Big),\qquad q_2(t)=\eta(t)D_0\Big(\partial_x\Big[\int_0^t W^{t-t'}F(u)dt'\Big]\Big).$$

In the following, we want to prove that the integral equation (8) has a unique solution in a suitable function space (given by Definition 1.1) on $\mathbb{R}\times\mathbb{R}$ for sufficiently small T. Note that the restriction of u to $\mathbb{R}^+\times[0,T]$ is a distributional solution of (1) whereas smooth solutions of the Eq. (8) are classical solutions of (1).

We implement contraction argument in $X^{s,b}(\mathbb{R}\times\mathbb{R})$ spaces:

$$\|u\|_{X^{s,b}}=\Big\|\langle\xi\rangle^s\langle\tau-\xi^4\rangle^b\widehat{u}(\xi,\tau)\Big\|_{L^2_\tau L^2_\xi}. \tag{9}$$

In order to carry out the contraction argument in the local theory we will need the following standard results from [37]

$$\text{for any } s \in \mathbb{R} \text{ and } b > \frac{1}{2}, \text{ we have } X^{s,b} \subset C_0^t H_x^s. \tag{10}$$

For any $s, b \in \mathbb{R}$,

$$\left\| \eta(t) W^t g \right\|_{X^{s,b}} \lesssim \|g\|_{H^s}. \tag{11}$$

For $T < 1$ and $-\frac{1}{2} < b_1 < b_2 < \frac{1}{2}$ we have

$$\|\eta(t/T) F\|_{X^{s,b_1}} \lesssim T^{b_2 - b_1} \|F\|_{X^{s,b_2}}. \tag{12}$$

We also need the following estimate whose proof can be obtained by adapting the proof of Lemma 3.12 in [15]. For any $s \in \mathbb{R}$, $0 \le b_1 < \frac{1}{2}$ and $b_2 = 1 - b_1$, we have

$$\left\| \eta(t) \int_0^t W^{t-t'} F dt' \right\|_{X^{s,b_2}} \lesssim \|F\|_{X^{s,-b_1}}. \tag{13}$$

Next for the boundary data h_1 and h_2, we need estimates on the sizes of $\left\| \chi_{(0,\infty)} h_1 \right\|_{H^{\frac{2s+3}{8}}(\mathbb{R})}$ and $\left\| \chi_{(0,\infty)} h_2 \right\|_{H^{\frac{2s+1}{8}}(\mathbb{R})}$ which is the content of the next lemma.

Lemma 2.2 (See [16]) *Assume $h \in H^s(\mathbb{R}^+)$ for some $s \in (-\frac{1}{2}, \frac{5}{2})$.*

1. *If $-\frac{1}{2} < s < \frac{1}{2}$, then $\left\| \chi_{(0,\infty)} h \right\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R}^+)}$.*
2. *If $\frac{1}{2} < s < \frac{3}{2}$ and $h(0) = 0$, then $\left\| \chi_{(0,\infty)} h \right\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R}^+)}$.*
3. *If $\frac{1}{2} < s < \frac{3}{2}$, then $\|h_{even}\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R}^+)}$.*
4. *If $\frac{1}{2} < s < \frac{5}{2}$, $s \neq \frac{3}{2}$ and $h(0) = 0$, then $\|h_{odd}\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R}^+)}$.*

where $h_{even}(x) = h(|x|)$ *and* $h_{odd}(x) = \begin{cases} h(|x|) & \text{if } x \ge 0 \\ -h(|x|) & \text{if } x \le 0 \end{cases}$.

As a final note following will be useful in establishing the Theorem 1.4.

Remark 2.3 By the definition of linear flow W^t and the Lemma 2.1 we may write

$$W_0^t(g, h_1, h_2) - W_0^t(\tilde{g}, h_1, h_2) = W_0^t(g - \tilde{g}, 0, 0).$$

Moreover, by writing $W_0^t(g, 0, 0)$ with the method of odd extension and then utilizing Lemma 4.3, Lemma 4.1 below and 4. of Lemma 2.2 we obtain the bound

$$\left\| W_0^t(g, 0, 0) \right\|_{H^s(\mathbb{R}^+)} \lesssim \left\| W^t g_{\text{odd}} \right\|_{H^s(\mathbb{R})} = \|g_{\text{odd}}\|_{H^s(\mathbb{R})} \lesssim \|g\|_{H^s(\mathbb{R}^+)}.$$

3 Proof of Lemma 2.1: boundary term

In this section we obtain explicit solution formula for the linear problem (6) by the application of the Laplace transform. So taking the Laplace transform of the Eq. (6) in t leads to the initial value problem in the spatial variable x

$$\begin{cases} \widetilde{u}_{xxxx} + i\lambda \widetilde{u} = 0 \\ \widetilde{u}(0, \lambda) = \widetilde{h}_1(\lambda), \quad \widetilde{u}_x(0, \lambda) = \widetilde{h}_2(\lambda) \end{cases} \tag{14}$$

where

$$\widetilde{u}(x,\lambda)=\int_0^{\infty}e^{-\lambda t}u(x,t)dt,\qquad \widetilde{h}_j(x,\lambda)=\int_0^{\infty}e^{-\lambda t}h_j(t)dt,\quad j=1,2.$$

The solution of (14) can be written as follows

$$\widetilde{u}(x,\lambda)=c_1(\lambda)e^{r_1(\lambda)x}+c_2(\lambda)e^{r_2(\lambda)x}$$

where $r_1(\lambda)$ and $r_2(\lambda)$ are solutions of the characteristic equation $r^4(\lambda)+i\lambda=0$ for which $\operatorname{Re} r_1<0, \operatorname{Re} r_2<0$. Employing the initial conditions and suppressing the λ dependence of $r_1(\lambda)$ and $r_2(\lambda)$ we have

$$c_1(\lambda)=\frac{\widetilde{h}_2(\lambda)-r_2\widetilde{h}_1(\lambda)}{r_1-r_2},\qquad c_2(\lambda)=\frac{r_1\widetilde{h}_1(\lambda)-\widetilde{h}_2(\lambda)}{r_1-r_2}.$$

Then by Mellin inversion we can express the solution as

$$u(x,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}}{r_1-r_2}\Big[\big(\widetilde{h}_2(\lambda)-r_2\widetilde{h}_1(\lambda)\big)e^{r_1x}+\big(r_1\widetilde{h}_1(\lambda)-\widetilde{h}_2(\lambda)\big)e^{r_2x}\Big]d\lambda$$

for $x,t>0$ and where $\gamma>0$ is fixed. Letting $\gamma\to 0$ we have

$$\begin{aligned}
u(x,t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\beta t}}{r_1-r_2}\Big[\big(\widetilde{h}_2(i\beta)-r_2\widetilde{h}_1(i\beta)\big)e^{r_1(i\beta)x}+\big(r_1\widetilde{h}_1(i\beta)-\widetilde{h}_2(i\beta)\big)e^{r_2(i\beta)x}\Big]d\beta\\
&=\frac{1}{2\pi}\int_{-\infty}^{0}\frac{e^{i\beta t}}{i\sqrt{2}\sqrt[4]{-\beta}}\Big[\widetilde{h}_2(i\beta)+\Big(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\Big)\sqrt[4]{-\beta}\,\widetilde{h}_1(i\beta)\Big]e^{\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)\sqrt[4]{-\beta}x}d\beta\\
&+\frac{1}{2\pi}\int_{-\infty}^{0}\frac{e^{i\beta t}}{i\sqrt{2}\sqrt[4]{-\beta}}\Big[\Big(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\Big)\sqrt[4]{-\beta}\,\widetilde{h}_1(i\beta)-\widetilde{h}_2(i\beta)\Big]e^{\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)\sqrt[4]{-\beta}x}d\beta\\
&+\frac{1}{2\pi}\int_{0}^{\infty}\frac{e^{i\beta t}}{-(1+i)\sqrt[4]{\beta}}\big[\widetilde{h}_2(i\beta)-i\sqrt[4]{\beta}\,\widetilde{h}_1(i\beta)\big]e^{-\sqrt[4]{\beta}x}d\beta\\
&+\frac{1}{2\pi}\int_{0}^{\infty}\frac{e^{i\beta t}}{-(1+i)\sqrt[4]{\beta}}\Big[-\sqrt[4]{\beta}\,\widetilde{h}_1(i\beta)-\widetilde{h}_2(i\beta)\Big]e^{i\sqrt[4]{\beta}x}d\beta\\
&=\frac{1}{2\pi}\int_{0}^{\infty}\frac{e^{-i\beta^4 t}}{i\sqrt{2}\beta}\Big[\widetilde{h}_2(-i\beta^4)+\Big(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\Big)\beta\,\widetilde{h}_1(-i\beta^4)\Big]e^{\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)\beta x}4\beta^3d\beta\\
&+\frac{1}{2\pi}\int_{0}^{\infty}\frac{e^{-i\beta^4 t}}{i\sqrt{2}\beta}\Big[\Big(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\Big)\beta\,\widetilde{h}_1(-i\beta^4)-\widetilde{h}_2(-i\beta^4)\Big]e^{\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)\beta x}4\beta^3d\beta\\
&+\frac{1}{2\pi}\int_{0}^{\infty}\frac{e^{i\beta^4 t}}{-(1+i)\beta}\big[\widetilde{h}_2(i\beta^4)-i\beta\widetilde{h}_1(i\beta^4)\big]e^{-\beta x}4\beta^3d\beta\\
&+\frac{1}{2\pi}\int_{0}^{\infty}\frac{e^{i\beta^4 t}}{-(1+i)\beta}\Big[-\beta\,\widetilde{h}_1(i\beta^4)-\widetilde{h}_2(i\beta^4)\Big]e^{i\beta x}4\beta^3d\beta,
\end{aligned}$$

by a slight abuse of notation after writing $\widehat{h}_j$ instead of $\widetilde{h}_j$ to denote the Fourier transform of $\chi_{(0,\infty)}h_j$, $j=1,2$, we obtain

$$\begin{aligned}
u(x,t)=&-\frac{\sqrt{2}}{\pi}i\int_0^{\infty}e^{-i\beta^4t}e^{\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)\beta x}\Big[\widehat{h}_2(-\beta^4)+\Big(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\Big)\beta\,\widehat{h}_1(-\beta^4)\Big]\beta^2d\beta\\
&-\frac{\sqrt{2}}{\pi}i\int_0^{\infty}e^{-i\beta^4t}e^{\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)\beta x}\Big[\Big(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\Big)\beta\,\widehat{h}_1(-\beta^4)-\widehat{h}_2(-\beta^4)\Big]\beta^2d\beta
\end{aligned}$$

$$+\frac{-1+i}{\pi}\int_0^\infty e^{i\beta^4 t-\beta x}\big[\widehat{h}_2(\beta^4)-i\beta\widehat{h}_1(\beta^4)\big]\beta^2 d\beta$$
$$+\frac{-1+i}{\pi}\int_0^\infty e^{i\beta^4 t+i\beta x}\big[-\beta\,\widehat{h}_1(\beta^4)-\widehat{h}_2(\beta^4)\big]\beta^2 d\beta.$$

Finally, we add the cut-off function ρ in the above integrals except the last one to extend the solution to all x. Note that with this choice the integrals converge for all x.

4 A priori estimates

4.1 Estimates for the linear terms

In this section we justify that the linear terms in (8) stay in the function space given in the Definition 1.1. First we begin with the Kato smoothing inequality depicting interaction between the space and time derivatives. Note that this affirms the selection of the spaces that the data g, h_1 and h_2 reside in.

Lemma 4.1 (Kato smoothing inequality) *For any $s\in\mathbb{R}$, $g\in H^s(\mathbb{R})$, we have $\eta(t)W^t g\in C^0_x H_t^{\frac{2s+3}{8}}(\mathbb{R}\times\mathbb{R})$ and $\eta(t)\partial_x[W^t g]\in C^0_x H_t^{\frac{2s+1}{8}}(\mathbb{R}\times\mathbb{R})$, moreover,*

$$\left\|\eta(t)W^t g\right\|_{L^\infty_x H_t^{\frac{2s+3}{8}}}\lesssim\|g\|_{H^s_x}$$
$$\left\|\eta(t)\partial_x[W^t g]\right\|_{L^\infty_x H_t^{\frac{2s+1}{8}}}\lesssim\|g\|_{H^s_x}.$$

Proof We start by writing that

$$\mathcal{F}_t(\eta W^t g)(\tau)=\int e^{ix\xi}\widehat{\eta}(\tau-\xi^4)\widehat{g}(\xi)d\xi$$
$$=\int_{|\xi|<1}e^{ix\xi}\widehat{\eta}(\tau-\xi^4)\widehat{g}(\xi)d\xi+\int_{|\xi|\ge1}e^{ix\xi}\widehat{\eta}(\tau-\xi^4)\widehat{g}(\xi)d\xi.$$

Using the fact that η is a Schwarz function, the contribution of the $H_t^{\frac{2s+3}{8}}$ norm of the first term above is bounded by

$$\int_{|\xi|<1}\left\|\langle\tau\rangle^{\frac{2s+3}{8}}\widehat{\eta}(\tau-\xi^4)\right\|_{L^2_\tau}|\widehat{g}(\xi)|d\xi\lesssim\int_{|\xi|<1}\left\|\langle\tau\rangle^{\frac{2s+3}{8}}\widehat{\eta}(\tau-\xi^4)\right\|_{L^2_\tau}\langle\xi\rangle^s|\widehat{g}(\xi)|d\xi$$
$$\lesssim\int_{|\xi|<1}\langle\xi\rangle^s|\widehat{g}(\xi)|d\xi\lesssim\|g\|_{H^s}.$$

Next by the inequality $\langle x+y\rangle^r\lesssim\langle x\rangle^{|r|}\langle y\rangle^r$ for any $r\in\mathbb{R}$, and a change of variable, the contribution for the second term is estimated by

$$\left\|\int_{|\xi|\ge1}\langle\tau\rangle^{\frac{2s+3}{8}}|\widehat{\eta}(\tau-\xi^4)||\widehat{g}(\xi)|d\xi\right\|_{L^2_\tau}$$
$$\lesssim\left\|\int_{|\xi|\ge1}\langle\tau-\xi^4\rangle^{\frac{|2s+3|}{8}}\langle\xi\rangle^{\frac{2s+3}{2}}|\widehat{\eta}(\tau-\xi^4)||\widehat{g}(\xi)|d\xi\right\|_{L^2_\tau}$$
$$\lesssim\left\|\int_{|\rho|\ge1}\langle\tau-\rho\rangle^{\frac{|2s+3|}{8}}\langle\rho\rangle^{\frac{2s-3}{8}}|\widehat{\eta}(\tau-\rho)||\widehat{g}(\pm\rho^{\frac14})|d\rho\right\|_{L^2_\tau}$$

$$\lesssim \left\| \langle \cdot \rangle^{\frac{|2s+3|}{8}} \widehat{\eta}(\cdot) \right\|_{L^1} \left\| \rho^{\frac{2s-3}{8}} \widehat{g}(\pm \rho^{\frac{1}{4}}) \right\|_{L^2_{\rho \geq 1}}$$
$$\lesssim \left\| \rho^{\frac{2s-3}{8}} \widehat{g}(\pm \rho^{\frac{1}{4}}) \right\|_{L^2_{\rho \geq 1}}$$

where we have used Young's inequality in the third inequality. Changing variable back to ξ this is bounded by

$$\left(\int_1^\infty \langle \xi \rangle^{2s-3} |\widehat{g}(\pm \xi)|^2 \xi^3 d\xi \right)^{\frac{1}{2}} \lesssim \left(\int \langle \xi \rangle^{2s} |\widehat{g}(\xi)|^2 d\xi \right)^{\frac{1}{2}} = \|g\|_{H^s}.$$

From these and the dominated convergence theorem continuity statement follows. Using the same argument we estimate $\left\| \eta(t) \partial_x [W^t g] \right\|_{H_t^{\frac{2s+1}{8}}}$ likewise. □

Proposition 4.2, Lemmas 4.3, and 4.4 below verify that the boundary operator belongs to the space from Definition 1.1.

Proposition 4.2 *For any $s \geq -\frac{1}{2}$, $b \leq \frac{1}{2}$ and h_1, h_2 satisfying $\chi_{(0,\infty)} h_1 \in H^{\frac{2s+3}{8}}$, $\chi_{(0,\infty)} h_2 \in H^{\frac{2s+1}{8}}$ we have*

$$\left\| \eta(t) W_0^t(0, h_1, h_2) \right\|_{X^{s,b}} \lesssim \left\| \chi_{(0,\infty)} h_1 \right\|_{H_t^{\frac{2s+3}{8}}} + \left\| \chi_{(0,\infty)} h_2 \right\|_{H_t^{\frac{2s+1}{8}}}.$$

Proof First recall that

$$W_0^t(0, h_1, h_2) = \frac{-1+i}{\pi} \Big[W_1 h_2 - i W_2 h_1 - W_3 h_1 - W_4 h_2 \Big]$$
$$- \frac{\sqrt{2} i}{\pi} \Big[W_5 h_2 + \Big(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \Big) W_6 h_1 + \Big(\frac{-\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \Big) W_7 h_1 - W_8 h_2 \Big]$$

where the terms $W_1 h_2$, $W_2 h_1$, $W_3 h_1$, $W_4 h_2$, $W_5 h_2$, $W_6 h_1$, $W_7 h_1$ and $W_8 h_2$ are given in Lemma 2.1, also recall the notation of expressing $\widehat{h}$ as $\mathcal{F}_t(\chi_{(0,\infty)} h)$. Note that

$$W_3 h_1 = W^t \psi_3 \text{ and } W_4 h_2 = W^t \psi_2$$

where

$$\widehat{\psi}_3(\beta) = \beta^3 \widehat{h}_1(\beta^4) \chi_{(0,\infty)}(\beta) \text{ and } \widehat{\psi}_4(\beta) = \beta^2 \widehat{h}_2(\beta^4) \chi_{(0,\infty)}(\beta). \tag{15}$$

By change of variables we have

$$\|\psi_3\|_{H^s} = \left\| \langle \beta \rangle^s \widehat{\psi}_3(\beta) \right\|_{L^2_\beta} = \left(\int_0^\infty \langle \beta \rangle^{2s} \beta^6 |\widehat{h}_1(\beta^4)|^2 d\beta \right)^{\frac{1}{2}}$$
$$\lesssim \left(\int_0^\infty \langle \rho \rangle^{\frac{2s+3}{4}} |\widehat{h}_1(\rho)|^2 d\rho \right)^{\frac{1}{2}} \lesssim \left\| \chi_{(0,\infty)} h_1 \right\|_{H^{\frac{2s+3}{8}}} \tag{16}$$

and similarly

$$\|\psi_4\|_{H^s} = \left\| \langle \beta \rangle^s \widehat{\psi}_4(\beta) \right\|_{L^2_\beta} = \left(\int_0^\infty \langle \beta \rangle^{2s} \beta^4 |\widehat{h}_2(\beta^4)|^2 d\beta \right)^{\frac{1}{2}}$$
$$\lesssim \left(\int_0^\infty \langle \rho \rangle^{\frac{2s+1}{4}} |\widehat{h}_2(\rho)|^2 d\rho \right)^{\frac{1}{2}} \lesssim \left\| \chi_{(0,\infty)} h_2 \right\|_{H^{\frac{2s+1}{8}}}. \tag{17}$$

Then using (11) together with the bounds (16) and (17) we have

$$\|\eta(t) W_3 h_1\|_{X^{s,b}} = \left\| \eta W^t \psi_3 \right\|_{X^{s,b}} \lesssim \|\psi_3\|_{H^s} \lesssim \left\| \chi_{(0,\infty)} h_1 \right\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R})}$$

and

$$\|\eta(t)W_4h_2\|_{X^{s,b}} = \left\|\eta W^t\psi_4\right\|_{X^{s,b}} \lesssim \|\psi_4\|_{H^s} \lesssim \left\|\chi_{(0,\infty)}h_2\right\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R})}.$$

For W_1h_2 and W_2h_1, set $f(x) = e^{-x}\rho(x)$. Note that f is a Schwarz function. Assume $s \in 4\mathbb{N}$, we can write

$$\begin{aligned}
\partial_x^s W_1h_2 &= \eta\int_0^\infty e^{i\beta^4 t} f^{(s)}(\beta x)\beta^{s+2}\widehat{h}_2(\beta^4)d\beta \\
&= (-i)^{s/4}\eta\int_0^\infty e^{i\beta^4 t} f^{(s)}(\beta x)\beta^2\mathcal{F}_t[\chi_{(0,\infty)}\partial_t^{(s/4)}h_2](\beta^4)d\beta
\end{aligned}$$

and

$$\partial_x^s W_2h_1 = (-i)^{s/4}\eta\int_0^\infty e^{i\beta^4 t} f^{(s)}(\beta x)\beta^3\mathcal{F}_t[\chi_{(0,\infty)}\partial_t^{(s/4)}h_1](\beta^4)d\beta.$$

Then using these with the interpolation it suffices to prove the bounds for $s = 0$. We have

$$\begin{aligned}
\widehat{\eta W_1h_2}(\xi,\tau) &= \mathcal{F}_t\Big(\eta(t)\int_0^\infty e^{i\beta^4 t}\beta^2\widehat{h}_2(\beta^4)\mathcal{F}_x(f(\beta x))d\beta\Big)(\tau) \\
&= \int_0^\infty \widehat{\eta}(\tau-\beta^4)\beta\widehat{h}_2(\beta^4)\widehat{f}(\xi/\beta)d\beta
\end{aligned}$$

and

$$\widehat{\eta W_2h_1}(\xi,\tau) = \int_0^\infty \widehat{\eta}(\tau-\beta^4)\beta^2\widehat{h}_1(\beta^4)\widehat{f}(\xi/\beta)d\beta.$$

Since f is a Schwarz function,

$$|\widehat{f}(\xi/\beta)| \lesssim \frac{1}{\langle\xi/\beta\rangle^4} \lesssim \frac{1}{1+(\xi/\beta)^4} = \frac{\beta^4}{\beta^4+\xi^4}$$

and as η is a compact supported C^∞ function we may write

$$|\widehat{\eta}(\tau-\beta^4)| \lesssim \langle\tau-\beta^4\rangle^{-3}$$

as well. Therefore

$$\|\eta W_1h_2\|_{X^{0,b}} \lesssim \left\|\langle\tau-\xi^4\rangle^b\int_0^\infty\langle\tau-\beta^4\rangle^{-3}\frac{\beta^5}{\beta^4+\xi^4}|\widehat{h}_2(\beta^4)|d\beta\right\|_{L^2_{\xi,\tau}}.$$

We separate the integral into regions where $\beta^4+\xi^4 \le 1$ and $\beta^4+\xi^4 > 1$. In the first case, we have

$$\begin{aligned}
\left\|\int_0^1\langle\tau\rangle^{b-3}\frac{\beta^5}{\beta^4+\xi^4}|\widehat{h}_2(\beta^4)|d\beta\right\|_{L^2_{|\xi|\le 1}L^2_\tau} &\lesssim \left\|\langle\tau\rangle^{b-3}\right\|_{L^2_\tau}\int_0^1\left\|\frac{\beta^5}{\beta^4+\xi^4}\right\|_{L^2_{|\xi|\le 1}}|\widehat{h}_2(\beta^4)|d\beta \\
&\lesssim \int_0^1\beta^{\frac{3}{2}}|\widehat{h}_2(\beta^4)|d\beta \\
&= \int_0^1\rho^{-\frac{3}{8}}|\widehat{h}_2(\rho)|d\rho \\
&\lesssim \left\|\chi_{(0,\infty)}h_2\right\|_{L^2(\mathbb{R})} \le \left\|\chi_{(0,\infty)}h_2\right\|_{H^{\frac{1}{8}}(\mathbb{R})}
\end{aligned}$$

where we have used Minkowski's and Cauchy-Schwarz inequalities in the first and third bounds respectively. For the other case where $\beta^4+\xi^4 > 1$, making use of relations $\langle \tau-\xi^4\rangle \lesssim \langle \tau-\beta^4\rangle\langle \beta^4+\xi^4\rangle$ and $\beta^4+\xi^4 \sim \langle \beta^4+\xi^4\rangle$ we have the bound

$$\begin{aligned}
&\left\| \int_0^\infty \langle \tau-\beta^4\rangle^{b-3} \frac{\beta^5}{(\beta^4+\xi^4)^{1-b}} |\widehat{h}_2(\beta^4)| d\beta \right\|_{L^2_{\xi,\tau}} \\
&\lesssim \left\| \int_0^\infty \langle \tau-\beta^4\rangle^{b-3} \left\| \frac{\beta^5}{(\beta^4+\xi^4)^{1-b}} \right\|_{L^2_\xi} |\widehat{h}_2(\beta^4)| d\beta \right\|_{L^2_\tau} \\
&\lesssim \left\| \int_0^\infty \langle \tau-\beta^4\rangle^{b-3} \beta^{\frac{3}{2}+4b} |\widehat{h}_2(\beta^4)| d\beta \right\|_{L^2_\tau} \\
&\lesssim \left\| \int_0^\infty \langle \tau-\rho\rangle^{b-3} \rho^{b-\frac{3}{8}} |\widehat{h}_2(\rho)| d\rho \right\|_{L^2_\tau} \\
&\lesssim \left\| \langle \tau\rangle^{b-3} \right\|_{L^1_\tau} \left\| \langle \rho\rangle^{\frac{1}{8}} \widehat{h}_2(\rho) \right\|_{L^2_\rho} \lesssim \left\| \chi_{(0,\infty)} h_2 \right\|_{H^{\frac{1}{8}}(\mathbb{R})}
\end{aligned}$$

where we have used Minkowski's and Young inequalities and note that we require $b \le \frac{1}{2}$ in the fourth inequality so that $b-\frac{3}{8} \le \frac{1}{8}$. Accordingly, using the similar arguments, we have

$$\begin{aligned}
\|\eta W_2 h_1\|_{X^{0,b}} &\lesssim \int_0^1 \beta^{\frac{5}{2}} |\widehat{h}_1(\rho)| d\beta + \left\| \int_0^\infty \langle \tau-\beta^4\rangle^{b-3} \beta^{\frac{5}{2}+4b} |\widehat{h}_1(\beta^4)| d\beta \right\|_{L^2_\tau} \\
&\lesssim \int_0^1 \rho^{-\frac{1}{8}} |\widehat{h}_1(\rho)| d\beta + \left\| \int_0^\infty \langle \tau-\rho\rangle^{b-3} \rho^{b-\frac{1}{8}} |\widehat{h}_1(\rho)| d\rho \right\|_{L^2_\tau} \\
&\lesssim \left\| \chi_{(0,\infty)} h_1 \right\|_{L^2_\rho} + \left\| \langle \tau\rangle^{b-3} \right\|_{L^1_\tau} \left\| \langle \rho\rangle^{b-\frac{1}{8}} \widehat{h}_1(\rho) \right\|_{L^2_\rho} \\
&\lesssim \left\| \chi_{(0,\infty)} h_1 \right\|_{H^{\frac{3}{8}}(\mathbb{R})} + \left\| \langle \rho\rangle^{\frac{3}{8}} \widehat{h}_1(\rho) \right\|_{L^2_\rho} \lesssim \left\| \chi_{(0,\infty)} h_1 \right\|_{H^{\frac{3}{8}}(\mathbb{R})}.
\end{aligned}$$

For the remaining terms of $W_0^t(0,h_1,h_2)$, estimates are similar; for $W_5 h_2$ and $W_6 h_1$ we let $f_1(x) = e^{(-\sqrt{2}/2+i\sqrt{2}/2)x}\rho(x)$ and for $W_7 h_1$ and $W_8 h_2$ set $f_2(x) = e^{(-\sqrt{2}/2-i\sqrt{2}/2)x}\rho(x)$ both of which are clearly Schwarz functions. So we adapt the previous estimates by swapping f with f_1 and f_2 for the terms $W_5 h_2$, $W_6 h_1$ and $W_7 h_1$, $W_8 h_2$ respectively. Eventually we have the bounds

$$\begin{aligned}
\left\| \eta W_j h_2 \right\|_{X^{0,b}} &\lesssim \left\| \chi_{(0,\infty)} h_2 \right\|_{H^{\frac{1}{8}}(\mathbb{R})} \quad \text{for j} = 5, 8, \\
\left\| \eta W_j h_1 \right\|_{X^{0,b}} &\lesssim \left\| \chi_{(0,\infty)} h_1 \right\|_{H^{\frac{3}{8}}(\mathbb{R})} \quad \text{for j} = 6, 7.
\end{aligned}$$

As before interpolating between the integers $s \in 4\mathbb{N}$ we obtain the bounds for any $s \ge 0$. To treat the $s < 0$ case we define the Fourier multiplier operator $\langle D\rangle_x^{-\frac{1}{2}}$ given by $\langle \xi\rangle^{-\frac{1}{2}}$ on the Fourier side. In this case,

$$\langle D\rangle_x^{-\frac{1}{2}} \big[\eta W_1 h_2\big](x,t) = \eta(t) \int_0^\infty e^{i\beta^4 t} \beta^2 \widehat{h}_2(\beta^4) \langle D\rangle_x^{-\frac{1}{2}} \big[f(\beta x)\big] d\beta$$

with similar formulas for the other terms of $W_0^t(0,h_1,h_2)$ other than W_3h_1 and W_4h_2. Note that

$$\begin{aligned}\mathcal{F}_{x,t}\Big(\langle D\rangle_x^{-\frac{1}{2}}\big[\eta W_1h_2\big]\Big)(\xi,\tau) &= \int_0^\infty \widehat{\eta}(\tau-\beta^4)\beta^2\widehat{h}_2(\beta^4)\mathcal{F}_x\big(\langle D\rangle_x^{-\frac{1}{2}}\big[f(\beta x)\big]\big)d\beta \\ &= \int_0^\infty \widehat{\eta}(\tau-\beta^4)\beta^2\widehat{h}_2(\beta^4)\mathcal{F}_x\big(\big[\langle D\rangle_x^{-\frac{1}{2}}f\big](\beta x)\big)(\xi)\langle\xi/\beta\rangle^{\frac{1}{2}}\langle\xi\rangle^{-\frac{1}{2}}d\beta.\end{aligned}$$

As $\langle D\rangle_x^{-\frac{1}{2}}f$ is a Schwarz function, we are free to establish the bounds

$$\Big|\mathcal{F}_x\big(\big[\langle D\rangle_x^{-\frac{1}{2}}f\big](\beta x)\big)(\xi)\Big| = \Big|\frac{1}{\beta}\widehat{\langle D\rangle_x^{-\frac{1}{2}}f}(\xi/\beta)\Big| \lesssim \frac{1}{|\beta|}\langle\xi/\beta\rangle^{-\frac{9}{2}}$$

and

$$|\widehat{\eta}(\tau-\beta^4)| \lesssim \langle\tau-\beta^4\rangle^{-3}.$$

This leads to the bound

$$\begin{aligned}\Big\|\langle D\rangle_x^{-\frac{1}{2}}\eta W_1h_2\Big\|_{X^{0,b}} &\lesssim \Big\|\langle\tau-\xi^4\rangle^b\int_0^\infty\langle\tau-\beta^4\rangle^{-3}\beta\langle\xi/\beta\rangle^{-4}\langle\xi\rangle^{-\frac{1}{2}}|\widehat{h}_2(\beta^4)|d\beta\Big\|_{L^2_{\xi,\tau}} \\ &\lesssim \Big\|\langle\tau-\xi^4\rangle^b\int_0^\infty\langle\tau-\beta^4\rangle^{-3}\frac{\beta^5}{\beta^4+\xi^4}|\widehat{h}_2(\beta^4)|d\beta\Big\|_{L^2_{\xi,\tau}}\end{aligned}$$

which has been treated above. Thus interpolation between $s=-\frac{1}{2}$ and $s=0$ yields the result. Other terms are handled similarly. □

Lemma 4.3 *For $s\geq -1$ and boundary data (h_1,h_2) satisfying $(\chi_{(0,\infty)}h_1,\chi_{(0,\infty)}h_2)\in H^{\frac{2s+3}{8}}(\mathbb{R})\times H^{\frac{2s+1}{8}}(\mathbb{R})$, we have*

$$W_0^t(0,h_1,h_2)\in C_t^0H_x^s(\mathbb{R}\times\mathbb{R}).$$

Proof We begin by showing that W_3h_1 and W_4h_2 belong to $C_t^0H_x^s(\mathbb{R}\times\mathbb{R})$. Since $W^t=e^{it\Delta^2}$ is unitary in H^s, we have

$$\|W_3h_1\|_{H_x^s} = \big\|W^t\psi_3\big\|_{H_x^s} = \|\psi_3\|_{H_x^s} \lesssim \big\|\chi_{(0,\infty)}h_1\big\|_{H^{\frac{2s+3}{8}}(\mathbb{R})}$$

and

$$\|W_4h_2\|_{H_x^s} = \big\|W^t\psi_4\big\|_{H_x^s} = \|\psi_4\|_{H_x^s} \lesssim \big\|\chi_{(0,\infty)}h_2\big\|_{H^{\frac{2s+1}{8}}(\mathbb{R})}$$

where we have used (16) and (17) in the above inequalities respectively, and ψ_3, ψ_4 are defined as in (15). Continuity in the temporal variable follows from these bounds and the continuity of the linear group W^t in H^s. To show that the remaining terms of $W_t^0(0,h_1,h_2)$ lie in $C_t^0H_x^s(\mathbb{R}\times\mathbb{R})$, recalling the explicit form of the boundary operator from Lemma 2.1, we rewrite the remaining terms as follows

$$\begin{aligned}W_1h_2(x,t) &= \int_{\mathbb{R}} f(\beta x)\mathcal{F}_x(e^{it\Delta^2}\psi_1)(\beta)d\beta,\ \ \widehat{\psi}_1(\beta)=\beta^2\widehat{h}_2(\beta^4)\chi_{(0,\infty)}(\beta), \\ W_2h_1(x,t) &= \int_{\mathbb{R}} f(\beta x)\mathcal{F}_x(e^{it\Delta^2}\psi_2)(\beta)d\beta,\ \ \widehat{\psi}_2(\beta)=\beta^3\widehat{h}_1(\beta^4)\chi_{(0,\infty)}(\beta), \\ W_5h_2(x,t) &= \int_{\mathbb{R}} f_1(\beta x)\mathcal{F}_x(e^{-it\Delta^2}\psi_5)(\beta)d\beta,\ \ \widehat{\psi}_5(\beta)=\beta^2\widehat{h}_2(-\beta^4)\chi_{(0,\infty)}(\beta),\end{aligned}$$

$$W_6h_1(x,t)=\int_{\mathbb{R}} f_1(\beta x)\mathcal{F}_x(e^{-it\Delta^2}\psi_6)(\beta)d\beta,\ \ \widehat{\psi}_6(\beta)=\beta^3\widehat{h}_1(-\beta^4)\chi_{(0,\infty)}(\beta),$$
$$W_7h_1(x,t)=\int_{\mathbb{R}} f_2(\beta x)\mathcal{F}_x(e^{-it\Delta^2}\psi_7)(\beta)d\beta,\ \ \widehat{\psi}_7(\beta)=\beta^3\widehat{h}_1(-\beta^4)\chi_{(0,\infty)}(\beta),$$
$$W_8h_2(x,t)=\int_{\mathbb{R}} f_2(\beta x)\mathcal{F}_x(e^{-it\Delta^2}\psi_8)(\beta)d\beta,\ \ \widehat{\psi}_8(\beta)=\beta^2\widehat{h}_2(-\beta^4)\chi_{(0,\infty)}(\beta),$$

where $f(x)=e^{-x}\rho(x)$, $f_1(x)=e^{(-\sqrt{2}/2+i\sqrt{2}/2)x}\rho(x)$ and $f_2(x)=e^{(-\sqrt{2}/2-i\sqrt{2}/2)x}\rho(x)$. Note that following the same computations done in (16) and (17), we have

$$\|\psi_j\|_{H_x^s}\lesssim\|\chi_{(0,\infty)}h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R})}\ \text{for } j=2,6,7,$$
$$\|\psi_j\|_{H_x^s}\lesssim\|\chi_{(0,\infty)}h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R})}\ \text{for } j=1,5,8.$$

Using these and the continuity of the group $e^{\pm it\Delta^2}$ on H^s it suffices to show that the maps

$$g\mapsto Tg=\int_{\mathbb{R}} f(\beta x)\widehat{g}(\beta)d\beta,\ \ g\mapsto T_1g=\int_{\mathbb{R}} f_1(\beta x)\widehat{g}(\beta)d\beta,\ \ g\mapsto T_2g=\int_{\mathbb{R}} f_2(\beta x)\widehat{g}(\beta)d\beta$$

are bounded in H^s. We show this for the map $g\mapsto Tg$ only as each f, f_1 and f_2 are Schwarz functions leading to the same result. Consider first $s=0$, we rewrite $Tg(x)$ by using the change of variable $\beta x\to\beta$ as follows

$$Tg(x)=\int_{\mathbb{R}} f(\beta)\widehat{g}(x^{-1}\beta)x^{-1}d\beta.$$

Therefore,

$$\begin{aligned}\|Tg\|_{L_x^2}&\le\int_{\mathbb{R}}|f(\beta)|\,\|x^{-1}\widehat{g}(x^{-1}\beta)\|_{L_x^2}d\beta=\int_{\mathbb{R}}|f(\beta)|\Big(\int x^{-2}|\widehat{g}(x^{-1}\beta)|^2dx\Big)^{\frac12}d\beta\\&=\int_{\mathbb{R}}|f(\beta)|\Big(\int\beta^{-1}|\widehat{g}(z)|^2dz\Big)^{\frac12}d\beta=\|g\|_{L^2}\int_{\mathbb{R}}\frac{|f(\beta)|}{\sqrt{\beta}}d\beta\lesssim\|g\|_{L^2}.\end{aligned}$$

Since f is a Schwarz function, the verification of the final inequality can be made as follows

$$\int_{\mathbb{R}}\frac{|f(\beta)|}{\sqrt{\beta}}d\beta\lesssim\int_{\mathbb{R}}\frac{d\beta}{\langle\beta\rangle^{\frac12+}\sqrt{\beta}}\lesssim\int_{|\beta|\le1}\frac{d\beta}{\sqrt{\beta}}+\int_{|\beta|>1}\frac{d\beta}{\langle\beta\rangle^{1+}}\lesssim1.$$

Note that for any $s\in\mathbb{N}$ we write

$$\partial_x^sTg(x)=\int_{\mathbb{R}} f^{(s)}(\beta x)\beta^s\widehat{g}(\beta)d\beta.$$

This with $s=0$ result implies that $\|Tg\|_{H^s}\lesssim\|g\|_{H^s}$, $s\in\mathbb{N}$. Hence by interpolation, $s\ge0$ case follows. As for $s=-1$, we pick ρ such that $\int f dx=0$ so that $\partial_x^{-1}f$ belongs to the Schwarz space. Then we write

$$\partial_x^{-1}Tg(x)=\int_{\mathbb{R}}\partial_x^{-1}\big(f(\beta x)\big)\widehat{g}(\beta)d\beta=\int_{\mathbb{R}}\partial_x^{-1}f(\beta x)\beta^{-1}\widehat{g}(\beta)d\beta.$$

Entegrating this with $s=0$ result and then applying the interpolation argument we get the bound for $s\ge-1$. □

Lemma 4.4 *For $s \geq -1$ and boundary data (h_1, h_2) satisfying $(\chi_{(0,\infty)}h_1, \chi_{(0,\infty)}h_2) \in H^{\frac{2s+3}{8}}(\mathbb{R}) \times H^{\frac{2s+1}{8}}(\mathbb{R})$, we have*

$$\eta(t)W_0^t(0, h_1, h_2) \in C_x^0 H_t^{\frac{2s+3}{8}}(\mathbb{R}\times\mathbb{R}).$$

Proof Recalling $W_3h_2 = W^t\psi_3$ and $W_4h_1 = W^t\psi_4$, the claim for these terms follows by (7), (16), (17), the continuity of W^t and the Kato smoothing inequality (Lemma 4.1). With the notations of the previous lemma we rewrite the remaining terms of $W_0^t(0, h_1, h_2)$ as follows

$$\begin{aligned}
W_1h_2(x,t) &= \int_{\mathbb{R}} \mathcal{F}_\beta\big(f(\beta x)\big)(z)W^t\psi_1(z)dz, \;\; W_2h_1(x,t) = \int_{\mathbb{R}} \mathcal{F}_\beta\big(f(\beta x)\big)(z)W^t\psi_2(z)dz,\\
W_5h_2(x,t) &= \int_{\mathbb{R}} \mathcal{F}_\beta\big(f_1(\beta x)\big)(z)W^{-t}\psi_5(z)dz, \;\; W_6h_1(x,t) = \int_{\mathbb{R}} \mathcal{F}_\beta\big(f_1(\beta x)\big)(z)W^{-t}\psi_6(z)dz,\\
W_7h_1(x,t) &= \int_{\mathbb{R}} \mathcal{F}_\beta\big(f_2(\beta x)\big)(z)W^{-t}\psi_7(z)dz, \;\; W_8h_2(x,t) = \int_{\mathbb{R}} \mathcal{F}_\beta\big(f_2(\beta x)\big)(z)W^{-t}\psi_8(z)dz.
\end{aligned}$$

We show only $\eta(t)W_1h_2 \in C_x^0 H_t^{\frac{2s+3}{8}}(\mathbb{R}\times\mathbb{R})$ since the estimates for the other terms follow by the same arguments. Hence

$$\begin{aligned}
W_1h_2(x,t) &= \int_{\mathbb{R}} \mathcal{F}_\beta\big(f(\beta x)\big)(z)W^t\psi_1(z)dz\\
&= \int_{\mathbb{R}} \frac{1}{x}\widehat{f}\Big(\frac{z}{x}\Big)W^t\psi_1(z)dz\\
&= \int_{\mathbb{R}} \widehat{f}(z)W^t\psi_1(xz)dz.
\end{aligned}$$

Then Minkowski's and Kato smoothing inequalities lead to the bound

$$\begin{aligned}
\|\eta W_1h_2\|_{H_t^{\frac{2s+3}{8}}} &\leq \int_{\mathbb{R}} |\widehat{f}(z)|\, \big\|\eta W^t\psi_1(xz)\big\|_{H_t^{\frac{2s+3}{8}}} dz\\
&\leq \|\widehat{f}\|_{L^1}\, \big\|\eta W^t\psi_1(xz)\big\|_{H_t^{\frac{2s+3}{8}} L_z^\infty}\\
&\lesssim \|\psi_1\|_{H_z^s} \lesssim \big\|\chi_{(0,\infty)}h_2\big\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R})}
\end{aligned}$$

since $\widehat{f} \in L^1$. Finally, continuity in the spatial variable follows from the dominated convergence theorem. □

4.2 Estimates for the nonlinear term

This section discusses the estimates for the nonlinear term in (8). These estimates will play crucial role in establishing the smoothing theorem and closing the fixed point argument.

Proposition 4.5 *For any compactly supported smooth function η and $\frac{1}{2} - b > 0$ sufficiently small, we have*

$$\Big\|\eta(t)\int_0^t W^{t-t'}Fdt'\Big\|_{C_x^0 H_t^{\frac{2s+3}{8}}(\mathbb{R}\times\mathbb{R})} + \Big\|\eta(t)\partial_x\Big(\int_0^t W^{t-t'}Fdt'\Big)\Big\|_{C_x^0 H_t^{\frac{2s+1}{8}}(\mathbb{R}\times\mathbb{R})}$$
$$\lesssim \begin{cases} \|F\|_{X^{s,-b}} & \text{if } -\frac{1}{2} \leq s \leq \frac{1}{2}\\ \|F\|_{X^{s,-b}} + \|F\|_{X^{\frac{1}{2}+,\frac{2s-5}{8}}} & \text{if } s > \frac{1}{2}. \end{cases}$$

Proof Assume first that $-\frac{1}{2} \le s \le \frac{1}{2}$, then

$$\begin{aligned}\int_0^t W^{t-t'} F dt' &= \int_0^t \int_{\mathbb{R}} e^{ix\xi} e^{i(t-t')\xi^4} \widehat{F}(\xi, t') d\xi dt' \\ &= \int_{\mathbb{R}} \int_0^t e^{ix\xi} e^{i(t-t')\xi^4} \Big(\int_{\mathbb{R}} e^{it'\tau} \widehat{F}(\xi, \tau) d\tau \Big) dt' d\xi \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} e^{it\xi^4} \Big(\int_0^t e^{it'(\tau-\xi^4)} dt' \Big) \widehat{F}(\xi, \tau) d\xi d\tau \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} \frac{e^{it\tau} - e^{it\xi^4}}{i(\tau - \xi^4)} \widehat{F}(\xi, \tau) d\xi d\tau.\end{aligned}$$

First we wish to bound

$$\left\| \eta \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} \frac{e^{it\tau} - e^{it\xi^4}}{i(\tau - \xi^4)} \widehat{F}(\xi, \tau) d\xi d\tau \right\|_{H_t^{\frac{2s+3}{8}}}.$$

Let φ be a smooth cut-off function such that $\varphi = 1$ on $[-1, 1]$ and $\operatorname{supp}\varphi \subset \{x : |x| \le 2\}$ and let $\varphi^c = 1 - \varphi$. We will proceed by writing

$$\begin{aligned}\eta(t) \int_0^t W^{t-t'} F dt' = \eta(t) &\int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} \frac{(e^{it\tau} - e^{it\xi^4})\varphi(\tau - \xi^4)}{i(\tau - \xi^4)} \widehat{F}(\xi, \tau) d\xi d\tau \\ &+ \eta(t) \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} \frac{e^{it\tau}\varphi^c(\tau - \xi^4)}{i(\tau - \xi^4)} \widehat{F}(\xi, \tau) d\xi d\tau \\ &- \eta(t) \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} \frac{e^{it\xi^4}\varphi^c(\tau - \xi^4)}{i(\tau - \xi^4)} \widehat{F}(\xi, \tau) d\xi d\tau \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

By Taylor expansion, we write

$$\frac{e^{it\tau} - e^{it\xi^4}}{i(\tau - \xi^4)} = ie^{it\tau} \frac{1}{\tau - \xi^4} \big(e^{-it(\tau - \xi^4)} - 1 \big) = ie^{it\tau} \sum_{k=1}^{\infty} \frac{(-it)^k}{k!} (\tau - \xi^4)^{k-1}.$$

For I, using Lemma 7.3, we have the bound

$$\begin{aligned}\|\mathrm{I}\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} &\lesssim \sum_{k=1}^{\infty} \frac{\|t^k \eta\|_{H^1}}{k!} \left\| \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} e^{it\tau} (\tau - \xi^4)^{k-1} \varphi(\tau - \xi^4) \widehat{F}(\xi, \tau) d\xi d\tau \right\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R})} \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \left\| \langle \tau \rangle^{\frac{2s+3}{8}} \int_{\mathbb{R}} e^{ix\xi} (\tau - \xi^4)^{k-1} \varphi(\tau - \xi^4) \widehat{F}(\xi, \tau) d\xi d\tau \right\|_{L^2_\tau} \\ &\lesssim \left\| \langle \tau \rangle^{\frac{2s+3}{8}} \int_{|\tau - \xi^4| < 1} \widehat{F}(\xi, \tau) d\xi \right\|_{L^2_\tau}\end{aligned}$$

where we have used

$$\left\| t^k \eta \right\|_{H^1} \sim \left\| t^k \eta \right\|_{L^2} + \left\| \partial_t (t^k \eta) \right\|_{L^2} \lesssim k \left\| t^{k-1} \eta \right\|_{L^2} + \left\| t^k \eta' \right\|_{L^2} \lesssim k.$$

Using Cauchy-Schwarz inequality in ξ, this is bounded by

$$\Big[\int_{\mathbb{R}} \langle \tau \rangle^{\frac{2s+3}{4}} \Big(\int_{|\tau - \xi^4| < 1} \langle \xi \rangle^{-2s} d\xi \Big) \Big(\int_{|\tau - \xi^4| < 1} \langle \xi \rangle^{2s} |\widehat{F}(\xi, \tau)|^2 d\xi \Big) d\tau \Big]^{\frac{1}{2}}$$

$$\lesssim \sup_{\tau} \left(\langle \tau \rangle^{\frac{2s+3}{4}} \int_{|\tau-\xi^4|<1} \langle \xi \rangle^{-2s} d\xi \right)^{\frac{1}{2}} \|F\|_{X^{s,-b}}$$
$$\lesssim \|F\|_{X^{s,-b}} .$$

For $|\tau| \lesssim 1$, the supremum is apparently bounded whereas for $|\tau| \gg 1$, by the change of variable $\rho = \xi^4$, it is bounded by

$$\langle \tau \rangle^{\frac{2s+3}{4}} \int_{|\tau|-1}^{|\tau|+1} \langle \rho \rangle^{\frac{-s}{2}} \frac{1}{|\rho|^{\frac{3}{4}}} d\rho \lesssim \langle \tau \rangle^{\frac{2s+3}{4}} \int_{|\tau|-1}^{|\tau|+1} \langle \rho \rangle^{\frac{-2s-3}{4}} d\rho \lesssim 1$$

since $|\rho| \sim |\tau| \gg 1$. Next we consider II. By using Lemma 7.3 we have

$$\|\mathrm{II}\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} \lesssim \|\eta\|_{H^1} \left\| \langle \tau \rangle^{\frac{2s+3}{8}} \int_{|\tau-\xi^4|\geq 1} \frac{|\widehat{F}(\xi,\tau)|}{\langle \tau - \xi^4 \rangle} d\xi \right\|_{L^2_\tau}$$
$$\lesssim \left[\int_{\mathbb{R}} \langle \tau \rangle^{\frac{2s+3}{4}} \left(\int \frac{d\xi}{\langle \xi \rangle^{2s} \langle \tau - \xi^4 \rangle^{2-2b}} \right) \left(\int \langle \xi \rangle^{2s} \langle \tau - \xi^4 \rangle^{-2b} |\widehat{F}(\xi,\tau)|^2 d\xi \right) d\tau \right]^{\frac{1}{2}}$$
$$\lesssim \sup_{\tau} \left[\langle \tau \rangle^{\frac{2s+3}{4}} \int \frac{d\xi}{\langle \tau - \xi^4 \rangle^{2-2b} \langle \xi \rangle^{2s}} \right]^{\frac{1}{2}} \|F\|_{X^{s,-b}}$$
$$\lesssim \|F\|_{X^{s,-b}}$$

where we have applied Cauchy–Schwarz inequality in the second line. To see that the supremum above is finite we write

$$\langle \tau \rangle^{\frac{2s+3}{4}} \left[\int_{|\xi|<1} \frac{d\xi}{\langle \tau - \xi^4 \rangle^{2-2b} \langle \xi \rangle^{2s}} + \int_{|\xi|\geq 1} \frac{d\xi}{\langle \tau - \xi^4 \rangle^{2-2b} \langle \xi \rangle^{2s}} \right]$$
$$\lesssim \langle \tau \rangle^{\frac{2s+3}{4}} \left[\langle \tau \rangle^{2b-2} \int_{|\xi|<1} \frac{d\xi}{\langle \xi \rangle^{2s}} + \int_{|\rho|\geq 1} \frac{d\rho}{\langle \tau - \rho \rangle^{2-2b} \langle \rho \rangle^{\frac{2s+3}{4}}} \right]$$
$$\lesssim \langle \tau \rangle^{\frac{2s+3}{4}+2b-2} + \langle \tau \rangle^{\frac{2s+3}{4}} \langle \tau \rangle^{\frac{-2s-3}{4}} \lesssim 1$$

where we have used Lemma 7.4 in the $\rho-$integral and $\frac{1}{2} \leq \frac{2s+3}{4} \leq 1$ with $b < \frac{1}{2}$. Next for III, we divide the region of integration into two pieces $|\xi| < 1$ and $|\xi| \geq 1$. For $|\xi| < 1$ using Minkowski's inequality and then Cauchy–Schwarz inequality we have

$$\left\|\mathrm{III}_{|\xi|<1}\right\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} := \left\| \eta(t) \int \int_{|\xi|<1} \frac{e^{ix\xi} e^{it\xi^4}}{i(\tau - \xi^4)} \varphi^c(\tau - \xi^4) |\widehat{F}(\xi,\tau)| d\xi d\tau \right\|_{H_t^{\frac{2s+3}{8}}}$$
$$\leq \int \int_{|\xi|<1} \left\| \eta(t) e^{it\xi^4} \right\|_{H_t^{\frac{2s+3}{8}}} \frac{\varphi^c(\tau - \xi^4)}{|\tau - \xi^4|} |\widehat{F}(\xi,\tau)| d\xi d\tau$$
$$\lesssim \int \int_{|\xi|<1} \frac{|\widehat{F}(\xi,\tau)|}{\langle \tau - \xi^4 \rangle} d\xi d\tau$$
$$\lesssim \left[\int \int_{|\xi|<1} \langle \tau \rangle^{2b-2} d\xi d\tau \right]^{\frac{1}{2}} \left[\iint \langle \xi \rangle^{2s} \langle \tau - \xi^4 \rangle^{-2b} |\widehat{F}(\xi,\tau)| d\xi d\tau \right]^{\frac{1}{2}}$$
$$\lesssim \|F\|_{X^{s,-b}}$$

since $2b - 2 < -1$ for $b < \frac{1}{2}$. To treat the case regarding the region $|\xi| \geq 1$, we use change of variable $\rho = \xi^4$ as before

$$\left\|\mathrm{III}_{|\xi|\geq 1}\right\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} := \left\| \eta(t) \int \int_{|\xi|\geq 1} \frac{e^{ix\xi} e^{it\xi^4}}{i(\tau - \xi^4)} \varphi^c(\tau - \xi^4) |\widehat{F}(\xi,\tau)| d\xi d\tau \right\|_{H_t^{\frac{2s+3}{8}}}$$

$$
\begin{aligned}
&\lesssim \|\eta\|_{H^1} \left\| \int_{|\tau-\rho|>1} \int_{|\rho|>1} \frac{e^{ix\sqrt[4]{\rho}} e^{it\rho}}{|\tau-\rho|} \widehat{F}(\sqrt[4]{\rho}, \tau) \frac{1}{|\rho|^{\frac{3}{4}}} d\rho d\tau \right\|_{H_t^{\frac{2s+3}{8}}} \\
&\lesssim \left\| \langle\rho\rangle^{\frac{2s+3}{8}} \mathcal{F}_t \circ \mathcal{F}_\rho^{-1} \Big(\int_{|\tau-\rho|>1,\, |\rho|>1} \frac{e^{ix\sqrt[4]{\rho}}}{|\tau-\rho|} \widehat{F}(\sqrt[4]{\rho}, \tau) \frac{1}{|\rho|^{\frac{3}{4}}} d\tau \Big)(\rho) \right\|_{L^2_\rho} \\
&\lesssim \left\| \langle\rho\rangle^{\frac{2s+3}{8}} \int \frac{\widehat{F}(\sqrt[4]{\rho}, \tau)}{\langle\tau-\rho\rangle |\rho|^{\frac{3}{4}}} d\tau \right\|_{L^2_{|\rho|\geq 1}} \\
&\lesssim \Big[\int \langle\rho\rangle^{\frac{2s+3}{4}} \langle\rho\rangle^{\frac{-3}{4}} \Big(\int \frac{d\tau}{\langle\tau-\rho\rangle^{2-2b}} \Big) \Big(\int \frac{|\widehat{F}(\sqrt[4]{\rho}, \tau)|^2}{\langle\tau-\rho\rangle^{2b}} d\tau \Big) \frac{1}{|\rho|^{\frac{3}{4}}} d\rho \Big]^{\frac{1}{2}} \\
&\lesssim \Big[\iint \langle\rho\rangle^{\frac{s}{2}} \langle\tau-\rho\rangle^{-2b} |\widehat{F}(\sqrt[4]{\rho}, \tau)|^2 \frac{1}{|\rho|^{\frac{3}{4}}} d\rho d\tau \Big]^{\frac{1}{2}} \\
&\lesssim \|F\|_{X^{s,-b}}
\end{aligned}
$$

where we used Cauchy-Schwarz inequality in the fifth line and changed variables back to ξ in the last line. This finishes the proof for $-\frac{1}{2} \leq s \leq \frac{1}{2}$. Next we consider $s > \frac{1}{2}$, in which case, instead of Lemma 7.3, proof makes use of algebra property of Sobolev spaces

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{H^s}$$

in order to extract the Sobolev norm of η. As η is a smooth compactly supported function, the proof proceeds along the same lines as with the case $-\frac{1}{2} \leq s \leq \frac{1}{2}$ except for the one for II just because we needed $s \leq \frac{1}{2}$ to obtain the bound $\|\mathrm{II}\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} \lesssim \|F\|_{X^{s,-b}}$. Thus to estimate II, we use the identity

$$\langle\tau\rangle^{\frac{2s+3}{8}} \lesssim \langle\tau-\xi^4\rangle^{\frac{2s+3}{8}} + |\xi|^{\frac{2s+3}{2}}$$

to write

$$
\begin{aligned}
\|\mathrm{II}\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} &\lesssim \|\eta\|_{H^1} \left\| \langle\tau\rangle^{\frac{2s+3}{8}} \int \frac{|\widehat{F}(\xi, \tau)|}{\langle\tau-\xi^4\rangle} d\xi \right\|_{L^2_\tau} \\
&\lesssim \left\| \int \langle\tau-\xi^4\rangle^{\frac{2s-5}{8}} |\widehat{F}(\xi, \tau)| d\xi \right\|_{L^2_\tau} + \left\| \int \frac{|\xi|^{\frac{2s+3}{2}}}{\langle\tau-\xi^4\rangle} |\widehat{F}(\xi, \tau)| d\xi \right\|_{L^2_\tau}.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality in the ξ–integral, the second term is bounded by

$$
\begin{aligned}
\left\| \int \frac{|\xi|^{\frac{2s+3}{2}}}{\langle\tau-\xi^4\rangle} |\widehat{F}(\xi, \tau)| d\xi \right\|_{L^2_\tau} &\lesssim \Big[\int \Big(\int \frac{|\xi|^3}{\langle\tau-\xi^4\rangle^{2-2b}} d\xi \Big) \Big(\int \frac{|\xi|^{2s}}{\langle\tau-\xi^4\rangle^{2b}} |\widehat{F}(\xi, \tau)|^2 d\xi \Big) d\tau \Big]^{\frac{1}{2}} \\
&\lesssim \sup_\tau \Big[\int \frac{|\xi|^3}{\langle\tau-\xi^4\rangle^{2-2b}} d\xi \Big]^{\frac{1}{2}} \|F\|_{X^{s,-b}} \\
&\lesssim \sup_\tau \Big[\int \frac{1}{\langle\tau-\rho\rangle^{2-2b}} d\rho \Big]^{\frac{1}{2}} \|F\|_{X^{s,-b}} \lesssim \|F\|_{X^{s,-b}}.
\end{aligned}
$$

since $2 - 2b > 1$. Applying the Cauchy-Schwarz inequality in the ξ–integral for the first term in this case

$$\left\| \int \langle\tau-\xi^4\rangle^{\frac{2s-5}{8}} |\widehat{F}(\xi, \tau)| d\xi \right\|_{L^2_\tau} \lesssim \Big[\int \Big(\int \frac{d\xi}{\langle\xi\rangle^{+1}} \Big) \Big(\int \langle\xi\rangle^{+1} \langle\tau-\xi^4\rangle^{\frac{2s-5}{4}} |\widehat{F}(\xi, \tau)|^2 d\xi \Big) d\tau \Big]^{\frac{1}{2}}$$

$$\lesssim \|F\|_{X^{\frac{1}{2}+,\frac{2s-5}{8}}}.$$

As a result we have obtained

$$\left\|\eta(t)\int_0^t W^{t-t'}Fdt'\right\|_{C_x^0 H_t^{\frac{2s+3}{8}}(\mathbb{R}\times\mathbb{R})} \lesssim \begin{cases} \|F\|_{X^{s,-b}} & \text{if } -\frac{1}{2}\le s\le\frac{1}{2} \\ \|F\|_{X^{s,-b}} + \|F\|_{X^{\frac{1}{2}+,\frac{2s-5}{8}}} & \text{if } s>\frac{1}{2} \end{cases}.$$

Now we move to the estimate on the derivative term where we take less time derivatives $\frac{2s+1}{8}$ while we have additional $i\xi$ factor coming from the spatial derivative. As before we divide the Duhamel integral into three pieces as follows

$$\begin{aligned}
\eta(t)\partial_x\Big(\int_0^t W^{t-t'}Fdt'\Big) &= \eta(t)\int_0^t \partial_x\big(W^{t-t'}F\big)dt' \\
&= \eta(t)\int_{\mathbb{R}}\int_{\mathbb{R}} \xi e^{ix\xi}\frac{\big(e^{it\tau}-e^{it\xi^4}\big)\varphi(\tau-\xi^4)}{\tau-\xi^4}\widehat{F}(\xi,\tau)d\xi d\tau \\
&\quad + \eta(t)\int_{\mathbb{R}}\int_{\mathbb{R}} \xi e^{ix\xi}\frac{e^{it\tau}\varphi^c(\tau-\xi^4)}{\tau-\xi^4}\widehat{F}(\xi,\tau)d\xi d\tau \\
&\quad - \eta(t)\int_{\mathbb{R}}\int_{\mathbb{R}} \xi e^{ix\xi}\frac{e^{it\xi^4}\varphi^c(\tau-\xi^4)}{\tau-\xi^4}\widehat{F}(\xi,\tau)d\xi d\tau \\
&=: \mathrm{I}^x + \mathrm{II}^x + \mathrm{III}^x.
\end{aligned}$$

To bound I^x, note that on the region of integration we have $|\tau|\approx\xi^4$ hence the additional factor ξ leads to the situation $\langle\tau\rangle^{\frac{2s+1}{8}}|\xi| \lesssim \langle\tau\rangle^{\frac{2s+3}{8}}$ which was examined before for the integral I. In order to estimate III^x we divide the region of integration as before into pieces $|\xi|<1$ and $|\xi|\ge 1$. For the former case, the bounds are identical to those obtained for III, for the latter case, we make the same change of variable $\rho=\xi^4$ as done for III so that the additional factor of ξ contributes the additional factor of $|\rho|^{\frac{1}{4}}$ to the integral III^x that brings us back to the situation handled in bounding III. Nevertheless estimation for the term II^x needs verification. When $-\frac{1}{2}\le s\le\frac{1}{2}$, using Cauchy-Schwarz inequality we have the bound:

$$\begin{aligned}
\|\mathrm{II}^x\|_{H^{\frac{2s+1}{8}}(\mathbb{R})} &\lesssim \|\eta\|_{H^1}\left\|\langle\tau\rangle^{\frac{2s+1}{8}}\int_{|\tau-\xi^4|\ge1}\frac{\xi}{\langle\tau-\xi^4\rangle}|\widehat{F}(\xi,\tau)|d\xi\right\|_{L^2_\tau} \\
&\lesssim \Big[\int\langle\tau\rangle^{\frac{2s+1}{4}}\Big(\int\frac{\xi^2}{\langle\xi\rangle^{2s}\langle\tau-\xi^4\rangle^{2-2b}}d\xi\Big)\Big(\int\langle\xi\rangle^{2s}\langle\tau-\xi^4\rangle^{-2b}|\widehat{F}(\xi,\tau)|^2d\xi\Big)d\tau\Big]^{\frac{1}{2}} \\
&\lesssim \sup_\tau\Big(\langle\tau\rangle^{\frac{2s+1}{4}}\int\frac{\xi^2}{\langle\xi\rangle^{2s}\langle\tau-\xi^4\rangle^{2-2b}}d\xi\Big)\|F\|_{X^{s,-b}}.
\end{aligned}$$

To see that the supremum above is bounded, we write the integral as

$$\begin{aligned}
&\langle\tau\rangle^{\frac{2s+1}{4}}\Big[\int_{|\xi|<1}\frac{\xi^2}{\langle\tau-\xi^4\rangle^{2-2b}\langle\xi\rangle^{2s}}d\xi + \int_{|\xi|\ge1}\frac{\xi^2}{\langle\tau-\xi^4\rangle^{2-2b}\langle\xi\rangle^{2s}}d\xi\Big] \\
&\lesssim \langle\tau\rangle^{\frac{2s+1}{4}}\Big[\langle\tau\rangle^{2b-2} + \int_{|\rho|\ge1}\frac{|\rho|^{\frac{1}{2}}}{\langle\tau-\rho\rangle^{2-2b}\langle\rho\rangle^{\frac{2s+3}{4}}}d\rho\Big] \\
&\lesssim \langle\tau\rangle^{\frac{2s+1}{4}+2b-2} + \langle\tau\rangle^{\frac{2s+1}{4}}\langle\tau\rangle^{\frac{-2s-1}{4}} \lesssim 1
\end{aligned}$$

where we have used Lemma 7.4 in the ρ-integral and the fact that $0 \le \frac{2s+1}{4} \le \frac{1}{2}$ with $b < \frac{1}{2}$. In the case $s > \frac{1}{2}$, we estimate II^x by

$$\|\mathrm{II}^x\|_{H^{\frac{2s+1}{8}}(\mathbb{R})} \lesssim \left\| \langle \tau \rangle^{\frac{2s+1}{8}} \int \frac{|\xi|}{\langle \tau - \xi^4 \rangle} |\widehat{F}(\xi, \tau)| d\xi \right\|_{L^2_\tau}.$$

We consider the cases $|\tau| \gtrsim \xi^4$ and $|\tau| \ll \xi^4$, in the first case, the above integral is bounded by

$$\left\| \langle \tau \rangle^{\frac{2s+3}{8}} \int \frac{|\widehat{F}(\xi, \tau)|}{\langle \tau - \xi^4 \rangle} d\xi \right\|_{L^2_\tau}$$

which was addressed before for II. For the second case notice that $|\tau - \xi^4| \approx \xi^4$ with which one has $|\xi| \lesssim \langle \tau - \xi^4 \rangle^{\frac{1}{4}}$. Thus we bound the integral by

$$\left\| \langle \tau \rangle^{\frac{2s+1}{8}} \int_{|\tau| \ll \xi^4} \frac{|\widehat{F}(\xi, \tau)|}{\langle \tau - \xi^4 \rangle^{\frac{3}{4}}} d\xi \right\|_{L^2_\tau}.$$

On the region where $|\tau| \ll \xi^4$, we have the relation

$$\langle \tau \rangle^{\frac{2s+1}{8}} \lesssim \langle \tau - \xi^4 \rangle^{\frac{2s+1}{8}} + |\xi|^{\frac{2s+1}{2}} \lesssim \langle \tau - \xi^4 \rangle^{\frac{2s+1}{8}}$$

through which we bound the above integral by

$$\left\| \int \langle \tau - \xi^4 \rangle^{\frac{2s-5}{8}} |\widehat{F}(\xi, \tau)| d\xi \right\|_{L^2_\tau}$$

which was handled in bounding II. □

Proposition 4.6 *For fixed $s > -\frac{1}{3}$ with $a < \min\{2s+1, 1\}$ and $\frac{1}{2} - b > 0$ sufficiently small, we have*

$$\|u_1 \overline{u}_2 u_3\|_{X^{s+a,-b}} \lesssim \prod_{j=1}^{3} \|u_j\|_{X^{s,b}}.$$

Proof Expressing the space time Fourier transform of $u_1 \overline{u}_2 u_3$ as a convolution

$$\mathcal{F}_{x,t}(u_1 \overline{u}_2 u_3)(\xi, \tau) = \int_{\xi_1, \xi_2} \int_{\tau_1, \tau_2} \widehat{u}_1(\xi_1, \tau_1) \overline{\widehat{u}_2(\xi_2, \tau_2)} \widehat{u}_3(\xi - \xi_1 + \xi_2, \tau - \tau_1 + \tau_2)$$

and then using the definition of $X^{s,b}$ norm we write

$$\|u_1 \overline{u}_2 u_3\|^2_{X^{s+a,-b}} = \left\| \int_{\xi_1, \xi_2} \int_{\tau_1, \tau_2} \frac{\langle \xi \rangle^{s+a} \widehat{u}_1(\xi_1, \tau_1) \overline{\widehat{u}_2(\xi_2, \tau_2)} \widehat{u}_3(\xi - \xi_1 + \xi_2, \tau - \tau_1 + \tau_2)}{\langle \tau - \xi_4 \rangle^b} \right\|^2_{L^2_{\xi,\tau}}.$$

Now define

$$f_j(\xi, \tau) = \langle \xi \rangle^s \langle \tau - \xi^4 \rangle^b |\widehat{u}_j(\xi, \tau)| \text{ for } j = 1, 2, 3.$$

Thus the desired bound is equivalent to showing that

$$
\begin{aligned}
&\left\| \int_{\xi_1,\xi_2} \int_{\tau_1,\tau_2} M(\xi,\xi_1,\xi_2,\tau,\tau_1,\tau_2) f_1(\xi_1,\tau_1) f_2(\xi_2,\tau_2) f_3(\xi-\xi_1+\xi_2,\tau-\tau_1+\tau_2) \right\|^2_{L^2_{\xi,\tau}} \\
&\quad \lesssim \prod_{j=1} \|f_j\|^2_{L^2} = \prod_{j=1} \|u_j\|^2_{X^{s,b}}
\end{aligned}
$$

where

$$
M(\xi,\xi_1,\xi_2,\tau,\tau_1,\tau_2) = \frac{\langle\xi\rangle^{s+a}\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}\langle\xi-\xi_1+\xi_2\rangle^{-s}}{\langle\tau-\xi^4\rangle\langle\tau_1-\xi_1^4\rangle\langle\tau_2-\xi_2^4\rangle\langle\tau-\tau_1+\tau_2-(\xi-\xi_1+\xi_2)^4\rangle}.
$$

By an application of the Cauchy-Schwarz inequality in the $\xi_1, \xi_2, \tau_1, \tau_2$ integrals and then using Hölder's and Young's inequalities respectively the norm above is majorized by

$$
\begin{aligned}
&\left\| \Big(\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} M^2\Big)^{\frac{1}{2}} \Big(\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} f_1^2(\xi_1,\tau_1) f_2^2(\xi_2,\tau_2) f_3^2(\xi-\xi_1+\xi_2,\tau-\tau_1+\tau_2)\Big)^{\frac{1}{2}} \right\|^2_{L^2_{\xi,\tau}} \\
&= \left\| \Big(\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} M^2\Big) \Big(\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} f_1^2(\xi_1,\tau_1) f_2^2(\xi_2,\tau_2) f_3^2(\xi-\xi_1+\xi_2,\tau-\tau_1+\tau_2)\Big) \right\|_{L^1_{\xi,\tau}} \\
&\le \sup_{\xi,\tau} \Big(\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} M^2\Big) \left\| \int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} f_1^2(\xi_1,\tau_1) f_2^2(\xi_2,\tau_2) f_3^2(\xi-\xi_1+\xi_2,\tau-\tau_1+\tau_2) \right\|_{L^1_{\xi,\tau}} \\
&= \sup_{\xi,\tau} \Big(\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} M^2\Big) \left\| f_1^2 * f_2^2 * f_3^2 \right\|_{L^1_{\xi,\tau}} \\
&\lesssim \sup_{\xi,\tau} \Big(\int_{\xi_1,\xi_2}\int_{\tau_1,\tau_2} M^2\Big) \prod_{j=1} \|f_j\|^2_{L^2}.
\end{aligned}
$$

Therefore, it suffices to show that the supremum above is finite. Application of Lemma 7.4 in the τ_1, τ_2 integrals bounds the supremum by

$$
\sup_{\xi,\tau} \int \frac{\langle\xi\rangle^{2s+2a}\langle\xi_1\rangle^{-2s}\langle\xi_2\rangle^{-2s}\langle\xi-\xi_1+\xi_2\rangle^{-2s}}{\langle\tau-\xi^4\rangle^{2b}\langle\tau-\xi_1^4+\xi_2^4-(\xi-\xi_1+\xi_2)^4\rangle^{6b-2}} d\xi_1 d\xi_2.
$$

Implementing the identity $\langle\alpha-\beta\rangle \lesssim \langle\tau-\alpha\rangle\langle\tau-\beta\rangle$ and then using Lemma 7.2, this is bounded by

$$
\begin{aligned}
&\sup_{\xi} \int \frac{\langle\xi\rangle^{2s+2a}\langle\xi_1\rangle^{-2s}\langle\xi_2\rangle^{-2s}\langle\xi-\xi_1+\xi_2\rangle^{-2s}}{\langle\xi^4-\xi_1^4+\xi_2^4-(\xi-\xi_1+\xi_2)^4\rangle^{6b-2}} d\xi_1 d\xi_2 \\
&\quad \lesssim \sup_{\xi} \int \frac{\langle\xi\rangle^{2s+2a}\langle\xi_1\rangle^{-2s}\langle\xi_2\rangle^{-2s}\langle\xi-\xi_1+\xi_2\rangle^{-2s}}{\langle(\xi_1^2+\xi_2^2+\xi^2)(\xi_1-\xi)(\xi_1-\xi_2)\rangle^{1-}} d\xi_1 d\xi_2.
\end{aligned}
$$

We divide the integration region into two pieces

$$
\begin{aligned}
R_1 &= \{(\xi_1,\xi_2) : |\xi_1-\xi| \ll 1 \text{ or } |\xi_1-\xi_2| \ll 1\} \text{ and} \\
R_2 &= \{(\xi_1,\xi_2) : |\xi_1-\xi| \gtrsim 1 \text{ and } |\xi_1-\xi_2| \gtrsim 1\}
\end{aligned}
$$

to control the supremum.

Clearly we have $\xi_1^2+\xi_2^2+\xi^2 \gtrsim 1$ on R_2, so the supremum on this region is estimated by

$$\int_{R_2} \frac{\langle \xi\rangle^{2s+2a}\langle \xi_1\rangle^{-2s}\langle \xi_2\rangle^{-2s}\langle \xi-\xi_1+\xi_2\rangle^{-2s}}{\langle (\xi_1^2+\xi_2^2+\xi^2)(\xi_1-\xi)(\xi_1-\xi_2)\rangle^{1-}} d\xi_1 d\xi_2$$
$$\lesssim \int \frac{\langle \xi\rangle^{2s+2a}\langle \xi_1\rangle^{-2s}\langle \xi_2\rangle^{-2s}\langle \xi-\xi_1+\xi_2\rangle^{-2s}}{\langle \xi_1^2+\xi_2^2+\xi^2\rangle^{1-}\langle \xi_1-\xi\rangle^{1-}\langle \xi_1-\xi_2\rangle^{1-}} d\xi_1 d\xi_2.$$

Since the sign of the Sobolev index s affects the way we follow in the proof, we begin with considering the case $s>0$ first. In this case, there are three separate cases to examine:

i) $|\xi-\xi_1+\xi_2| \gtrsim |\xi|$

$$\int \frac{\langle \xi\rangle^{2s+2a}\langle \xi_1\rangle^{-2s}\langle \xi_2\rangle^{-2s}\langle \xi-\xi_1+\xi_2\rangle^{-2s}}{\langle \xi_1^2+\xi_2^2+\xi^2\rangle^{1-}\langle \xi_1-\xi\rangle^{1-}\langle \xi_1-\xi_2\rangle^{1-}} d\xi_1 d\xi_2$$
$$\lesssim \int \frac{\langle \xi\rangle^{2a-2+}\langle \xi_1\rangle^{-2s}\langle \xi_2\rangle^{-2s}}{\langle \xi_1-\xi\rangle^{1-}\langle \xi_2-\xi_1\rangle^{1-}} d\xi_2 d\xi_1$$
$$\lesssim \langle \xi\rangle^{2a-2+} \int \frac{\phi_{\max(2s,1-)}(\xi_1)}{\langle \xi_1-\xi\rangle^{1-}\langle \xi_1\rangle^{2s+\min(2s,1-)}} d\xi_1$$

where we have used the Lemma 7.4 in the last line above. For $s \geq \frac{1}{2}$, using the Lemma 7.4, this is bounded by

$$\langle \xi\rangle^{2a-2+} \int \frac{\log(1+\langle \xi_1\rangle)}{\langle \xi_1-\xi\rangle^{1-}\langle \xi_1\rangle^{2s+1-}} d\xi_1 \lesssim \langle \xi\rangle^{2a-3+} \lesssim 1$$

provided that $a<\frac{3}{2}$. As for $0<s<\frac{1}{2}$, the Lemma 7.4 yields the bound

$$\langle \xi\rangle^{2a-2+} \int \frac{d\xi_1}{\langle \xi_1-\xi\rangle^{1-}\langle \xi_1\rangle^{4s-}} \lesssim \begin{cases} \langle \xi\rangle^{2a-2-4s+} & \text{for } 0<s\leq \frac{1}{4} \\ \langle \xi\rangle^{2a-3+} & \text{for } \frac{1}{4}<s<\frac{1}{2} \end{cases}$$

which is finite as long as $a<\min\{2s+1,\frac{3}{2}\}$.

ii) $|\xi_1| \gtrsim |\xi|$

$$\int \frac{\langle \xi\rangle^{2s+2a}\langle \xi_1\rangle^{-2s}\langle \xi_2\rangle^{-2s}\langle \xi-\xi_1+\xi_2\rangle^{-2s}}{\langle \xi_1^2+\xi_2^2+\xi^2\rangle^{1-}\langle \xi_1-\xi\rangle^{1-}\langle \xi_1-\xi_2\rangle^{1-}} d\xi_1 d\xi_2$$
$$\lesssim \int \frac{\langle \xi\rangle^{2a-2+}\langle \xi-\xi_1+\xi_2\rangle^{-2s}\langle \xi_2\rangle^{-2s}}{\langle \xi_1-\xi\rangle^{1-}\langle \xi_1-\xi_2\rangle^{1-}} d\xi_2 d\xi_1.$$

From the substitutions $x_1=\xi-\xi_1+\xi_2$ and $x_2=\xi_2$ the integral above is replaced by

$$\int \frac{\langle \xi\rangle^{2a-2+}\langle x_1\rangle^{-2s}\langle x_2\rangle^{-2s}}{\langle x_1-\xi\rangle^{1-}\langle x_2-x_1\rangle^{1-}} dx_2 dx_1$$

which is identical to the integral estimated in the previous case.

iii) $|\xi_2| \gtrsim |\xi|$

$$\int \frac{\langle \xi\rangle^{2s+2a}\langle \xi_1\rangle^{-2s}\langle \xi_2\rangle^{-2s}\langle \xi-\xi_1+\xi_2\rangle^{-2s}}{\langle \xi_1^2+\xi_2^2+\xi^2\rangle^{1-}\langle \xi_1-\xi\rangle^{1-}\langle \xi_1-\xi_2\rangle^{1-}} d\xi_1 d\xi_2$$
$$\lesssim \int \frac{\langle \xi\rangle^{2a-2+}\langle \xi-\xi_1+\xi_2\rangle^{-2s}\langle \xi_1\rangle^{-2s}}{\langle \xi_1-\xi\rangle^{1-}\langle \xi_1-\xi_2\rangle^{1-}} d\xi_2 d\xi_1.$$

In this case after making change of variables $x_1 = \xi_1$ and $x_2 = \xi - \xi_1 + \xi_2$ in the above integral and then applying the Lemma 7.4 we have the bound

$$\int \frac{\langle \xi \rangle^{2a-2+} \langle x_1 \rangle^{-2s} \langle x_2 \rangle^{-2s}}{\langle x_1 - \xi \rangle^{1-} \langle x_2 - \xi \rangle^{1-}} dx_1 dx_2 = \langle \xi \rangle^{2a-2+} \Big(\int \frac{dx}{\langle x - \xi \rangle^{1-} \langle x \rangle^{2s}} \Big)^2$$
$$\lesssim \langle \xi \rangle^{2a-2-2\min(2s,1-)} \phi^2_{\max(2s,1-)}(\xi) \lesssim \begin{cases} \langle \xi \rangle^{2a-2-4s+} & \text{for } 0 < s < \frac{1}{2} \\ \langle \xi \rangle^{2a-4+} & \text{for } s \geq \frac{1}{2} \end{cases}$$

which is bounded provided that $a < \min\{2s+1, 2\}$. Next we focus on the case $-\frac{1}{3} < s \leq 0$. In this case, since $\langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s} \lesssim \langle \xi_1^2 + \xi_2^2 + \xi^2 \rangle^{-3s}$

$$\int_{R_2} \frac{\langle \xi \rangle^{2s+2a} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle (\xi_1^2 + \xi_2^2 + \xi^2)(\xi_1 - \xi)(\xi_1 - \xi_2) \rangle^{1-}} d\xi_1 d\xi_2$$
$$\lesssim \int \frac{\langle \xi \rangle^{2s+2a}}{\langle \xi_1^2 + \xi_2^2 + \xi^2 \rangle^{1+3s-} \langle \xi_1 - \xi \rangle^{1-} \langle \xi_1 - \xi_2 \rangle^{1-}} d\xi_1 d\xi_2$$
$$\lesssim \int \frac{\langle \xi \rangle^{2s+2a}}{\langle \xi_2 + \xi^2 \rangle^{1+3s-} \langle \xi_1 - \xi \rangle^{1-} \langle \xi_1 - \xi_2 \rangle^{1-}} d\xi_1 d\xi_2.$$

Since $\frac{1}{2} - b > 0$ was taken sufficiently small, using Lemma 7.4 twice this integral is bounded by

$$\langle \xi \rangle^{2s+2a} \int \frac{d\xi_1}{\langle \xi_1 + \xi^2 \rangle^{1+3s-} \langle \xi_1 - \xi \rangle^{1-}} \lesssim \langle \xi \rangle^{2a-4s-2+} \lesssim 1$$

provided that $a < 2s + 1$. Next we move on estimating the supremum on the region R_1. In this region notice that

$$\langle \xi - \xi_1 + \xi_2 \rangle \langle \xi_1 \rangle \approx \langle \xi_2 \rangle \langle \xi \rangle,$$

thus

$$\int_{R_1} \frac{\langle \xi \rangle^{2s+2a} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle (\xi_1^2 + \xi_2^2 + \xi^2)(\xi_1 - \xi)(\xi_1 - \xi_2) \rangle^{1-}} d\xi_1 d\xi_2$$
$$\lesssim \int \frac{\langle \xi \rangle^{2a} \langle \xi_2 \rangle^{-4s}}{\langle (\xi_2^2 + \xi^2)(\xi_1 - \xi)(\xi_1 - \xi_2) \rangle^{1-}} d\xi_1 d\xi_2.$$

Note that on R_1 the relation $\xi_2^2 + \xi^2 \lesssim 1$ implies that $|\xi| \lesssim 1$, $|\xi_j| \lesssim 1$ for $j = 1, 2$ in which case the integral above turns out to be finite at once. So for a nontrivial situation we assume that $\xi_2^2 + \xi^2 \gtrsim 1$. Then making substitution $x = (\xi_2^2 + \xi^2)(\xi_1 - \xi_2)(\xi_1 - \xi)$ in the ξ_1 integral and using the relations

$$2\xi_1 = \xi + \xi_2 \pm (\xi_2^2 + \xi^2)^{-\frac{1}{2}} \sqrt{4x + (\xi - \xi_2)^2 (\xi_2^2 + \xi^2)} \text{ and } \frac{dx}{\xi_2^2 + \xi^2} = (2\xi_1 - \xi_2 - \xi) d\xi_1$$

along with the Lemma 7.5, we have the bound

$$\int \frac{\langle \xi \rangle^{2a} \langle \xi_2 \rangle^{-4s}}{\langle \xi_2^2 + \xi^2 \rangle^{\frac{1}{2}} \langle x \rangle^{1-} \sqrt{|4x + (\xi_2 - \xi)^2 (\xi_2^2 + \xi^2)|}} dx d\xi_2$$
$$\lesssim \int \frac{\langle \xi \rangle^{2a} \langle \xi_2 \rangle^{-4s}}{\langle \xi_2^2 + \xi^2 \rangle^{\frac{1}{2}} \langle (\xi_2 - \xi)^2 (\xi_2^2 + \xi^2) \rangle^{\frac{1}{2}-}} d\xi_2.$$

We estimate the integral in the separate regions $|\xi_2| \lesssim 1$ and $|\xi_2| \gg 1$. In the former region this is bounded by

$$\int_{|\xi_2|\lesssim 1} \frac{\langle \xi \rangle^{2a} \langle \xi_2 \rangle^{-4s}}{\langle \xi_2^2 + \xi^2 \rangle^{\frac{1}{2}} \langle (\xi_2 - \xi)^2 (\xi_2^2 + \xi^2) \rangle^{\frac{1}{2}-}} d\xi_2 \lesssim \langle \xi \rangle^{2a-3+} \lesssim 1$$

provided that $a < \frac{3}{2}$. As regards to the latter region, we use the relation $(\xi_2 - \xi)^2(\xi_2^2 + \xi^2) \gtrsim (\xi_2^2 - \xi^2)^2$, and then making the change of variable $x = \xi_2^2$ we obtain the bound

$$\begin{aligned}
&\int_{|\xi_2|\gg 1} \frac{\langle \xi \rangle^{2a} \langle \xi_2 \rangle^{-4s}}{\langle \xi_2^2 + \xi^2 \rangle^{\frac{1}{2}} \langle (\xi_2 - \xi)^2 (\xi_2^2 + \xi^2) \rangle^{\frac{1}{2}-}} d\xi_2 \lesssim \int_{|\xi_2|\gg 1} \frac{\langle \xi \rangle^{2a}}{\langle \xi_2^2 \rangle^{2s+\frac{1}{2}} \langle \xi_2^2 - \xi^2 \rangle^{1-}} d\xi_2 \\
&\approx \int_{|x|\gg 1} \frac{\langle \xi \rangle^{2a}}{\langle x \rangle^{2s+\frac{1}{2}} \langle x - \xi^2 \rangle^{1-} |x|^{\frac{1}{2}}} dx \lesssim \int \frac{\langle \xi \rangle^{2a}}{\langle x \rangle^{2s+1} \langle x - \xi^2 \rangle^{1-}} dx \\
&\lesssim \begin{cases} \langle \xi \rangle^{2a-2-4s+} & \text{for } -\frac{1}{2} < s < 0 \\ \langle \xi \rangle^{2a-2+} & \text{for } s \geq 0 \end{cases}
\end{aligned}$$

which is finite provided that $a < \min\{1, 2s + 1\}$. □

We take $\frac{2s+2a-1}{8} - b = \frac{2s+2a-5}{8} + (\frac{1}{2} - b)$ rather than $\frac{2s+2a-5}{8}$ in the Proposition 4.7 so as to extract a positive power of T in the contraction argument below in the local theory.

Proposition 4.7 *For fixed* $-\frac{1}{3} < s < \frac{9}{2}$, $0 \leq a < \min\{1, 2s + 1, \frac{9}{2} - s\}$, *and* $\frac{1}{2} - b > 0$ *sufficiently small, we have*

$$\text{for } -\frac{1}{3} < s + a \leq \frac{1}{2}, \; \|u_1 \overline{u}_2 u_3\|_{X^{s+a,-b}} \lesssim \prod_{j=1}^{3} \|u_j\|_{X^{s,b}},$$

$$\text{for } \frac{1}{2} < s + a < \frac{9}{2}, \; \|u_1 \overline{u}_2 u_3\|_{X^{\frac{1}{2}+, \frac{2s+2a-1}{8}-b}} \lesssim \prod_{j=1}^{3} \|u_j\|_{X^{s,b}}.$$

Proof When $-\frac{1}{3} < s + a \leq \frac{1}{2}$, given statement follows from Proposition 4.6. So we only take account of the case $\frac{1}{2} < s + a \leq \frac{9}{2}$ here. In this case, using the fact that $a < 2s + 1$ we take $s > -\frac{1}{6}$ all along. Next let

$$\mathrm{I} := \int \frac{\langle \tau - \xi^4 \rangle^{\frac{2s+2a-8b-1}{4}} \langle \xi \rangle^{1+} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle \tau - \xi_1^4 + \xi_2^4 - (\xi - \xi_1 + \xi_2)^4 \rangle^{6b-2}} d\xi_1 d\xi_2.$$

Thus using the arguments of Proposition 4.6 we are required to show that

$$\sup_{\xi,\tau} \mathrm{I} < \infty.$$

We will demonstrate this in the separate cases $\frac{1}{2} < s + a < \frac{5}{2}$ and $\frac{5}{2} \leq s + a < \frac{9}{2}$.

Case 1) $\frac{1}{2} < s + a < \frac{5}{2}$.

Note that taking $\frac{1}{2} - b > 0$ sufficiently small we infer that $\frac{1}{2}(s + a) - 2b - \frac{1}{4} < 0$, also $s + a > \frac{1}{2}$ implies that $2b + \frac{1}{4} - \frac{1}{2}(s + a) < 6b - 2$. Hence using these, the identity $\langle \tau - a \rangle \langle \tau - b \rangle \gtrsim \langle a - b \rangle$ and then Lemma 7.2 we have

$$\begin{aligned} \mathrm{I} &\lesssim \int \frac{\langle \xi \rangle^{1+} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle \xi^4 - \xi_1^4 + \xi_2^4 - (\xi - \xi_1 + \xi_2)^4 \rangle^{2b + \frac{1}{4} - \frac{1}{2}(s+a)}} d\xi_1 d\xi_2 \\ &\lesssim \int \frac{\langle \xi \rangle^{1+} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle (\xi_1^2 + \xi_2^2 + \xi^2)(\xi_1 - \xi)(\xi_1 - \xi_2) \rangle^{2b + \frac{1}{4} - \frac{1}{2}(s+a)}} d\xi_1 d\xi_2 \end{aligned}$$

which is easily estimated, for $s > \frac{1}{2}$, by

$$\int \frac{\langle \xi \rangle^{1+}}{\langle \xi_1 \rangle^{2s} \langle \xi_2 \rangle^{2s} \langle \xi - \xi_1 + \xi_2 \rangle^{2s}} d\xi_1 d\xi_2 \lesssim \int \frac{\langle \xi \rangle^{1+}}{\langle \xi_2 \rangle^{2s} \langle \xi + \xi_2 \rangle^{2s}} d\xi_2 \lesssim \langle \xi \rangle^{1-2s+} \lesssim 1$$

by using Lemma 7.4 twice. It is left to treat the case $-\frac{1}{6} < s \le \frac{1}{2}$. For this case, we will analyze the integral on the sets $\mathrm{R}_1 = \{(\xi_1, \xi_2) : |\xi_1 - \xi| \ll 1 \text{ or } |\xi_1 - \xi_2| \ll 1\}$ and $\mathrm{R}_2 = \{(\xi_1, \xi_2) : |\xi_1 - \xi| \gtrsim 1 \text{ and } |\xi_1 - \xi_2| \gtrsim 1\}$ as before.

Recalling the identity $\langle \xi - \xi_1 + \xi_2 \rangle \langle \xi_1 \rangle \approx \langle \xi \rangle \langle \xi_2 \rangle$ that holds on the set R_1, we have the bound

$$\begin{aligned} &\int_{R_1} \frac{\langle \xi \rangle^{1+} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle (\xi_1^2 + \xi_2^2 + \xi^2)(\xi_1 - \xi)(\xi_1 - \xi_2) \rangle^{2b + \frac{1}{4} - \frac{1}{2}(s+a)}} d\xi_1 d\xi_2 \\ &\lesssim \int \frac{\langle \xi \rangle^{1-2s+} \langle \xi_2 \rangle^{-4s}}{\langle (\xi_2^2 + \xi^2)(\xi_1 - \xi)(\xi_1 - \xi_2)) \rangle^{2b + \frac{1}{4} - \frac{1}{2}(s+a)}} d\xi_1 d\xi_2. \end{aligned}$$

Making substitution $x = (\xi^2 + \xi_2^2)(\xi_1 - \xi)(\xi_1 - \xi_2)$ in the ξ_1 integral and assuming $\xi^2 + \xi_2^2 \gtrsim 1$ as in the Proposition 4.6, the integral above is bounded by

$$\begin{aligned} &\int \frac{\langle \xi \rangle^{1-2s+} \langle \xi_2 \rangle^{-4s}}{\langle \xi^2 + \xi_2^2 \rangle^{\frac{1}{2}} \langle x \rangle^{2b + \frac{1}{4} - \frac{1}{2}(s+a)} \sqrt{|4x + (\xi - \xi_2)^2 (\xi^2 + \xi_2^2)|}} dx d\xi_2 \\ &\lesssim \int \frac{\langle \xi \rangle^{1-2s+} \langle \xi_2 \rangle^{-4s}}{\langle \xi^2 + \xi_2^2 \rangle^{\frac{1}{2}} \langle (\xi - \xi_2)^2 (\xi^2 + \xi_2^2) \rangle^{2b - \frac{1}{4} - \frac{1}{2}(s+a)}} d\xi_2 \end{aligned}$$

where we have used Lemma 7.4 which is applicable due to the fact that $\frac{1}{2} - b > 0$ is sufficiently small, and $a < \min\{2s + 1, 1\}$. So we estimate this by

$$\begin{aligned} &\int \frac{\langle \xi \rangle^{1-2s+} \langle \xi_2 \rangle^{-4s}}{\langle \xi^2 + \xi_2^2 \rangle^{\frac{1}{2}} \langle (\xi - \xi_2)^2 (\xi + \xi_2)^2 \rangle^{2b - \frac{1}{4} - \frac{1}{2}(s+a)}} d\xi_2 \\ &\lesssim \int \frac{\langle \xi \rangle^{1-2s+} \langle \xi_2^2 \rangle^{-2s}}{\langle \xi^2 + \xi_2^2 \rangle^{\frac{1}{2}} \langle \xi_2^2 - \xi^2 \rangle^{4b - \frac{1}{2} - s - a}} d\xi_2. \end{aligned}$$

In the case $|\xi_2| \lesssim 1$, the integral is bounded by

$$\langle \xi \rangle^{1+2a-8b+} \lesssim 1$$

provided that $a < \frac{3}{2}$, whereas for the other case $|\xi_2| \gg 1$, we change variable $x = \xi_2^2$ to bound the integral, using Lemma 7.4, by

$$\int \frac{\langle \xi \rangle^{1-2s+}}{\langle x \rangle^{1+2s} \langle x - \xi^2 \rangle^{4b - \frac{1}{2} - s - a}} dx \lesssim \begin{cases} \langle \xi \rangle^{2+2a-8b+} & \text{for } 0 \le s \le \frac{1}{2} \\ \langle \xi \rangle^{2-4s+2a-8b+} & \text{for } -\frac{1}{6} < s < 0 \end{cases}$$

which is bounded since $a < \min\{2s + 1, 1\}$ and $\frac{1}{2} - b > 0$ is sufficiently small. Next we estimate the integral on the set R_2 by

$$\int \frac{\langle \xi \rangle^{1+} \langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s}}{\langle \xi_1^2 + \xi_2^2 + \xi^2 \rangle^{2b + \frac{1}{4} - \frac{1}{2}(s+a)} \langle \xi_1 - \xi \rangle^{2b + \frac{1}{4} - \frac{1}{2}(s+a)} \langle \xi_1 - \xi_2 \rangle^{2b + \frac{1}{4} - \frac{1}{2}(s+a)}} d\xi_1 d\xi_2.$$

We bound this in the separate cases $-\frac{1}{6} < s \le 0$ and $0 < s \le \frac{1}{2}$. In the former case, using the identity

$$\langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s} \lesssim \langle \xi_1^2 + \xi_2^2 + \xi^2 \rangle^{-3s},$$

we obtain the bound

$$\int \frac{\langle \xi \rangle^{1+}}{\langle \xi_1 \rangle^{2b+\frac{1}{4}+\frac{1}{2}(5s-a)} \langle \xi - \xi_1 + \xi_2 \rangle^{2b+\frac{1}{4}+\frac{1}{2}(5s-a)} \langle \xi_1 - \xi \rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)} \langle \xi_1 - \xi_2 \rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)}} d\xi_1 d\xi_2.$$

By a change of variable $\xi_2 \to \xi_1 + \xi_2$, $\xi_1 \to \xi_1 + \xi$, it suffices to estimate

$$\langle \xi \rangle^{1+} \Big(\int \frac{d\xi_1}{\langle \xi_1 + \xi \rangle^{2b+\frac{1}{4}+\frac{1}{2}(5s-a)} \langle \xi_1 \rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)}} \Big)^2.$$

Nothing that $2b + \frac{1}{4} - \frac{1}{2}(s+a) < 1$ and $2b + \frac{1}{4} + \frac{1}{2}(5s - a) < 1$ and then applying Lemma 7.4 this is bounded by

$$\langle \xi \rangle^{2-8b-4s+2a+},$$

which is finite for $a < \min\{2s+1, 1\}$ and $\frac{1}{2} - b > 0$ sufficiently small. Now for the latter case $0 < s \le \frac{1}{2}$, after using the bound $\langle \xi_1 \rangle^{-2s} \langle \xi_2 \rangle^{-2s} \langle \xi - \xi_1 + \xi_2 \rangle^{-2s} \lesssim \langle \xi \rangle^{-2s}$ and applying the same change of variables as above, the integral is bounded by

$$\int \frac{\langle \xi \rangle^{1-2s+}}{\langle \xi_1 \rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)} \langle \xi_1 + \xi \rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)} \langle \xi_2 \rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)} \langle \xi_2 + \xi \rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)}} d\xi_1 d\xi_2$$
$$\lesssim \langle \xi \rangle^{1-2s+} \Big(\int \frac{d\xi_1}{\langle \xi_1 \rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)} \langle \xi_1 + \xi \rangle^{2b+\frac{1}{4}-\frac{1}{2}(s+a)}} \Big)^2 \lesssim \langle \xi \rangle^{2-8b+2a+} \lesssim 1$$

by using the Lemma 7.4, $a < \min\{2s+1, 1\}$ and the assumption that $\frac{1}{2} - b > 0$ is sufficiently small.

Case 2) $\frac{5}{2} \le s + a < \frac{9}{2}$

Note in this case $0 \le \frac{1}{2}(s+a) - 2b - \frac{1}{4} < 6b - 2$. Making use of the proof of Lemma 7.2, we write

$$\begin{aligned} &\xi^4 - \xi_1^4 + \xi_2^4 - (\xi - \xi_1 + \xi_2)^4 \\ &= (\xi - \xi_1)(\xi_1 - \xi_2) \Big[\frac{5}{2}(\xi + \xi_2)^2 + \xi^2 + \xi_2^2 + 2(\xi_1 - \frac{1}{2}\xi - \frac{1}{2}\xi_2)^2 \Big] \\ &=: g(\xi, \xi_1, \xi_2). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \langle \tau - \xi^4 \rangle &= \langle \tau - \xi_1^4 + \xi_2^4 - (\xi - \xi_1 + \xi_2)^4 - g(\xi, \xi_1, \xi_2) \rangle \\ &\lesssim \langle \tau - \xi_1^4 + \xi_2^4 - (\xi - \xi_1 + \xi_2)^4 \rangle + \langle g(\xi, \xi_1, \xi_2) \rangle \\ &\lesssim \langle \tau - \xi_1^4 + \xi_2^4 - (\xi - \xi_1 + \xi_2)^4 \rangle + \langle \xi_1 \rangle^2 \langle \xi_2 \rangle^2 \langle \xi \rangle^2 \langle \xi_1 - \xi \rangle \langle \xi_1 - \xi_2 \rangle. \end{aligned}$$

From this identity we obtain

$$\mathrm{I} \lesssim \int \frac{\langle \xi_1 - \xi \rangle^{\frac{1}{2}(s+a)-\frac{1}{4}-2b} \langle \xi_1 - \xi_2 \rangle^{\frac{1}{2}(s+a)-\frac{1}{4}-2b} \langle \xi \rangle^{\frac{1}{2}+s+a-4b+}}{\langle \xi_1 \rangle^{s-a+\frac{1}{2}+4b} \langle \xi_2 \rangle^{s-a+\frac{1}{2}+4b} \langle \xi - \xi_1 + \xi_2 \rangle^{2s}} d\xi_1 d\xi_2.$$

Substitutions $\xi_2 \to \xi_2 + \xi_1$ and $\xi_1 \to \xi_1 + \xi$ in the above integral lead to

$$\langle \xi \rangle^{\frac{1}{2}+s+a-4b+} \int \frac{\langle \xi_1 \rangle^{\frac{1}{2}(s+a)-\frac{1}{4}-2b} \langle \xi_2 \rangle^{\frac{1}{2}(s+a)-\frac{1}{4}-2b}}{\langle \xi_1 + \xi \rangle^{s-a+\frac{1}{2}+4b} \langle \xi_1 + \xi_2 + \xi \rangle^{s-a+\frac{1}{2}+4b} \langle \xi + \xi_2 \rangle^{2s}} d\xi_1 d\xi_2$$
$$\lesssim \langle \xi \rangle^{\frac{1}{2}+s+a-4b+} \int \frac{\langle \xi_1 \rangle^{\frac{1}{2}(s+a)-\frac{1}{4}-2b} \langle \xi_2 \rangle^{\frac{1}{2}(s+a)-\frac{1}{4}-2b}}{\langle \xi_1 + \xi \rangle^{s-a+\frac{1}{2}+4b} \langle \xi_1 + \xi_2 + \xi \rangle^{s-a+\frac{1}{2}+4b} \langle \xi + \xi_2 \rangle^{s-a+\frac{1}{2}+4b}} d\xi_1 d\xi_2$$

where we have used $s + a - 4b - \frac{1}{2} \geq 0$ in the last line above. Since $a < \min\{2s + 1, 1\}$, we note that $s > \frac{3}{2}$. Now by symmetry we have two cases to consider $|\xi + \xi_1 + \xi_2| \gtrsim |\xi|$ and $|\xi + \xi_1| \gtrsim |\xi|$. For the first one, using $\langle \xi_1 \rangle \lesssim \langle \xi_1 + \xi \rangle \langle \xi \rangle$ we have the bound

$$\langle \xi \rangle^{2a-8b+} \Big(\int \langle \xi_1 \rangle^{\frac{1}{2}(s+a)-\frac{1}{4}-2b} \langle \xi_1 + \xi \rangle^{-s+a-\frac{1}{2}-4b} d\xi_1 \Big)^2 \lesssim \langle \xi \rangle^{3a+s-\frac{1}{2}-12b+} \lesssim 1$$

owing to the the restrictions on a, b and s. For the second one, the integral is bounded by

$$\langle \xi \rangle^{2a-8b+} \int \langle \xi_1 \rangle^{\frac{1}{2}(s+a)-\frac{1}{4}-2b} \langle \xi_2 \rangle^{\frac{1}{2}(s+a)-\frac{1}{4}-2b} \langle \xi_2 + \xi \rangle^{-s+a-\frac{1}{2}-4b} \langle \xi_1 + \xi_2 + \xi \rangle^{-s+a-\frac{1}{2}-4b} d\xi_1 d\xi_2.$$

The inequalities $\langle \xi_1 \rangle \lesssim \langle \xi_1 + \xi_2 + \xi \rangle \langle \xi_2 + \xi \rangle$ and $\langle \xi_2 \rangle \lesssim \langle \xi_2 + \xi \rangle \langle \xi \rangle$ give rise to the bound

$$\langle \xi \rangle^{\frac{s}{2}+\frac{5a}{2}-\frac{1}{4}-10b+} \int \langle \xi_2 + \xi \rangle^{2a-8b-1} \langle \xi_1 + \xi_2 + \xi \rangle^{-\frac{s}{2}+\frac{3a}{2}-\frac{3}{4}-6b} d\xi_1 d\xi_2$$

which can be easily verified to be finite by the restrictions on a, b and s.

□

5 Local theory: the Proof of Theorem 1.2

In this section, we establish the local existence of solutions to (18). Firstly we aim to show that Γ defined by

$$\Gamma u(t) = \eta(t) W^t g_e + \eta(t) \int_0^t W^{t-s} F(u) ds + \eta(t) W_0^t(0, h_1 - p_1 - q_1, h_2 - p_2 - q_2) \tag{18}$$

has a fixed point in the space $X^{s,b}$, and recall where $g_e \in H^s(\mathbb{R})$ is the extension of g such that $\|g_e\|_{H^s(\mathbb{R})} \lesssim \|g\|_{H^s(\mathbb{R}^+)}$ and

$$F(u) = \eta(t/T)|u|^2 u, \qquad p_1(t) = \eta(t) D_0(W^t g_e), \qquad p_2(t) = \eta(t) D_0\big(\partial_x \big(W^t g_e\big)\big),$$
$$q_1(t) = \eta(t) D_0\Big(\int_0^t W^{t-t'} F(u) dt' \Big), \qquad q_2(t) = \eta(t) D_0\Big(\partial_x \Big[\int_0^t W^{t-t'} F(u) dt' \Big] \Big).$$

We also recall that $s \in (-\frac{1}{3}, \frac{9}{2})$, $s \neq \frac{1}{2}, \frac{3}{2}$ and $\frac{1}{2} - b > 0$ is sufficiently small. We start with showing that Γ is a bounded operator on $X^{s,b}$. To do so, we gather necessary bounds we have so far. Using (11) we have

$$\big\| \eta(t) W^t g_e \big\|_{X^{s,b}} \lesssim \|g_e\|_{H^s(\mathbb{R})} \lesssim \|g\|_{H^s(\mathbb{R}^+)} .$$

Next by (13), (12) followed by Proposition 4.6 we obtain

$$\left\|\eta\int_0^t W^{t-s}F(u)ds\right\|_{X^{s,b}} \leq \left\|\eta\int_0^t W^{t-s}F(u)ds\right\|_{X^{s,\frac{1}{2}+}} \lesssim \|F(u)\|_{X^{s,-\frac{1}{2}+}}$$
$$\lesssim T^{\frac{1}{2}-b-}\left\||u|^2u\right\|_{X^{s,-b}} \lesssim T^{\frac{1}{2}-b-}\|u\|^3_{X^{s,b}}.$$

By using Proposition 4.2 and Lemma 2.2

$$\left\|\eta(t)W_0^t(0,h_1-p_1-q_1,h_2-p_2-q_2)\right\|_{X^{s,b}} \lesssim \left\|\chi_{(0,\infty)}(h_1-p_1-q_1)\right\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R})}$$
$$+\left\|\chi_{(0,\infty)}(h_2-p_2-q_2)\right\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R})} \lesssim \|h_1-p_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2-p_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)}$$
$$+\|q_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|q_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)}$$
$$\lesssim \|h_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)} + \|p_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R})}$$
$$+\|p_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R})} + \|q_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R})} + \|q_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R})}$$

by the Kato smoothing estimate

$$\|p_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R})} + \|p_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R})} \lesssim \|g_e\|_{H^s(\mathbb{R})} \lesssim \|g\|_{H^s(\mathbb{R}^+)}.$$

To bound the q_i norms we use Proposition 4.5, (12) and Proposition 4.7

$$\|q_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R})} + \|q_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R})} \lesssim \begin{cases} \|F\|_{X^{s,-\frac{1}{2}+}} & \text{for } -\frac{1}{3}<s\leq\frac{1}{2} \\ \|F\|_{X^{s,-\frac{1}{2}+}} + \|F\|_{X^{\frac{1}{2}+,\frac{2s-5}{8}+}} & \text{for } \frac{1}{2}<s<\frac{9}{2} \end{cases}$$
$$\lesssim T^{\frac{1}{2}-b-}\begin{cases} \left\||u|^2u\right\|_{X^{s,-b}} & \text{for } -\frac{1}{3}<s\leq\frac{1}{2} \\ \left\||u|^2u\right\|_{X^{s,-b}} + \left\||u|^2u\right\|_{X^{\frac{1}{2}+,\frac{2s-1}{8}-b}} & \text{for } \frac{1}{2}<s<\frac{9}{2} \end{cases}$$
$$\lesssim T^{\frac{1}{2}-b-}\|u\|^3_{X^{s,b}}$$

putting these estimates together in estimating (18) we have

$$\|\Gamma u\|_{X^{s,b}} \lesssim \|g\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)} + T^{\frac{1}{2}-b-}\|u\|^3_{X^{s,b}}.$$

Having shown that Γ is bounded, our next objective is to reveal that Γ is indeed a contraction. To achieve this we implement the similar calculations for the difference $\Gamma u - \Gamma\tilde{u}$ as follows

$$\|\Gamma u - \Gamma\tilde{u}\|_{X^{s,b}} \leq \left\|\eta\int_0^t W^{t-s}[F(u)-F(\widetilde{u})]ds\right\|_{X^{s,b}} + \left\|\eta W_0^t(0,\widetilde{q_1}-q_1,\widetilde{q_2}-q_2)\right\|_{X^{s,b}}$$
$$\lesssim \left\|\eta\int_0^t W^{t-s}[F(u)-F(\widetilde{u})]ds\right\|_{X^{s,\frac{1}{2}+}} + \left\|\eta\int_0^t W^{t-s}[F(u)-F(\widetilde{u})]ds\right\|_{L_x^\infty H_t^{\frac{2s+3}{8}}}$$
$$+\left\|\eta\partial_x\Big(\int_0^t W^{t-s}[F(u)-F(\widetilde{u})]ds\Big)\right\|_{L_x^\infty H_t^{\frac{2s+1}{8}}}$$
$$\lesssim T^{\frac{1}{2}-b-}\Big(\left\||u|^2u-|\widetilde{u}|^2\widetilde{u}\right\|_{X^{s,-b}} + \chi_{(\frac{1}{2},\frac{9}{2})}(s)\left\||u|^2u-|\widetilde{u}|^2\widetilde{u}\right\|_{X^{\frac{1}{2}+,\frac{2s-1}{8}-b}}\Big)$$
$$\lesssim T^{\frac{1}{2}-b-}\Big(\|u\|^2_{X^{s,b}} + \|\widetilde{u}\|^2_{X^{s,b}}\Big)\|u-\widetilde{u}\|_{X^{s,b}}.$$

In the last line we have used Proposition 4.7 along with the inequality

$$||f|^\alpha f - |g|^\alpha g| \leq C(|f|^\alpha + |g|^\alpha)|f - g|$$

for absolute constant C and $\alpha \geq 0$. Therefore taking $0 < T < 1$ sufficiently small, Γ is a contraction on the ball $B = \left\{u \in X^{s,b} : \|u\|_{X^{s,b}} \leq C\left(\|g\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)} \right)\right\}$ with radius depending on the initial and boundary data. Hence by the Banach fixed point theorem, this ensures the existence of a solution to (1) in $X^{s,b}$ spaces. Next we establish that the fixed point of Γ lies in $C_t^0 H_x^s([0,T] \times \mathbb{R})$. Since the operator $W^t = e^{it\Delta^2}$ is unitary on $H^s(\mathbb{R})$ we have

$$\left\|\eta W^t g_e\right\|_{C_t^0 H_x^s} \lesssim \|g_e\|_{H^s(\mathbb{R})} \lesssim \|g\|_{H^s(\mathbb{R}^+)} .$$

By the embedding (10) and the contraction argument

$$\left\|\eta \int_0^t W^{t-s} F(u) ds\right\|_{C_t^0 H_x^s} \lesssim \left\|\eta \int_0^t W^{t-s} F(u) ds\right\|_{X^{s,\frac{1}{2}+}} \lesssim \cdots \lesssim T^{\frac{1}{2}-b-} \|u\|^3_{X^{s,b}} .$$

Next from Lemma 4.3 and the previous estimates in the contraction argument

$$\begin{aligned}
&\left\|\eta W_0^t(0, h_1 - p_1 - q_1, h_2 - p_2 - q_2)\right\|_{C_t^0 H_x^s} \\
&\lesssim \left\|\chi_{(0,\infty)}(h_1 - p_1 - q_1)\right\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R})} \\
&\quad + \left\|\chi_{(0,\infty)}(h_2 - p_2 - q_2)\right\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R})} \\
&\lesssim \cdots \lesssim \|g\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)} + T^{\frac{1}{2}-b-} \|u\|^3_{X^{s,b}} .
\end{aligned}$$

We also show that $u = \Gamma u$ belongs to the space $C_x^0 H_t^{\frac{2s+3}{8}}([0,T] \times \mathbb{R})$. We have already obtained the following bounds in the contraction argument

$$\left\|\eta W^t g_e\right\|_{C_x^0 H_t^{\frac{2s+3}{8}}} \lesssim \|g_e\|_{H^s(\mathbb{R})} \lesssim \|g\|_{H^s(\mathbb{R}^+)}$$

and

$$\left\|\eta \int_0^t W^{t-s} F(u) ds\right\|_{C_x^0 H_t^{\frac{2s+3}{8}}} \lesssim T^{\frac{1}{2}-b-} \|u\|^3_{X^{s,b}} .$$

For the remaining term of Γ we exploit the Lemma 4.4 and the contraction argument to get

$$\begin{aligned}
\left\|\eta W_0^t(0, h_1 - p_1 - q_1, h_2 - p_2 - q_2)\right\|_{C_x^0 H_t^{\frac{2s+3}{8}}} &\lesssim \|g\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R}^+)} \\
&\quad + \|h_2\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)} + T^{\frac{1}{2}-b-} \|u\|^3_{X^{s,b}} .
\end{aligned}$$

As a result we have established that $u = \Gamma u$ lies in the Banach space of the definition 1.1. Therefore this finishes the proof of local existence of solutions to (1). The uniqueness of these solutions will be treated in the subsequent Sect. 5.1 below. The continuous dependence of these local solutions on the initial and boundary data follows from the fixed point argument and the a priori estimates as well. To see this let u and u_n be solutions of (1) with initial and boundary data g, h_1, h_2 and g_n, h_{n1}, h_{n2} respectively. Then from what we have already shown in the contraction argument, we have

$$\|u-u_n\|_{X^{s,b}} \le C_0\big(\|g-g_n\|_{H^s(\mathbb{R}^+)} + \|h_1-h_{n1}\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2-h_{n2}\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)} \big) + C_1 T^{\frac{1}{2}-b-} \|u-u_n\|_{X^{s,b}}$$

where $C_0 > 0$ is a positive constant and C_1 depends on the radius of the ball in the fixed point argument and hence on the initial and boundary data. By means of contraction argument, we may take existence time $T < 1$ so that $C_1 T^{\frac{1}{2}-b-} < 1$. So by the inequality

$$\|u-u_n\|_{X^{s,b}} \le \frac{C_0}{\big(1-C_1T^{\frac{1}{2}-b-}\big)}\big(\|g-g_n\|_{H^s(\mathbb{R}^+)} + \|h_1-h_{n1}\|_{H_t^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2-h_{n2}\|_{H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)} \big)$$

the continuous dependence in $X^{s,b}$ follows. In a similar manner we prove the continuous dependence in the spaces $C_t^0H_x^s$ and $C_x^0H_t^{\frac{2s+3}{8}}$ as well. In order to complete the proof of the Theorem 1.2, it is left to establish the quantification of the dependence of existence time T to the initial and boundary data. By a scaling argument, we easily see that if u solves the Eq. (1) with data g, h_1 and h_2 on $[0, \lambda^{-4}]$, then $u^\lambda(x,t) = \lambda^{-2}u(\lambda^{-1}x, \lambda^{-4}t)$ solves the Eq. (1) with data $g^\lambda(x) = \lambda^{-2}g(\lambda^{-1}x)$, $h_1^\lambda(t) = \lambda^{-2}h_1(\lambda^{-4}t)$ and $h_2^\lambda(t) = \lambda^{-3}h_1(\lambda^{-4}t)$ on $[0, 1]$. Therefore for $\lambda > 1$,

$$\big\|h_1^\lambda\big\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} \lesssim \|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)},$$
$$\big\|h_2^\lambda\big\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)} \lesssim \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)},$$

$$\begin{aligned}\big\|g^\lambda\big\|_{H^s(\mathbb{R}^+)} &\le \big\|g^\lambda\big\|_{L^2(\mathbb{R}^+)} + \big\|g^\lambda\big\|_{\dot{H}^s(\mathbb{R}^+)} \\ &\le \lambda^{-\frac{3}{2}}\|g\|_{L^2(\mathbb{R}^+)} + \lambda^{-\frac{3}{2}-s}\|g\|_{\dot{H}^s(\mathbb{R}^+)} \le \|g\|_{L^2} + \lambda^{-\frac{3}{2}-s}\|g\|_{H^s(\mathbb{R}^+)}.\end{aligned}$$

Then for $\lambda^{-\frac{3}{2}-s}\|g\|_{H^s(\mathbb{R}^+)} \approx 1$, the solution is defined up to the local existence time

$$T \approx \big(C + \|g\|_{H^s(\mathbb{R}^+)}\big)^{-\frac{8}{2s+3}}$$

where the constant C depends on $\|g\|_{L^2} + \|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)}$. Moreover, in order to have local existence interval without implicit dependence on $\|g\|_{L^2}$ (to be used later in Sect. 6), we make use of the following bound

$$\big\|g^\lambda\big\|_{H^s} \le \big\|g^\lambda\big\|_{L^2} + \big\|g^\lambda\big\|_{\dot{H}^s} \le \lambda^{-\frac{3}{2}}\|g\|_{L^2} + \lambda^{-\frac{3}{2}-s}\|g\|_{\dot{H}^s} \le \lambda^{-\frac{7}{6}}\|g\|_{H^s}$$

that gives rise to the local existence time $T \approx \big(C + \|g\|_{H^s}\big)^{-\frac{24}{7}}$, in this case, with constant C dependent to $\|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)}$.

5.1 Uniqueness of solutions

In this section, we exhibit that the solutions to the Eq. (1) constructed above are unique. The uniqueness statement of the Theorem 1.2 for $s > \frac{1}{2}$ follows from an energy argument which we want to illustrate next, and then using the smoothing theorem we will extend the uniqueness argument to the whole well-posedness range. Hence first consider the smooth solutions u and v of (1) with sufficient decay. Then using $u(x,0)=v(x,0)$, $u(0,t)=v(0,t)$ and $u_x(0,t)=v_x(0,t)$, we compute

$$\partial_t \|u-v\|^2_{L^2(\mathbb{R}^+)} = 2\,\mathrm{Re}\, i\mu \int_0^\infty \big(|u|^2u - |v|^2v\big)\overline{(u-v)}dx,$$

hence, for any $t>0$, integrating this and then using Sobolev embedding $H^s(\mathbb{R}^+)\subset L^\infty(\mathbb{R}^+)$, $s>\frac{1}{2}$ we obtain

$$\begin{aligned}\|(u-v)(t)\|^2_{L^2_x(\mathbb{R}^+)} &\le 2|\mu|\big(\|u\|^2_{L^\infty_{t\in[0,T]}L^\infty_x(\mathbb{R}^+)} + \|v\|^2_{L^\infty_{t\in[0,T]}L^\infty_x(\mathbb{R}^+)}\big)\int_0^t \|(u-v)(s)\|^2_{L^2_x(\mathbb{R}^+)}ds\\ &\lesssim \big(\|u\|^2_{L^\infty_{t\in[0,T]}H^s_x(\mathbb{R}^+)} + \|v\|^2_{L^\infty_{t\in[0,T]}H^s_x(\mathbb{R}^+)}\big)\int_0^t \|(u-v)(s)\|^2_{L^2_x(\mathbb{R}^+)}ds.\end{aligned}$$

Since, by the local theory, the solutions u and v belong to $C^0_tH^s_x([0,T]\times\mathbb{R}^+)$, this with the Gronwall's inequality imply that $u=v$. The uniqueness of rougher solutions follows from taking convolution of $u-v$ with smooth approximate identities and then carrying out a limiting argument as usual, see for instance [26]. Also since the norms are taken on $\mathbb{R}^+$ in the energy estimate above, the restriction of solution to the right half line is independent of the choice of extension of the initial data. Next we will prove the uniqueness of the local solutions in the case $s\in(-\frac{1}{3},\frac{1}{2})$ by utilizing the uniqueness obtained above for $s>\frac{1}{2}$ and the smoothing estimates from Theorem 1.3. Here we follow the arguments of [8]. We get started by considering data $(g,h_1,h_2)\in H^s_x(\mathbb{R}^+)\times H_t^{\frac{2s+3}{8}}(\mathbb{R}^+)\times H_t^{\frac{2s+1}{8}}(\mathbb{R}^+)$ for $s\in(0,\frac{1}{2})$. Let g_e and $\widetilde{g_e}$ be two $H^s(\mathbb{R})$ extensions of $g\in H^s(\mathbb{R}^+)$. Associated to these extensions let u and $\widetilde{u}$ be the fixed points of Γ defined in (18). Next pick a sequence $g^k\in H^{\frac{1}{2}+}(\mathbb{R}^+)$ converging to g in $H^s(\mathbb{R}^+)$. Then, by Lemma 5.1 below, we may assume that g_e^k and $\widetilde{g_e}^k$ are $H^{\frac{1}{2}+}(\mathbb{R})$ extensions of g^k that converge respectively to g_e and $\widetilde{g_e}$ in $H^r(\mathbb{R})$ for $r<s<\frac{1}{2}$. Running a contraction argument on the set $B_1\cap B_2$ where

$$B_1=\Big\{u:\|u\|_{X^{\frac{1}{2}+,b}}\le C\Big(\big\|g^k\big\|_{H^{\frac{1}{2}+}(\mathbb{R}^+)}+\|h_1\|_{H^{\frac{1}{2}+}(\mathbb{R}^+)}+\|h_2\|_{H^{\frac{1}{4}+}(\mathbb{R}^+)}\Big)\Big\}$$

and

$$B_2=\Big\{u:\|u\|_{X^{s,b}}\le C\Big(\|g\|_{H^s(\mathbb{R}^+)}+\|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)}+\|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)}\Big)\Big\}$$

we construct $H^{\frac{1}{2}+}(\mathbb{R})$ solutions u^k and $\widetilde{u}^k$ to the Eq. (1) associated to the extensions g_e^k and $\widetilde{g_e}^k$ respectively. At this juncture we make use of the smoothing estimate of Theorem 1.3 to obtain local existence time $T=T\Big(\|g\|_{H^s(\mathbb{R}^+)},\|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)},\|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)}\Big)$ for $s<\frac{1}{2}$. By the uniqueness of $H^{\frac{1}{2}+}$ solutions obtained above, the restrictions of solutions u^k and $\widetilde{u}^k$ to $\mathbb{R}^+$ are the same. Since, by the fixed point argument, $u^k\to u$ and $\widetilde{u}^k\to\widetilde{u}$ in $H^{s-}(\mathbb{R})$, we then have $u|_{\mathbb{R}^+}=\widetilde{u}|_{\mathbb{R}^+}$. Iterating this argument the uniqueness for $s>-\frac{1}{3}$ follows.

Lemma 5.1 (See [17]) *Fix $-\frac{1}{2}<s<\frac{1}{2}$ and $k>s$. Let $f\in H^s(\mathbb{R}^+)$ and $g\in H^k(\mathbb{R}^+)$. Let f^e be an H^s extension of f to $\mathbb{R}$. Then there is an H^k extension g^e of g to $\mathbb{R}$ such that*

$$\|f^e-g^e\|_{H^r(\mathbb{R})}\lesssim\|f-g\|_{H^s(\mathbb{R}^+)}\qquad \text{for } r<s.$$

6 Proofs of Theorem 1.3 and Theorem 1.4

Proof of Theorem 1.3 By (18), for $t \in [0, T]$ we write the difference of nonlinear and linear solutions as

$$u(t) - W_0^t(0, h_1 - p_1, h_2 - p_2)(t) = \eta(t) \int_0^t W^{t-t'} \eta(t'/T)|u|^2 u dt' - \eta(t) W_0^t(0, q_1, q_2)(t)$$

where

$$q_1(t) = \eta(t) D_0\Big(\int_0^t W^{t-t'} \eta(t'/T)|u|^2 u dt'\Big), \;\; q_2(t) = \eta(t) D_0\Big(\partial_x\Big[\int_0^t W^{t-t'} \eta(t'/T)|u|^2 u dt'\Big]\Big).$$

Therefore using the embedding $X^{s,\frac{1}{2}+} \subset C_t^0 H_x^s$ in (10), (13), Lemma 4.3 and then Proposition 4.5, we have

$$\begin{aligned}
&\big\| u - W_0^t(0, h_1 - p_1, h_2 - p_2)\big\|_{C^0_{t\in[0,T]} H^{s+a}_{x\in\mathbb{R}+}} \\
&\lesssim \Big\| \eta \int_0^t W^{t-t'} \eta(t'/T)|u|^2 u dt' \Big\|_{X^{s+a,\frac{1}{2}+}} \\
&\quad + \big\| W_0^t(0, q_1, q_2)\big\|_{C_t^0 H_x^{s+a}} \lesssim \big\|\eta |u|^2 u\big\|_{X^{s+a,-\frac{1}{2}+}} + \|q_1\|_{H_t^{\frac{2s+2a+3}{8}}} + \|q_2\|_{H_t^{\frac{2s+2a+1}{8}}} \\
&\lesssim \big\|\eta |u|^2 u\big\|_{X^{s+a,-\frac{1}{2}+}} + \begin{cases} \big\|\eta |u|^2 u\big\|_{X^{s+a,-\frac{1}{2}+}} & \text{for } -\frac{1}{3} < s+a \le \frac{1}{2} \\ \big\|\eta |u|^2 u\big\|_{X^{s+a,-\frac{1}{2}+}} + \big\|\eta |u|^2 u\big\|_{X^{\frac{1}{2}+,\frac{2s+2a-5}{8}}} & \text{for } \frac{1}{2} < s+a < \frac{9}{2} \end{cases}.
\end{aligned}$$

By Propositions 4.6, Proposition 4.7 and Theorem 1.2 along with the local theory, this is bounded by

$$\|u\|^3_{X^{s,b}} \lesssim \big(\|g\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)} \big)^3,$$

so the claim follows. □

Proof of Theorem 1.4 Fix $T > 0$ and assume the growth bound $\|u\|_{H^s(\mathbb{R}^+)} \le f(T)$ for f depending on $\|g\|_{H^s(\mathbb{R}^+)}$, $\|h_1\|_{H^{s_1}(\mathbb{R}^+)}$ and $\|h_2\|_{H^{s_2}(\mathbb{R}^+)}$, for some $s_1 \ge \frac{2s+3}{8}$, $s_2 \ge \frac{2s+1}{8}$. Using the final claim of the proof of Theorem 1.2, we may pick the local existence time based on $f(T)$: $\delta \approx (C + f(T))^{-\frac{24}{7}}$ where C is a constant proportional to $\|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)}$. Therefore for $J \approx T/\delta$

$$\begin{aligned}
&\Big\| u(J\delta) - W_0^{J\delta}(g, h_1, h_2)\Big\|_{H^{s+a}_{x\in\mathbb{R}+}} \\
&= \Big\| \sum_{k=1}^J W_{k\delta}^{J\delta}(u(k\delta), h_1, h_2) - W_{(k-1)\delta}^{J\delta}(u((k-1)\delta), h_1, h_2)\Big\|_{H^{s+a}_{x\in\mathbb{R}+}} \\
&\le \sum_{k=1}^J \Big\| W_{k\delta}^{J\delta}(u(k\delta), h_1, h_2) - W_{(k-1)\delta}^{J\delta}(u((k-1)\delta), h_1, h_2)\Big\|_{H^{s+a}_{x\in\mathbb{R}+}}
\end{aligned}$$

$$\leq \sum_{k=1}^{J} \left\| W_{k\delta}^{J\delta}\big([u(k\delta) - W_{(k-1)\delta}^{k\delta}(u((k-1)\delta), h_1, h_2)], 0, 0\big) \right\|_{H^{s+a}_{x\in\mathbb{R}^+}}$$
$$\leq \sum_{k=1}^{J} \left\| u(k\delta) - W_{(k-1)\delta}^{k\delta}(u((k-1)\delta), h_1, h_2) \right\|_{H^{s+a}_{x\in\mathbb{R}^+}} \lesssim Jf(T)^3 \lesssim \langle T\rangle f(T)^{\frac{45}{7}}$$

where we have used Remark 2.3 in the second and third inequalities. Also the implicit constants just depend on $\|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)}$, $\|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)}$. Then we have

$$\|u(T)\|_{H^{s+a}(\mathbb{R}^+)} \lesssim \langle T\rangle f(T)^{\frac{45}{7}} + \left\| W_0^T(g, h_1, h_2) \right\|_{H^{s+a}(\mathbb{R}^+)}.$$

To bound this, first recall that

$$W_0^T(g, h_1, h_2) = W^T g_e + W_0^T(g, h_1 - p_1, h_2 - p_2)$$

where $p_1(t) = \eta(t/\langle T\rangle)D_0(W^t g_e)$, $p_2(t) = \eta(t/\langle T\rangle)D_0(\partial_x[W^t g_e])$. Then by Lemma 2.2

$$\begin{aligned}
\left\| W_0^T(g, h_1, h_2) \right\|_{H^s(\mathbb{R})} &\lesssim \|g_e\|_{H^s(\mathbb{R})} + \left\| \chi_{(0,\infty)}(h_1 - p_1) \right\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} \\
&\quad + \left\| \chi_{(0,\infty)}(h_2 - p_2) \right\|_{H^{\frac{2s+1}{8}}(\mathbb{R})} \\
&\lesssim \|g\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)} + \|p_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} \\
&\quad + \|p_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R})}.
\end{aligned}$$

We estimate p_1 and p_2 by writing $\eta(t/\langle T\rangle) = \sum_{j=1}^{\langle T\rangle} \eta_j(t)$ and then using Kato smoothing inequality (Lemma 4.1) as follows

$$\|p_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R})} + \|p_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R})} \lesssim \langle T\rangle \|g_e\|_{H^s(\mathbb{R})} \lesssim \langle T\rangle \|g\|_{H^s(\mathbb{R}^+)}.$$

So then we have

$$\left\| W_0^T(g, h_1, h_2) \right\|_{H^s(\mathbb{R}^+)} \lesssim \langle T\rangle \|g\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)}$$

which leads to the bound

$$\|u(T)\|_{H^{s+a}(\mathbb{R}^+)} \lesssim \langle T\rangle \big[f(T)^{\frac{45}{7}} + \|g\|_{H^{s+a}(\mathbb{R}^+)}\big] + \|h_1\|_{H^{\frac{2s+2a+3}{8}}(\mathbb{R}^+)} + \|h_2\|_{H^{\frac{2s+2a+1}{8}}(\mathbb{R}^+)}.$$

When $s = 2$ and $s_1 = s_2 = 1$, Lemma 7.1 below implies that $f(t) \approx 1$. As a result this and above bound yield that $\|u(t)\|_{H^s(\mathbb{R}^+)} \lesssim \langle T\rangle$ for $2 < s < \frac{5}{2}$. □

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7 Appendix

In this section, we reserve some useful inequalities to be used in the text when necessary. Firstly we start with the lemma which is a consequence of the proof of Theorem 1.3 in [34].

Lemma 7.1 *when $\mu = 1$ (defocusing nonlinearity), the solutions of the Eq. (1) satisfy the following a priori estimate*

$$\|u\|_{H^2(\mathbb{R}^+)} \leq C(\|g\|_{H^2}, \|h_1\|_{H^1}, \|h_2\|_{H^1}).$$

Next Lemma is useful in the proofs of Proposition 4.6 and Proposition 4.7.

Lemma 7.2 *For $m, n, k \in \mathbb{R}$ we have*

$$|m^4 - n^4 + k^4 - (m-n+k)^4| \gtrsim |m-n||n-k|(m^2+n^2+k^2).$$

Proof Let $g(m,n,k) := m^4 - n^4 + k^4 - (m-n+k)^4$. Then

$$\begin{aligned} g(m,n,k) &= (m-n)\big[(m^2+n^2)(m+n) - (m-n)^3 - 4(m-n)^2k - 6(m-n)k^2 - 4k^3\big] \\ &= (m-n)(n-k)\big[4m^2 + 2n^2 + 4k^2 - 2mn - 2nk\big] \\ &= (m-n)(n-k)\Big[\frac{5}{2}(m+n)^2 + m^2 + k^2 + 2(n - \frac{1}{2}m - \frac{1}{2}k)^2\Big] \end{aligned}$$

which gives the desired estimate. □

Lemma 7.3 (See [6]) *For $-\frac{1}{2} \leq s \leq \frac{1}{2}$, we have*

$$\|fg\|_{H^s} \lesssim \|f\|_{H^{\frac{1}{2}+}} \|g\|_{H^s}$$

Finally we have the following lemmas we use throughout the text. For proofs of the first and the second of these, see [13] and [16] respectively.

Lemma 7.4 *If $\beta \geq \gamma \geq 0$ and $\beta + \gamma > 1$ then*

$$\int_{\mathbb{R}} \frac{dx}{\langle x - a_1\rangle^\beta \langle x - a_2\rangle^\gamma} \lesssim \langle a_1 - a_2\rangle^{-\gamma} \varphi_\beta(a_1 - a_2)$$

where

$$\varphi_\beta(a) = \sum_{|n| \leq |a|} \frac{1}{\langle n\rangle^\beta} \sim \begin{cases} 1 & \beta > 1 \\ \log(1 + \langle a\rangle) & \beta = 1 \\ \langle a\rangle^{1-\beta} & \beta < 1. \end{cases}$$

Lemma 7.5 *For fixed $\rho \in (\frac{1}{2}, 1)$, we have*

$$\int \frac{1}{\langle x\rangle^\rho \sqrt{|x-a|}} dx \lesssim \frac{1}{\langle a\rangle^{\rho - \frac{1}{2}}}.$$

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