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ORIGINAL PAPER

Local existence and nonexistence for fractional in time weakly coupled reaction-diffusion systems

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Abstract

We study a fractional in time weakly coupled reaction-diffusion system in a bounded domain with the Dirichlet boundary condition. The domain is imbedded in an N-dimensional space and it has C^2 boundary, and fractional derivatives are meant in a generalized Caputo sense. The system can be referred to as a standard reaction-diffusion system in two components with polynomial growth. We obtain integrability conditions on the initial state functions which determine the existence/nonexistence of a local in time mild solution.

Keywords Local in time solutions · Caputo fractional derivative · Singular initial functions · Uniqueness · Cauchy sequences · Semigroup estimates

Mathematics Subject Classification Primary · 35K51 · 35A01 · 35R11 · Secondary · 26A33 · 46E35

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with C^2 boundary. We study existence and nonexistence of a local in time solution of the fractional in time weakly coupled reaction-diffusion system

$$\begin{cases} \partial_t^{\alpha_1} u = \Delta u + f_1(x,t,v) & \text{in } \Omega \times (0,T), \\ \partial_t^{\alpha_2} v = \Delta v + f_2(x,t,u) & \text{in } \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $0 < \alpha_1 \leq \alpha_2 < 1$ and $T > 0$. The fractional derivatives $\partial_t^{\alpha_1}$ and $\partial_t^{\alpha_2}$ are meant in a generalized Caputo sense, i.e.,

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$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \partial_s u(s) ds \quad \text{for } 0 < \alpha < 1,$$

where Γ denotes the usual Gamma function. In the present paper we suppose the following:

Assumption A Let $1 < p_1, p_2 < \infty$, $q_1, q_2 \in (1, \infty] \cap \left(\frac{N}{2}, \infty\right]$, $m_1 \in [0, \alpha_1)$ and $m_2 \in [0, \alpha_2)$ be given constants. There exist nonnegative functions $c_1 \in L^{q_1}(\Omega)$ and $c_2 \in L^{q_2}(\Omega)$ such that the following hold:

(**F**1) for $i = 1, 2$, $f_i(x, t, \cdot) : \mathbb{R} \to \mathbb{R}$ is a measurable function such that

$$|f_i(x,t,\xi)| \le c_i(x) \cdot t^{-m_i}(1 + |\xi|)^{p_i} \text{ for } \xi \in \mathbb{R}, \text{ a.e. } (x,t) \in \Omega \times (0, \infty),$$

(**F**2) for $i = 1, 2$, f_i satisfies the local Lipschitz condition

$$|f_i(x,t,\xi) - f_i(x,t,\eta)| \le c_i(x) \cdot t^{-m_i}(1 + |\xi| + |\eta|)^{p_i - 1}|\xi - \eta| \text{ for } \xi, \eta \in \mathbb{R}, \\ \text{a.e. } (x,t) \in \Omega \times (0, \infty).$$

Let us start with classical equations, $\alpha = 1$. We consider the scalar problem

$$\begin{cases} \partial_t u = \Delta u + f(u) & \text{in } \Omega \times (0, T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $f \in C^1$ and Ω is a (possibly unbounded) smooth domain. It is well known that the problem (1.2) possesses a local in time classical solution for a general nonlinear term f if $u_0 \in L^\infty(\Omega)$ (cf. [7, 16]). On the other hand, in the case where $u_0 \notin L^\infty(\Omega)$, the existence of solutions heavily depends on the balance between the growth rate of f and the singularity of u_0 (cf. [7]). In Weissler [20], (1.2) was studied when $f(u) = |u|^{p-1}u$, $p > 1$, and Ω is bounded. A local in time solution was constructed when $u_0 \in L^r(\Omega)$ for $r > \frac{N}{2}(p-1)$ and $r \ge 1$, or $r = \frac{N}{2}(p-1)$ and $r > 1$. It was also shown that if $1 \le r < \frac{N}{2}(p-1)$, then there exists a nonnegative initial function $u_0 \in L^r(\Omega)$ such that, for every $T > 0$, (1.2) admits no nonnegative solution.

Next, we consider the reaction-diffusion system

$$\begin{cases} \partial_t u = \Delta u + f_1(u, v) & \text{in } \Omega \times (0, T), \\ \partial_t v = \Delta v + f_2(u, v) & \text{in } \Omega \times (0, T), \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{1.3}$$

Quittner–Souplet [15] studied (1.3), where $f_1(u, v) = |v|^{p_1 - 1}v$, $f_2(u, v) = |u|^{p_2 - 1}u$ ($p_1, p_2 > 0$) and Ω is bounded. This is called a weakly coupled system. The existence (resp. nonexistence) of a local in time solution was proved when $(u_0, v_0) \in L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ and

$$\frac{N}{2}\max\left\{\frac{p_1}{r_2}-\frac{1}{r_1},\frac{p_2}{r_1}-\frac{1}{r_2}\right\}\le 1,\ p_1,p_2>1 \text{ and } r_1,r_2>1$$

(resp. $\frac{N}{2}\max\left\{\frac{p_1}{r_2}-\frac{1}{r_1},\frac{p_2}{r_1}-\frac{1}{r_2}\right\}>1,\ p_1,p_2>0$ and $r_1,r_2\ge 1$). Moreover, weakly and strongly coupled variants of system (1.3) were studied in [4–6, 11, 13, 19]. They assumed that initial functions may have singularities.

We obtain integrability conditions of (u_0,v_0) which determine the existence/nonexistence of a local in time solution of (1.1). In the proof we combine methods of Gal–Warma [9] and Quittner–Souplet [15, 16].

To define a mild solution, we recall the Wright type function [10] defined by

$$\Phi_\alpha(z):=\sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma(-\alpha n+1-\alpha)} \text{ for } 0<\alpha<1 \text{ and } z\in\mathbb{C}.$$

This is also sometimes called the Mainardi function and studied in [14, 18, 22]. It follows from [2, 10] that

$$\Phi_\alpha(t)\ge 0 \text{ for } t\ge 0 \text{ and } \int_0^\infty \Phi_\alpha(t)dt=1. \tag{1.4}$$

We prove our nonexistence result using (1.4). It is a key point that the function $\Phi_\alpha(t)$ is nonnegative and integrable. Moreover, it is well known ([10]) that

$$\int_0^\infty t^p\Phi_\alpha(t)dt=\frac{\Gamma(p+1)}{\Gamma(\alpha p+1)} \text{ for } p>-1 \text{ and } 0<\alpha<1. \tag{1.5}$$

Next, we consider functional spaces. Let $1<r<\infty$. We denote by $W^{k,r}(\Omega)$ the Sobolev space (resp. the Sobolev–Slobodeckii space) if k is an integer (resp. if k is not an integer). We also denote by $\mathcal{D}(\Omega)$ the space of C^∞-functions with compact support in Ω. Put $X_0(r):=L^r(\Omega)$ and $X_1(r):=W^{2,r}(\Omega)\cap W_0^{1,r}(\Omega)$, where $W_0^{1,r}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,r}(\Omega)$. Let Δ be the Laplace operator with the domain $D(\Delta)=X_1(r)$. Then Δ generates a C^0 analytic semigroup in X_0 by [16, Examples 51.4 (i)]. Let $X_{-1}(r)$ be the completion of $X_0(r)$ endowed with the norm $|x|_{X_{-1}(r)}:=|(\omega+\Delta)^{-1}x|_{X_0(r)}$, where $\omega\in\mathbb{R}$ satisfies that $\omega+\Delta: X_1(r)\to X_0(r)$ is an isomorphism ([16, p.466, 467]). For $0<\theta<1$, set $X_\theta(r):=(X_0(r),X_1(r))_\theta$ and $X_{-1+\theta}(r):=(X_{-1}(r),X_0(r))_\theta$, where $(\cdot,\cdot)_\theta$ is the complex interpolation functor if $\theta=\frac{1}{2}$ and the real interpolation functor $(\cdot,\cdot)_{\theta,r}$ otherwise. Due to [16, Theorem 51. 1 (i) and Examples 51.4 (i)], we have $X_{\theta_1}(r)\hookrightarrow X_{\theta_2}(r)$ if $-1\le\theta_2\le\theta_1\le 1$,

$$X_\theta(r)\hookrightarrow W^{2\theta,r}(\Omega) \text{ if } \theta\ge 0, \text{ and } X_\theta(r)\doteq(X_{-\theta}(r'))' \text{ if } \theta<0, \tag{1.6}$$

where r' is the conjugate exponent of r, i.e., $\frac{1}{r}+\frac{1}{r'}=1$. We denote X' by the (topological) dual space if X is a Banach space. We write $X\hookrightarrow Y$ if X is continuously embedded in Y. Moreover, $X\doteq Y$ means that $X\hookrightarrow Y$ and $Y\hookrightarrow X$.

We are ready to introduce some operators related to fractional derivatives. Let $0\le\theta\le 1$ and $1<r<\infty$. We observe from [16, Theorem 51.1 (iv)] that the operator Δ also generates a C^0 analytic semigroup in $X_\theta(r)$, which is denoted by $S(t)$. For $0<\alpha<1$ and $t>0$, we define

$$S_\alpha(t) : X_\theta(r) \to X_\theta(r),\ P_\alpha(t) : X_\theta(r) \to X_\theta(r)$$

by

$$\begin{cases} S_\alpha(t)w := \int_0^\infty \Phi_\alpha(\tau)S(\tau t^\alpha)wd\tau, \\ P_\alpha(t)w := \alpha t^{\alpha-1}\int_0^\infty \tau\Phi_\alpha(\tau)S(\tau t^\alpha)wd\tau \end{cases} \tag{1.7}$$

for $w \in X_\theta(r)$. Note that if $w \in L^r(\Omega)$ for some $1 \le r < \infty$, then we can also define in the same way as (1.7) (cf.[9]). Moreover, by definition the operator S_α is strongly continuous, i.e.,

$$\lim_{t\to 0} \|S_\alpha(t)w - w\|_X = 0 \text{ for } w \in X, \tag{1.8}$$

where $X = X_\theta(r)$, $0 \le \theta \le 1$ and $1 < r < \infty$, or $X = L^r(\Omega)$, $1 \le r < \infty$.

Definition 1.1 (*Mild solution*) By a mild solution of (1.1) on [0, *T*) we mean that the measurable functions (*u*, *v*) have the following properties:

(a) $u(t) = u(\cdot,t) \in L^1(\Omega)$ and $v(t) = v(\cdot,t) \in L^1(\Omega)$ for $t \in (0,T)$,
(b) $f_1(t,v(t)) = f_1(\cdot,t,v(\cdot,t)) \in L^1(\Omega)$ and $f_2(t,u(t)) = f_2(\cdot,t,u(\cdot,t)) \in L^1(\Omega)$ for almost all $t \in (0,T)$,
(c) $\int_0^t \|f_1(s,v(s))\|_{L^1(\Omega)}ds < \infty$ and $\int_0^t \|f_2(s,u(s))\|_{L^1(\Omega)}ds < \infty$ for $t \in (0,T)$,
(d) the functions (*u*, *v*) satisfy

$$u(t) = S_{\alpha_1}(t)u_0 + \int_0^t P_{\alpha_1}(t-s)f_1(s,v(s))ds \text{ in } \Omega \times (0,T),$$
$$v(t) = S_{\alpha_2}(t)v_0 + \int_0^t P_{\alpha_2}(t-s)f_2(s,u(s))ds \text{ in } \Omega \times (0,T),$$

where the integral terms are absolutely converging Bochner integrals in $L^1(\Omega)$,
(e) the initial functions (u_0, v_0) satisfy

$$\|u(t) - u_0\|_{L^{r_1}(\Omega)} \to 0 \quad (t \to 0) \text{ and } \|v(t) - v_0\|_{L^{r_2}(\Omega)} \to 0 \quad (t \to 0)$$

for $(u_0, v_0) \in L^{r_1}(\Omega) \times L^{r_2}(\Omega)$, if $1 \le r_1, r_2 < \infty$.

It follows from (1.8) and [9, Remark 3.1.2] that the property of Definition 1.1 (e) holds if and only if

$$\lim_{t\to 0} \|u(t) - S_{\alpha_1}(t)u_0\|_{L^{r_1}(\Omega)} = 0 \quad \text{and} \quad \lim_{t\to 0} \|v(t) - S_{\alpha_2}(t)v_0\|_{L^{r_2}(\Omega)} = 0, \tag{1.9}$$

which are equivalent to the convergence in the norms of the integral terms in Definition 1.1 (d) to 0.

We are ready to state our main results.

Theorem 1.2 (Local in time existence) *Let* $N \ge 1$, $0 < \alpha_1 \le \alpha_2 < 1$, $1 < p_1, p_2 < \infty$, $q_1, q_2 \in (1,\infty] \cap \left(\frac{N}{2}, \infty\right]$, $1 < r_1, r_2 < \infty$, $m_1 \in [0, \alpha_1)$ *and* $m_2 \in [0, \alpha_2)$. *Suppose that Assumption A holds. Put*

$$\mathcal{P} := \frac{N}{2}\left(\frac{p_1}{r_2} - \frac{1}{r_1} + \frac{1}{q_1}\right) + \frac{m_1}{\alpha_1}, \quad \mathcal{Q} := \frac{N}{2}\left(\frac{p_2}{r_1} - \frac{1}{r_2} + \frac{1}{q_2}\right) + \frac{m_2}{\alpha_2} \text{ and}$$
$$\mathcal{R} := \frac{N}{2}\left(\frac{p_1}{r_2} - \frac{1}{r_1} + \frac{1}{q_1}\right) + \frac{m_1}{\alpha_2} + \left(1 - \frac{\alpha_1}{\alpha_2}\right)\left\{1 - \frac{N}{2}\left(1 - \frac{1}{r_1}\right)\right\}.$$

Suppose that one of the following holds:

(a) $\max\{\mathcal{P}, \mathcal{Q}, \mathcal{R}\} < 1$,
(b) $\alpha_1 = \alpha_2$ *and* $\max\{\mathcal{P}, \mathcal{Q}, \mathcal{R}\} = 1$,
(c) $\alpha_1 < \alpha_2$ *and* $\max\{\mathcal{P}, \mathcal{Q}, \mathcal{R}\} = 1$, *where we also suppose that* $m_1 \in (0, \alpha_1)$ *when* $\mathcal{P} = 1$.

Then for any $(u_0, v_0) \in L^{r_1}(\Omega) \times L^{r_2}(\Omega)$, *there exist* $T > 0$ *and a unique local in time mild solution* (u, v) *of* (1.1) *in the sense of Definition* 1.1 *on the interval* $[0, T)$.

We also obtain the following nonexistence result.

Theorem 1.3 (Local in time nonexistence) *Let* $N \geq 1$, $0 < \alpha_1 \leq \alpha_2 < 1$, $0 < p_1, p_2 < \infty$, $q_1, q_2 \in [1, \infty]$, $1 \leq r_1, r_2 < \infty$, $m_1 \in [0, \alpha_1)$ *and* $m_2 \in [0, \alpha_2)$. *Put*

$$\mathcal{Q}_1 := \frac{N\alpha_1}{2\alpha_2}\left(\frac{p_2}{r_1} - \frac{1}{r_2} + \frac{1}{q_2}\right) + \frac{m_2}{\alpha_2} \text{ and } \mathcal{Q}_2 := \frac{N}{2}\left(\frac{\alpha_1 p_2}{\alpha_2 r_1} - \frac{1}{r_2} + \frac{1}{q_2}\right) + \frac{m_2}{\alpha_2}.$$

Put $\mathcal{P}$ *and* $\mathcal{R}$ *in the same way as in Theorem* 1.2. *Suppose that* $\max\{\mathcal{P}, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{R}\} > 1$. *Then there exist nonnegative functions* $(c_1, c_2, u_0, v_0) \in L^{q_1}(\Omega) \times L^{q_2}(\Omega) \times L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ *such that, for every* $T > 0$, *the problem* (1.1) *with* $f_1(x, t, v) = c_1(x) \cdot t^{-m_1} v^{p_1}$ *and* $f_2(x, t, u) = c_2(x) \cdot t^{-m_2} u^{p_2}$ *admits no local in time nonnegative mild solution* (u, v) *in the sense of Definition* 1.1 *on the interval* $[0, T)$.

If $1 < p_1, p_2 < \infty$, then the nonlinear terms $f_1(x, t, v) = c_1(x) \cdot t^{-m_1} v^{p_1}$ and $f_2(x, t, u) = c_2(x) \cdot t^{-m_2} u^{p_2}$ mentioned in Theorem 1.3 satisfy Assumption A with $\mathbb{R}$ replaced by $[0, \infty)$.

We deduce the following corollary from Theorems 1.2 and 1.3.

Corollary 1.4 *Let* $N \geq 1$ *and* $0 < \alpha_1 = \alpha_2 < 1$. *Then the following are true*:

(i) *Let* $1 < p_1, p_2 < \infty$, $q_1, q_2 \in (1, \infty] \cap \left(\frac{N}{2}, \infty\right]$, $1 < r_1, r_2 < \infty$, $m_1 \in [0, \alpha_1)$ *and* $m_2 \in [0, \alpha_2)$. *Suppose that Assumption A holds. Put* $\mathcal{P}$ *and* $\mathcal{Q}$ *in the same way as in Theorem* 1.2. *If* $\max\{\mathcal{P}, \mathcal{Q}\} \leq 1$, *then for any* $(u_0, v_0) \in L^{r_1}(\Omega) \times L^{r_2}(\Omega)$, *there exist* $T > 0$ *and a unique local in time mild solution* (u, v) *of* (1.1).

(ii) *Let* $0 < p_1, p_2 < \infty$, $q_1, q_2 \in [1, \infty]$, $1 \leq r_1, r_2 < \infty$, $m_1 \in [0, \alpha_1)$ *and* $m_2 \in [0, \alpha_2)$. *If* $\max\{\mathcal{P}, \mathcal{Q}\} > 1$, *then there exist nonnegative functions* $(c_1, c_2, u_0, v_0) \in L^{q_1}(\Omega) \times L^{q_2}(\Omega) \times L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ *such that, for every* $T > 0$, *the problem* (1.1) *with* $f_1(x, t, v) = c_1(x) \cdot t^{-m_1} v^{p_1}$ *and* $f_2(x, t, u) = c_2(x) \cdot t^{-m_2} u^{p_2}$ *admits no local in time nonnegative mild solution* (u, v).

Corollary 1.4 implies that when $\alpha_1 = \alpha_2$, we can *explicitly* determine the existence/nonexistence of a solution. Our conditions cover all the cases $(p_1, p_2, q_1, q_2, m_1, m_2)$ in Assumption A. Moreover, Corollary 1.4 leads to the following pure power case result, which corresponds to [15].

Corollary 1.5 *Let* $N \geq 1$ *and* $0 < \alpha_1 = \alpha_2 < 1$. *Then the following are true*:

(i) *Let* $1 < p_1, p_2 < \infty$ *and* $1 < r_1, r_2 < \infty$. *Put*

$$\widetilde{\mathcal{P}} := \frac{N}{2}\left(\frac{p_1}{r_2} - \frac{1}{r_1}\right) \text{ and } \widetilde{\mathcal{Q}} := \frac{N}{2}\left(\frac{p_2}{r_1} - \frac{1}{r_2}\right).$$

If $\max\{\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}}\} \leq 1$, *then for any* $(u_0, v_0) \in L^{r_1}(\Omega) \times L^{r_2}(\Omega)$, *there exist* $T > 0$ *and a unique local in time mild solution* (u, v) *of* (1.1) *with* $f_1(x,t,v) = |v|^{p_1-1}v$ *and* $f_2(x,t,u) = |u|^{p_2-1}u$.

(ii) *Let* $0 < p_1, p_2 < \infty$ *and* $1 \leq r_1, r_2 < \infty$. *If* $\max\{\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}}\} > 1$, *then there are nonnegative functions* $(u_0, v_0) \in L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ *such that, the problem* (1.1) *with* $f_1(x,t,v) = v^{p_1}$ *and* $f_2(x,t,u) = u^{p_2}$ *has no local in time nonnegative mild solution on any time interval.*

Let us recall fundamental properties of scalar problems. Fractional in time parabolic equations with nonlinear terms have not been well studied until recently. Gal–Warma [9] has studied the fractional in time scalar problem

$$\begin{cases} \partial_t^\alpha u = Au + f(x,t,u) & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.10}$$

where A is a differential operator which generates a strongly continuous semigroup on $L^2(\Omega)$. Detailed results can be found in [1, 3, 8, 12]. Let $1 \leq p < \infty$ and $q_1, q_2 \in [1, \infty]$ be given constants. In [9] the authors assumed that there exists a nonnegative function $c \in L_{q_1,q_2}$ such that the following hold:

(F1') $f(x,t,\cdot) : \mathbb{R} \to \mathbb{R}$ is a measurable function such that

$$|f(x,t,\xi)| \leq c(x,t)\ (1 + |\xi|)^p \text{ for } \xi \in \mathbb{R},\ \text{a.e.}(x,t) \in \Omega \times (0,\infty),$$

(F2') f satisfies the local Lipschitz condition

$$|f(x,t,\xi) - f(x,t,\eta)| \leq c(x,t)\ (1 + |\xi| + |\eta|)^{p-1}|\xi - \eta| \text{ for } \xi, \eta \in \mathbb{R}, \\ \text{a.e. } (x,t) \in \Omega \times (0,\infty).$$

Here L_{q_1,q_2} denotes the Banach space defined by

$$L_{q_1,q_2} := \left\{ c : \Omega \times (0,\infty) \to \mathbb{R} \ \text{ measurable}, \ \|c\|_{L_{q_1,q_2}} := \sup_{\substack{t_1,t_2 \in (0,\infty), \\ 0 \leq t_2 - t_1 \leq 1}} \left(\int_{t_1}^{t_2} \|c(\cdot,s)\|_{L^{q_1}(\Omega)}^{q_2} ds \right)^{\frac{1}{q_2}} < \infty \right\}$$

for $q_1 \in [1,\infty]$ and $q_2 \in [1,\infty)$ with the obvious modifications when $q_2 = \infty$. Proposition 2.2.2 of [9, p.21, 22] states the following existence result. Let $0 < \alpha < 1$, $1 \leq p < \infty$, $q_1 \in [1,\infty] \cap (\beta_A, \infty]$ and $q_2 \in \left(\frac{1}{\alpha}, \infty\right]$. The constant β_A is related to the L^p-L^q estimate of the semigroup generated by A. Assume **(F1')** and **(F2')**. If

$$\beta_A\left(\frac{p-1}{r}+\frac{1}{q_1}\right)+\frac{1}{\alpha q_2}<1 \text{ and } 1\leq p,r<\infty, \text{ or}$$
$$\beta_A\left(\frac{p-1}{r}+\frac{1}{q_1}\right)+\frac{1}{\alpha q_2}=1 \text{ and } 1<p,r<\infty,$$

then for any $u_0\in L^r(\Omega)$, there exist $T>0$ and a unique local in time solution of (1.10). Note that if Ω is a bounded domain in $\mathbb{R}^N$, $N\geq 1$, with C^2 boundary, $A=\Delta$ and $f(x,t,u)=|u|^{p-1}u$, $p>1$, then $\beta_A=\frac{N}{2}$, $q_1=q_2=\infty$ and hence this result corresponds to the existence part in [20].

In [9, Remark 5.0.2], based on [20, 21], they conjectured the nonexistence of a local in time solution of (1.10) in the super-critical case

$$\beta_A\left(\frac{p-1}{r}+\frac{1}{q_1}\right)+\frac{1}{\alpha q_2}>1. \tag{1.11}$$

We give an affirmative answer to the conjecture when $A=\Delta$. Sect. 5 is devoted to this nonexistence result.

Let us explain a sketch of the proofs. The main points of the proofs are Cauchy sequences for the existence part, including

$$u_n(t)=S_{\alpha_1}(t)u_0+\int_0^t P_{\alpha_1}(t-s)f_1(s,v_{n-1}(s))ds, \tag{1.12}$$

$$v_n(t)=S_{\alpha_2}(t)v_0+\int_0^t P_{\alpha_2}(t-s)f_2(s,u_{n-1}(s))ds \tag{1.13}$$

for $n\geq 2$, $u_1=v_1=0$, and the contradiction argument for the nonexistence part.

For the existence part by induction method we can show that if $T>0$ is sufficiently small, then $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are Cauchy sequences. Then limits u and v of the sequences exist and (u, v) is a mild solution of (1.1) in the sense of Definition 1.1. In order to show that these are indeed Cauchy sequences, it is crucial to find various exponents including θ_1, θ_2 and θ_3. However, it is not obvious how to find these exponents. Section 3 addresses this aspect.

For the nonexistence part we construct initial data (u_0,v_0). Assume that (1.1) has a local in time nonnegative mild solution (u, v). Since the singularity of the constructed functions are strong, the norm of at least one integral term in Definition 1.1 (d) diverges as $t\to 0$, which follows from estimates of $S_\alpha(t)$ and $P_\alpha(t)$. This is a contradiction. It is known (cf. [16, p.440]) that there exists a positive C^∞-function $G_\Omega:\Omega\times\Omega\times(0,\infty)\to\mathbb{R}$ (Dirichlet heat kernel) such that

$$(S(t)\phi)(x)=\int_\Omega G_\Omega(x,y,t)\phi(y)dy$$

for $\phi\in L^r(\Omega)$, $1\leq r\leq\infty$. After that, we abbreviate G_Ω as G. Since G has a lower bound with respect to t, we obtain estimates of $S_\alpha(t)$ and $P_\alpha(t)$ from (1.7).

This paper is organized as follows. In Sect. 2 we give and recall some properties of $S_\alpha(t)$, $P_\alpha(t)$ and the Dirichlet heat kernel. In Sects. 3 and 4 we use these properties and prove Theorems 1.2 and 1.3, respectively. In Sect. 5 we give a nonexistence result for scalar problems. In Sect. 6 we discuss our results and explain possible future problems ensuing from the current analysis.

2 Preliminaries

For any set $\mathcal{X}$ and the mappings $a = a(x)$ and $b = b(x)$ from $\mathcal{X}$ to $[0, \infty)$, we say

$$a(x) \lesssim b(x) \text{ for all } x \in \mathcal{X}$$

if there exists a positive constant C such that $a(x) \leq Cb(x)$ for all $x \in \mathcal{X}$.

Proposition 2.1 *Let $0 < \alpha < 1$, $1 < r < \infty$ and $-1 \leq \theta_1 \leq \theta_2 \leq 1$. Then the following are true:*

(i) If $\theta_2 - \theta_1 < 1$, then there exists $C > 0$ such that

$$|S_\alpha(t)w|_{X_{\theta_2}(r)} \leq Ct^{\alpha(\theta_1 - \theta_2)}|w|_{X_{\theta_1}(r)}$$

for $t > 0$ and $w \in X_{\theta_1}(r)$.

(ii) If $\theta_1 > -1$ or $\theta_2 < 1$, then there exists $C > 0$ such that

$$|t^{1-\alpha}P_\alpha(t)w|_{X_{\theta_2}(r)} \leq Ct^{\alpha(\theta_1 - \theta_2)}|w|_{X_{\theta_1}(r)}$$

for $t > 0$ and $w \in X_{\theta_1}(r)$.

Proof We prove (i) in a similar manner to [9, Proposition 2.2.2]. Using (1.5), (1.7) and [15, Theorem 51.1 (iv)], we have

$$\begin{aligned}
|S_\alpha(t)w|_{X_{\theta_2}(r)} &\leq \int_0^\infty \Phi_\alpha(\tau)|S(\tau t^\alpha)w|_{X_{\theta_2}(r)}d\tau \\
&\lesssim \int_0^\infty \Phi_\alpha(\tau)\tau^{\theta_1-\theta_2}t^{\alpha(\theta_1-\theta_2)}|w|_{X_{\theta_1}(r)}d\tau \\
&= t^{\alpha(\theta_1-\theta_2)}|w|_{X_{\theta_1}(r)}\int_0^\infty \Phi_\alpha(\tau)\tau^{\theta_1-\theta_2}d\tau \\
&= \frac{\Gamma(1+\theta_1-\theta_2)}{\Gamma(1+\alpha(\theta_1-\theta_2))}t^{\alpha(\theta_1-\theta_2)}|w|_{X_{\theta_1}(r)}.
\end{aligned}$$

We can obtain the assertion (ii) in the same way as the assertion (i). □

Lemma 2.2 *Let $0 < \alpha_1 \leq \alpha_2 < 1$ and $0 < T < \infty$. For $i = 1, 2$, let $0 < \theta_i < 1$, $1 < r_i < \infty$ and $\Pi_i \subset X_0(r_i)$ $(= L^{r_i}(\Omega))$. Suppose that for $i = 1, 2$,*

$$\kappa(\Pi_i) := \left\{u|u|^{-1}_{X_0(r_i)} : u \in \Pi_i,\ u \neq 0\right\}$$

is precompact in $X_0(r_i)$. Then there exists a continuous and nondecreasing function $g : (0, T) \to (0, \infty)$, depending on α_i, θ_i, r_i and Π_i $(i = 1, 2)$ such that the following are true:

(i) For $i = 1, 2$, the following is true:

$$|S_{\alpha_i}(t)u|_{X_{\theta_i}(r_i)} \lesssim g(t) \cdot t^{-\alpha_2\theta_i}|u|_{X_0(r_i)} \text{ for } 0 < t < T \text{ and } u \in \Pi_i.$$

(ii) We have $\lim_{t\to 0} g(t) = 0$. For $i = 1, 2$, the function $w_i = w_i(t)$ defined by

$$(w_i(t))^{-\alpha_2\theta_i} = g(t) \cdot t^{-\alpha_2\theta_i}$$

has the properties

$$\lim_{t\to 0} w_i(t) = 0 \text{ and } \min\left\{t, t^{1-\frac{\min\{\theta_1,\theta_2\}}{2\theta_i}}\right\} \le w_i(t) \le t^{1-\frac{\min\{\theta_1,\theta_2\}}{2\max\{\theta_1,\theta_2\}}}.$$

Proof Define

$$\begin{cases} h_1(t,u) := |S_{\alpha_1}(t)u|_{X_{\theta_1}(r_1)} C_0^{-1} t^{\alpha_1\theta_1} |u|_{X_0(r_1)}^{-1} & \text{for } (t,u) \in (0,T)\times \Pi_1\backslash\{0\}, \\ \overline{h_1}(t) := \sup\{h_1(s,u) : s \in (0,t],\ u \in \Pi_1\backslash\{0\}\} & \text{for } t \in (0,T), \end{cases}$$

where $C_0(= C) > 0$ is the constant from Proposition 2.1 (i). We observe from [9, Lemma A.0.2] that $0 \le \overline{h_1} \le 1$, $\lim_{t\to 0}\overline{h_1}(t) = 0$ and

$$\begin{aligned} |S_{\alpha_1}(t)u|_{X_{\theta_1}(r_1)} &\le C_0\overline{h_1}(t)\cdot t^{-\alpha_1\theta_1}|u|_{X_0(r_1)} \\ &\lesssim C_0\overline{h_1}(t)\cdot t^{-\alpha_2\theta_1}|u|_{X_0(r_1)} \text{ for } 0<t<T \text{ and } u \in \Pi_1. \end{aligned}$$

Put $\overline{h_2}$ in the same way. We set

$$g(t) := \max\{\overline{h_1}(t), \overline{h_2}(t), t^{\delta\alpha_2\theta_1}, t^{\delta\alpha_2\theta_2}\},$$

where $\delta := \dfrac{\min\{\theta_1,\theta_2\}}{2\max\{\theta_1,\theta_2\}}$.

It suffices to prove the estimates of $w_i(t)$ for $i = 1, 2$. Since $g(t) \ge t^{\delta\alpha_2\theta_1}$, we obtain

$$w_1(t) = g(t)^{-\frac{1}{\alpha_2\theta_1}}\cdot t \le t^{1-\delta} \text{ for } t > 0.$$

On the other hand, let $0<t<\min\{T,1\}$. Due to $\overline{h_1}, \overline{h_2} \le 1$, we have $g(t) \le 1$ and hence $w_1(t) \ge t$. If $t \ge 1$, then $g(t) = t^{\delta\alpha_2\max\{\theta_1,\theta_2\}}$. Thus it follows that

$$w_1(t) = t^{1-\frac{\delta\max\{\theta_1,\theta_2\}}{\theta_1}} = t^{1-\frac{\min\{\theta_1,\theta_2\}}{2\theta_1}}.$$

Therefore, we deduce the desired estimate of $w_1(t)$. We can obtain the estimate of $w_2(t)$ in the same way. □

Proposition 2.3 ([16, Proposition 49.10]) *Let $N \ge 1$ and Ω be an arbitrary domain in $\mathbb{R}^N$. There exist constants $c_1 > 0$ and $c_2 \ge 2$ depending only on N, such that the Dirichlet heat kernel $G(x, y, t)$ in Ω satisfies*

$$G(x,y,t) \ge c_1 t^{-\frac{N}{2}}$$

for $t > 0$ and $x, y \in \Omega$ such that

$$dist(x, \partial\Omega) \ge c_2\sqrt{t} \text{ and } |x-y| \le \sqrt{t}.$$

3 Existence result

Proposition 3.1 *Let $N \ge 1$, $0<\alpha_1 \le \alpha_2 < 1$, $1<p_1,p_2<\infty$, $q_1, q_2 \in (1,\infty] \cap \left(\frac{N}{2},\infty\right]$, $1<r_1,r_2<\infty$, $m_1 \in [0,\alpha_1)$ and $m_2 \in [0,\alpha_2)$. Put $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ in the same way as in*

Theorem 1.2. *If* $\max\{\mathcal{P},\mathcal{Q},\mathcal{R}\}<1$, *then there exist* (s_1,θ_2,z_1,z_2), $(s_2,\theta_1,z_3,z_4)\in(0,1)\times(0,1)\times(1,\infty)\times(1,\infty)$ *such that*

$$\frac{1}{z_1}-\frac{2}{N}(1-s_1)=\frac{1}{r_1},\ \frac{1}{q_1}+\frac{p_1}{z_2}=\frac{1}{z_1}\ \text{and}\ \frac{1}{r_2}-\frac{2}{N}\theta_2\le\frac{1}{z_2},\tag{3.1}$$

$$\alpha_1 s_1-m_1-p_1\alpha_2\theta_2+(\alpha_2-\alpha_1)\theta_1>0,\tag{3.2}$$

$$m_1+p_1\alpha_2\theta_2<1,\tag{3.3}$$

$$\theta_1<s_1,\tag{3.4}$$

$$\frac{1}{z_3}-\frac{2}{N}(1-s_2)=\frac{1}{r_2},\ \frac{1}{q_2}+\frac{p_2}{z_4}=\frac{1}{z_3}\ \text{and}\ \frac{1}{r_1}-\frac{2}{N}\theta_1\le\frac{1}{z_4},\tag{3.5}$$

$$\alpha_2 s_2-m_2-p_2\alpha_2\theta_1>0,\tag{3.6}$$

$$\theta_2<s_2.\tag{3.7}$$

Moreover, if $\max\{\mathcal{P},\mathcal{Q},\mathcal{R}\}=1$, *then there exist* (s_1,θ_2,z_1,z_2), $(s_2,\theta_1,z_3,z_4)\in(0,1)\times(0,1)\times(1,\infty)\times(1,\infty)$ *such that* (3.1)–(3.7) *hold with* (3.2) *and* (3.6) *replaced by* $\alpha_1 s_1-m_1-p_1\alpha_2\theta_2+(\alpha_2-\alpha_1)\theta_1\ge 0$ *and* $\alpha_2 s_2-m_2-p_2\alpha_2\theta_1\ge 0$, *respectively.*

Proof Suppose that $\max\{\mathcal{P},\mathcal{Q},\mathcal{R}\}<1$. For $1-\frac{N}{2}\left(1-\frac{1}{r_1}\right)<s<\min\left\{1,1-\frac{N}{2}\left(\frac{1}{q_1}-\frac{1}{r_1}\right)\right\}$, put

$$\begin{aligned}\theta_2(s)&:=\frac{N}{2p_1}\left(\frac{p_1}{r_2}-\frac{1}{r_1}+\frac{1}{q_1}\right)-\frac{1}{p_1}(1-s),\\ \frac{1}{z_1(s)}&:=\frac{2}{N}(1-s)+\frac{1}{r_1}\ \text{and}\\ \frac{1}{z_2(s)}&:=\frac{1}{p_1}\left\{\frac{1}{r_1}-\frac{1}{q_1}+\frac{2}{N}(1-s)\right\}.\end{aligned}$$

We see that

$$\max\left\{\frac{1}{q_1},\frac{1}{r_1}\right\}<\frac{1}{z_1(s)}<1\ \text{and}\ \max\left\{\frac{1}{p_1}\left(\frac{1}{r_1}-\frac{1}{q_1}\right),0\right\}<\frac{1}{z_2(s)}<\frac{1}{p_1}\left(1-\frac{1}{q_1}\right).$$

Hence, $z_1(s)\in(1,\infty)$ and $z_2(s)\in(1,\infty)$. Let $(s_1,\theta_2,z_1,z_2)=(s,\theta_2(s),z_1(s),z_2(s))$. By direct calculation we have (3.1). In the same way we derive (3.5) with $(s_2,\theta_1,z_3,z_4)=(\tilde{s},\theta_1(\tilde{s}),z_3(\tilde{s}),z_4(\tilde{s}))$, where

$$\theta_1(\tilde{s}) := \frac{N}{2p_2}\left(\frac{p_2}{r_1} - \frac{1}{r_2} + \frac{1}{q_2}\right) - \frac{1}{p_2}(1-\tilde{s}),$$
$$\frac{1}{z_3(\tilde{s})} := \frac{2}{N}(1-\tilde{s}) + \frac{1}{r_2} \text{ and}$$
$$\frac{1}{z_4(\tilde{s})} := \frac{1}{p_2}\left\{\frac{1}{r_2} - \frac{1}{q_2} + \frac{2}{N}(1-\tilde{s})\right\}$$

for $1 - \frac{N}{2}\left(1 - \frac{1}{r_2}\right) < \tilde{s} < \min\left\{1, 1 - \frac{N}{2}\left(\frac{1}{q_2} - \frac{1}{r_2}\right)\right\}$. Moreover, we see that (3.3) and (3.6) follow from $\mathcal{P} < 1$ and $\mathcal{Q} < 1$, respectively.

Substituting $(s_1, \theta_2, s_2, \theta_1) = (s, \theta_2(s), \tilde{s}, \theta_1(\tilde{s}))$ into (3.4) and (3.7), we obtain

$$\frac{N}{2p_1}\left(\frac{p_1}{r_2} - \frac{1}{r_1} + \frac{1}{q_1}\right) - \frac{1}{p_1}(1-s) < \tilde{s}, \tag{3.8}$$

$$\frac{N}{2p_2}\left(\frac{p_2}{r_1} - \frac{1}{r_2} + \frac{1}{q_2}\right) - \frac{1}{p_2}(1-\tilde{s}) < s. \tag{3.9}$$

We show that (3.8) holds with $(s, \tilde{s}) = \left(\max\left\{1 - \frac{N}{2}\left(1 - \frac{1}{r_1}\right), \frac{m_1}{\alpha_1}\right\}, \min\left\{1, 1 - \frac{N}{2}\left(\frac{1}{q_2} - \frac{1}{r_2}\right)\right\}\right)$. Since $\mathcal{P} < 1$, we have

$$\frac{N}{2p_1}\left(\frac{p_1}{r_2} - 1 + \frac{1}{q_1}\right) < \frac{N}{2p_1}\left(\frac{p_1}{r_2} - \frac{1}{r_1} + \frac{1}{q_1}\right) < \frac{1}{p_1}\left(1 - \frac{m_1}{\alpha_1}\right) < 1.$$

Moreover, it follows from $q_1, q_2 \in (1, \infty] \cap \left(\frac{N}{2}, \infty\right]$ that $\frac{N}{2p_1}\left(\frac{p_1}{r_2} - 1 + \frac{1}{q_1}\right) < \frac{N}{2r_2} < 1 - \frac{N}{2}\left(\frac{1}{q_2} - \frac{1}{r_2}\right)$. Thus (3.8) holds when $s = 1 - \frac{N}{2}\left(1 - \frac{1}{r_1}\right)$. Since $\mathcal{P} < 1$, the left hand side of (3.8) is negative when $s = \frac{m_1}{\alpha_1}$. Therefore, (3.8) holds with $(s, \tilde{s}) = \left(\max\left\{1 - \frac{N}{2}\left(1 - \frac{1}{r_1}\right), \frac{m_1}{\alpha_1}\right\}, \min\left\{1, 1 - \frac{N}{2}\left(\frac{1}{q_2} - \frac{1}{r_2}\right)\right\}\right)$. In the same way (3.9) holds with $(s, \tilde{s}) = \left(\min\left\{1, 1 - \frac{N}{2}\left(\frac{1}{q_1} - \frac{1}{r_1}\right)\right\}, \max\left\{1 - \frac{N}{2}\left(1 - \frac{1}{r_2}\right), \frac{m_2}{\alpha_2}\right\}\right)$.

We divide the possibilities into two cases: $\alpha_1 = \alpha_2$ and $\alpha_1 < \alpha_2$.

Let $\alpha_1 = \alpha_2$. Since $\mathcal{P} < 1$, we obtain (3.2). Put

$$I_1 := \left(\max\left\{1 - \frac{N}{2}\left(1 - \frac{1}{r_1}\right), \frac{m_1}{\alpha_1}\right\}, \min\left\{1, 1 - \frac{N}{2}\left(\frac{1}{q_1} - \frac{1}{r_1}\right)\right\}\right),$$
$$I_2 := \left(\max\left\{1 - \frac{N}{2}\left(1 - \frac{1}{r_2}\right), \frac{m_2}{\alpha_2}\right\}, \min\left\{1, 1 - \frac{N}{2}\left(\frac{1}{q_2} - \frac{1}{r_2}\right)\right\}\right) \text{ and } R := I_1 \times I_2.$$

The area of $(s, \tilde{s})$ satisfying (3.8) and (3.9) is the shaded portion on the graphs. See Fig. 1. Due to the above calculation results, the position of R is as shown in the left figure, but not as shown in the right one. Thus there exists $(s, \tilde{s}) \in R$ such that (3.8) and (3.9) hold.

It remains to consider θ_1 and θ_2. We observe from (3.8) and (3.9) that $\theta_2(s) < 1$ and $\theta_1(\tilde{s}) < 1$. If $\theta_2(s) \leq 0$, then we replace $\theta_2 = \theta_2(s)$ with $\theta_2 = \varepsilon > 0$, where ε is sufficiently

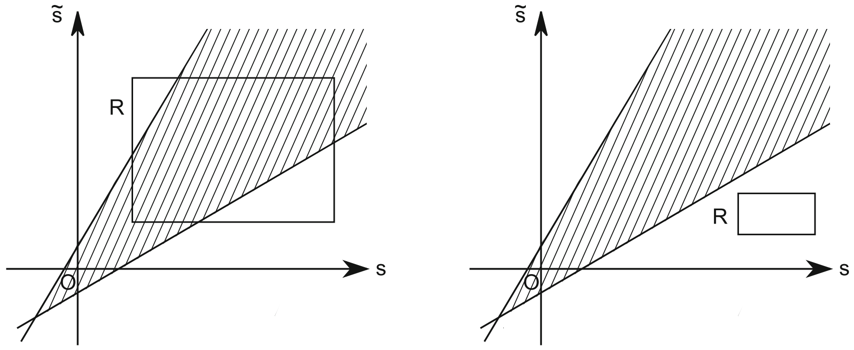

Fig. 1 Possible relative positions of the rectangle R and sector defined by (3.8)–(3.9)

small such that $\alpha_1 s_1 - m_1 - p_1\alpha_2\varepsilon > 0$ and $\varepsilon < \tilde{s}$. We see that (3.1), (3.2), (3.3) and (3.7) hold. If $\theta_1(\tilde{s}) \le 0$, then we can replace $\theta_1(\tilde{s})$ with $\theta_1 \in (0,1)$ in the same way.

Let $\alpha_1 < \alpha_2$. We recall that

$$(s_1, \theta_2, z_1, z_2) = (s, \theta_2(s), z_1(s), z_2(s)) \text{ and } (s_2, \theta_1, z_3, z_4) = (\tilde{s}, \theta_1(\tilde{s}), z_3(\tilde{s}), z_4(\tilde{s})).$$

We divide the possibilities into two cases: (i) $\frac{p_2}{r_1} - \frac{1}{r_2} + \frac{1}{q_2} > 0$ and (ii) $\frac{p_2}{r_1} - \frac{1}{r_2} + \frac{1}{q_2} \le 0$.

We prove (i). It follows that (3.1), (3.3), (3.5) and (3.6) hold. The inequality (3.2) is equivalent to

$$\frac{N}{2p_2}\left(\frac{p_2}{r_1} - \frac{1}{r_2} + \frac{1}{q_2}\right) - \frac{1}{p_2}(1-\tilde{s}) > s + \frac{\frac{N}{2}\left(\frac{p_1}{r_2} - \frac{1}{r_1} + \frac{1}{q_1}\right) + \frac{m_1}{\alpha_2} - 1}{1 - \frac{\alpha_1}{\alpha_2}}, \tag{3.10}$$

which holds with $(s, \tilde{s}) = \left(\max\left\{1 - \frac{N}{2}\left(1 - \frac{1}{r_1}\right), \frac{m_1}{\alpha_1}\right\}, \min\left\{1, 1 - \frac{N}{2}\left(\frac{1}{q_2} - \frac{1}{r_2}\right)\right\}\right)$. Indeed, the right hand side of (3.10) is negative when $s = \frac{m_1}{\alpha_1}$ (resp. when $s = 1 - \frac{N}{2}\left(1 - \frac{1}{r_1}\right)$), since $\mathcal{P} < 1$ (resp. since $\mathcal{R} < 1$). On the other hand, since $\frac{p_2}{r_1} - \frac{1}{r_2} + \frac{1}{q_2} > 0$, the left hand side of (3.10) is positive.

The area of $(s, \tilde{s})$ satisfying (3.8), (3.9) and (3.10) is the shaded portion on the graphs. See Fig. 2. Here, R is defined in the same way as when $\alpha_1 = \alpha_2$. Due to the above calculation results, the position of R is as shown in the left figure, but not as shown in the right one. Thus there exists $(s, \tilde{s}) \in R$ such that (3.8), (3.9) and (3.10) hold.

It remains to consider θ_1 and θ_2. We observe from (3.8) and (3.9) that $\theta_2(s) < 1$ and $\theta_1(\tilde{s}) < 1$. If $\theta_2(s) \le 0$ or $\theta_1(\tilde{s}) \le 0$, then it is sufficient to replace it/them in the same way as when $\alpha_1 = \alpha_2$.

We prove (ii). Let $\varepsilon > 0$ be sufficiently small. Put

$$s := \max\left\{1 - \frac{N}{2}\left(1 - \frac{1}{r_1}\right), \frac{m_1}{\alpha_1}\right\} + \varepsilon \in I_1 \quad \text{and} \quad \tilde{s} := \min\left\{1, 1 - \frac{N}{2}\left(\frac{1}{q_2} - \frac{1}{r_2}\right)\right\} - \varepsilon \in I_2$$

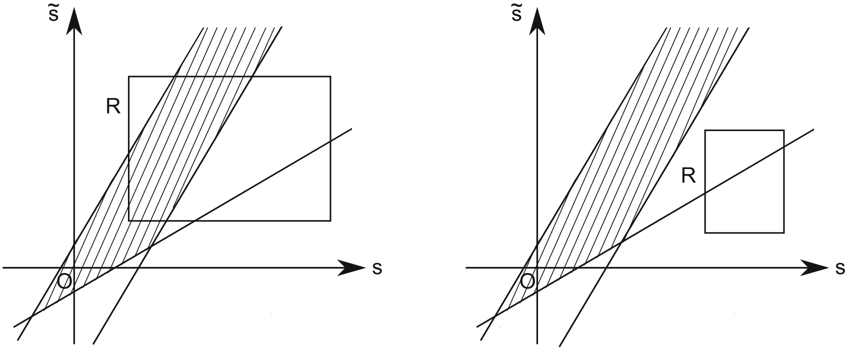

Fig. 2 Possible relative positions of the rectangle R and the strip defined by (3.8)–(3.10)

Due to $\theta_1(\tilde{s})<0$ in this case, we replace $\theta_1=\theta_1(\tilde{s})$ with $\theta_1=\varepsilon$. It follows that (3.1), (3.3) and (3.5) hold. Since $\theta_1=\varepsilon$ is sufficiently small, $s\geq\varepsilon>0$ and $\tilde{s}>\dfrac{m_2}{\alpha_2}$, we obtain (3.4) and (3.6). The inequality (3.2) is equivalent to (3.10) with the left hand side replaced by ε. Then by $\max\{\mathcal{P},\mathcal{R}\}<1$ we deduce (3.2). Moreover, since (3.8) holds with $(s,\tilde{s})=\left(\max\left\{1-\dfrac{N}{2}\left(1-\dfrac{1}{r_1}\right),\dfrac{m_1}{\alpha_1}\right\},\min\left\{1,1-\dfrac{N}{2}\left(\dfrac{1}{q_2}-\dfrac{1}{r_2}\right)\right\}\right)$, (3.8) also holds with $(s,\tilde{s})=\left(\max\left\{1-\dfrac{N}{2}\left(1-\dfrac{1}{r_1}\right),\dfrac{m_1}{\alpha_1}\right\}+\varepsilon,\min\left\{1,1-\dfrac{N}{2}\left(\dfrac{1}{q_2}-\dfrac{1}{r_2}\right)\right\}-\varepsilon\right)$.

It remains to consider θ_2. We observe from (3.8) that $\theta_2(s)<1$. If $\theta_2(s)\leq 0$, then it is sufficient to replace it in the same way as when $\alpha_1=\alpha_2$.

Since we can prove the case where $\max\{\mathcal{P},\mathcal{Q},\mathcal{R}\}=1$ in a similar way, we omit the proof. □

Lemma 3.2 *Suppose that* $\alpha_1<\alpha_2$ *and* $\max\{\mathcal{P},\mathcal{Q},\mathcal{R}\}<1$ *hold in particular in the assumptions of Proposition* 3.1. *Let* $s_1\in(0,1)$ *be chosen in Proposition* 3.1. *Put*

$$\mathcal{S}:=\frac{1-\frac{N}{2}\left(\frac{p_1}{r_2}-\frac{1}{r_1}+\frac{1}{q_1}\right)-\frac{m_1}{\alpha_2}}{1-\frac{\alpha_1}{\alpha_2}}.$$

If $s_1\geq\mathcal{S}$, *then the following are true*:

- *(i)* *There exists* $s_3\in(0,1)$ *such that* $1-\dfrac{N}{2}\left(1-\dfrac{1}{r_1}\right)<s_3<\mathcal{S}$ *and that* $\theta_3:=\theta_2(s_3)>0$, *where* $\theta_2(s)$ *is defined in the proof of Proposition* 3.1.
- *(ii)* $\theta_2(s_1)>0$.

Proof By direct calculation we have $p_1\alpha_2\theta_2(\mathcal{S})=\alpha_1\mathcal{S}-m_1$. It follows from $\mathcal{P}<1$ that $\mathcal{S}>\dfrac{m_1}{\alpha_1}$. Then $\theta_2(\mathcal{S})>0$ holds. Note that $\mathcal{R}<1$ yields $1-\dfrac{N}{2}\left(1-\dfrac{1}{r_1}\right)<\mathcal{S}$. Choosing $\max\left\{1-\dfrac{N}{2}\left(1-\dfrac{1}{r_1}\right),\dfrac{m_1}{\alpha_1}\right\}<s_3<\mathcal{S}$ sufficiently large, we obtain $\theta_2(s_3)>0$. Moreover, we see that

$$\max\left\{1-\frac{N}{2}\left(1-\frac{1}{r_1}\right),\frac{m_1}{\alpha_1}\right\}<s_3<\mathcal{S}\le s_1<1.$$

Thus $s_3\in(0,1)$ holds. Since $\theta_2(s)$ is increasing with respect to s, we have $\theta_2(s_1)>\theta_2(s_3)>0$. The proof is complete. □

Proof of Theorem 1.2 Set $u_1:=0$ and $v_1:=0$. For $n\ge 2$, define the functions u_n and v_n by (1.12) and (1.13), respectively. We introduce the Banach space defined by

$$Y_{\theta,r,T}:=\left\{u\in L^\infty_{loc}((0,T],X_\theta(r)) : \|u\|_{Y_{\theta,r,T}}:=\sup_{t\in(0,T)}t^{\alpha_2\theta}|u(t)|_{X_\theta(r)}<\infty\right\}$$

for $0<\theta<1$, $1<r<\infty$ and $T>0$.

We prove the case (a). Choose (s_1,θ_2,z_1,z_2) and (s_2,θ_1,z_3,z_4) as in Proposition 3.1.

We consider the existence part. Let $T>0$. Assume that $(u_{n-1},v_{n-1})\in Y_{\theta_1,r_1,T}\times Y_{\theta_2,r_2,T}$. We obtain from Proposition 2.1 that for $0<t<T$,

$$\begin{aligned}t^{\alpha_2\theta_1}|u_n|_{X_{\theta_1}(r_1)}\le{}& t^{\alpha_2\theta_1}|S_{\alpha_1}(t)u_0|_{X_{\theta_1}(r_1)}+t^{\alpha_2\theta_1}\int_0^t|P_{\alpha_1}(t-s)f_1(s,v_{n-1}(s))|_{X_{\theta_1}(r_1)}ds\\ \lesssim{}& |u_0|_{X_0(r_1)}+t^{\alpha_2\theta_1}\int_0^t(t-s)^{\alpha_1(s_1-\theta_1)-1}|f_1(s,v_{n-1}(s))|_{X_{s_1-1}(r_1)}ds.\end{aligned}\tag{3.11}$$

It follows from (1.6) and the first equality of (3.1) that $X_{1-s_1}(r_1')\hookrightarrow L^{z_1'}(\Omega)$, which implies that $L^{z_1}(\Omega)\hookrightarrow X_{s_1-1}(r_1)$. By the last inequality of (3.1) we have $X_{\theta_2}(r_2)\hookrightarrow L^{z_2}(\Omega)$. Then we can deduce from Assumption A and (3.1) that for $0<s<t$,

$$\begin{aligned}|f_1(s,v_{n-1}(s))|_{X_{s_1-1}(r_1)}\lesssim{}& \|f_1(s,v_{n-1}(s))\|_{L^{z_1}(\Omega)}\le\|c_1\cdot s^{-m_1}(1+|v_{n-1}(s)|)^{p_1}\|_{L^{z_1}(\Omega)}\\ \le{}& s^{-m_1}\cdot\|c_1\|_{L^{q_1}(\Omega)}\|(1+|v_{n-1}(s)|)^{p_1}\|_{L^{\frac{z_2}{p_1}}(\Omega)}\\ ={}& s^{-m_1}\cdot\|c_1\|_{L^{q_1}(\Omega)}\|1+|v_{n-1}(s)|\|^{p_1}_{L^{z_2}(\Omega)}\\ \lesssim{}& s^{-m_1}\cdot\|c_1\|_{L^{q_1}(\Omega)}|1+|v_{n-1}(s)||^{p_1}_{X_{\theta_2}(r_2)}\\ \le{}& s^{-m_1-p_1\alpha_2\theta_2}\cdot\|c_1\|_{L^{q_1}(\Omega)}\|1+|v_{n-1}|\|^{p_1}_{Y_{\theta_2,r_2,T}},\end{aligned}\tag{3.12}$$

which yields

$$\begin{aligned}&t^{\alpha_2\theta_1}\int_0^t(t-s)^{\alpha_1(s_1-\theta_1)-1}|f_1(s,v_{n-1}(s))|_{X_{s_1-1}(r_1)}ds\\ &\lesssim t^{\alpha_2\theta_1}\int_0^t(t-s)^{\alpha_1(s_1-\theta_1)-1}s^{-m_1-p_1\alpha_2\theta_2}\cdot\|c_1\|_{L^{q_1}(\Omega)}\|1+|v_{n-1}|\|^{p_1}_{Y_{\theta_2,r_2,T}}ds\\ &\le\|c_1\|_{L^{q_1}(\Omega)}\|1+|v_{n-1}|\|^{p_1}_{Y_{\theta_2,r_2,T}}\cdot t^{\alpha_1s_1-m_1-p_1\alpha_2\theta_2+(\alpha_2-\alpha_1)\theta_1}\\ &\quad\times\int_0^1(1-s)^{\alpha_1(s_1-\theta_1)-1}s^{-m_1-p_1\alpha_2\theta_2}ds\end{aligned}\tag{3.13}$$

for $0<t<T$. Here we use $\alpha_1(s_1-\theta_1)-1>-1$ and $0<m_1+p_1\alpha_2\theta_2<1$, which follow from (3.4) and (3.3), respectively. Combining (3.2), (3.11) and (3.13), we have $u_n\in Y_{\theta_1,r_1,T}$. We can obtain $v_n\in Y_{\theta_2,r_2,T}$ in the same way. By induction we derive $(u_n,v_n)\in Y_{\theta_1,r_1,T}\times Y_{\theta_2,r_2,T}$ for $n\ge 1$.

We divide the possibilities into two cases: (i) $\alpha_1 = \alpha_2$, or $\alpha_1 < \alpha_2$ and $s_1 < \mathcal{S}$ and (ii) $\alpha_1 < \alpha_2$ and $s_1 \geq \mathcal{S}$. In the case (ii) we also choose s_3 and put θ_3 as in Lemma 3.2. Moreover, let us introduce the Banach space defined by

$$Y_{\theta_2,\theta_3,r_2,T} := \Big\{ u \in L^\infty_{loc}((0,T], X_{\theta_2}(r_2) \cap X_{\theta_3}(r_2)) : \|u\|_{Y_{\theta_2,\theta_3,r_2,T}} : \\ = \max\Big\{ \|u\|_{Y_{\theta_2,r_2,T}}, \|u\|_{Y_{\theta_3,r_2,T}} \Big\} < \infty \Big\}.$$

In a similar way to (3.11), (3.12) and (3.13) we obtain

$$\begin{aligned} & t^{\alpha_2\theta_3} |v_n|_{X_{\theta_3}(r_2)} \\ & \leq t^{\alpha_2\theta_3} |S_{\alpha_2}(t)v_0|_{X_{\theta_3}(r_2)} + t^{\alpha_2\theta_3} \int_0^t |P_{\alpha_2}(t-s) f_2(s, u_{n-1}(s))|_{X_{\theta_3}(r_2)} ds \\ & \lesssim |v_0|_{X_0(r_2)} + t^{\alpha_2\theta_3} \int_0^t (t-s)^{\alpha_2(s_2-\theta_3)-1} |f_2(s,u_{n-1}(s))|_{X_{s_2-1}(r_2)} ds \\ & \lesssim |v_0|_{X_0(r_2)} + t^{\alpha_2\theta_3} \int_0^t (t-s)^{\alpha_2(s_2-\theta_3)-1} s^{-m_2-p_2\alpha_2\theta_1} \cdot \|c_2\|_{L^{q_2}(\Omega)} \|1 + |u_{n-1}|\|^{p_2}_{Y_{\theta_1,r_1,T}} ds \\ & \leq |v_0|_{X_0(r_2)} + \|c_2\|_{L^{q_2}(\Omega)} \|1 + |u_{n-1}|\|^{p_2}_{Y_{\theta_1,r_1,T}} \cdot t^{\alpha_2 s_2 - m_2 - p_2\alpha_2\theta_1} \\ & \quad \times \int_0^1 (1-s)^{\alpha_2(s_2-\theta_3)-1} s^{-m_2-p_2\alpha_2\theta_1} ds. \end{aligned} \tag{3.14}$$

Here we apply $s_2 > \theta_2 = \theta_2(s_1) > \theta_2(s_3) = \theta_3$, which follows from (3.7) and Lemma 3.2. Then $u_{n-1} \in Y_{\theta_1,r_1,T}$ leads not only to $v_n \in Y_{\theta_2,r_2,T}$ but also to $v_n \in Y_{\theta_3,r_2,T}$. Thus $v_n \in Y_{\theta_2,\theta_3,r_2,T}$ for $n \geq 1$.

We consider the rest of the existence part only in the case (i). In the case (ii) it can be proved by replacing $Y_{\theta_2,r_2,T}$ and Y_{θ_2,r_2,T_*} with $Y_{\theta_2,\theta_3,r_2,T}$ and $Y_{\theta_2,\theta_3,r_2,T_*}$, respectively.

In a similar way there exist $C_1 > 0$ and $C_2 > 0$ such that for $n \geq 2$,

$$\begin{aligned} \|u_{n+1} - u_n\|_{Y_{\theta_1,r_1,T}} &\leq C_1 T^{\alpha_1 s_1 - m_1 - p_1\alpha_2\theta_2 + (\alpha_2-\alpha_1)\theta_1} \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v_n| + |v_{n-1}|\|^{p_1-1}_{Y_{\theta_2,r_2,T}} \|v_n - v_{n-1}\|_{Y_{\theta_2,r_2,T}}, \\ \|v_{n+1} - v_n\|_{Y_{\theta_2,r_2,T}} &\leq C_2 T^{\alpha_2 s_2 - m_2 - p_2\alpha_2\theta_1} \|c_2\|_{L^{q_2}(\Omega)} \|1 + |u_n| + |u_{n-1}|\|^{p_2-1}_{Y_{\theta_1,r_1,T}} \|u_n - u_{n-1}\|_{Y_{\theta_1,r_1,T}}. \end{aligned} \tag{3.15}$$

Put $U := 2\max\Big\{ \|u_2\|_{Y_{\theta_1,r_1,T}}, \|v_2\|_{Y_{\theta_2,r_2,T}} \Big\}$ and choose a sufficiently small time $T_* > 0$ such that

$$\begin{aligned} C_1 {T_*}^{\alpha_1 s_1 - m_1 - p_1\alpha_2\theta_2 + (\alpha_2-\alpha_1)\theta_1} \|c_1\|_{L^{q_1}(\Omega)} (1+2U)^{p_1-1} \leq \frac{1}{2}, \\ C_2 {T_*}^{\alpha_2 s_2 - m_2 - p_2\alpha_2\theta_1} \|c_2\|_{L^{q_2}(\Omega)} (1+2U)^{p_2-1} \leq \frac{1}{2}. \end{aligned} \tag{3.16}$$

Note that T_* is independent of n. By induction together with (3.15) and (3.16) we have

$$\begin{cases} \|u_n\|_{Y_{\theta_1,r_1,T_*}} \le U & \text{for } n \ge 1, \\ \|v_n\|_{Y_{\theta_2,r_2,T_*}} \le U & \text{for } n \ge 1, \\ \|u_{n+1} - u_n\|_{Y_{\theta_1,r_1,T_*}} \le \frac{1}{2}\|v_n - v_{n-1}\|_{Y_{\theta_2,r_2,T_*}} & \text{for } n \ge 2, \\ \|v_{n+1} - v_n\|_{Y_{\theta_2,r_2,T_*}} \le \frac{1}{2}\|u_n - u_{n-1}\|_{Y_{\theta_1,r_1,T_*}} & \text{for } n \ge 2. \end{cases} \tag{3.17}$$

By iteration in (3.17) the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are Cauchy in the Banach spaces Y_{θ_1,r_1,T_*} and Y_{θ_2,r_2,T_*}, respectively. Consequently, there exist limits $u \in Y_{\theta_1,r_1,T_*}$ and $v \in Y_{\theta_2,r_2,T_*}$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{Y_{\theta_1,r_1,T_*}} = 0 \text{ and } \lim_{n\to\infty} \|v_n - v\|_{Y_{\theta_2,r_2,T_*}} = 0. \tag{3.18}$$

We show that the limits u and v have all the properties of Definition 1.1 on $[0, T_*)$. Property (a) immediately follows from $u \in Y_{\theta_1,r_1,T_*}$ and $v \in Y_{\theta_2,r_2,T_*}$. We obtain in the same way as we deduce (3.12) that

$$\begin{aligned} \int_0^{T_*} \|f_1(s, v(s))\|_{L^1(\Omega)} ds &\lesssim \int_0^{T_*} \|f_1(s, v(s))\|_{L^{z_1}(\Omega)} ds \\ &\lesssim \int_0^{T_*} s^{-m_1 - p_1\alpha\theta_2} \cdot \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v|\|_{Y_{\theta_2,r_2,T_*}}^{p_1} ds \\ &= \frac{(T_*)^{1-m_1-p_1\alpha_2\theta_2}}{1 - m_1 - p_1\alpha_2\theta_2} \cdot \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v|\|_{Y_{\theta_2,r_2,T_*}}^{p_1}. \end{aligned}$$

Here we use $m_1 + p_1\alpha_2\theta_2 < 1$ by (3.3). Since the same is true for f_2, (u, v) satisfies the properties (b) and (c).

We mention the properties (d) and (e). We divide the possibilities into the same two cases as in the existence part.

In the case (i) it follows that

$$\alpha_1 s_1 - m_1 - p_1\alpha_2\theta_2 > 0. \tag{3.19}$$

Indeed, when $\theta_2 = \theta_2(s_1)$, (3.19) follows from $\mathcal{P} < 1$ (resp. $s_1 < \mathcal{S}$) if $\alpha_1 = \alpha_2$ (resp. if $\alpha_1 < \alpha_2$). On the other hand, when $\theta_2 = \varepsilon$, i.e., $\theta_2(s_1) \le 0$, (3.19) follows from the choice of ε in the proof of Proposition 3.1. Then in the same way as (3.15) with θ_1 replaced by 0 we can evaluate

$$\begin{aligned} &\left\| \int_0^t P_{\alpha_1}(t-s)(f_1(s, v_n(s)) - f_1(s, v(s))) ds \right\|_{L^{r_1}(\Omega)} \\ &\le C_1 (T_*)^{\alpha_1 s_1 - m_1 - p_1\alpha_2\theta_2} \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v_n| + |v|\|_{Y_{\theta_2,r_2,T_*}}^{p_1 - 1} \|v_n - v\|_{Y_{\theta_2,r_2,T_*}}, \end{aligned} \tag{3.20}$$

which converges to zero as $n \to \infty$, by (3.18).

In the case (ii) it follows from Lemma 3.2 (ii) that $\theta_2 = \theta_2(s_1)$. Since $s_3 < \mathcal{S}$, we have

$$\alpha_1 s_3 - m_1 - p_1\alpha_2\theta_3 = \alpha_1 s_3 - m_1 - p_1\alpha_2\theta_2(s_3) > 0. \tag{3.21}$$

Then we obtain in the same way as (3.11), (3.12) and (3.13) with $(s_1, \theta_2, z_1, z_2) = (s_1, \theta_2(s_1), z_1(s_1), z_2(s_1))$ and θ_1 replaced by $(s_3, \theta_3, z_1(s_3), z_2(s_3))$ and 0, respectively that

$$
\begin{aligned}
&|u_{n+1} - u_n|_{X_{\theta_1}(r_1)} \\
&\quad \le C_1 (T_*)^{\alpha_1 s_3 - m_1 - p_1 \alpha_2 \theta_3} \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v_n| + |v_{n-1}|\|^{p_1 - 1}_{Y_{\theta_3, r_2, T_*}} \|v_n - v_{n-1}\|_{Y_{\theta_3, r_2, T_*}} \\
&\quad \le C_1 (T_*)^{\alpha_1 s_3 - m_1 - p_1 \alpha_2 \theta_3} \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v_n| + |v_{n-1}|\|^{p_1 - 1}_{Y_{\theta_2, \theta_3, r_2, T_*}} \|v_n - v_{n-1}\|_{Y_{\theta_2, \theta_3, r_2, T_*}},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\left\| \int_0^t P_{\alpha_1}(t-s)(f_1(s, v_n(s)) - f_1(s, v(s)))ds \right\|_{L^{r_1}(\Omega)} \\
&\quad \le C_1 (T_*)^{\alpha_1 s_3 - m_1 - p_1 \alpha_2 \theta_3} \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v_n| + |v|\|^{p_1 - 1}_{Y_{\theta_2, \theta_3, r_2, T_*}} \|v_n - v\|_{Y_{\theta_2, \theta_3, r_2, T_*}}.
\end{aligned}
\tag{3.22}
$$

Both (3.18) and (3.20) (or (3.22)) enable us to take the limit of (1.12) as $n \to \infty$ in $L^1(\Omega)$. Since we can take the limit of (1.13) in a similar way, we deduce the integral system in Definition 1.1 (d). For the last property (e), it suffices to prove (1.9). In the case (i) it follows from (3.13) with θ_1 and v_{n-1} replaced by 0 and v, respectively and (3.19) that

$$
\begin{aligned}
&\|u(t) - S_{\alpha_1}(t)u_0\|_{L^{r_1}(\Omega)} \\
&\lesssim \int_0^t (t-s)^{\alpha_1 s_1 - 1} |f_1(s, v(s))|_{X_{s_1 - 1}(r_1)} ds \\
&\lesssim \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v|\|^{p_1}_{Y_{\theta_2, r_2, T_*}} \cdot t^{\alpha_1 s_1 - m_1 - p_1 \alpha_2 \theta_2} \int_0^1 (1-s)^{\alpha_1 s_1 - 1} s^{-m_1 - p_1 \alpha_2 \theta_2} ds \\
&\to 0 \quad (t \to 0).
\end{aligned}
\tag{3.23}
$$

In the case (ii) we observe from considering (3.13) and (3.19) that (3.23) also holds with s_1 and θ_2 replaced by s_3 and θ_3, respectively. Moreover, in both cases (i) and (ii) since we can obtain $\lim_{t \to 0} \|v(t) - S_{\alpha_2}(t)v_0\|_{L^{r_2}(\Omega)} = 0$ in a similar way, (1.9) holds. Thus (u, v) possesses the desired properties of Definition 1.1.

We consider the uniqueness of the mild solution only in the case (i). In the case (ii) it can be proved by replacing $Y_{\theta_2, r_2, T}$ with $Y_{\theta_2, \theta_3, r_2, T}$. We can derive the uniqueness from calculations similar to (3.15). Indeed, let $T \in (0, T_*]$ and let $(u, v), (\tilde{u}, \tilde{v}) \in Y_{\theta_1, r_1, T} \times Y_{\theta_2, r_2, T}$ be any two mild solutions of (1.1) with the same initial data (u_0, v_0). In a similar way to deduce (3.15) it follows that

$$
\begin{aligned}
&\|u - \tilde{u}\|_{Y_{\theta_1, r_1, T}} \le C_1 T^{\alpha_1 s_1 - m_1 - p_1 \alpha_2 \theta_2 + (\alpha_2 - \alpha_1)\theta_1} \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v| + |\tilde{v}|\|^{p_1 - 1}_{Y_{\theta_2, r_2, T}} \|v - \tilde{v}\|_{Y_{\theta_2, r_2, T}}, \\
&\|v - \tilde{v}\|_{Y_{\theta_2, r_2, T}} \le C_2 T^{\alpha_2 s_2 - m_2 - p_2 \alpha_2 \theta_1} \|c_2\|_{L^{q_2}(\Omega)} \|1 + |u| + |\tilde{u}|\|^{p_2 - 1}_{Y_{\theta_1, r_1, T}} \|u - \tilde{u}\|_{Y_{\theta_1, r_1, T}}.
\end{aligned}
\tag{3.24}
$$

We see that there exists a sufficiently small time $\widehat{T} \in (0, T]$ such that

$$
\begin{aligned}
&C_1 \widehat{T}^{\alpha_1 s_1 - m_1 - p_1 \alpha_2 \theta_2 + (\alpha_2 - \alpha_1)\theta_1} \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v| + |\tilde{v}|\|^{p_1 - 1}_{Y_{\theta_2, r_2, T}} \le \frac{1}{2}, \\
&C_2 \widehat{T}^{\alpha_2 s_2 - m_2 - p_2 \alpha_2 \theta_1} \|c_2\|_{L^{q_2}(\Omega)} \|1 + |u| + |\tilde{u}|\|^{p_2 - 1}_{Y_{\theta_1, r_1, T}} \le \frac{1}{2}.
\end{aligned}
$$

By this together with (3.24) we have $u(\cdot, t) \equiv \tilde{u}(\cdot, t)$ and $v(\cdot, t) \equiv \tilde{v}(\cdot, t)$ for $t \in [0, \widehat{T}]$. We obtain from a standard continuation argument (cf.[17]) that the uniqueness over the whole interval $[0, T_*]$ holds.

We prove the cases (b) and (c). Choose $(s_1, \theta_2, z_1, z_2)$ and $(s_2, \theta_1, z_3, z_4)$ as in Proposition 3.1. We divide the possibilities into two cases: (i) $\alpha_1 = \alpha_2$, or $\alpha_1 < \alpha_2$ and $s_1 \le \mathcal{S}$ and (ii) $\alpha_1 < \alpha_2$ and $s_1 > \mathcal{S}$.

We consider the existence part in the case (i). Set $\Pi_1 := \{u_0\} \subset L^{r_1}(\Omega)$ and $\Pi_2 := \{v_0\} \subset L^{r_2}(\Omega)$. Let $0 < T < \infty$. We can apply Lemma 2.2 and consider the constructed functions g and w_i $(i = 1, 2)$. For $i = 1, 2$, put

$$Y_{w_i,\theta_i,r_i,T} := \left\{ u \in L^\infty_{loc}((0,T], X_{\theta_i}(r_i)) : \|u\|_{Y_{w_i,\theta_i,r_i,T}} := \sup_{t\in(0,T)} (w_i(t))^{\alpha_2\theta_i} |u(t)|_{X_{\theta_i}(r_i)} < \infty \right\}. \tag{3.25}$$

Lemma 2.2 (ii) implies that the spaces $Y_{w_i,\theta_i,r_i,T}$, $i = 1, 2$, are Banach spaces. Note that for $i = 1, 2$, $Y_{w_i,\theta_i,r_i,T} \subset Y_{\theta_i,r_i,T}$ follows if $0 < T < \infty$. Define the functions (u_n, v_n) in the same way as in the case (a). Assume that $(u_{n-1}, v_{n-1}) \in Y_{w_1,\theta_1,r_1,T} \times Y_{w_2,\theta_2,r_2,T}$. In the same way as (3.11) and (3.13) we have for $0 < t < T$,

$$\begin{aligned}(w_1(t))^{\alpha_2\theta_1} |u_n|_{X_{\theta_1}(r_1)} &\le (w_1(t))^{\alpha_2\theta_1} |S_{\alpha_1}(t)u_0|_{X_{\theta_1}(r_1)} + (w_1(t))^{\alpha_2\theta_1} \int_0^t |P_{\alpha_1}(t-s) f_1(s, v_{n-1}(s))|_{X_{\theta_1}(r_1)} ds \\ &\lesssim |u_0|_{X_0(r_1)} + (w_1(t))^{\alpha_2\theta_1} \int_0^t (t-s)^{\alpha_1(s_1-\theta_1)-1} |f_1(s, v_{n-1}(s))|_{X_{s_1-1}(r_1)} ds\end{aligned}$$

and

$$\begin{aligned}&(w_1(t))^{\alpha_2\theta_1} \int_0^t (t-s)^{\alpha_1(s_1-\theta_1)-1} |f_1(s, v_{n-1}(s))|_{X_{s_1-1}(r_1)} ds \\ &\lesssim (w_1(t))^{\alpha_2\theta_1} \int_0^t (t-s)^{\alpha_1(s_1-\theta_1)-1} s^{-m_1} (w_2(s))^{-p_1\alpha_2\theta_2} \cdot \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v_{n-1}|\|^{p_1}_{Y_{w_2,\theta_2,r_2,T}} ds \\ &\le \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v_{n-1}|\|^{p_1}_{Y_{w_2,\theta_2,r_2,T}} \cdot (g(t))^{p_1-1} \cdot t^{\alpha_2\theta_1} \int_0^t (t-s)^{\alpha_1(s_1-\theta_1)-1} s^{-m_1-p_1\alpha_2\theta_2} ds \\ &= \|c_1\|_{L^{q_1}(\Omega)} \|1 + |v_{n-1}|\|^{p_1}_{Y_{w_2,\theta_2,r_2,T}} \\ &\quad \cdot (g(t))^{p_1-1} \cdot t^{\alpha_1 s_1 - m_1 - p_1\alpha_2\theta_2 + (\alpha_2-\alpha_1)\theta_1} \int_0^1 (1-s)^{\alpha_1(s_1-\theta_1)-1} s^{-m_1-p_1\alpha_2\theta_2} ds.\end{aligned} \tag{3.26}$$

We recall that $\lim_{t\to 0} g(t) = 0$ and $\alpha_1 s_1 - m_1 - p_1\alpha_2\theta_2 + (\alpha_2 - \alpha_1)\theta_1 \ge 0$. Hence, $u_n \in Y_{w_1,\theta_1,r_1,T}$. We can obtain $v_n \in Y_{w_2,\theta_2,r_2,T}$ in the same way. By induction we derive $(u_n, v_n) \in Y_{w_1,\theta_1,r_1,T} \times Y_{w_2,\theta_2,r_2,T}$ for $n \ge 1$. In a similar way there exist $C_3 > 0$ and $C_4 > 0$ such that for $n \ge 2$,

$$\|u_{n+1}-u_n\|_{Y_{w_1,\theta_1,r_1,T}} \le C_3(g(T))^{p_1-1}T^{\alpha_1 s_1-m_1-p_1\alpha_2\theta_2+(\alpha_2-\alpha_1)\theta_1}$$
$$\cdot \|c_1\|_{L^{q_1}(\Omega)}\|1+|v_n|+|v_{n-1}|\|^{p_1-1}_{Y_{w_2,\theta_2,r_2,T}}\|v_n-v_{n-1}\|_{Y_{w_2,\theta_2,r_2,T}},$$
$$\|v_{n+1}-v_n\|_{Y_{w_2,\theta_2,r_2,T}} \le C_4(g(T))^{p_2-1}T^{\alpha_2 s_2-m_2-p_2\alpha_2\theta_1}$$
$$\cdot \|c_2\|_{L^{q_2}(\Omega)}\|1+|u_n|+|u_{n-1}|\|^{p_2-1}_{Y_{w_1,\theta_1,r_1,T}}\|u_n-u_{n-1}\|_{Y_{w_1,\theta_1,r_1,T}}.$$

Put $V := 2\max\left\{\|u_2\|_{Y_{w_1,\theta_1,r_1,T}}, \|v_2\|_{Y_{w_2,\theta_2,r_2,T}}\right\}$ and choose a sufficiently small time $T_{**}>0$ such that

$$C_3(g(T_{**}))^{p_1-1}{T_{**}}^{\alpha_1 s_1-m_1-p_1\alpha_2\theta_2+(\alpha_2-\alpha_1)\theta_1}\|c_1\|_{L^{q_1}(\Omega)}(1+2V)^{p_1-1}\le\frac{1}{2},$$
$$C_4(g(T_{**}))^{p_2-1}{T_{**}}^{\alpha_2 s_2-m_2-p_2\alpha_2\theta_1}\|c_2\|_{L^{q_2}(\Omega)}(1+2V)^{p_2-1}\le\frac{1}{2}.$$

We deduce the analogue of (3.17) in $Y_{w_1,\theta_1,r_1,T_{**}}\times Y_{w_2,\theta_2,r_2,T_{**}}$ instead of $Y_{\theta_1,r_1,T_*}\times Y_{\theta_2,r_2,T_*}$. Therefore, there exist limits $u\in Y_{w_1,\theta_1,r_1,T_{**}}$ and $v\in Y_{w_2,\theta_2,r_2,T_{**}}$ such that

$$\lim_{n\to\infty}\|u_n-u\|_{Y_{w_1,\theta_1,r_1,T_{**}}}=0 \text{ and } \lim_{n\to\infty}\|v_n-v\|_{Y_{w_2,\theta_2,r_2,T_{**}}}=0.$$

We consider the case (ii). Let $s_3=\mathcal{S}$ and $\theta_3=\theta_2(s_3)$, where $\theta_2(s)$ is defined in the proof of Proposition 3.1. It follows from $\mathcal{P}<1$, or $\mathcal{P}=1$ and $m_1>0$ that $s_3>0$ and $\theta_3\ge 0$ hold. By this together with $\mathcal{S}<s_1<1$ and $\theta(s_1)<1$ we have $s_3\in(0,1)$ and $\theta_3\in[0,1)$. Let $0<T<\infty$. In a similar way to Lemma 2.2 we can construct a continuous and nondecreasing function $g:(0,T)\to(0,\infty)$ such that

$$|S_{\alpha_1}(t)u_0|_{X_{\theta_1}(r_1)}\lesssim g(t)\cdot t^{-\alpha_2\theta_1}|u_0|_{X_0(r_1)},\quad |S_{\alpha_2}(t)v_0|_{X_{\theta_2}(r_2)}\lesssim g(t)\cdot t^{-\alpha_2\theta_2}|v_0|_{X_0(r_2)} \text{ and}$$
$$|S_{\alpha_3}(t)v_0|_{X_{\theta_3}(r_2)}\lesssim g(t)\cdot t^{-\alpha_2\theta_3}|v_0|_{X_0(r_2)} \text{ for} 0<t<T,$$

$\lim_{t\to 0}g(t)=0$, and $\lim_{t\to 0}w_i(t)=0$ for $i=1,2,3$, where $(w_i(t))^{-\alpha_2\theta_i}=g(t)\cdot t^{-\alpha_2\theta_i}$. Put $Y_{w_1,\theta_1,r_1,T}$, $Y_{w_2,\theta_2,r_2,T}$ and $Y_{w_3,\theta_3,r_2,T}$ in the same way as (3.25). Then it follows that $(u_n,v_n)\in Y_{w_1,\theta_1,r_1,T}\times Y_{w_2,\theta_2,r_2,T}$ for $n\ge 1$. Let us introduce the Banach space defined by

$$Y_{w_2,\theta_2,w_3,\theta_3,r_2,T}$$
$$:=\left\{u\in L^\infty_{loc}((0,T],X_{\theta_2}(r_2)\cap X_{\theta_3}(r_2)):\|u\|_{Y_{w_2,\theta_2,w_3,\theta_3,r_2,T}}:=\max\left\{\|u\|_{Y_{w_2,\theta_2,r_2,T}},\|u\|_{Y_{w_3,\theta_3,r_2,T}}\right\}<\infty\right\}.$$

We observe from a similar manner to (3.14) that $u_{n-1}\in Y_{w_1,\theta_1,r_1,T}$ leads not only to $v_n\in Y_{w_2,\theta_2,r_2,T}$ but also to $v_n\in Y_{w_3,\theta_3,r_2,T}$. Thus $v_n\in Y_{w_2,\theta_2,w_3,\theta_3,r_2,T}$ for $n\ge 1$. We omit the rest of the existence part, since it can be proved by replacing $Y_{w_2,\theta_2,r_2,T}$ and Y_{w_2,θ_2,r_2,T_*} with $Y_{w_2,\theta_2,w_3,\theta_3,r_2,T}$ and $Y_{w_2,\theta_2,w_3,\theta_3,r_2,T_*}$, respectively.

We can obtain in a similar way to the case (a) that u and v have the properties (a)–(d) of Definition 1.1 on $[0,T_{**})$. In the case (i) it follows that $\alpha_1 s_1-m_1-p_1\alpha_2\theta_2\ge 0$. Then we deduce from (3.26) by replacing θ_1 and v_{n-1} with 0 and v, respectively that

$$\begin{aligned}
&\|u(\cdot,t)-S_{\alpha_1}(t)u_0\|_{L^{r_1}(\Omega)}\\
&\lesssim \int_0^t (t-s)^{\alpha_1 s_1-1}|f_1(s,v(s))|_{X_{s_1-1}(r_1)}ds\\
&\lesssim \|c_1\|_{L^{q_1}(\Omega)}\|1+|v|\|^{p_1}_{Y_{w_2,\theta_2,r_2,T_{**}}}\cdot (g(t))^{p_1-1}\cdot t^{\alpha_1 s_1-m_1-p_1\alpha_2\theta_2}\int_0^1 (1-s)^{\alpha_1 s_1-1}s^{-m_1-p_1\alpha_2\theta_2}ds\\
&\to 0 \quad (t\to 0).
\end{aligned} \tag{3.27}$$

In the same way it follows that $\lim_{t\to 0}\|v(t)-S_{\alpha_2}(t)v_0\|_{L^{r_2}(\Omega)}=0$. In the case (ii) we observe from considering (3.26) that (3.27) also holds with s_1, θ_2 and $Y_{w_2,\theta_2,r_2,T_{**}}$ replaced by s_3, θ_3 and $Y_{w_3,\theta_3,r_2,T_{**}}$, respectively. Thus (u, v) is a mild solution of (1.1) in the sense of Definition 1.1 on $[0, T_{**})$.

Note that in the case of (ii) how to derive the limit corresponding to (3.27) differs between when $\mathcal{R}<1$ and when $\mathcal{R}=1$. When $\mathcal{R}<1$, by considering (3.11) and (3.12) we can evaluate

$$\begin{aligned}
\|u(\cdot,t)-S_{\alpha_1}(t)u_0\|_{L^{r_1}(\Omega)} &\lesssim \int_0^t (t-s)^{\alpha_1 s_3-1}|f_1(s,v(s))|_{X_{s_3-1}(r_1)}ds\\
&\lesssim \int_0^t (t-s)^{\alpha_1 s_3-1}\|f_1(s,v(s))\|_{L^{z_1}(\Omega)}ds,
\end{aligned}$$

where $z_1=z_1(s_3)\in(1,\infty)$. On the other hand, when $\mathcal{R}=1$, since $z_1=z_1(s_3)=1$, the inequality $|f_1(s,v(s))|_{X_{s_3-1}(r_1)}\lesssim\|f_1(s,v(s))\|_{L^{z_1}(\Omega)}$ does not hold. Then we use the L^p-L^q estimate of the heat semigroup ([16, Proposition 48.4 (c)-(e)]) and deduce that

$$\begin{aligned}
&\|P_{\alpha_1}(t-s)f_1(s,v(s))\|_{L^{r_1}(\Omega)}\\
&\le \alpha_1(t-s)^{\alpha_1-1}\int_0^\infty \tau\Phi_{\alpha_1}(\tau)\|S(\tau(t-s)^{\alpha_1})f_1(s,v(s))\|_{L^{r_1}(\Omega)}d\tau\\
&\lesssim \alpha_1(t-s)^{\alpha_1-1}\int_0^\infty \tau\Phi_{\alpha_1}(\tau)\cdot\tau^{-\frac{N}{2}\left(1-\frac{1}{r_1}\right)}(t-s)^{-\frac{N}{2}\alpha_1\left(1-\frac{1}{r_1}\right)}\|f_1(s,v(s))\|_{L^1(\Omega)}d\tau\\
&=\alpha_1(t-s)^{\alpha_1 s_3-1}\|f_1(s,v(s))\|_{L^1(\Omega)}\int_0^\infty \tau^{s_3}\Phi_{\alpha_1}(\tau)d\tau\\
&=\frac{\alpha_1\Gamma(s_3+1)}{\Gamma(\alpha_1 s_3+1)}(t-s)^{\alpha_1 s_3-1}\|f_1(s,v(s))\|_{L^1(\Omega)}.
\end{aligned}$$

Here we apply (1.5) and $\mathcal{S}=1-\dfrac{N}{2}\left(1-\dfrac{1}{r_1}\right)$, which follows from $\mathcal{R}=1$. Thus we have

$$\|u(\cdot,t)-S_{\alpha_1}(t)u_0\|_{L^{r_1}(\Omega)}\lesssim\int_0^t (t-s)^{\alpha_1 s_3-1}\|f_1(s,v(s))\|_{L^1(\Omega)}ds.$$

We also obtain the uniqueness of the mild solution in the space $Y_{w_1,\theta_1,r_1,T_{**}}\times Y_{w_2,\theta_2,r_2,T_{**}}$ through an argument similar to (3.24). We omit the details. □

4 Nonexistence result

Proof of Theorem 1.3 Without loss of generality, we suppose that $0 \in \Omega$. Choose $\rho > 0$ such that $B(0, \rho) \subset \Omega$, where $B(0, \rho)$ denotes the ball of radius $\rho > 0$ centered at 0.

We prove the case where $\max\{\mathcal{P}, \mathcal{R}\} > 1$. Let $0 < l < \frac{N}{q_1}$ if $1 \leq q_1 < \infty$, $l = 0$ if $q_1 = \infty$ and $0 < k < \frac{N}{r_2}$. Note that k is chosen to be sufficiently large and the same is valid for l when $1 \leq q_1 < \infty$. Put

$$c_1(x) := |x|^{-l},\ c_2(x) := 1,\ u_0(x) := 0 \text{ and } v_0(x) := |x|^{-k}\chi_{B(0,\rho)}(x),$$

where $\chi_{B(0,\rho)}$ is a characteristic function. Then $(c_1, c_2, u_0, v_0) \in L^{q_1}(\Omega) \times L^{q_2}(\Omega) \times L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ holds.

The proof is by contradiction. Assume that there exists $T > 0$ such that the problem (1.1) with $f_1(x, t, v) = c_1(x) \cdot t^{-m_1} v^{p_1}$ and $f_2(x, t, u) = c_2(x) \cdot t^{-m_2} u^{p_2}$ possesses a local in time nonnegative mild solution (u, v) in the sense of Definition 1.1 on $[0, T)$. Let $1 < \tau < 2$ and let $s > 0$ be sufficiently small. Let $|x| \leq \frac{\sqrt{s^{\tilde{\alpha}}}}{2}$, where $\tilde{\alpha} = \alpha_1$ or $\tilde{\alpha} = \alpha_2$. Then we can apply Proposition 2.3 and obtain

$$\begin{aligned}
(S(\tau s^{\alpha_2})v_0)(x) &= \int_{\{|y|<\rho\}} G(x, y, \tau s^{\alpha_2})|y|^{-k} dy \geq \int_{\{|y-x|<\sqrt{\tau s^{\alpha_2}}\}} G(x, y, \tau s^{\alpha_2})|y|^{-k} dy \\
&\geq c_1 \int_{\{|y-x|<\sqrt{\tau s^{\alpha_2}}\}} (\tau s^{\alpha_2})^{-\frac{N}{2}} |y|^{-k} dy \gtrsim s^{-\frac{N}{2}\alpha_2} \int_{\{|y-x|<\sqrt{\tau s^{\alpha_2}}\}} |y|^{-k} dy \\
&\geq s^{-\frac{N}{2}\alpha_2} \left(\sqrt{\tau s^{\alpha_2}} + \frac{\sqrt{s^{\tilde{\alpha}}}}{2}\right)^{-k} \int_{\{|y-x|<\sqrt{\tau s^{\alpha_2}}\}} dy \gtrsim s^{-\frac{k}{2}\tilde{\alpha}}.
\end{aligned}$$

Due to (1.4) and (1.7), we have

$$(S_{\alpha_2}(s)v_0)(x) \geq \int_1^2 \Phi_{\alpha_2}(\tau)(S(\tau s^{\alpha_2})v_0)(x) d\tau \gtrsim \int_1^2 \Phi_{\alpha_2}(\tau) d\tau \cdot s^{-\frac{k}{2}\tilde{\alpha}}$$

and hence

$$S_{\alpha_2}(s)v_0 \gtrsim s^{-\frac{k}{2}\tilde{\alpha}} \chi_{B\left(0, \frac{\sqrt{s^{\tilde{\alpha}}}}{2}\right)} \tag{4.1}$$

for sufficiently small $s > 0$.

Let $t > 0$ be sufficiently small and let $\frac{t}{3} \leq s \leq \frac{t}{2}$, $|x| < \frac{\sqrt{s^{\alpha_1}}}{2}$ and $1 < \tau < 2$. Then for $|y| < \frac{\sqrt{s^{\tilde{\alpha}}}}{2}$,

$$|x - y| < \sqrt{s^{\alpha_1}} \leq \sqrt{(t - s)^{\alpha_1}} < \sqrt{\tau (t - s)^{\alpha_1}}.$$

Using Proposition 2.3, we obtain

$$G(x,y,\tau(t-s)^{\alpha_1}) \gtrsim \tau^{-\frac{N}{2}}(t-s)^{-\frac{N}{2}\alpha_1} \gtrsim s^{-\frac{N}{2}\alpha_1} \tag{4.2}$$

for $|y| < \frac{\sqrt{s^{\tilde{\alpha}}}}{2}$. It follows from (4.1) and (4.2) that

$$\begin{aligned}(S(\tau(t-s)^{\alpha_1})f_1(s,S_{\alpha_2}(s)v_0))(x) &= \int_\Omega G(x,y,\tau(t-s)^{\alpha_1})|y|^{-l}s^{-m_1}((S_{\alpha_2}(s)v_0)(y))^{p_1}dy \\ &\gtrsim s^{-m_1-\frac{k}{2}p_1\tilde{\alpha}} \int_{\left\{|y|<\frac{\sqrt{s^{\tilde{\alpha}}}}{2}\right\}} G(x,y,\tau(t-s)^{\alpha_1})|y|^{-l}dy \\ &\gtrsim s^{-\frac{l}{2}\tilde{\alpha}-m_1-\frac{k}{2}p_1\tilde{\alpha}-\frac{N}{2}\alpha_1} \int_{\left\{|y|<\frac{\sqrt{s^{\tilde{\alpha}}}}{2}\right\}} dy \\ &\gtrsim s^{-\frac{l}{2}\tilde{\alpha}-m_1-\frac{k}{2}p_1\tilde{\alpha}-\frac{N}{2}\alpha_1+\frac{N}{2}\tilde{\alpha}},\end{aligned}$$

which yields

$$\begin{aligned}(P_{\alpha_1}(t-s)f_1(s,S_{\alpha_2}(s)v_0))(x) &\geq \alpha_1(t-s)^{\alpha_1-1}\int_1^2 \tau\Phi_{\alpha_1}(\tau)(S(\tau(t-s)^{\alpha_1})f_1(s,S_{\alpha_2}(s)v_0))(x)d\tau \\ &\gtrsim (t-s)^{\alpha_1-1}s^{-\frac{l}{2}\tilde{\alpha}-m_1-\frac{k}{2}p_1\tilde{\alpha}-\frac{N}{2}\alpha_1+\frac{N}{2}\tilde{\alpha}} \\ &\gtrsim (t-s)^{\alpha_1-1}t^{-\frac{l}{2}\tilde{\alpha}-m_1-\frac{k}{2}p_1\tilde{\alpha}-\frac{N}{2}\alpha_1+\frac{N}{2}\tilde{\alpha}}.\end{aligned}$$

Here we use $s \leq \frac{t}{2}$. By direct calculation we have

$$\int_0^t (P_{\alpha_1}(t-s)f_1(s,S_{\alpha_2}(s)v_0))(x)ds \geq \int_{\frac{t}{3}}^{\frac{t}{2}} (P_{\alpha_1}(t-s)f_1(s,S_{\alpha_2}(s)v_0))(x)ds \gtrsim t^{\alpha_1-\frac{l}{2}\tilde{\alpha}-m_1-\frac{k}{2}p_1\tilde{\alpha}-\frac{N}{2}\alpha_1+\frac{N}{2}\tilde{\alpha}}$$

and

$$\begin{aligned}\left\|\int_0^t P_{\alpha_1}(t-s)f_1(s,S_{\alpha_2}(s)v_0)ds\right\|_{L^{r_1}(\Omega)}^{r_1} &\gtrsim \int_{\left\{|x|<\frac{\sqrt{s^{\alpha_1}}}{2}\right\}} t^{r_1\left(\alpha_1-\frac{l}{2}\tilde{\alpha}-m_1-\frac{k}{2}p_1\tilde{\alpha}-\frac{N}{2}\alpha_1+\frac{N}{2}\tilde{\alpha}\right)}dx \\ &\gtrsim t^{\frac{N}{2}\alpha_1+r_1\left(\alpha_1-\frac{l}{2}\tilde{\alpha}-m_1-\frac{k}{2}p_1\tilde{\alpha}-\frac{N}{2}\alpha_1+\frac{N}{2}\tilde{\alpha}\right)}.\end{aligned}$$

Taking the limit as $l \to \frac{N}{q_1}$ (if $1 \leq q_1 < \infty$) and $k \to \frac{N}{r_2}$, we deduce that

$$\frac{N}{2}\alpha_1 + r_1\left(\alpha_1 - \frac{l}{2}\tilde{\alpha} - m_1 - \frac{k}{2}p_1\tilde{\alpha} - \frac{N}{2}\alpha_1 + \frac{N}{2}\tilde{\alpha}\right) \to \begin{cases} \alpha_1 r_1(1-\mathcal{P}) & \text{when } \tilde{\alpha} = \alpha_1, \\ \alpha_2 r_1(1-\mathcal{R}) & \text{when } \tilde{\alpha} = \alpha_2. \end{cases}$$

Thus we obtain $\left\|\int_0^t P_{\alpha_1}(t-s)f_1(s,S_{\alpha_2}(s)v_0)ds\right\|_{L^{r_1}(\Omega)}^{r_1} \to \infty$ as $t \to 0$. By Definition 1.1 (d) we have $v(t) \geq S_{\alpha_2}(t)v_0$ in $\Omega \times (0,T)$, which yields

$$\left\|\int_0^t P_{\alpha_1}(t-s)f_1(s,v(s))ds\right\|_{L^{r_1}(\Omega)} \geq \left\|\int_0^t P_{\alpha_1}(t-s)f_1(s,S_{\alpha_2}(s)v_0)ds\right\|_{L^{r_1}(\Omega)} \to \infty(t \to 0).$$

This contradicts the property of Definition 1.1 (e).

We prove the case where $\max\{\mathcal{Q}_1,\mathcal{Q}_2\} > 1$. Let $0 < \tilde{l} < \frac{N}{q_2}$ if $1 \leq q_2 < \infty$, $\tilde{l} = 0$ if

$q_2 = \infty$ and $0 < \tilde{k} < \dfrac{N}{r_1}$. Note that $\tilde{k}$ is chosen to be sufficiently large and the same is valid for $\tilde{l}$ when $1 \le q_2 < \infty$. Put

$$c_1(x) := 1,\ c_2(x) := |x|^{-\tilde{l}},\ u_0(x) := |x|^{-\tilde{k}}\chi_{B(0,\rho)}(x) \text{ and } v_0(x) := 0.$$

Then $(c_1, c_2, u_0, v_0) \in L^{q_1}(\Omega) \times L^{q_2}(\Omega) \times L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ holds.

The proof is also by contradiction. As in the case where $\max\{\mathcal{P}, \mathcal{R}\} > 1$, we assume that there exists a local in time nonnegative mild solution (u, v) in the sense of Definition 1.1. We obtain in the same way as we deduce (4.1) that

$$S_{\alpha_1}(s)u_0 \gtrsim s^{-\frac{\tilde{k}}{2}\alpha_1}\chi_{B\left(0,\frac{\sqrt{s^{\alpha_1}}}{2}\right)} \tag{4.3}$$

for sufficiently small $s > 0$.

We prove the case where $\mathcal{Q}_1 > 1$. Let $t > 0$ be sufficiently small and let $\dfrac{t}{3} \le s \le \dfrac{t}{2}$, $|x| < \dfrac{\sqrt{s^{\alpha_1}}}{3}$ and $1 < \tau < 2$. It follows from Proposition 2.3 and (4.3) that

$$\begin{aligned}
(S(\tau(t-s)^{\alpha_2})f_2(s, S_{\alpha_1}(s)u_0))(x) &= \int_\Omega G(x, y, \tau(t-s)^{\alpha_2})|y|^{-\tilde{l}}s^{-m_2}((S_{\alpha_1}(s)u_0)(y))^{p_2}dy \\
&\gtrsim s^{-\frac{N}{2}\alpha_2}\int_{\{|y-x|<\sqrt{\tau s^{\alpha_2}}\}}|y|^{-\tilde{l}}s^{-m_2}((S_{\alpha_1}(s)u_0)(y))^{p_2}dy \\
&\gtrsim s^{-\frac{N}{2}\alpha_2-\frac{\tilde{l}}{2}\alpha_1-m_2-\frac{\tilde{k}}{2}p_2\alpha_1}\int_{\{|y-x|<\sqrt{\tau s^{\alpha_2}}\}}dy \\
&\gtrsim s^{-\frac{\tilde{l}}{2}\alpha_1-m_2-\frac{\tilde{k}}{2}p_2\alpha_1}.
\end{aligned}$$

Note that we apply $|y| \le |y-x| + |x| < \sqrt{\tau s^{\alpha_2}} + \dfrac{\sqrt{s^{\alpha_1}}}{3} \le \dfrac{\sqrt{s^{\alpha_1}}}{2}$, since s is sufficiently small. Then by $s \le \dfrac{t}{2}$ we have

$$\begin{aligned}
(P_{\alpha_2}(t-s)f_2(s, S_{\alpha_1}(s)u_0))(x) &\ge \alpha_2(t-s)^{\alpha_2-1}\int_1^2 \tau\Phi_{\alpha_2}(\tau)(S(\tau(t-s)^{\alpha_2})f_1(s, S_{\alpha_1}(s)u_0))(x)d\tau \\
&\gtrsim (t-s)^{\alpha_2-1}s^{-\frac{\tilde{l}}{2}\alpha_1-m_2-\frac{\tilde{k}}{2}p_2\alpha_1} \\
&\gtrsim (t-s)^{\alpha_2-1}t^{-\frac{\tilde{l}}{2}\alpha_1-m_2-\frac{\tilde{k}}{2}p_2\alpha_1}.
\end{aligned}$$

By direct calculation we get

$$\int_0^t (P_{\alpha_2}(t-s)f_2(s, S_{\alpha_1}(s)u_0))(x)ds \ge \int_{\frac{t}{3}}^{\frac{t}{2}} (P_{\alpha_2}(t-s)f_2(s, S_{\alpha_1}(s)u_0))(x)ds \gtrsim t^{\alpha_2-\frac{\tilde{l}}{2}\alpha_1-m_2-\frac{\tilde{k}}{2}p_2\alpha_1}$$

and

$$\left\|\int_0^t P_{\alpha_2}(t-s)f_2(s,S_{\alpha_1}(s)u_0)ds\right\|_{L^{r_2}(\Omega)}^{r_2} \gtrsim \int_{\left\{|x|<\frac{\sqrt{s^{\alpha_1}}}{3}\right\}} t^{r_2\left(\alpha_2-\frac{\tilde{l}}{2}\alpha_1-m_2-\frac{\tilde{k}}{2}p_2\alpha_1\right)}dx$$
$$\gtrsim t^{\frac{N}{2}\alpha_1+r_2\left(\alpha_2-\frac{\tilde{l}}{2}\alpha_1-m_2-\frac{\tilde{k}}{2}p_2\alpha_1\right)}.$$

Taking the limit as $\tilde{l}\to\frac{N}{q_2}$ (if $1\le q_2<\infty$) and $\tilde{k}\to\frac{N}{r_1}$, we deduce that

$$\frac{N}{2}\alpha_1+r_2\left(\alpha_2-\frac{\tilde{l}}{2}\alpha_1-m_2-\frac{\tilde{k}}{2}p_2\alpha_1\right)\to\alpha_2 r_2(1-\mathcal{Q}_1)<0.$$

Thus we obtain $\left\|\int_0^t P_{\alpha_2}(t-s)f_2(s,S_{\alpha_1}(s)u_0)ds\right\|_{L^{r_2}(\Omega)}^{r_2}\to\infty$ as $t\to 0$. This contradicts the property of Definition 1.1(e) in a similar way to the case where $\max\{\mathcal{P},\mathcal{R}\}>1$.

We prove the case where $\mathcal{Q}_2>1$. Let $t>0$ be sufficiently small and let $\frac{t}{3}\le s\le\frac{t}{2}$, $|x|<\frac{\sqrt{s^{\alpha_2}}}{2}$ and $1<\tau<2$. Then for $|y|<\frac{\sqrt{s^{\alpha_2}}}{2}$,

$$|x-y|<\sqrt{s^{\alpha_2}}\le\sqrt{(t-s)^{\alpha_2}}<\sqrt{\tau(t-s)^{\alpha_2}}.$$

Due to Proposition 2.3, we have

$$G(x,y,\tau(t-s)^{\alpha_2})\gtrsim\tau^{-\frac{N}{2}}(t-s)^{-\frac{N}{2}\alpha_2}\gtrsim s^{-\frac{N}{2}\alpha_2} \tag{4.4}$$

for $|y|<\frac{\sqrt{s^{\alpha_2}}}{2}$. Considering in the same way as in the case where $\max\{\mathcal{P},\mathcal{R}\}>1$, we obtain from (4.3) and (4.4) that

$$(S(\tau(t-s)^{\alpha_2})f_2(s,S_{\alpha_1}(s)u_0))(x)\gtrsim s^{-\frac{\tilde{l}}{2}\alpha_2-m_2-\frac{\tilde{k}}{2}p_2\alpha_1},$$

which yields

$$(P_{\alpha_2}(t-s)f_2(s,S_{\alpha_1}(s)u_0))(x)\gtrsim(t-s)^{\alpha_2-1}t^{-\frac{\tilde{l}}{2}\alpha_2-m_2-\frac{\tilde{k}}{2}p_2\alpha_1}.$$

Moreover, this leads to

$$\left\|\int_0^t P_{\alpha_2}(t-s)f_2(s,S_{\alpha_1}(s)u_0)ds\right\|_{L^{r_2}(\Omega)}^{r_2}\gtrsim t^{\frac{N}{2}\alpha_2+r_2\left(\alpha_2-\frac{\tilde{l}}{2}\alpha_2-m_2-\frac{\tilde{k}}{2}p_2\alpha_1\right)}.$$

Taking the limit as $\tilde{l}\to\frac{N}{q_2}$ (if $1\le q_2<\infty$) and $\tilde{k}\to\frac{N}{r_1}$, we deduce that

$$\frac{N}{2}\alpha_2+r_2\left(\alpha_2-\frac{\tilde{l}}{2}\alpha_2-m_2-\frac{\tilde{k}}{2}p_2\alpha_1\right)\to\alpha_2 r_2(1-\mathcal{Q}_2)<0.$$

Thus we obtain $\left\|\int_0^t P_{\alpha_2}(t-s)f_2(s,S_{\alpha_1}(s)u_0)ds\right\|_{L^{r_2}(\Omega)}^{r_2}\to\infty$ as $t\to 0$. This contradicts the property of Definition 1.1 (e). Therefore, the proof is complete. □

5 Nonexistence result for scalar problems

In this section we apply our study to the nonexistence of a local in time solution of the scalar problem

$$\begin{cases} \partial_t^\alpha u = \Delta u + f(x,t,u) & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{5.1}$$

We obtain the following nonexistence theorem.

Theorem 5.1 *Let* $N \geq 1$, $0 < \alpha < 1$, $0 < p < \infty$, $q_1 \in [1, \infty]$, $q_2 \in \left(\frac{1}{\alpha}, \infty\right]$ *and* $1 \leq r < \infty$. *Suppose that* (1.11) *holds with* $\beta_A = \frac{N}{2}$. *Then there exist nonnegative functions* $c(x,t) \in L_{q_1,q_2}$ *and* $u_0 \in L^r(\Omega)$ *such that, for every* $T > 0$, *the problem* (5.1) *with* $f(x,t,u) = c(x,t) \cdot u^p$ *admits no local in time nonnegative mild solution u in the sense of Definition* 1.1 (*more precisely*, *in the sense of* [9, Definition 3.1.1]) *on the interval* [0, *T*).

Since we can prove in the same way as in the proof of Theorem 1.3, we leave the proof to readers. Note that we only solve the nonexistence conjecture in [9] when the nonlinear term f is separable with respect to x, t and u.

Remark that if $1 \leq r < \frac{N}{2}(p-1)$, then there exists a nonnegative initial function $u_0 \in L^r(\Omega)$ such that, the problem (5.1) with $f(x,t,u) = u^p$ has no local in time nonnegative mild solution on any time interval. Hence, our nonexistence result corresponds to [20]. In conclusion, for scalar problems and systems with pure power nonlinear terms, the existence/nonexistence results correspond to [20] and [15], respectively.

6 Discussion

In this paper we consider a local in time solution of a time fractional weakly coupled reaction-diffusion system in two components with possibly distinct fractional orders. In Theorems 1.2 and 1.3 we derive the integrability conditions on the initial state functions for the local in time existence and nonexistence results. The parameters $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{R}$ describe the balance between the factors: the growth rates (resp. the singularities) of the nonlinear terms with respect to u or v (resp. x and t), the singularities of the initial data, and the fractional exponents. For instance, the larger the growth rates p_1 and p_2 become, the larger these five parameters become. Then Theorems 1.2 and 1.3 imply that the existence result is less likely to hold, and that the nonexistence result is more likely to hold. The integrability is determined by $\max\{\mathcal{P}, \mathcal{Q}, \mathcal{R}\}$ and $\max\{\mathcal{P}, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{R}\}$ in the existence and nonexistence part, respectively.

When $\alpha_1 = \alpha_2$, the equalities $\mathcal{P} = \mathcal{R}$ and $\mathcal{Q} = \mathcal{Q}_1 = \mathcal{Q}_2$ hold. Therefore, as seen in Corollary 1.4, we can explicitly determine the existence/nonexistence of a solution. The threshold integrability condition on initial data, which is a pair (r_1, r_2), is defined by $\max\{\mathcal{P}, \mathcal{Q}\} = 1$. When $m_1 > 0$ and $m_2 > 0$, the larger α_1 and α_2 become, the wider the space of initial data for the existence result becomes. On the other hand, when $\alpha_1 < \alpha_2$, the inequalities $\max\{\mathcal{Q}_1, \mathcal{Q}_2\} \leq 1 < \mathcal{Q}$ and $\max\{\mathcal{P}, \mathcal{R}\} \leq 1$ can occur. In this case since

Theorems 1.2 and 1.3 cannot be applied, we cannot determine whether the problem (1.1) possesses a local in time solution or not. We mention the following points:

1. In Theorem 1.3 even if we assume $\mathcal{Q} > 1$, we cannot obtain the nonexistence result, since the inequality (4.3) does not hold with α_1 replaced by α_2 on the right hand side. If this is true, we can evaluate $(S(\tau(t-s)^{\alpha_2})f_2(s, S_{\alpha_1}(s)u_0))(x)$ in the same way as in the case where $\max\{\mathcal{P}, \mathcal{R}\} > 1$. Hence, we can get the nonexistence result even if $\mathcal{Q} > 1$.
2. If we assume $\max\{\mathcal{Q}_1, \mathcal{Q}_2\} < 1$ instead of $\mathcal{Q} < 1$, then Proposition 3.1 does not hold. In particular, the inequality $\max\{\mathcal{Q}_1, \mathcal{Q}_2\} < 1$ does not lead to (3.6) and (3.9) with $(s, \tilde{s}) = \left(\min\left\{1, 1 - \frac{N}{2}\left(\frac{1}{q_1} - \frac{1}{r_1}\right)\right\}, \max\left\{1 - \frac{N}{2}\left(1 - \frac{1}{r_2}\right), \frac{m_2}{\alpha_2}\right\}\right)$. Thus we cannot obtain the existence result under this assumption.

The author conjectures that the nonexistence result does not hold in the above case, since the former point is a greater reason. In the existence result it seems that the solvability of the problem (1.1) in the above case may hold by using a functional space different from the Banach spaces introduced in the proof of Theorem 1.2.

Possible future problems ensuing from the current analysis are as follows:

1. What are the consequences of the problem (1.1) with a different boundary condition or situation, e.g. the Neumann boundary condition, the boundary condition where u and v are non-zero positive bounded functions, and the situation where the boundary is broken into parts with a condition of a different type set on each?
2. What happens to a local in time solution of (1.1) as $\alpha \to 1^-$? Given that (1.3) is the limit of (1.1) as $\alpha \to 1^-$, is it possible to show that the estimates obtained by this paper approach those known for (1.3)? If not, which is more conservative and why?
3. For the problem (1.1), what is the solvability when one equation has an integer in time derivative and the other has a fractional in time derivative?

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