SN Partial Differ. Equ. Appl. (2020) 1:48
https://doi.org/10.1007/s42985-020-00050-y

ORIGINAL PAPER

Dynamic boundary conditions and the Carslaw-Jaeger constitutive relation in heat transfer

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Received: 14 May 2020 / Accepted: 7 October 2020 / Published online: 10 November 2020

Abstract

We study a dynamic boundary condition problem in heat transfer which represents the interaction between a conducting solid enclosed by a conducting shell. Both the solid and the shell are thermally inhomogeneous and anisotropic. Interaction is modelled by considering the solid as a source of thermal energy to the shell. A constitutive equation proposed by Carslaw and Jaeger establishes a relation between temperature in the shell and the boundary value of temperature in the solid. This gives rise to a dynamic boundary condition problem that has not been studied in the recent literature. The system of equations so obtained is presented as an implicit evolution equation which involves a pair of unbounded linear operators that map between two different spaces. We extend the operators to a jointly closed pair for which the implicit equation makes sense. The solution of the initial value problem is constructed by means of a holomorphic family of solution operators. The class of admissible initial states is surprisingly large.

Keywords Dynamic boundary condition · Heat transfer · Carslaw-Jaeger relation

Mathematics Subject Classification 34G10 · 35K15 · 58J35

1 Introduction

In the 1947-edition of their scholarly book, Carslaw and Jaeger laid down a fundamental constitutive relation for thermal contact between heat-conducting materials. It states that *at a point of contact between two bodies* [*the normal component of*] *thermal flux is proportional to the difference in temperature at the given point and directed toward the lower temperature*. This constitutive equation introduces the notion of *perfect/imperfect* contact by means of a function that is zero at points of perfect contact and positive when contact is imperfect.

This article is part of the section "Theory of PDEs" edited by Eduardo Teixeira.

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This paper is devoted to the study of heat transfer in a solid, represented by a bounded open set $\Omega \subset \mathbb{R}^3$, enclosed by a thin shell modelled as the boundary $\Gamma = \partial\Omega$. It is assumed that the shell internally conducts thermal energy in a tangential direction and that, according to the Carslaw-Jaeger relation, contact is everywhere imperfect. If $u(x, t)$ denotes the temperature at $x \in \Omega$ and $U(x', t)$ at $x' \in \Gamma$ at time $t > 0$, the following equations arise after scaling to dimensionless variables:

$$u_t + Lu = 0 \text{ in } \Omega; \tag{1}$$

$$U_t + \Lambda U + \gamma_L u = 0 \text{ in } \Gamma. \tag{2}$$

Here the the operators L and Λ are defined by the differential expressions $Lu = -\nabla \cdot [a(x)\nabla u]$ and $\Lambda U = -\nabla_S \cdot [b(x')\nabla_S U]$ with a and b suitable symmetric matrix functions. The operator L represents internal heat transfer in the solid and Λ heat transfer in the shell. In (2), $\gamma_L u$ denotes the co-normal derivative associated with L. It signifies flux of thermal energy between solid and shell. We use $\nabla_S \cdot$ and ∇_S to denote the surface divergence and surface (tangential/covariant) gradient on Γ. When the shell does not internally conduct heat, $\Lambda U = 0$.

The *Carslaw-Jaeger relation* is

$$U(x', t) - \gamma_0 u(x', t) = k(x')\gamma_L u(x', t) \tag{3}$$

with γ_0 the trace operator that assigns boundary values of u. The non-negative function k expresses the quality of contact. If $k \equiv 0$ we talk of *perfect contact* in which case $U = \gamma_0 u$.

The system of Eqs. (1)–(3) comprises the dynamic boundary condition problem treated here under the condition that $k(x') > 0$ everywhere on Γ.

Recent work on dynamic boundary conditions in heat transfer (or diffusion) largely focused on the Wentzell boundary condition (first introduced by William Feller). Here heat transfer/diffusion in a solid is represented by the Laplace operator (Δ) and a boundary operator of the form $bu = \gamma_0[\Delta u] +$ *lower order terms* is involved. The original *Wentzell boundary condition* is $bu = g$ for given g and the dynamic boundary condition is in the form $\partial_t[\gamma_0 u] = bu$ (*e.g.*, Favini, Goldstein et al. [5]). The boundary operator was later replaced by $bu = \Delta_L[\gamma_0 u] + \cdots$ with Δ_L the Laplace-Beltrami operator and the term *generalized Wentzell boundary condition* made its appearance (e.g., Vázquez and Vitillaro [27]). In Goldstein et al. [9] the Laplacian is replaced by a general symmetric strongly elliptic operator of second order and the Laplace-Beltrami operator by a similar, unrelated, general elliptic operator on the boundary manifold. Here the phrase *general Wentzell boundary condition* was born. In Coclite, Goldstein et al. [4] and Gal [8] nonlinear operators are considered, the latter without diffusion in the boundary. The dynamic boundary condition in the studies mentioned above corresponds to the case $U = \gamma_0 u$ in (2) i.e., $k \equiv 0$; perfect contact. The system we study here cannot be considered as related to boundary conditions of Wentzell type.

Some earlier papers deal with the case of no internal heat transfer in the boundary shell ($\Lambda U \equiv 0$) and perfect contact (Sauer [18] and Hintermann [13], who considers higher order elliptic operators and Dirichlet boundary operators). Everywhere imperfect contact, again with $\Lambda U \equiv 0$, and the singular transition from imperfect to perfect is treated by van der Merwe [24, 25].

This paper systematically explores the situation of a thermally conducting boundary shell with the Carslaw-Jaeger relation for contact. To understand the equations (1)–(3) we discuss the physical background of the problem from the viewpoint of balance/conservation laws (principles) to arrive at a system of implicit evolution equations from which it is difficult to escape. It provides the physical significance of mathematical concepts. Then we go on to a mathematical analysis based on the notion that implicit evolution equations involve a description where initial states live in a world different from the one in which solutions are sought (see e.g., Favini-Yagi [6], Sauer [18]).

Section 2 is devoted to a 'rational' way of deriving the heat equation for inhomogeneous, anisotropic materials. It lays down the fundamental concepts for understanding heat transfer in a shell and thermal interaction between the shell and the solid it encloses. This is where the Carslaw-Jaeger relation enters. The discussion here builds on the detailed work of Rossouw [17], specifically on the notions of *thin boundary models* and *constitutive equations of contact*. The boundary equation with tangential heat conduction and perfect contact, also features in van Rensburg [26]. To be noted is that the dynamic boundary condition obtained reflects the dynamics of the shell and the dynamical interaction between solid and shell. This is the true nature of dynamic boundary conditions. A recent paper of Goldstein [10] presents some of the thoughts involved, although with less than adequate attention to heat transfer in shells and interactions as a source in boundary operators, with perfect contact tacitly assumed. It is of interest to note that in his antecedent to the Wentzell boundary condition, Feller [7] gives a physical interpretation which is akin to the approach we present here.

With the physical background in hand, we go on in Sect. 3 to a mathematical formulation of the derived system of equations. This involves precise requirements about 'conductivity' matrices, smoothness of functions and the boundary. For the necessary precision, and as a reminder that boundary operators should be seen as limits, the formulation continues to use trace operators as in Lions-Magenes [16], for example. Since the ultimate formulation will involve Sobolev spaces over three and two-dimensional manifolds, scaling to dimensionless quantities is introduced in Sect. 4. This essential step is all too often ignored in mathematical texts.

The final formulation comes in Sect. 5 as an implicit evolution equation of the form $\frac{d}{dt}[Bu(t)] + Au(t) = 0$ with A and B unbounded linear operators defined on a domain $\mathfrak{D} \subset X = L^2(\Omega)$ mapping to the space $Y := L^2(\Omega) \times L^2(\Gamma)$ where Ω represents the solid, and its boundary Γ the enclosing shell. The natural initial condition is $\lim_{t\to 0^+}[Bu(t)] = y \in Y$. Thus the solutions $u(t)$ we look for are in a space different from the one in which the given initial state y finds itself. We introduce two operators A_0 and B_0 to be extended in a specific way to the operators A and B.

For the purpose of the extension we introduce in Sect. 6 bilinear forms tailored to the the demands of the problem at hand and develop some of their properties. In particular, we introduce a related sectorial form (Kato [14, Chap.6, p.319 *ff*.]) which plays a key role in defining a family of (generalized) resolvent operators. This is crucial for the extension of the operator pair $\langle A_0, B_0 \rangle$ to an operator pair $\langle A, B \rangle$ that will feature in the implicit equation. Also, towards this goal, we obtain in Sect. 7 some intricate results on the density of the ranges of operators in the space $Y = L^2(\Omega) \times L^2(\Gamma)$. To obtain these results we consider a system of two elliptic equations, the one defined on the open set Ω and the other on the boundary Γ. The subtle link between the two equations is the Carslaw-Jaeger relation, viewed as a constraint, and the thermal interaction between solid and shell. Existence (and uniqueness) is achieved by the introduction of a nonlinear mapping and use of the Leray-Schauder principle—a construct, not necessary for problems of the Wentzell kind. Once this is established, we construct in Sect. 8 the Friedrichs extension of the operator pair. This extension, analogous to its famous namesake for a single operator as presented by Lax-Milgram [15], is jointly, not separately. It was introduced in [19] and later expanded in [24]. Use of elliptic boundary value problems in this context goes back to [24, 25] and is also employed in [9] for the L^p-setting. See also Grubb [11].

In Sect. 9 we (finally) prove that the mathematical problem developed earlier is well-posed. To achieve this, we directly construct a holomorphic family of solution operators $S(t)$ for t in a positive cone of the complex plane. These operators map arbitrary initial states $y \in Y = L^2(\Omega) \times L^2(\Gamma)$ to solutions of the form $u(t) = S(t)y \in X = L^2(\Omega)$. It is remarkable that this is done without explicit recourse to semigroup theory. The approach is partly in accordance with the ideas of Arendt and ter Elst [2] which also ties in with the sectorial form introduced in Sect. 6. There is a semigroup lurking in the background, though. This is briefly discussed in Sect. 10.

2 Heat transfer in solids and shells

In this section we give a systematic account of the physical model that underpins Eqs. (1)–(3). In the process the basic assumptions and the meaning of mathematical notions such as co-normal derivative will become clear. We do this under various subheadings.

In a solid. We represent the solid under consideration as a simply connected open set $\Omega \subset \mathbb{R}^3$ with a C^∞ boundary Γ. To formulate the principle of *balance of thermal energy* we need the scalar quantity $q(x, t)$, thermal density (Joule m^{-3}) and the vector quantity $\boldsymbol{\varphi}$, thermal flux density (Watt m^{-2}). When there are no external sources the principle is expressed mathematically as

$$\frac{d}{dt}\int_{\mathscr{G}} q(x,t)\mathrm{d}x = -\int_{\partial\mathscr{G}} \boldsymbol{\varphi}\cdot\boldsymbol{\nu}\mathrm{d}S, \tag{4}$$

with $\mathscr{G} \subset \overline{\mathscr{G}} \subset \Omega$ a suitable, but arbitrary open set with boundary $\partial\mathscr{G}$ and $\boldsymbol{\nu}$ denoting the unit exterior normal to $\partial\mathscr{G}$ (as it will always do from now on). In words the principle reads: *In the absence of external sources the rate of increase of thermal energy in an arbitrary part $\mathscr{G}$ of the body is balanced by the netto flow-rate (flux) of thermal energy over its boundary*. The minus-sign in (4) indicates flow into $\mathscr{G}$ from $\partial\mathscr{G}$.

Under some differentiability and related assumptions (about functions we do not know), by use of the divergence theorem, the statement (4) can be re-written in the form

$$\int_{\mathscr{G}} \left[q_t(x,t) + \nabla\cdot\boldsymbol{\varphi}(x,t)\right]\mathrm{d}x = 0. \tag{5}$$

Since $\mathscr{G}$ is arbitrary, the general balance equation

$$q_t(x,t) + \nabla\cdot\boldsymbol{\varphi}(x,t) = 0; \quad x \in \Omega,\ t > 0, \tag{6}$$

follows from (5). We note that the gradient operation ∇ only involves differentiation with respect to spatial variables.

The single general equation (6) has four unknowns. They are augmented by *constitutive equations* that relate to the specific nature of the material under consideration. It is customary to express these equations in terms of the temperature $u(x, t)$ at $x \in \Omega$ and time $t > 0$. We begin with thermal density q. Let $c > 0$ be the *volume-specific heat capacity* (Joule K^{-1} m^{-3}; K = Kelvin) of the material. It is assumed to be constant. The first constitutive equation is

$$q(x,t) = cu(x,t). \tag{7}$$

Next we formulate a constitutive relation for the flux density vector in which the material can be thermally inhomogeneous and anisotropic. Conductivity depends on position and has varying directions. The relation is

$$\boldsymbol{\varphi}(x,t) = -a(x)\nabla u(x,t). \tag{8}$$

Here $a(x)$ is a symmetric, real-valued matrix which is assumed to be positive, i.e. $\boldsymbol{\xi} \cdot a(x)\boldsymbol{\xi} > 0$ for all $x \in \Omega$ and all nonzero $\boldsymbol{\xi} \in \mathbb{R}^3$. This ensures that thermal energy will not flow from low to high temperatures. The components of a have as unit Watt $K^{-1}\,m^{-1}$.

Combination of (6)–(8) gives the equation

$$cu_t(x,t) - \nabla \cdot [a(x)\nabla u(x,t)] = 0 \tag{9}$$

which resembles the familiar heat equation. The traditional heat equation is obtained when $a(x) = \kappa I$ with the conductivity κ a positive constant and I the 3×3 identity matrix. This means that the material is thermally homogeneous and isotropic. It is then also customary to divide throughout by c to obtain the equation $u_t - K\Delta u = 0$ with Δ the Laplacian.

In a shell. To describe the bounding surface Γ of a conducting body we consider it as a two-dimensional orientable differentiable manifold embedded in $\mathbb{R}^3$. The physical entities corresponding to thermal density and flux density will be denoted by Q and $\boldsymbol{\Phi}(x',t)$ defined for $x' \in \Gamma$. Their units will have one spatial dimension less than their 'solid' counterparts. Without special notation we shall assume a parametrization of the surface.

Let $\mathscr{B}$ be a submanifold of Γ bounded by a smooth curve $\partial\mathscr{B}$ and let $\boldsymbol{\mu}(x')$ denote the unit exterior normal to $\partial\mathscr{B}$, tangential to Γ at x'. We consider an exterior source $g(x',t)$ at $x' \in \Gamma$ (with unit Watt m^{-2}). Balance of thermal energy is expressed as follows:

$$\frac{d}{dt}\int_{\mathscr{B}} Q(x',t)\mathrm{d}S(x') = -\int_{\partial\mathscr{B}} \boldsymbol{\Phi}(x',t) \cdot \boldsymbol{\mu}(x')\mathrm{d}\ell(x') + \int_{\mathscr{B}} g(x',t)\mathrm{d}S(x'). \tag{10}$$

It is reasonable to require that the flux $\boldsymbol{\Phi}$ be tangential to Γ in which case the divergence theorem on 2-dimensional manifolds allows us to express the line integral in (10) as the surface integral of $\nabla_S \cdot \boldsymbol{\Phi}$ where $\nabla_S \cdot$ denotes the 'surface divergence' on Γ (e.g., Weatherburn [28, Sect.122, p.238 *ff.*]). As before, we obtain the counterpart of (6) for heat transfer in the shell:

$$Q_t(x',t) + \nabla_S \cdot \boldsymbol{\Phi}(x',t) = g(x',t); \quad x' \in \Gamma,\ t > 0. \tag{11}$$

The constitutive equations are similar in form to those for a solid. We denote the temperature in Γ by $U(x',t)$. Thus, for thermal density the equation is

$$Q(x',t) = CU(x',t); \quad x' \in \Gamma. \tag{12}$$

Again we require that the heat capacity $C > 0$ be constant and note that its unit is Joule $K^{-1}\,m^{-2}$. The constitutive equation for flux density is more complicated. Let $T(x',\Gamma)$ denote the tangent space at $x' \in \Gamma$ and let $b(x')$ be a 3×3 real, symmetric matrix that maps $T(x',\Gamma)$ into itself. The constitutive equation is

$$\boldsymbol{\Phi}(x',t) = -b(x')\nabla_S U(x',t), \tag{13}$$

with ∇_S the surface (tangential/covariant) gradient in Γ, so that $\nabla_S U$ is tangential to Γ. In addition it is required that b is positive in the sense that $\boldsymbol{\eta} \cdot b(x')\boldsymbol{\eta} > 0$ for $\boldsymbol{\eta} \in T(x',\Gamma)$. If the shell does not conduct thermal energy the flux density $\boldsymbol{\Phi}$ is taken as zero. The unit for components of b is Watt K^{-1}. Combination of the expressions (11)–(13) yields

$$CU_t(x',t) - \nabla_S \cdot \left[b(x')\nabla_S U(x',t)\right] = g(x',t). \tag{14}$$

As before, we remark that for the case where $b(x') = KI$ with I the 3×3 identity matrix, $\nabla_S \cdot [b(x')\nabla_S U] = K\Delta_S U$ with Δ_S the Laplace-Beltrami operator.

A SOLID AND A SHELL INTERACTING. Here we discuss the situation of a heat-conducting solid which we model as an open set $\Omega \subset \mathbb{R}^3$ enclosed in a heat conducting shell modelled as the boundary $\Gamma = \partial\Omega$. The orientation of Γ as a differentiable manifold is so chosen that the exterior normal $\boldsymbol{v}$ to $\partial\Omega$ is also the unit exterior normal to the manifold Γ.

Interaction between solid and shell is described by a choice of the source term in (14). First we identify two differential expressions that occur in (9) and (14):

$$Lu(x,t) := -\nabla \cdot [a(x)\nabla u(x,t)]; \quad x \in \Omega, \tag{15}$$

$$\Lambda U(x',t) := -\nabla_S \cdot \left[b(x')\nabla_S U(x',t)\right]; \quad x' \in \Gamma. \tag{16}$$

Associated with the 'operator' L is the co-normal operator

$$\gamma_L u(x',t) := -\boldsymbol{\varphi}|_{x' \in \Gamma} \cdot \boldsymbol{v}(x') = [a(x')\nabla u(x',t)] \cdot \boldsymbol{v}(x'). \tag{17}$$

This is the normal component of flux into the solid at a boundary point x'. Now we postulate: *The external source of thermal energy to the boundary shell is from internal flux at the boundary*. This means that $g(x',t) = \boldsymbol{\varphi} \cdot \boldsymbol{v} = -\gamma_L u$. The Eqs. (9) and (14) obtained in the two previous sections can now be re-phrased:

$$cu_t(x,t) + Lu(x,t) = 0; \quad x \in \Omega, \tag{18}$$

$$CU_t(x',t) + \Lambda U(x',t) + \gamma_L u(x',t) = 0; \quad y \in \Gamma. \tag{19}$$

This, however, does not relate the temperatures u and U. For that we need a *contact constitutive equation* that reflects the nature of contact between the solid and the shell surrounding it. We shall use the one proposed by Carslaw and Jaeger [3, pp.18 & 23]:

$$\begin{aligned} U(x',t) - \lim_{\substack{x \to x' \\ x \in \Omega}} u(x,t) &= U(x',t) - \gamma_0 u(x',t) \\ &= k(x')\gamma_L u(x',t); \quad x' \in \Gamma, \end{aligned} \tag{20}$$

with γ_0 the trace operator that assigns boundary values and $k(x') > 0$, defined on Γ, the *contact function*. An extreme case is $k(x') = 0$, so that $U(x',t) = \gamma_0 u(x',t)$ at $x' \in \Gamma$. This describes *perfect contact*. Wentzell boundary conditions deal with perfect contact over the whole of Γ. Another extreme case is when the contact function $k(x')$ is large. Then thermal insulation is approached ($\gamma_L u(x',t) \approx 0$). What we deal with here is the case of *imperfect* *contact* over all of Γ. The system of Eqs. (18), (19), (20), already mentioned in Sect. 1, will be the concern of the rest of the paper.

3 Mathematical setting

In the foregoing section we derived a system of equations without giving strict mathematical requirements for objects to exist or operations to be valid. Let Ω be a bounded open subset of $\mathbb{R}^3$ with boundary Γ of class C^∞ and unit exterior normal $\boldsymbol{\nu}$. Γ is taken to be a compact infinitely differentiable manifold without boundary, orientated as described above.

For the matrix-valued functions, a and b that occur implicitly in (18) and (19) we require that $a \in C^\infty(\overline{\Omega})$ and $b \in C^\infty(\Gamma)$ (the components are). We require, as stated before, that a and b are real-valued and symmetric. Positivity will be replaced by the stronger requirement of *uniform positive definiteness*:

$$\boldsymbol{\xi} \cdot a(x)\boldsymbol{\xi} \geq c_\Omega |\boldsymbol{\xi}|^2 \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^3,\ x \in \overline{\Omega}, \tag{21}$$

$$\boldsymbol{\eta} \cdot b(x')\boldsymbol{\eta} \geq c_\Gamma |\boldsymbol{\eta}|^2 \quad \text{for all } \boldsymbol{\eta} \in T(x', \Gamma),\ x' \in \Gamma, \tag{22}$$

with c_Ω and c_Γ positive constants. The operators L and Λ are now strongly elliptic. Of course, the matrix b is required to keep tangential vectors tangential. Also note that c_Ω has the same unit as the components of a, and similarly for c_Γ.

To give a precise setting for the boundary operators γ_0, γ_L to be defined we need to introduce appropriate Sobolev spaces and trace operators. The spaces we have in mind are $H^2(\Omega)$ and $H^2(\Gamma)$ embedded in the complex Lebesgue space $L^2(\Omega)$. We note that within the present context the boundary space $H^2(\Gamma)$ needs only to be defined in terms of tangential derivatives (see [16, Remark 7.6, p.37]).

For $u \in H^2(\Omega)$ the co-normal operator γ_L is defined as $\gamma_L u(x') = [a(x')\gamma_0 \nabla u(x')] \cdot \boldsymbol{\nu}(x')$, with γ_0 the trace operator that assigns boundary values. The contact constitutive equation (20) may now be formulated for $u \in H^2(\Omega)$ (after some re-arrangement) as

$$U = \gamma u := \gamma_0 u + k\gamma_L u, \tag{23}$$

and we require that the contact function k be sufficiently smooth; in $C^\infty(\Gamma)$, say. We note that the boundary operator γ equals the customary one only if $k \equiv 0$ in which case $U = \gamma u = \gamma_0 u$.

The system of Eqs. (18), (19), (20) may now be expressed as a dynamic boundary condition problem: *find* $u(t) = u(\cdot, t) \in \mathfrak{D} \subset L^2(\Omega)$ *that satisfies*

$$\left.\begin{aligned} &cu_t + Lu = 0 \text{ in } \Omega; \\ &CU_t + \Lambda U + \gamma_L u = 0 \text{ on } \Gamma; \\ &U = \gamma u = \gamma_0 u + k\gamma_L u, \text{ on } . \end{aligned}\right\} \tag{24}$$

The dynamic boundary condition here is of Wentzell-type only if $k \equiv 0$ that is, when $\gamma u = \gamma_0 u$. In the papers [24, 25] it is assumed that the boundary (shell) does not internally conduct thermal energy. Thus the elliptic boundary operator Λ is taken to be zero and the second Eq. in (24) is replaced by $CU_t + \gamma_L u = 0$. The presence of the operator Λ requires an analysis much deeper than that used before.

Our further investigation will be to identify the domain $\mathfrak{D}$ and appropriate initial conditions. Since the operators involved may not be closeable, this is delicate.

4 Scaling

The heat capacities c, C and the matrices $a(x)$, $b(x)$ (hidden in L and Λ) that occur in Eqs. (23) and (24) are in different physical units. The reason for this is that the set Ω is open in $\mathbb{R}^3$ while the manifold Γ is locally represented in $\mathbb{R}^2$. This may lead to incomparable quantities being compared. We can, however, scale the equations to dimensionless form and the difficulty will exist no more.

One way of doing this is as follows: Let $\vartheta := C/c$ be the chosen unit of length. It may be thought of as the 'thermal thickness' of Γ. As unit of time we choose $T := \vartheta^2 [c/c_\Omega]$ with the requirements (21), (22) in mind. Scaling is by the replacements $t \to t/T$, $x \to x/\vartheta$, $x' \to x'/\vartheta$, $a \to c_\Omega^{-1} a$ and $b \to [\vartheta c_\Omega]^{-1} b$. The function k in (23) is replaced by $[c_\Omega/\vartheta]k$.

Under this scaling (with abuse of notation) the system (23), (24), in dimensionless form, becomes

$$\left.\begin{aligned} &u_t + Lu = 0;\\ &U_t + \Lambda U + \gamma_L u = 0;\\ &U = \gamma u = \gamma_0 u + k\gamma_L u. \end{aligned}\right\} \tag{25}$$

These are the Eqs. (1)–(3).

5 An implicit evolution equation

It is tempting to eliminate the boundary temperature U in (25) to obtain the (seemingly) familiar dynamic boundary equation $[\gamma u]_t + \Lambda[\gamma u] + \gamma_L u = 0$. We shall resist this temptation, but keep in mind that U is intricately related to u. Thus we write the system in vector form as follows:

$$\begin{aligned} &\frac{\partial}{\partial t}\begin{pmatrix} u \\ U \end{pmatrix} + \begin{pmatrix} L & 0 \\ \gamma & \Lambda \end{pmatrix}\begin{pmatrix} u \\ U \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};\\ &U = \gamma u = \gamma_0 u + k\gamma_L u. \end{aligned} \tag{26}$$

This suggests the setting of an implicit evolution equation of the form $\frac{d}{dt}[Bu(t)] + Au(t) = 0$, that involves unbounded linear operators A and B between (complex) Banach spaces X and Y. More precisely, we consider a domain $\mathfrak{D} \subset X$ and $A, B : \mathfrak{D} \to Y$. The natural initial condition would be $\lim_{t\to 0^+}[Bu(t)] = y \in Y$, with y given. At the core of the investigation are the 'resolvent operators' $P(\lambda) = (\lambda B + A)^{-1} : Y \to \mathfrak{D}$ for complex λ, taken as bounded linear operators. It has been shown that if $P(\lambda)$ exists for two distinct values of λ, the operator pair $\langle A, B\rangle : u \in \mathfrak{D} \to \langle Au, Bu\rangle \in Y \times Y$ is closed. If A and B are both closed, this is certainly the case, but the converse is not necessarily true. A counterexample within the context of dynamic boundary conditions can be found in [21].

Let us work towards this setting for the Eq. (25). Let $X = L^2(\Omega)$ and let $\mathfrak{D}_0 = C^\infty(\overline{\Omega})$ be a preliminary domain for which the operations in the equations are well-defined. Further, we take $Y = L^2(\Omega) \times L^2(\Gamma)$. Elements of Y will be denoted by $\langle f, F\rangle$; $f \in L^2(\Omega), F \in L^2(\Gamma)$. With the vector form (26) in mind, we define the operators $A_0, B_0 : \mathfrak{D}_0 \to Y$ as follows:

$$A_0 u = \langle Lu, \Lambda U + \gamma_L u\rangle; \tag{27}$$

$$B_0 u = \langle u, U\rangle; \tag{28}$$

$$U = \gamma u = \gamma_0 u + k\gamma_L u. \tag{29}$$

In the next three sections we prepare for an extension of the operators A_0, B_0 to a closed pair that will fulfill our needs. To achieve this we define a third operator $C_0 : \mathfrak{D}_0 \to Y$ by

$$C_0 u = A_0 u + B_0 u = \langle Lu + u, \Lambda U + U + \gamma_L u\rangle. \tag{30}$$

6 Some bilinear forms

For the purpose of extending the operators A_0 and B_0 defined in Sect. 5, we define in this section some bilinear (sesquilinear) forms on the domain $\mathfrak{D}_0 \subset X = L^2(\Omega)$ related to these operators which map to the product space $Y = L^2(\Omega) \times L^2(\Gamma)$. Inner products in $L^2(\Omega)$ and $L^2(\Gamma)$ are denoted by subscripts such as $(\ ,\ )_\Omega$ and $(\ ,\ )_\Gamma$ and likewise for the norms. The bilinear forms are set out by means of the inner product $(\langle f, F\rangle, \langle g, G\rangle)_Y := (f, g)_\Omega + (F, G)_\Gamma$ in Y. We begin with

$$R_0(u, v) = (A_0 u, C_0 v)_Y = (A_0 u, A_0 v)_Y + (A_0 u, B_0 v)_Y; \tag{31}$$

$$\begin{aligned} S_0(u, v) &= (B_0 u, C_0 v)_Y = (B_0 u, A_0 v)_Y + (B_0 u, B_0 v)_Y; \\ [u, v] &= R_0(u, v) + S_0(u, v) = (C_0 u, C_0 v)_Y, \end{aligned} \tag{32}$$

having used the definitions (27)–(30) to expand. These forms may be expressed more explicitly by doing integration by parts, mindful of the fact that on Γ this is valid because $b\nabla_S U$ is tangential to Γ. With the definitions (15), (16) and (17) in mind, we find with $V = \gamma v = \gamma_0 v + k\gamma_L v$,

$$\left.\begin{aligned} (A_0 u, B_0 v)_Y &= (a\nabla u, \nabla v)_\Omega + (b\nabla_S U, \nabla_S V)_\Gamma + (\gamma_L u, k\gamma_L v)_\Gamma; \\ (B_0 u, A_0 v)_Y &= (\nabla u, a\nabla v)_\Omega + (\nabla_S U, b\nabla_S V)_\Gamma + (k\gamma_L u, \gamma_L v)_\Gamma. \end{aligned}\right\} \tag{33}$$

Since the matrices a, b are real-symmetric and the function k is real-valued, it follows from (33) that

$$(A_0 u, B_0 v)_Y = (B_0 u, A_0 v)_Y = (a\nabla u, \nabla v)_\Omega + (b\nabla_S U, \nabla_S V)_\Gamma + (k\gamma_L u, \gamma_L v)_\Gamma.$$

The bilinear forms in question may now be expressed in expanded form:

$$R_0(u,v) = (A_0 u, A_0 v)_Y + (a\nabla u, \nabla v)_\Omega + (b\nabla_S U, \nabla_S V)_\Gamma + (k\gamma_L u, \gamma_L v)_\Gamma; \tag{34}$$

$$S_0(u,v) = (B_0 u, B_0 v)_Y + (a\nabla u, \nabla v)_\Omega + (b\nabla_S U, \nabla_S V)_\Gamma + (k\gamma_L u, \gamma_L v)_\Gamma; \tag{35}$$

$$\begin{aligned}[u,v] =& (A_0 u, A_0 v)_Y + (B_0 u, B_0 v)_Y \\ &+ 2\Big[(a\nabla u, \nabla v)_\Omega + (b\nabla_S U, \nabla_S V)_\Gamma + (k\gamma_L u, \gamma_L v)_\Gamma\Big].\end{aligned} \tag{36}$$

We explore some properties of the bilinear forms. For this we denote by $\widehat{R_0}(u) := R_0(u,u)$ and $\widehat{S_0}(u) := S_0(u,u)$ the associated quadratic forms. From (34) and (35) we see that R_0 and S_0 are symmetric and therefore the quadratic forms are real-valued. The same, and more, is true for [,]. Indeed, if we observe that $(B_0 u, B_0 v)_Y = (u,v)_\Omega + (U,V)_\Gamma$, it follows from (36) that

$$\begin{aligned}|[u]|^2 :=& [u,u] = \|A_0 u\|_Y^2 + \|u\|_\Omega^2 + \|U\|_\Gamma^2 \\ &+ 2[(a\nabla u, \nabla u)_\Omega + (b\nabla_S U, \nabla_S U)_\Gamma + (k\gamma_L u, \gamma_L u)_\Gamma] \geq \|u\|_\Omega^2.\end{aligned} \tag{37}$$

Thus [,] is an inner product and defines a norm $|[\ ]|$ on $\mathfrak{D}_0$. From the identity (36), $\|A_0 u\|_Y^2 \leq |[u]|^2$. The same holds for $\|B_0 u\|_Y^2$. Thus we have

Theorem 1 *The operators $A_0, B_0, C_0 : \mathfrak{D}_0 \to Y$ are bounded in $|[\ ]|$.*

Theorem 2 *The mapping $u \in \langle \mathfrak{D}_0, \|\ \|_\Omega \rangle \to u \in \langle \mathfrak{D}_0, |[\ ]| \rangle$ is injective in the sense that if $\{u_n\} \subset \mathfrak{D}_0$ is a Cauchy sequence in $|[\ ]|$ and $\|u_n\|_\Omega \to 0$, then $|[u_n]| \to 0$.*

Proof From (37) we see that

1. $\{u_n\}$ is a Cauchy-sequence in $H^1(\Omega)$ since $(a\nabla u_n, \nabla u_n)_\Omega \geq c_\Omega \|\nabla u_n\|_\Omega^2$. Hence, $u_n \to 0$ in $H^1(\Omega)$. By the trace theorem, $\gamma_0 u_n \to 0$ in $H^{1/2}(\Gamma)$.
2. Likewise, $\{U_n\}$ is a Cauchy-sequence in $H^1(\Gamma)$. Let $U \in H^1(\Gamma)$ be its limit.
3. $\{\gamma_L u_n\}$ is a Cauchy-sequence in the weighted space $L^2(\Gamma, k\mathrm{d}S)$. Let $U_{L,k}$ be its limit.
4. $\{A_0 u_n\}$ is a Cauchy-sequence in Y. This means that $\{Lu_n\}$ is a Cauchy-sequence in $L^2(\Omega)$ and so is $\{\Lambda U_n + \gamma_L u_n\}$ in $L^2(\Gamma)$.

By the Aronszajn coerciveness-estimates (see Agmon [1, Sects 10,11]) there is a constant $C_A > 0$ such that

$$\|u_n\|_{H^2(\Omega)} \leq C_A\Big[\|Lu_n\|_\Omega + \|u_n\|_\Omega\Big]. \tag{38}$$

Thus $u_n \to 0$ in $H^2(\Omega)$. By the trace theorem, therefore, $\gamma_L u_n \to 0$ in $H^{1/2}(\Gamma) \subset L^2(\Gamma)$. Since the function k is bounded, $\|\gamma_L u_n\|_{L^2(\Gamma, k\mathrm{dS})}$ is dominated by $\|\gamma_L u_n\|_\Gamma$ and it follows that $U = \sqrt{k} U_{L,k} = 0$.

Finally an estimate similar to (38) for ΛU_n shows that $U_n \to 0$ in $H^2(\Gamma)$. Scrutiny of (37) shows that indeed $|[u_n]| \to 0$. □

For later purposes we define a family of bilinear forms that depends on a complex parameter λ:

$$Q_0(u,v;\lambda) := R_0(u,v) + (\lambda+1)S_0(u,v) = [u,v] + \lambda S_0(u,v);\ u,v \in \mathfrak{D}_0, \tag{39}$$

and write $\widehat{Q_0}(u;\lambda) := Q_0(u,u;\lambda)$. By (35) $\widehat{S_0}(u) \geq 0$ and it follows immediately that for real $\lambda > 0$, $\widehat{Q_0}(u;\lambda) \geq |[u]|^2$. For such λ, $Q_0(\cdot;\lambda)$ is *positive definite*. But there is more. Let $\phi \in (0,\pi/2)$ and let $\Sigma_\phi := \{\lambda \in \mathbb{C} : |\arg\lambda| < \phi + \pi/2\}$. The following result, a more general version of which can be found in [24], will often be used:

Lemma 1 *For $r,s \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, let $z_\lambda = r\lambda + s$. If $\lambda \in \Sigma_\phi$,*

$$\text{(i)} \qquad |z_\lambda|^2 \geq s^2 \cos^2\phi,$$

and

$$\text{(ii)} \qquad |z_\lambda|^2 \geq r^2|\lambda|^2\cos^2\phi.$$

Proof This follows from $|z_\lambda|^2 = r^2|\lambda|^2 + s^2 + 2rs|\lambda|\cos(\arg\lambda) \geq r^2\lambda|^2 + s^2 - 2rs|\lambda|\sin\phi$ and the inequalities $2rs|\lambda|\sin\phi \leq r^2|\lambda|^2 + s^2\sin^2\phi;\ 2rs|\lambda|\sin\phi \leq r^2|\lambda|^2\sin^2\phi + s^2$. □

From (i) in Lemma 1, we have with $z_\lambda := \widehat{Q_0}(u;\lambda) = \lambda\widehat{S_0}(u) + |[u]|^2$, taken from (39),

Theorem 3 *For $\lambda \in \Sigma_\phi$ and $u \in \mathfrak{D}_0$, $|\widehat{Q_0}(u;\lambda)| \geq \cos\phi|[u]|^2$.*

7 Density theorems

We proceed to show that the ranges of the operators C_0 and B_0 are dense in $Y = L^2(\Omega) \times L^2(\Gamma)$.

For given $\langle f,g\rangle \in Y$ consider the system of elliptic equations

$$\left.\begin{aligned} &Lu + u = f \text{ in } \Omega;\\ &\gamma u = \gamma_0 u + k\gamma_L u = U \text{ on } \Gamma. \end{aligned}\right\} \tag{40}$$

$$\Lambda U + U + \gamma_L u = g \text{ in } \Gamma. \tag{41}$$

We note that the boundary condition in (40) is the Carslaw-Jaeger relation. Since the manifold Γ is without boundary, there is no boundary condition to accompany (41).

It has been shown in [23] that the boundary operator γ is normal, covers L (see [16, p.113]) and that the problem (40) has, for given $U \in H^{1/2}(\Gamma)$, a unique solution in $H^2(\Omega)$. Similarly the Eq. (41) has, for given $u \in H^2(\Omega)$, a unique solution in $H^2(\Gamma)$ (Taylor [22], for example). We need to bring the two equations together.

Theorem 4 *There exists a unique $u \in H^2(\Omega)$ with $\gamma u \in H^2(\Gamma)$ so that the equations* (40), (41) *are satisfied.*

Proof We begin by making the equations as homogeneous as possible. Let $w \in H^2(\Omega)$ be the solution of the problem $Lw + w = f$; $\gamma w = 0$ and let W be the solution of $\Lambda W + W = g - \gamma_L w$. Then solving the problem (40), (41) reduces to solving the following:

$$\left.\begin{aligned} Lv + v &= 0 \text{ in } \Omega; \\ \gamma v &= W + V \text{ on } \Gamma. \end{aligned}\right\} \tag{42}$$

$$\Lambda V + V = -\gamma_L v \text{ in } \Gamma. \tag{43}$$

Indeed, the solution we look for, would be of the form $u = w + v$.

We show that a unique solution of (42), (43) exists. For this purpose let v in $H^2(\Omega)$ be chosen and let $V \in H^2(\Gamma)$ be the solution of (43). We then use the V so obtained in the boundary condition of (42). The problem (42) has a unique solution which we denote by $T(v) \in H^2(\Omega)$. Formally, $T : v \in H^2(\Omega) \to V \in H^2(\Gamma) \to T(v) \in H^2(\Omega)$.

The next task is to show that the (nonlinear) operator T has a fixed point, a solution of (42), (43). For this we apply the Leray-Schauder principle.

First we show that T is compact. Consider $v, v^\dagger \in H^2(\Omega)$ and let $V, V^\dagger \in H^2(\Gamma)$ be the corresponding solutions of (43). Then $L[T(v) - T(v^\dagger)] + [T(v) - T(v^\dagger)] = 0$, $\gamma[T(v) - T(v^\dagger)] = V - V^\dagger$ and $\Lambda[V - V^\dagger] = -\gamma_L[v - v^\dagger]$. From the standard apriori estimate [16, Thm. 5.1, p.149*ff.*]

$$\|T(v) - T(v^\dagger)\|_{H^2(\Omega)} \leq \text{const.}\|V - V^\dagger\|_{H^{1/2}(\Gamma)}. \tag{44}$$

From the same estimates on manifolds (e.g. [22, Chap. 5]) we have

$$\|V - V^\dagger\|_{H^2(\Gamma)} \leq \text{const.}\|\gamma_L v - \gamma_L v^\dagger\|_{H^{1/2}(\Gamma)},$$

so that the mapping $\gamma_L v \in H^{1/2}(\Gamma) \to V \in H^2(\Gamma)$ is continuous. Let $\{v_n\}$ be a weakly convergent sequence in $H^2(\Omega)$ and V_n the corresponding V. Then from the continuity of the mapping $v \in H^2(\Omega) \to \gamma_L v \in H^{1/2}(\Gamma)$, the sequence $\{V_n\}$ is weakly convergent in $H^2(\Gamma)$. By the Rellich embedding (Hebey [12, Chap. 3]), $\{V_n\}$ is norm-convergent in $H^{1/2}(\Gamma)$. From (44) we now have $\|T(v_n) - T(v_m)\|_{H^2(\Omega)} \leq \text{const.}\|V_n - V_m\|_{H^{1/2}(\Gamma)}$ which proves the compactness of T.

Next, suppose that for $0 \leq \lambda \leq 1$, $v_\lambda = \lambda T(v_\lambda)$ and let V_λ be the solution of (43) corresponding to v_λ. Now we have, after some manipulation,

$$\left.\begin{aligned} Lv_\lambda + v_\lambda &= 0; \\ \gamma v_\lambda &= \lambda[W + V_\lambda]; \\ \Lambda V_\lambda + V_\lambda &= -\gamma_L v_\lambda. \end{aligned}\right\} \tag{45}$$

To estimate v_λ we consider the first of the equations in (45) and expand to obtain

$$\|Lv_\lambda + v_\lambda\|_\Omega^2 = \|Lv_\lambda\|_\Omega^2 + \|v_\lambda\|_\Omega^2 + (Lv_\lambda, v_\lambda)_\Omega + (v_\lambda, Lv_\lambda)_\Omega = 0. \tag{46}$$

The last two terms in (46) may, as we have done in Sect. 6, be integrated by parts to obtain

$$\|Lv_\lambda\|^2_\Omega + \|v_\lambda\|^2_\Omega + 2(a\nabla v_\lambda, \nabla v_\lambda)_\Omega - 2\mathrm{Re}\,(\gamma_0 v_\lambda, \gamma_L v_\lambda)_\Gamma = 0. \tag{47}$$

From the second equation in (45) we obtain $\gamma_0 v_\lambda = \lambda[W + V_\lambda] - k\gamma_L v_\lambda$. This, substituted in (47), gives

$$\begin{aligned} \|Lv_\lambda\|^2_\Omega + \|v_\lambda\|^2_\Omega + 2(a\nabla v_\lambda, \nabla v_\lambda)_\Omega + 2\|\gamma_L v_\lambda\|^2_{L^2(\Gamma,k\mathrm{dS})} \\ - 2\lambda\mathrm{Re}\,(V_\lambda, \gamma_L v_\lambda)_\Gamma = 2\lambda\mathrm{Re}\,(W, \gamma_L v_\lambda)_\Gamma. \end{aligned} \tag{48}$$

Now we obtain from the third equation in (45), again after integration by parts, $-\mathrm{Re}\,(V_\lambda, \gamma_L v_\lambda)_\Gamma = (b\nabla_s V_\lambda, \nabla_s V_\lambda)_\Gamma + \|V_\lambda\|^2_\Gamma$, which, together with (48), leads to the identity

$$\begin{aligned} \|Lv_\lambda\|^2_\Omega + \|v_\lambda\|^2_\Omega + 2(a\nabla v_\lambda, \nabla v_\lambda)_\Omega + 2\|\gamma_L v_\lambda\|^2_{L^2(\Gamma,k\mathrm{ds})} \\ + 2\lambda[(b\nabla_s V_\lambda, \nabla_s V_\lambda)_\Gamma + \|V_\lambda\|^2_\Gamma] = 2\lambda\mathrm{Re}\,(W, \gamma_L v_\lambda)_\Gamma. \end{aligned} \tag{49}$$

The well-known coercivity estimate for elliptic operators is now invoked and combined with with (49). We also note that none of the terms on the left of (49) are negative. The result is

$$\begin{aligned} C_A\|v_\lambda\|^2_{H^2(\Omega)} &\le [\|Lv_\lambda\|^2_\Omega + \|v_\lambda\|^2_\Omega] \\ &\le \|Lv_\lambda\|^2_\Omega + \|v_\lambda\|^2_\Omega + 2(a\nabla v_\lambda, \nabla v_\lambda)_\Omega + 2\|\gamma_L v_\lambda\|^2_{L^2(\Gamma,k\mathrm{dS})} \\ &\quad + 2\lambda[(b\nabla_s V_\lambda, \nabla_s V_\lambda)_\Gamma + \|V_\lambda\|^2_\Gamma] = 2\lambda\mathrm{Re}\,(W, \gamma_L v_\lambda)_\Gamma. \end{aligned} \tag{50}$$

The right of (50) can now be estimated with the aid of the trace theorem (with $c > 0$ a constant). For $\varepsilon > 0$,

$$\begin{aligned} 2\lambda\mathrm{Re}\,(W, \gamma_L v_\lambda)_\Gamma &\le 2\|W\|_\Gamma\|\gamma_L v_\lambda\|_\Gamma \\ &\le 2c\|W\|_\Gamma\|v_\lambda\|_{H^2(\Omega)} \le \frac{c^2}{\varepsilon}\|W\|^2_\Gamma + \varepsilon\|v_\lambda\|^2_{H^2(\Omega)}. \end{aligned} \tag{51}$$

Combination of (50) and (51) gives

$$(C_A - \varepsilon)\|v_\lambda\|^2_{H^2(\Omega)} \le \frac{c^2}{\varepsilon}\|W\|^2_\Gamma. \tag{52}$$

By choosing $0<\varepsilon<C_A$, we see from the Leray-Schauder principle that the mapping T indeed has a fixed point v which solves the system (42), (43). That the solution is unique can be seen by noticing that v also obeys the inequality (52) so that $W = 0$ implies that $v = 0$. The regularity of γu is evident. □

Theorem 5 $C_0[\mathfrak{D}_0]$ *is dense in* Y.

Proof Let us approximate f by $f_n \in C^\infty(\overline{\Omega})$ and g by $g_n \in C^\infty(\Gamma)$. Let W_n and v_n denote the corresponding entities when f is replaced by f_n and g by g_n and set $u_n = w_n + v_n$. By standard regularity of solutions of elliptic equations, $u_n \in \mathfrak{D}_0$.

From the standard apriori estimates, $w_n \to w$ in $H^2(\Omega)$ and, consequently, $W_n \to W$ in $H^2(\Gamma)$. From (52) we see that $v_n \to v$ in $H^2(\Omega)$. Thus $u_n = w_n + v_n \to u = w + v$. Therefore $C_0 u_n = \langle f_n, g_n\rangle \to \langle f, g\rangle$ in Y and density is established. □

For later use we also need the following result:

Theorem 6 *$B_0[\mathfrak{D}_0]$ is dense in Y.*

Proof Suppose that $F = \langle f, g\rangle \in Y$ is orthogonal to $B_0[\mathfrak{D}_0]$. That is $(f, u)_\Omega + (g, \gamma u)_\Gamma = 0$ for every $u \in \mathfrak{D}_0$. In particular, $(f, u)_\Omega = 0$ for every $u \in C_0^\infty(\Omega)$, so that $f = 0$. Thus, $(g, \gamma u)_\Gamma = 0$ for all $u \in \mathfrak{D}_0$.

Let $u \in H^2(\Omega)$ be the solution of the system (40), (41) with $f = 0$. Take $g_n \in C^\infty(\Gamma)$ that approximates g, and let u_n be the solution of the same system of equations with g replaced by g_n. From the equation $Lu + u = 0$ in Ω we obtain in the familiar way

$$(a\nabla u, \nabla u_n)_\Omega + (u, u_n)_\Omega + (k\gamma_L u, \gamma_L u_n)_\Gamma - (\gamma_L u, U_n)_\Gamma = 0. \tag{53}$$

From the equation $\Lambda U + U = g - \gamma_L u$, since $(g, U_n)_\Gamma = 0$, we obtain similarly

$$-(\gamma_L u, U_n)_\Gamma = (b\nabla_S U, \nabla_S U_n)_\Gamma + (U, U_n)_\Gamma. \tag{54}$$

Combination of (53) and (54) leads to the identity

$$(a\nabla u, \nabla u_n)_\Omega + (u, u_n)_\Omega + (k\gamma_L u, \gamma_L u_n)_\Gamma + (b\nabla_S U, \nabla_S U_n)_\Gamma + (U, U_n)_\Gamma = 0. \tag{55}$$

We have proved above that $u_n \to u$ in $H^2(\Omega)$ and, consequently, that $U_n \to U$ in $H^2(\Gamma)$. So if we take limits in (55) the conclusion is that $u = 0$ and $U = 0$. But then, by (41), $g = 0$. □

8 The Friedrichs extension

We are now in a position to extend the operators A_0, B_0 to a domain $\mathfrak{D} \subset L^2(\Omega)$ in such a way that crucial properties are kept intact.

The first step is completion of the domain $\mathfrak{D}_0$ with respect to the norm $|[\ \ ]|$ to a Hilbert space $\mathfrak{D}_1$. From (37) it is clear that the norm $|[\ \ ]|$ is stronger than the $L^2(\Omega)$ norm and therefore an embedding of $J : \mathfrak{D}_1 \hookrightarrow L^2(\Omega)$ should be possible. Indeed, suppose $\{u_n\} \subset \mathfrak{D}_0$ is a Cauchy-sequence in $|[\ \ ]|$ and $u' \in \mathfrak{D}_1$ is associated with it. By (37), it is also a Cauchy-sequence in $L^2(\Omega)$ with limit $u \in L^2(\Omega)$. The embedding is $Ju' = u$ and $\|Ju'\|_\Omega \le |[u']|$.

From Theorem 2 we see that the operator J is bijective so that the elements of $\mathfrak{D}_1$ may be identified with elements of $L^2(\Omega)$. Theorem 1 allows us to extend by continuity the operators A_0, B_0 and C_0 to bounded linear operators A_1, B_1 and C_1 on $\mathfrak{D}_1$. From the definitions (31), (32) and (39) we see that the bilinear forms R_0, S_0 and $Q_0(.\ ,\ .;\lambda)$ may also be extended to bounded bilinear forms R, S and $Q(.\ ,\ .;\lambda)$ defined on $\mathfrak{D}_1$.

Unfortunately, operators extended by continuity may lose some of their properties. The next step is to restrict the extended operators in such a way that desired properties are retained. For this purpose we consider the variational problem: *Given $y \in Y$, find $u \in \mathfrak{D}_1$ such that for all $v \in \mathfrak{D}_1$*

$$Q(u, v; \lambda) = (y, C_1 v)_Y. \tag{56}$$

On account of Theorem 3 the Lax-Milgram lemma ensures that for every λ in the sectorial domain Σ_ϕ there is a unique solution $u_y \in \mathfrak{D}_1$. We consider u_y as an element of $L^2(\Omega)$ and the mapping $y \rightarrow u_y$ from Y to $X = L^2(\Omega)$, but immediately take note of the fact that the topology of X is weaker than that of $\mathfrak{D}_1$.

Theorem 7 *The linear operators* $P(\lambda) : y \in Y \rightarrow u_y \in L^2(\Omega)$; $\lambda \in \Sigma_\phi$, *are bounded and invertible.*

Proof With $u = v = u_y$ in (56), use of Theorem 3 leads to $\cos\phi|[u_y]|^2 \leq |\widehat{Q}(u_y;\lambda)| \leq \|y\|_Y.\|C_1 u_y\| = \|y\|_Y.|[u_y]|$. Thus $\cos\phi|[u_y]| \leq \|y\|_Y$. From (37) we see that $\|u_y\|_\Omega \leq |[u_y]|$ and we arrive at the inequality $\|P(\lambda)y\|_\Omega \leq \|y\|_Y / \cos\phi$ which establishes boundedness.

Further, if $P(\lambda)y = u_y = 0$, it follows from (56) that $(y, C_1 v)_Y = 0$ for all $v \in \mathfrak{D}_1$. From Theorem 5 it follows that $C_1[\mathfrak{D}_1]$ is dense in Y and hence $y = 0$ so that $P(\lambda)$ is invertible. □

Theorem 8 *The resolvent equation*

$$P(\lambda) - P(\mu) = (\mu - \lambda)P(\mu)B_1 P(\lambda) \tag{57}$$

holds for λ, $\mu \in \Sigma_\phi$. *In addition,*

$$P(\mu)B_1 P(\lambda) = P(\lambda)B_1 P(\mu). \tag{58}$$

The range of $P(\lambda)$ *does not depend on* λ.

Proof Suppose $u_y = P(\lambda)y$. From the identities

$$\begin{aligned}(y, C_1 v)_Y = Q(u_y, v; \lambda) &= R(u_y, v) + (\lambda + 1)S(u_y, v)\\ &= Q(u_y, v; \mu) - (\mu - \lambda)S(u_y, v)\\ &= Q(u_y, v; \mu) - (\mu - \lambda)(B_1 u_y, C_1 v)_Y,\end{aligned}$$

we conclude that $Q(u_y, v; \mu) = (y + (\mu - \lambda)B_1 u_y, C_1 v)_Y$ for all $v \in \mathfrak{D}_1$. Therefore, $P(\lambda)y = u_y = P(\mu)[y + (\mu - \lambda)B_1 u_y]$ which translates directly into (57) and proves the invariance of the range. The commutation rule (58) is obtained by interchange of the roles of μ and λ in (57). □

We may now, without reservations, set $\mathfrak{D} := P(\lambda)[Y]$; $\lambda \in \Sigma_\phi$. Evidently, $\mathfrak{D}_0 \subset \mathfrak{D} \subset \mathfrak{D}_1$. Let us denote by A, B and C the restrictions of A_1, B_1 and C_1 to $\mathfrak{D}$ and notice immediately that in the expressions (57) and (58), B_1 can be replaced by B. From now on we consider $\mathfrak{D}$ as a linear subspace of $L^2(\Omega)$. As a matter of fact, the operators A and B are restrictions of operators bounded in a stronger topology than that of $L^2(\Omega)$.

The solution u_y satisfies a system of equations so similar to (40), (41) that Theorem 4 applies to it. This leads to

Theorem 9 $\mathfrak{D} = \{u \in H^2(\Omega) : \gamma u = \gamma_0 u + k\gamma_L u \in H^2(\Gamma)\}$.

Thus the operators A and B retain their original meaning, at least in the sense of regular distributions. Moreover, the domain $\mathfrak{D}$ is determined by the contact function k. Therefore transition from imperfect to perfect contact represents a singular perturbation.

Theorem 10 *For $\lambda \in \Sigma_\phi$, $(\lambda+1)B + A$ is invertible and*

$$P(\lambda) = [(\lambda+1)B + A]^{-1}. \tag{59}$$

The operator pair $\langle A, B\rangle : \mathfrak{D} \subset L^2(\Omega) \to Y \times Y$ is closed.

Proof From the definitions (31) and (39) it is seen that $Q(u, v; \lambda) = ([A + (\lambda + 1)B]u, Cv)_Y$ for $u,\ v \in \mathfrak{D}$. Thus $([A + (\lambda+1)B]u_y, Cv)_Y = (y, Cv)_Y$ for all $v \in \mathfrak{D}$. By Theorem 5 $C[\mathfrak{D}]$ is dense in Y. Therefore, $[A + (\lambda+1)B]P(\lambda)y = [A + (\lambda+1)B]u_y = y$.

By Theorem 7, $A + (\lambda+1)B$, being the inverse of a bounded operator, is closed for all $\lambda \in \Sigma_\phi$, that is, for at least two distinct values of λ. □

The operator pair $\langle A, B\rangle$ is called the *Friedrichs extension* of $\langle A_0, B_0\rangle$. We shall use the notation $\mathfrak{D}_Y := B[\mathfrak{D}]$. From the definition (28) it is seen that $\|Bu\|_Y^2 = \|u\|_\Omega^2 + \|U\|_\Gamma^2 \geq \|u\|_\Omega^2$. We therefore have

Theorem 11 *The operator $B : \mathfrak{D} \subset L^2(\Omega) \to \mathfrak{D}_Y \subset Y$ has a bounded inverse B^{-1}.*

9 The solution operators

The ultimate step is to construct the solution of the Cauchy-problem

$$\left.\begin{aligned} &\frac{d}{dt}[Bu(t)] + Au(t) = 0; \\ &\lim_{t\to 0^+}[Bu(t)] = y, \end{aligned}\right\} \tag{60}$$

in which $\langle A, B\rangle$ is the extended pair constructed in Sect. 8. One burning issue is to identify the class of initial states $y \in Y$ for which the problem can be solved.

Our approach is to represent the solution in the form $u(t) = S(t)y$ with the solution operators $S(t) : Y \to L^2(\Omega)$ to be constructed. For this purpose we need some important estimates.

Theorem 12 *For $\lambda \in \Sigma_\phi$ and $y \in Y$ the following holds*:

$$\|BP(\lambda)y\|_Y \leq \frac{1}{|\lambda|\cos\phi}\|y\|_Y. \tag{61}$$

$$\|AP(\lambda)y\|_Y \leq \left[1 + \frac{|\lambda|+1}{|\lambda|\cos\phi}\right]\|y\|_Y. \tag{62}$$

$$\|P(\lambda)y\|_\Omega \leq \frac{1}{\cos\phi}\|y\|_Y. \tag{63}$$

Proof As in Sect. 8, let $u_y = P(\lambda)y$. Then, from (59), $[(\lambda+1)B + A]u_y = \lambda Bu_y + Cu_y = y$. Hence,

$$\lambda\|Bu_y\|_Y^2 + (Cu_y, Bu_y)_Y = (y, Bu_y)_Y. \tag{64}$$

From the expressions (32) and (35) we see that the term $(Cu_y, Bu_y)_Y \geq 0$ and therefore, with $z_\lambda = \lambda\|Bu_y\|_Y^2 + (Cu_y, Bu_y)_Y$, Lemma 1 (ii) gives $|z_\lambda| \geq \|Bu_y\|_Y^2.|\lambda|\cos\phi$. The right of (64) can be estimated by the Schwarz inequality so that in the end $\|Bu_y\|_Y.|\lambda|\cos\phi \leq \|y\|_Y$, which is the same as the inequality (61).

The identity $AP(\lambda)y = y - (\lambda+1)BP(\lambda)y$ together with (61) yields the inequality (62).

To derive the inequality (63), we use Lemma 1 (i) to obtain $|z_\lambda| \geq (Cu_y, Bu_y)_Y.\cos\phi$. But from (35) we see that $(Cu_y, Bu_y)_Y \geq \|Bu_y\|_Y^2$ so that indeed, $\|Bu_y\|_Y.\cos\phi \leq \|y\|_Y$. As we have noticed before, $\|Bu_y\|_Y \geq \|u_y\|_\Omega$ and that concludes the proof. □

To construct the solution operators, we fix an angle $\phi \in (0, \pi/2)$ and choose a contour $\mathscr{G}$ in the following way: For $0 < \varepsilon < \phi/2$, let $\psi = \phi - 2\varepsilon$ and let $\mathscr{G}$ be defined by the lines $z_+(r) = r\exp\{i(\psi + \pi/2\} + i$, $z_-(r) = -r\exp\{-i(\psi + \pi/2\} - i$; $r > 0$, and the semicircle $z(\theta) = \exp\{i\theta\}$; $-\pi/2 \leq \theta \leq \pi/2$. We define for complex t the family of operators $S^\dagger(t) : Y \to L^2(\Omega)$ by the integral

$$S^\dagger(t)y = \frac{1}{2\pi i}\int_{\mathscr{G}} \exp\{\lambda t\}P(\lambda)y\mathrm{d}\lambda; \quad t \neq 0,\ |\arg t| < \varepsilon,\ y \in Y. \tag{65}$$

It follows from (63) that the integral is well-defined. From the identities (57) and (58) we see that the mapping $\lambda \to P(\lambda)y$ is an analytic function so that the integral does not depend on the contour chosen. The change of variable $\lambda|t| = \mu$ leads to the representation $S^\dagger(t)y = \frac{1}{2\pi i|t|}\int_{\mathscr{H}} \exp\{\mu\xi\}P(\mu)y\mathrm{d}\mu$ with $\mathscr{H} = |t|\mathscr{G}$ and $\arg\xi = \arg t$, $|\xi| = 1$. The contour $\mathscr{H}$ can be 'deformed' back to $\mathscr{G}$ to obtain

$$S^\dagger(t)y = \frac{1}{2\pi i|t|}\int_{\mathscr{G}} \exp\{\mu\xi\}P(\mu/|t|)y\mathrm{d}\mu. \tag{66}$$

Theorem 13 *The operators $S^\dagger(t)$ are bounded and map Y to $\mathfrak{D}$. The operators A and B can be interchanged with the integral in* (65).

Proof From the inequality (63) and (66) we obtain an estimate of the form

$$\|S^\dagger(t)y\|_\Omega \leq \mathrm{Const.}|t|^{-1}\|y\|_Y$$

so that the operators $S^\dagger(t)$ are bounded. To prove the other assertions we need to verify integrability of the integrands in (65) or (66) after A and B had acted on them. From the estimate (61) we obtain

$$\||t|^{-1}\exp\{\mu\xi\}BP(\mu/|t|)y\| \leq (|\exp\{\mu\xi\}|/|\mu|)\|y\|_Y$$

so that the integral exists. From the identity $AP(\lambda)y = y - (\lambda+1)BP(\lambda)y$, and the estimate (62) we see that the integral of $\exp\{\mu\xi\}AP(\mu/|t|)$ exists. Since the operator pair $\langle A, B\rangle$ is closed (Theorem 10) it follows that $S^\dagger(t) : Y \to \mathfrak{D}$ and

$$AS^{\dagger}(t)y = \frac{1}{2\pi i}\int_{\mathscr{G}} \exp\{\lambda t\}AP(\lambda)\mathrm{d}\lambda; \tag{67}$$

$$BS^{\dagger}(t)y = \frac{1}{2\pi i}\int_{\mathscr{G}} \exp\{\lambda t\}BP(\lambda)\mathrm{d}\lambda. \tag{68}$$

Thus ends the proof. □

Theorem 14 *For all* $y \in Y$, $\|BS^{\dagger}(t)y - y\|_{Y} \to 0$ *as* $|t| \to 0$.

Proof For $y \in \mathfrak{D}_Y$ we have $y = BP(\lambda)[BP(\lambda)]^{-1}y = \lambda BP(\lambda)y + BP(\lambda)[y + AB^{-1}y]$. Since

$$y = \frac{1}{2\pi i}\int_{\mathscr{G}} \lambda^{-1}\exp\{\lambda t\}y\mathrm{d}\lambda,$$

we have, with the help of (61),

$$BS^{\dagger}(t)y - y = -\frac{1}{2\pi i}\int_{\mathscr{G}} \lambda^{-1}\exp\{\lambda t\}BP(\lambda)[y + AB^{-1}y]\mathrm{d}\lambda.$$

Once again, the substitution $\mu = |t|\lambda$ and the inequality (61) yields an estimate of the form

$$\|BS^{\dagger}(t)y - y\|_{Y} \le \mathrm{Const.}|t|\|y + AB^{-1}y\|_{Y}$$

which converges to zero as $|t| \to 0$ for $y \in \mathfrak{D}_Y$. From the proof of Theorem 13 we note that $BS^{\dagger}(t)y$ is uniformly bounded in t. Since $\mathfrak{D}_Y$ is dense in Y (Theorem 6) the final conclusion is reached. □

Theorem 15 *For* $y \in Y$, $u(t) := \exp\{-t\}S^{\dagger}(t)y$ *solves the Cauchy-problem* (60).

Proof From the identity (68) and the dominated convergence theorem we see that $\frac{d}{dt}\left[BS^{\dagger}(t)y\right] = \frac{1}{2\pi i}\int_{\mathscr{G}}\exp\{\lambda t\}\lambda BP(\lambda)y\mathrm{d}\lambda$. From the identity $(\lambda B + B + A)P(\lambda)y = y$, (67) and (68) we now obtain $\frac{d}{dt}\left[BS^{\dagger}(t)y\right] = \frac{1}{2\pi i}\int_{\mathscr{G}}\exp\{\lambda t\}[y - BP(\lambda)y - AP(\lambda)y]\mathrm{d}\lambda$, which, by virtue of Theorem 13, translates to the equation $\frac{d}{dt}\left[BS^{\dagger}(t)y\right] + BS^{\dagger}(t)y + AS^{\dagger}(t)y = 0$. This, together with Theorem 14, concludes the argument. □

The solution operators alluded to are then $S(t) = \exp\{-t\}S^{\dagger}(t)$ and can be represented in integral form as

$$S(t)y = \frac{1}{2\pi i}\int_{\mathscr{G}} \exp\{(\lambda - 1)t\}P(\lambda)y\mathrm{d}\lambda; \quad t \neq 0,\ |\arg t| < \varepsilon,\ y \in Y,$$

with a slight modification of the contour $\mathscr{G}$.

10 An implicit semigroup

In Sect. 9 the operators $R(\lambda) := BP(\lambda) : Y \to \mathfrak{D}_Y$ played a crucial role. From the identities (57) and (58) we readily see that $R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu)$ and $R(\lambda)R(\mu) = R(\mu)R(\lambda)$ for $\lambda \in \Sigma_\phi$. Hence we can define the holomorphic family $E(t) : Y \to Y$ by $E(t)y = \frac{1}{2\pi i}\int_{\mathscr{G}} \exp\{\lambda t\}R(\lambda)y\mathrm{d}\lambda$. From the estimate (61) it follows that $E(t)$ is indeed a holomorphic semigroup defined on Y. Moreover, a somewhat delicate calculation with the use of (57) in the form $P(\lambda) - P(\mu) = (\mu - \lambda)P(\lambda)R(\mu)$, leads to the *empathy relation* $S(t+s) = S(t)E(s)$ which is a general framework for implicit equations [20]. In the present discussion, however, neither the semigroup property nor the empathy relation is important.

Acknowledgements The thoughts and insights of my students from years gone by, Nic van Rensburg (1982), Wessel Rossouw (1983) and Alna van der Merwe (1993), echo in this work.

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