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ORIGINAL PAPER

A logarithmically improved regularity criterion for the Boussinesq equations in a bounded domain

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Abstract

The paper is concerned with the regularity of solutions of the Boussinesq equations for incompressible fluids without heat conductivity. The main goal is to prove a regularity criterion in terms of the vorticity for the initial boundary value problem in a bounded domain Ω of $\mathbb{R}^3$ with Navier-type boundary conditions and we prove that if

$$\int_0^T \frac{\|\omega(\cdot,t)\|_{BMO(\Omega)}}{\log\left(e+\|\omega(\cdot,t)\|_{BMO(\Omega)}\right)}dt<\infty,$$

where $\omega :=$curl u is the vorticity, then the unique local in time smooth solution of the 3D Boussinesq equations can be prolonged up to any finite but arbitrary time.

Keywords Boussinesq equations · Bounded domain · Smooth solutions · BMO spaces

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1 Introduction and main result.

Let Ω be a bounded, simply connected domain in $\mathbb{R}^3$ with $\partial\Omega \in C^\infty$ and $n = (n^1, n^2, n^3)$ be the outward unit normal vector field along boundary $\partial\Omega$. In this note, we consider the classical problem of regularity conditions for fluid mechanics equations. Precisely, we consider the initial boundary value problem for the 3D Boussinesq equations without heat conductivity modeling the flow of an incompressible fluid with Navier-type boundary conditions :

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \Delta u + \nabla\pi = \theta e_3, & \text{in } \ \Omega \times (0,\infty), \\ \partial_t \theta + (u \cdot \nabla)\theta = 0, & \text{in } \ \Omega \times (0,\infty), \\ \nabla \cdot u = 0, & \text{in } \ \Omega \times (0,\infty), \\ u \cdot n = 0, \quad (\nabla \times u) \times n = 0, & \text{on } \ \partial\Omega, \\ (u,\theta)(x,0) = (u_0,\theta_0) & \text{in } \ \Omega, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ and $\theta = \theta(x,t)$ denote the unknown velocity vector field and the scalar temperature. Initial data u_0 is assumed to satisfy a compatibility condition : $\nabla \cdot u_0(x) = 0$ in Ω. $e_3 = (0,0,1)^t$. $\pi = \pi(x,t)$ is the pressure of fluid at the point $(x,t) \in \Omega \times (0,\infty)$. The Boussinesq system has important roles in atmospheric sciences (see, e.g., [18]).

When $\theta = 0$, $(1.1)_1$ and $(1.1)_3$ are the well-known Navier-Stokes system. Giga [13], Kim [15] and Kang and Kim [14] have proved some Serrin type regularity criteria.

These type of regularity results are very well-known in literature and they all started with the improvement of Kozono and Taniuchi [16] of the Beale–Kato–Majda criterion for the 3D Euler equations, namely

$$\int_0^T \|\omega(\cdot,t)\|_{BMO(\mathbb{R}^3)}\, dt < \infty.$$

Here *BMO* stands for the space of the bounded mean oscillation.

This paper is an improvement on the results contained in the paper [7, 20]. Precisely, in the Reference [20] it is proved that if

$$\int_0^T \frac{\|\omega(\cdot,t)\|_{BMO(\mathbb{R}^3)}}{\sqrt{\ln\left(e + \|\omega(\cdot,t)\|_{BMO(\mathbb{R}^3)}\right)}}\, dt < \infty,$$

holds then the unique local in time smooth solution of the Cauchy problem for the 3D Boussinesq equations with thermal diffusivity can be prolonged up to any finite but arbitrary time. On the other hand in the reference [7] it is proved that if

$$\int_0^T \|\nabla u(\cdot,t)\|_{BMO(\Omega)} dt < \infty, \tag{1.2}$$

then the unique local in time smooth solution of the initial boundary value problem for the 3D Boussinesq equations without thermal diffusivity and with Navier boundary conditions can be prolonged up to any finite but arbitrary time.

One may also refer to some interesting results are devoted to find regularity criteria or prove partial regularity for these equations, such as [8] for Boussinesq equations, and [9–12, 21] for system, in a bounded domain.

Motivated by the result in [7], we will improve (1.2) as

$$\int_0^T \frac{\|\omega(\cdot,t)\|_{BMO(\Omega)}}{\ln\left(e+\|\omega(\cdot,t)\|_{BMO(\Omega)}\right)}\,dt<\infty, \tag{1.3}$$

where $\omega = \nabla \times u$ is the vorticity. More precisely, we will prove

Theorem 1.1 *Let* $(u_0, \theta_0) \in H^3(\Omega) \times W^{1,q}(\Omega)$ *with* $3<q\leq 6$ *and* $\nabla \cdot u_0 = 0$ *in* Ω *and* $u_0 \cdot n = 0$, $(\nabla \times u_0) \times n = 0$ *on* $\partial\Omega$. *Let* (u, θ) *be a strong solution of problem* (1.1). *If* (1.3) *holds, then the solution* (u, θ) *can be extended beyond* $T > 0$.

Remark 1.1 This result says that the velocity field of the fluids plays a more dominant role than the temperature θ in the regularity theory of the system (1.1). So our theorem is a complement and improvement of the previous results. Therefore, if $\theta = 0$, Theorem 1.1 directly yields an alternative proof of logarithmically improved Beale-Kato-Majda type extension criterion for smooth solutions to the incompressible Navier-Stokes equations, which improves the result in [19].

2 Proof of Theorem 1.1

In this section we prove our main result and to this end, we recall some preliminary results which will be used in the proof.

Lemma 2.1 [4] *Let* $\Omega \subset \mathbb{R}^3$ *be a domain with smooth connected boundary* $\partial\Omega$. *Let* $w : \Omega \to \mathbb{R}^3$ *be a smooth vector field and let* $1<s<\infty$. Then

$$\begin{aligned} &-\int_\Omega \Delta w \cdot w|w|^{s-2}dx \\ &= \frac{1}{2}\int_\Omega |w|^{s-2}|\nabla w|^2 dx + \frac{4(s-2)}{s^2}\int_\Omega \left|\nabla |w|^{\frac{s}{2}}\right|^2 dx - \int_{\partial\Omega} |w|^{s-2}(n\cdot\nabla)w\cdot w\,d\sigma. \end{aligned} \tag{2.1}$$

Here $d\sigma$ *denotes the surface measure sur* $\partial\Omega$.

In addition to the classical integration by parts, in some calculations we will also use the following Gauss–Green formula, where $\omega = \nabla \times u$.

Lemma 2.2 [5] *Assume that u is divergence-free and that on* $\partial\Omega$ *condition* $(1.1)_4$ *holds, i.e.* $u \cdot n = 0$ *and* $\omega \times n = 0$. *Then*

$$-\frac{\partial\omega}{\partial n}\cdot\omega = \left(\epsilon_{1jk}\epsilon_{1\beta\gamma} + \epsilon_{2jk}\epsilon_{2\beta\gamma} + \epsilon_{3jk}\epsilon_{3\beta\gamma}\right)\omega_j\omega_\beta\partial_k n_\gamma, \tag{2.2}$$

where ϵ_{ijk} *denotes the totally anti-symmetric tensor such that* $(a\times b)_i = \epsilon_{ijk}a_j b_k$.

We shall often make use of the following

Lemma 2.3 (See [1], Lemma 7.44 and [17], Corollary 1.7) *Let* Ω *be a smooth and bounded open set in* $\mathbb{R}^3$. *Then, there is a constant C depending on* Ω, *such that if* $f \in L^q(\Omega) \cap W^{1,q}(\Omega)$ *with* $1<q<\infty$, *then*

$$\|f\|_{L^q(\partial\Omega)} \leq C\|f\|_{L^q(\Omega)}^{1-\frac{1}{q}}\|f\|_{W^{1,q}(\Omega)}^{\frac{1}{q}}. \tag{2.3}$$

We need the following Gagliardo–Nirenberg inequality using *BMO*-norm.

Lemma 2.4 [3] *Let $1 \leq r < q < \infty$. There exists a constant C depending on Ω, such that for every $f \in L^r(\Omega) \cap BMO(\Omega)$,*

$$\|f\|_{L^q(\Omega)} \leq C\|f\|_{L^r(\Omega)}^{\frac{r}{q}}\|f\|_{BMO(\Omega)}^{1-\frac{r}{q}}. \tag{2.4}$$

In the sequel, we will use the classical regularity result for the following Stokes problem, see for instance [2].

Lemma 2.5 *Let Ω be a bounded, simply connected domain in $\mathbb{R}^3$ with $\partial\Omega \in C^\infty$ and let $m \in \mathbb{Z}$ so that $m \geq -1$ and $q \in (1, \infty)$. For any $f \in W^{m,q}(\Omega)$, there exists a unique solution (u, π) of the following Stokes system*

$$\begin{cases} -\Delta u + \nabla\pi = f \quad \text{in } \Omega, \\ \nabla \cdot u = 0 \quad \text{in} \quad \Omega, \\ u \cdot n = 0, \quad (\nabla \times u) \times n = 0 \quad \text{on } \partial\Omega, \end{cases}$$

such that $u \in W^{m+2,q}(\Omega)$ and $\pi \in W^{m+1,q}(\Omega)$. The solution satisfies the estimate

$$\|u\|_{W^{m+2,q}(\Omega)} + \|\pi\|_{W^{m+1,q}(\Omega)} \leq C\|f\|_{W^{m,q}(\Omega)},$$

for some constant C depending only on Ω and q. In particular, for $m = 0$, we have

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)} \tag{2.5}$$

and

$$\|\nabla\pi\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}. \tag{2.6}$$

We need also the following lemma due to [6], (see Theorem 2.6).

Lemma 2.6 [6] *Let s be a non-negative real. If $u \in H^2(\Omega)$ such that $\Delta u \in H^s(\Omega)$ and such that*

$$\begin{cases} \nabla \cdot u = 0 \quad \text{in} \quad \Omega, \\ u \cdot n = 0, \quad (\nabla \times u) \times n = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

then $u \in H^{s+2}(\Omega)$ and there is a positive constant C independent of u such that

$$\|u\|_{H^{s+2}(\Omega)} \leq C\Big(\|u\|_{L^2(\Omega)} + \|\Delta u\|_{H^s(\Omega)}\Big). \tag{2.7}$$

In particular, for $s = 1$, we have

$$\begin{aligned} \|u\|_{H^3(\Omega)} &\leq C\Big(\|u\|_{L^2(\Omega)} + \|\Delta u\|_{H^1(\Omega)}\Big) \\ &= C\Big(\|u\|_{L^2(\Omega)} + \|\nabla \times (\nabla \times u)\|_{H^1(\Omega)}\Big) \\ &\leq C\Big(\|u\|_{L^2(\Omega)} + \|\omega\|_{H^2(\Omega)}\Big) \\ &\leq C\Big(\|u\|_{L^2(\Omega)} + \|\omega\|_{L^2(\Omega)} + \|\Delta\omega\|_{L^2(\Omega)}\Big). \end{aligned} \tag{2.8}$$

Now we are in a position to prove our main result.

Proof We only need to establish a priori estimates. We will show that under the hypotheses of Theorem 1.1, the $H^3 \times W^{1,q}$−norms of both velocity and temperature remain uniformly bounded, hence we can uniquely continue the solution beyond T, contradicting its maximality.

First we observe that, by standard energy method we have

$$\theta \in L^\infty((0,T); L^q(\Omega)) \qquad \text{for any } 1 \leq q \leq \infty, \tag{2.9}$$

and

$$u \in L^\infty\big((0,T); L^2(\Omega)\big) \cap L^2\big((0,T); H^1(\Omega)\big).$$

Next, we consider the equation for the vorticity ω :

$$\partial_t\omega - \Delta\omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nabla \times (\theta e_3). \tag{2.10}$$

By multiplying (2.10) by $\omega|\omega|^{s-2}$ $(1 < s < \infty)$, using $(1.1)_3$, (2.1), (2.2), (2.3) and (2.4), we obtain

$$
\begin{aligned}
&\frac{1}{s}\frac{d}{dt}\int_\Omega |\omega|^s dx + \int_\Omega |\omega|^{s-2}|\nabla\omega|^2 dx + \frac{4(s-2)}{s^2}\int_\Omega \left|\nabla|\omega|^{\frac{s}{2}}\right|^2 dx \\
&= \int_{\partial\Omega} |\omega|^{s-2}(n\cdot\nabla)\omega\cdot\omega d\sigma + \int_\Omega \omega\cdot\nabla u\cdot\omega|\omega|^{s-2}dx + \int_\Omega \nabla\times(\theta e_3)\cdot\omega|\omega|^{s-2}dx \\
&= -\int_{\partial\Omega} |\omega|^{s-2}\left(\sum_{i=1}^3 \epsilon_{ijk}\epsilon_{i\beta\gamma}\right)\omega_j\omega_\beta\partial_k n_\gamma d\sigma + \int_\Omega (\omega\cdot\nabla)u\cdot\omega|\omega|^{s-2}dx \\
&+ \int_{\partial\Omega} (\theta e_3\times n)\cdot\omega|\omega|^{s-2}d\sigma + \int_\Omega (\theta e_3)\cdot\nabla\times(\omega|\omega|^{s-2})dx \\
&= -\int_{\partial\Omega} |\omega|^{s-2}\left(\sum_{i=1}^3 \epsilon_{ijk}\epsilon_{i\beta\gamma}\right)\omega_j\omega_\beta\partial_k n_\gamma d\sigma + \int_\Omega (\omega\cdot\nabla)u\cdot\omega|\omega|^{s-2}dx \\
&+ \int_\Omega (\theta e_3)\cdot|\omega|^{s-2}(\nabla\times\omega)dx + (s-2)\int_\Omega (\theta e_3)\cdot|\omega|^{s-3}\nabla|\omega|\times\omega dx \\
&\leq C\int_{\partial\Omega} |\omega|^s d\sigma + \|\nabla u\|_{L^{s+1}}\|\omega\|_{L^{s+1}}^s + (s-1)\int_\Omega |\theta||\omega|^{s-2}|\nabla\omega|dx \\
&\leq C\left\||\omega|^{\frac{s}{2}}\right\|_{L^2(\partial\Omega)}^2 + C\|\omega\|_{L^{s+1}}^{s+1} \\
&+ (s-1)\|\theta\|_{L^s}\left\||\omega|^{\frac{s}{2}-1}\right\|_{L^{\frac{2s}{s-2}}}\left\||\omega|^{\frac{s}{2}-1}|\nabla\omega|\right\|_{L^2} \\
&\leq C\left\||\omega|^{\frac{s}{2}}\right\|_{L^2}\left\|\nabla|\omega|^{\frac{s}{2}}\right\|_{L^2} + C\|\omega\|_{L^s}^s\|\omega\|_{BMO} + 2(s-1)^2\|\theta\|_{L^s}^2\|\omega\|_{L^s}^{s-2} \\
&+ \frac{1}{4}\left\||\omega|^{\frac{s}{2}-1}|\nabla\omega|\right\|_{L^2}^2 \\
&\leq \frac{2(s-2)}{s^2}\int_\Omega \left|\nabla|\omega|^{\frac{s}{2}}\right|^2 dx + C\|\omega\|_{L^s}^s\left(1+\|\omega\|_{BMO}\right) \\
&+ C\|\theta\|_{L^s}^s + C\|\omega\|_{L^s}^s + \frac{1}{4}\left\||\omega|^{\frac{s}{2}-1}|\nabla\omega|\right\|_{L^2}^2,
\end{aligned}
$$

which gives

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega |\omega|^s dx &\leq C\left(1+\|\omega\|_{L^s}^s\right)\left(1+\|\omega\|_{BMO}\right) \\
&=.C\left(1+\|\omega\|_{L^s}^s\right)\frac{1+\|\omega\|_{BMO}}{\log\left(e+\|\omega\|_{BMO}\right)}\log\left(e+\|\omega\|_{BMO}\right) \\
&\leq C\left(1+\|\omega\|_{L^s}^s\right)\frac{1+\|\omega\|_{BMO}}{\log\left(e+\|\omega\|_{BMO}\right)}\log\left(e+\|\omega\|_{H^2}\right) \\
&\leq C\left(1+\|\omega\|_{L^s}^s\right)\frac{1+\|\omega\|_{BMO}}{\log\left(e+\|\omega\|_{BMO}\right)}\log\left(e+\|\nabla\Delta u\|_{L^2}\right).
\end{aligned} \tag{2.11}
$$

Defining

$$\mathcal{Z}(t) = \sup_{T_* \le t \le T} \|\nabla \Delta u(\cdot, t)\|_{L^2},$$

the inequality (2.11) implies that

$$\sup_{T_* \le t \le T} \|\omega(\cdot, t)\|_{L^s}^s \le C_*(e + \mathcal{Z}(t))^{C\epsilon}, \tag{2.12}$$

where ϵ is a small constant, such that

$$\int_{T_*}^{T} \frac{\|\omega\|_{BMO}}{\log(e + \|\omega\|_{BMO})} dt \le \epsilon.$$

By using the standard energy estimate, we can deduce that

$$\|\omega(\cdot, T_*)\|_{L^s} < \infty.$$

Now, testing $(1.1)_1$ by $\partial_t u$, using $(1.1)_3$ and (2.12), we see that

$$\begin{aligned}
&\|\partial_t u\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla \times u\|_{L^2}^2 \\
&\quad = -\int_\Omega (u \cdot \nabla) u \cdot \partial_t u dx + \int_\Omega \nabla \pi \cdot \partial_t u dx + \int_\Omega \theta e_3 \cdot \partial_t u dx \\
&\quad \le \|u\|_{L^6} \|\nabla u\|_{L^3} \|\partial_t u\|_{L^2} + \|\theta\|_{L^2} \|\partial_t u\|_{L^2} \\
&\quad \le \frac{1}{4}\|\partial_t u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^3}^2 + C\|\theta\|_{L^2}^2 + \frac{1}{4}\|\partial_t u\|_{L^2}^2 \\
&\quad \le \frac{1}{2}\|\partial_t u\|_{L^2}^2 + C\|\omega\|_{L^2}^2 \|\omega\|_{L^3}^2 + C \\
&\quad \le \frac{1}{2}\|\partial_t u\|_{L^2}^2 + C_*(e + \mathcal{Z}(t))^{C\epsilon} + C,
\end{aligned}$$

which implies

$$\int_{T_*}^{t} \|\partial_t u(\cdot, \tau)\|_{L^2}^2 d\tau \le C_*(e + \mathcal{Z}(t))^{C\epsilon}. \tag{2.13}$$

Here we have used the facts

$$-\int_\Omega \Delta u \cdot \partial_t u dx = \int_\Omega \nabla \times (\nabla \times u) \cdot \partial_t u dx = \int_\Omega (\nabla \times u) \cdot \nabla \times (\partial_t u) dx = \frac{1}{2}\frac{d}{dt}\|\nabla \times u\|_{L^2}^2,$$

$$\int_\Omega \nabla \pi \cdot \partial_t u dx = -\int_\Omega \pi \cdot \partial_t (\nabla \cdot u) dx = 0.$$

Now we want to estimate $\|\partial_t u\|_{L^2}$. We take the time derivative of first equation in (1.1), we get

$$\partial_t^2 u + (u \cdot \nabla)\partial_t u - \Delta(\partial_t u) + \nabla(\partial_t \pi) = -(\partial_t u \cdot \nabla) u + \partial_t(\theta e_3). \tag{2.14}$$

Multiplying (2.14) by $\partial_t u$, using $(1.1)_3$, (2.9), (2.12) and (2.13), it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega}|\partial_t u|^2 dx + \int_{\Omega}|\nabla \times (\partial_t u)|^2 dx \\
&\leq \left|\int_{\Omega}(\partial_t u\cdot\nabla)u\cdot\partial_t u dx\right| + \left|\int_{\Omega}(u\cdot\nabla)\partial_t u\cdot\theta e_3 dx\right| \\
&\leq \|\partial_t u\|_{L^3}^2\|\nabla u\|_{L^3} + C\|u\|_{L^6}\|\nabla(\partial_t u)\|_{L^2}\|\theta\|_{L^3} \\
&\leq \|\partial_t u\|_{L^2}\|\nabla(\partial_t u)\|_{L^2}\|\nabla u\|_{L^3} + \frac{1}{4}\int_{\Omega}|\nabla\times(\partial_t u)|^2 dx + C\|\theta\|_{L^3}^2\|\nabla u\|_{L^2}^2 \\
&\leq \frac{1}{2}\|\nabla\times(\partial_t u)\|_{L^2}^2 + C\|\partial_t u\|_{L^2}^2\|\omega\|_{L^3}^2 + C\|\omega\|_{L^2}^2 \\
&\leq \frac{1}{2}\|\nabla\times(\partial_t u)\|_{L^2}^2 + C(1+\|\partial_t u\|_{L^2}^2)\|\omega\|_{L^3}^2.
\end{aligned}
$$

Here we have used the facts

$$\int_{\Omega}(u\cdot\nabla)(\partial_t u)\cdot(\partial_t u)dx = 0,$$

and also we use the continuity equation (to substitute $\partial_t\theta$ by $-(u\cdot\nabla)\theta$)

$$
\begin{aligned}
-\int_{\Omega}\partial_t\theta e_3\cdot\partial_t u dx &= \int_{\Omega}(u\cdot\nabla)\theta e_3\cdot\partial_t u dx \\
&= -\int_{\Omega}(u\cdot\nabla)\partial_t u\cdot\theta e_3 dx + \int_{\partial\Omega}(u\cdot n)(\theta e_3\cdot\partial_t u)d\sigma,
\end{aligned}
$$

where in the integration by parts we used the fact that u is divergence-free. In addition, the boundary term vanishes since $(u\cdot n)=0$ on $\partial\Omega$.

Integrating over $[T_*, t]$, using (2.12) and (2.13), we obtain

$$\int_{\Omega}|\partial_t u|^2 dx + \int_{T_*}^{t}\int_{\Omega}|\nabla\times(\partial_t u)(\cdot,\tau)|^2 dx d\tau \leq C_*(e+\mathcal{Z}(t))^{C\epsilon}. \tag{2.15}$$

On the other hand, since (u,π) is a solution of the Stokes system :

$$-\Delta u + \nabla\pi = -\partial_t u - (u\cdot\nabla)u + \theta e_3,$$

thanks to the H^2-theory of the Stokes system, we obtain by Hölder's inequality, Sobolev's inequality, (2.12) and (2.15)

$$
\begin{aligned}
\|u\|_{H^2} &\leq C\|-\Delta u+\nabla\pi\|_{L^2} = C\|\partial_t u + (u\cdot\nabla)u - \theta e_3\|_{L^2} \\
&\leq C\|\partial_t u\|_{L^2} + C\|u\|_{L^6}\|\nabla u\|_{L^3} + C\|\theta\|_{L^2} \\
&\leq C\|\partial_t u\|_{L^2} + C\|\nabla u\|_{L^2}\|\nabla u\|_{L^3} + C \\
&\leq C_*(e+\mathcal{Z}(t))^{C\epsilon}.
\end{aligned} \tag{2.16}
$$

Multiplying (2.14) by $[\nabla(\partial_t\pi) - \Delta(\partial_t u)]$, using $(1.1)_3$, (2.15) and (2.16), we derive

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega}|\nabla\times(\partial_t u)|^2dx \\
&\quad+\int_{\Omega}|\nabla\partial_t\pi-\Delta\partial_t u|^2dx \\
&\quad=\int_{\Omega}[(\partial_t\theta)e_3-(u\cdot\nabla)\partial_t u-(\partial_t u\cdot\nabla)u]\cdot(\nabla\partial_t\pi-\Delta\partial_t u)dx \\
&\quad=-\int_{\Omega}[(u\cdot\nabla\theta)e_3+(u\cdot\nabla)\partial_t u+(\partial_t u\cdot\nabla)u]\cdot(\nabla\partial_t\pi-\Delta\partial_t u)dx \\
&\quad\le\big(\|u\|_{L^\infty}\|\nabla\partial_t u\|_{L^2}+\|\partial_t u\|_{L^6}\|\nabla u\|_{L^3} \\
&\quad+\|u\|_{L^\infty}\|\nabla\theta\|_{L^2}\big)\|\nabla\partial_t\pi-\Delta\partial_t u\|_{L^2} \\
&\quad\le C(\|u\|_{H^2}\|\nabla\times(\partial_t u)\|_{L^2}+\|\nabla\times(\partial_t u)\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{H^1}^{\frac{1}{2}} \\
&\quad+C\|u\|_{H^2})\|\nabla\partial_t\pi-\Delta\partial_t u\|_{L^2} \\
&\quad\le C\big(\|u\|_{H^2}\|\nabla\times(\partial_t u)\|_{L^2} \\
&\quad+\|\nabla\times(\partial_t u)\|_{L^2}\|\nabla u\|_{H^1}+C\|u\|_{H^2}\big)\|\nabla\partial_t\pi-\Delta\partial_t u\|_{L^2} \\
&\quad\le C\|u\|_{H^2}\big(\|\nabla\times(\partial_t u)\|_{L^2}+1\big)\|\nabla\partial_t\pi-\Delta\partial_t u\|_{L^2} \\
&\quad\le\frac{1}{2}\|\nabla\partial_t\pi-\Delta\partial_t u\|_{L^2}^2+C\|u\|_{H^2}^2\Big(\|\nabla\times(\partial_t u)\|_{L^2}^2+1\Big),
\end{aligned}
\tag{2.17}
$$

where we have used the fact : since Ω is simply connected and $u\cdot n=0$ on $\partial\Omega$, then

$$
\|\nabla\partial_t u\|_{L^2}\le C\big(\|\nabla\cdot(\partial_t u)\|_{L^2}+\|\nabla\times(\partial_t u)\|_{L^2}\big)=C\|\nabla\times(\partial_t u)\|_{L^2}.
$$

Integrating (2.17) over (T_*,t) and using (2.15) and (2.16), we have

$$
\int_{\Omega}|\nabla\times(\partial_t u)|^2dx+\int_{T_*}^{t}\int_{\Omega}|(\nabla\partial_t\pi-\Delta\partial_t u)(x,\tau)|^2dxd\tau\le C_*(e+\mathcal{Z}(t))^{4C\epsilon}. \tag{2.18}
$$

Here we have used the fact

$$
\int_{\Omega}|\nabla\times(\partial_t u)(x,T_*)|^2dx<\infty,
$$

by the standard energy method.

On the other hand, it follows from (2.8), (2.10), (2.12), (2.16) and (2.18) that

$$
\begin{aligned}
\|u\|_{H^3}&\le C\big(\|u\|_{L^2}+\|\omega\|_{L^2}+\|\Delta\omega\|_{L^2}\big)\le C\big(1+\|\Delta\omega\|_{L^2}\big) \\
&\le C(1+\|\partial_t\omega+(u\cdot\nabla)\omega-(\omega\cdot\nabla)u+\nabla\times(\theta e_3)\|_{L^2} \\
&\le C\big(1+\|\partial_t\omega\|_{L^2}+\|u\|_{L^\infty}\|\nabla\omega\|_{L^2}+\|\omega\|_{L^6}\|\nabla u\|_{L^3}+\|\nabla\theta\|_{L^2}\big) \\
&\le C+C\|\partial_t\omega\|_{L^2}+C\|u\|_{H^2}^2 \\
&\le C_*(e+\mathcal{Z}(t))^{4C\epsilon}.
\end{aligned}
$$

Thus we conclude that

$$\|u\|_{L^\infty(0,T;H^3(\Omega))} \leq C. \tag{2.19}$$

By taking the gradient of the continuity equation in (1.1), we get the equation

$$\nabla(\partial_t \theta) + (u \cdot \nabla)(\nabla\theta) = -(\nabla u)(\nabla\theta). \tag{2.20}$$

By multiplying (2.20) by $\nabla\theta|\nabla\theta|^{q-2}$ for $q > 3$ and by integrating by parts, we obtain the following differential inequality

$$\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla\theta|^q dx \leq \|\nabla u\|_{L^\infty}\int_\Omega |\nabla\theta|^q dx \leq C\|u\|_{H^3}\int_\Omega |\nabla\theta|^q dx. \tag{2.21}$$

Then, by integrating the above differential inequality with respect to time and by using (2.19), we get that there exists a constant C independent of q such that

$$\|\nabla\theta\|_{L^\infty(0,T;L^q(\Omega))} \leq C \quad \text{with } 3 < q \leq 6.$$

This completes the proof of Theorem 1.1.

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