SN Partial Differ. Equ. Appl. (2020) 1:34
https://doi.org/10.1007/s42985-020-00036-w

ORIGINAL PAPER

Normalized solutions for a class of nonlinear Choquard equations

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Received: 23 May 2020 / Accepted: 31 July 2020 / Published online: 4 September 2020

Abstract
We prove the existence of a least energy solution to the problem

$$-\Delta u - (I_\alpha * F(u))f(u) = \lambda u \quad \text{in} \quad \mathbb{R}^N, \qquad \int_{\mathbb{R}^N} u^2(x)dx = a^2,$$

where $N \geq 1$, $\alpha \in (0, N)$, $F(s) := \int_0^s f(t)dt$, and $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential. If f is odd in u then we prove the existence of infinitely many normalized solutions.

Keywords Choquard equation · Stretched functional · Normalized solution

Mathematics Subject Classification 35B38 · 35J20 · 35J60 · 35P30

1 Introduction

We consider the equation

$$-\Delta u - (I_\alpha * F(u))f(u) = \lambda u \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

This article is part of the topical collection dedicated to Prof. Dajun Guo for his 85th birthday, edited by Yihong Du, Zhaoli Liu, Xingbin Pan, and Zhitao Zhang.

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where $N \geq 1$, $\alpha \in (0, N)$, $F(s) := \int_0^s f(t)dt$, and $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{N/2}2^\alpha|x|^{N-\alpha}} \quad \text{for } x \in \mathbb{R}^N \backslash \{0\}.$$

Equation (1.1) is usually called the nonlinear Choquard equation. It is a semilinear elliptic equation with a nonlocal nonlinearity. For the physical case $N = 3$, $\alpha = 2$ and $f(s) = s$, (1.1) is the Choquard–Pekar equation which goes back to the 1954's work by S. Pekar on quantum theory of a polaron at rest [28]. The Choquard equation was also introduced by P. Choquard in 1976 in the modelling of a one-component plasma [20]. Mathematically, in [20–22], Lieb and Lions opened the way to an intensive study of (1.1). The existence and qualitative properties of solutions of the Choquard equation (1.1) have been studied for a few decades by variational methods; see [9, 10, 17, 25–27] and further references therein, for instance. Almost all papers deal with the case of fixed λ.

From a physical point of view, it is interesting to find solutions of (1.1) with prescribed L^2-norms, λ appearing as Lagrange multiplier. Solutions of this type are often referred to as normalized solutions. The present paper is devoted to such solutions. We study the existence and multiplicity of solutions to the problem

$$\begin{cases} -\Delta u - (I_\alpha * F(u))f(u) = \lambda u & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = a^2, & u \in H^1(\mathbb{R}^N), \quad \lambda \in \mathbb{R}, \end{cases} \tag{1.2}$$

where a is a positive constant. By a solution we mean a couple (λ, u) which satisfies (1.2) with λ being a Lagrangian multiplier.

The existence of normalized solutions (λ, u) to the semilinear elliptic equation

$$-\Delta u - g(u) = \lambda u \quad \text{in } \mathbb{R}^N, \ \lambda \in \mathbb{R} \tag{1.3}$$

has achieved considerable attention recently. Mathematically, this is a more challenging issue than the existence of solutions of (1.3) with a prescribed frequency λ, and is much less understood. Normalized solutions can be obtained as critical points of a functional constrained to an L^2 sphere. The difficulty here is that even for subcritical nonlinearities the functional does not satisfy the Palais-Smale condition. In fact, a Palais-Smale sequence ((PS) sequence for short in the following) does not even need to be bounded. In order to overcome the difficulties in dealing with normalized solutions, L. Jeanjean in [12] introduced a *stretched functional* whose critical points are solutions of (1.3) and whose (PS) sequences carry information of the Pohozaev identity, which can be used to prove boundedness. Then another difficulty appears, namely that a weak limit of a bounded (PS) sequence need not lie on the prescribed L^2 sphere. In order to control this it is important to bound the Lagrange multipliers. Using a mountain pass structure for the stretched functional L. Jeanjean in [12] proved the existence of at least one normalized solution of (1.3). The existence of infinitely many normalized solutions of (1.3) was later proved by T. Bartsch and S. de Valeriola in [1] using a new linking geometry for the stretched functional (see also the papers by T. Bartsch and N. Soave [4, 5]). More results on normalized solutions for scalar equations and systems can be found in [2, 3, 6, 7, 13–15, 19, 24, 30, 31].

We know only a few papers dealing with the existence of normalized solutions of the Choquard equation. In the case $N \geq 3$, G. Li and H. Ye in [19] obtained a ground state solution (λ_a, u_a) of (1.2) under a set of assumptions on f, which when f takes the special form $f(s) = C_1|s|^{r-2}s + C_2|s|^{p-2}s$ requires that $\frac{N+\alpha+2}{N} < r \leq p < \frac{N+\alpha}{N-2}$. For a monomial nonlinearity $f(s) = |s|^{p-2}s$, qualitative properties including existence and nonexistence of minimizers of the functional associated to the nonlinear Choquard equation were discussed by H. Ye in [32].

The goal of this paper is to first prove the existence of a least energy solution of (1.2) in all dimensions $N \geq 1$, including $N = 1, 2$. We believe that our proof is simpler and more transparent than the one from [19]. In addition we prove the existence of infinitely many solutions of (1.2) if f is odd. This will be based on a linking argument which in turn depends on a certain topological intersection property. This has been proved in [1] using a cohomological index theory. Here we present a new and more elementary proof of this property using only the classical Borsuk-Ulam theorem.

We now present our results. Let $2^*_\alpha := (N+\alpha)/(N-2)$ if $N \geq 3$, and $2^*_\alpha := +\infty$ if $N = 1, 2$. We assume the following hypotheses on the nonlinearity.

(f_1) $f \in C^0(\mathbb{R})$ and there exist $r, p \in \mathbb{R}$ satisfying

$$\frac{N+\alpha+2}{N} < r \leq p < 2^*_\alpha$$

such that

$$0 < rF(s) \leq f(s)s \leq pF(s) \quad \text{for } s \neq 0.$$

(f_2) The function $\widetilde{F}(s) := f(s)s - \frac{N+\alpha}{N}F(s)$ satisfies:

$$\frac{\widetilde{F}(s)}{|s|^{(N+\alpha+2)/N}} \quad \text{is nonincreasing in } (-\infty, 0) \text{ and nondecreasing in } (0, +\infty).$$

Let

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)dx \tag{1.4}$$

be the corresponding variational functional of (1.2) defined on the constraint

$$S(a) = \{u \in H^1(\mathbb{R}^N) : \ \|u\|_{L^2(\mathbb{R}^N)} = a\}.$$

Setting

$$I(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{N+\alpha}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)dx - \frac{N}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))f(u)u\, dx \tag{1.5}$$

every solution of (1.2) lies on the Pohozaev manifold

$$V(a) = \{u \in S(a) : \ I(u) = 0\}.$$

Our first main result states that

$$m(a) := \inf_{u \in V(a)} J(u)$$

is achieved by a solution.

Theorem 1.1 *If (f_1) and (f_2) hold then, for any $a > 0$, problem (1.2) possesses a solution $(\lambda_a, u_a) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ such that $\lambda_a < 0$ and $J(u_a) = m(a)$.*

Remark 1.2

(a) Due to the radial symmetry of (1.2) we may also work on the space $E := H^1_{rad}(\mathbb{R}^N)$ of radial functions. A critical point of $J|_{S(a)\cap E}$ is also a critical point of $J|_{S(a)}$ by the principle of symmetic criticality. It is even simpler to obtain a solution $(\lambda_a^{rad}, u_a^{rad}) \in \mathbb{R}\times E$ such that

$$J(u_a^{rad}) = m_{rad}(a) := \inf_{u\in V(a)\cap E} J(u)$$

because E embeds compactly into $L^q(\mathbb{R}^N)$ for $2<q<2^*$. It is an open problem whether $m(a) = m_{rad}(a)$. Observe that $V(a)$ is not invariant under symmetrization.

(b) Replacing f by $f^+ := \max\{f, 0\}$ one obtains by a symmetrization argument a least energy solution $(\widetilde{\lambda}_a, \widetilde{u}_a)$ of

$$\begin{cases} -\Delta u - (I_\alpha * F^+(u))f^+(u) = \lambda u & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = a^2, \quad u\in H^1(\mathbb{R}^N), & \lambda\in\mathbb{R}, \end{cases}$$

with $\widetilde{\lambda}_a<0$ and $\widetilde{u}_a > 0$ being radial, where $F^+(s) := \int_0^s f^+(t)dt$. This is also a solution of (1.2). Similarly (1.2) has a solution $(\widehat{\lambda}_a, \widehat{u}_a)$ such that $\widehat{\lambda}_a<0$ and $\widehat{u}_a<0$ being radial. This suggests of course that there should exist a third solution of mountain pass type on $V(a)$, i.e. of Morse index 2 on $S(a)$. To obtain this one may work in the space of radial functions. If f is odd then $\widetilde{u}_a = -\widehat{u}_a$ are least energy solutions. In general the relation between $\widetilde{u}_a$, $\widehat{u}_a$ and the least energy solution u_a from Theorem 1.1 is not clear but we conjecture that u_a does not change sign.

(c) For $N\geq 3$, a similar result has been proved in [19]. Here we include the dimensions $N = 1, 2$. The idea of the argument in [19] is as follows. The stretched functional method from [12] was first used to find a (PS) sequence $\{v_n\}$ for $J|_{S(a)}$ at the level $m(a)$ such that $I(v_n)\to 0$. Then it was proved that there exists $\lambda_a\in\mathbb{R}$ such that $\langle J'(v_n), v_n\rangle/\|v_n\|^2_{L^2(\mathbb{R}^N)} \to \lambda_a$. After that, the key step was to prove that $\lambda_a<0$. In order to achieve this, it was proved that $\{v_n\}$ is a (PS) sequence of the functional Ψ defined by

$$\Psi(u) = J(u) - \frac{1}{2}\lambda_a\int_{\mathbb{R}^N} u^2 dx$$

and that $m(a)$ has a strict subadditivity property. This strict subadditivity property together with decomposition properties of (PS) sequences of Ψ make it possible to prove $\lambda_a<0$ and that there exists $u_a\in H^1(\mathbb{R}^N)\setminus\{0\}$ such that (λ_a, u_a) is a solution of (1.2). We will also use the stretched functional method from [12] to obtain a (PS) sequence $\{v_n\}$ for $J|_{S(a)}$ at the level $m(a)$ with the property $I(v_n) = o(1)$. But unlike [19], we use a concentration compactness argument to find a solution (λ_a, u_a) with $\lambda_a<0$ and $u_a\neq 0$ of (1.1) from the (PS) sequence $\{v_n\}$ directly. Then we prove that $m(a)$ is strictly decreasing in a. This property combined with the fact that $I(u_a) = \lim_{n\to\infty} I(v_n) = 0$ makes it possible for us to prove that (λ_a, u_a) is indeed a solution of (1.2) and $J(u_a) = m(a)$. The proof presented here is shorter, easier, and more transparent than the one in [19].

Our second main result deals with odd nonlinearities.

Theorem 1.3 *If $N \geq 2$, $(f_1) - (f_2)$ hold and f is odd, then for any $a > 0$, problem* (1.2) *possesses an unbounded sequence of pairs of radial solutions $(\lambda_k, \pm u_k)$ with $\lambda_k < 0$ and $J(u_k) \to \infty$ as $k \to \infty$.*

When we say that (λ, u) is a radial solution, we mean that $u \in E = H^1_{rad}(\mathbb{R}^N)$.

Remark 1.4 For the semilinear equation (1.3) infinitely many normalized solutions have been obtained via two different approaches in [1] and in [4]. In [1], the stretched functional method incorporating a new linking structure of the associated functional produces a bounded (PS) sequence and leads to normalized solutions at a sequence of energy levels $c_n \to \infty$ which is constructed by a minimax procedure. In [4], see also [5] for systems, the authors considered the functional corresponding to J, constrained to the Pohozaev manifold corresponding to $V(a)$. In this way they avoided the introduction of the stretched functional. In the present paper we extend the stretched functional method to deal with (1.2).

The paper is organized as follows. In Sect. 2 we prove Theorem 1.1. We first show that $m(a)$ is the mountain pass level of $J|_{S(a)}$, and we use the stretched functional method to obtain a (PS) sequence $\{v_n\}$ for $J|_{S(a)}$, which satisfies $J(v_n) \to m(a)$ and $I(v_n) \to 0$. Then we use a concentration compactness argument, which does not rely on the compactness from radial symmetry [16, 26], to show that, up to translations, $v_n \to u_a$ weakly in $H^1(\mathbb{R}^N)$ for some $u_a \neq 0$ and that there exists $\lambda_a < 0$ such that (λ_a, u_a) is a weak solution of (1.1). Next we show that $m(a)$ is strictly decreasing. This fact is used to show that $\|u_a\|_{L^2(\mathbb{R}^N)} = a$ and $J(u_a) = m(a)$. Therefore, (λ_a, u_a) is a weak solution of (1.2). We prove Theorem 1.3 in Sect. 3, working in the subspace $H^1_{\text{rad}}(\mathbb{R}^N)$ of $H^1(\mathbb{R}^N)$ consisting of radially symmetric functions. Here we first present a new and more elementary proof of the intersection lemma from [1]. This and a suitable equivariant pseudogradient vector field for the stretched functional will be used to construct an unbounded sequence of minimax values c_k for $J|_{S(a)}$ together with $(\text{PS})_{c_k}$ sequences $\{v_{k,n}\}_{n=1}^{\infty}$ at each minimax value c_k satisfying $\lim_{n\to\infty} I(v_{k,n}) = 0$, which is used to show that $\{v_{k,n}\}_{n=1}^{\infty}$ is bounded. Then we use the compactness of the imbedding $H^1_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for $q \in (2, 2^*)$ to show that $\{v_{k,n}\}_{n=1}^{\infty}$ converges along a subsequence strongly in E to some $u_k \in S(a)$ as $n \to \infty$.

2 Proof of Theorem 1.1

Recall the definition of J and I in (1.4) and (1.5), respectively. Observe that (f_1) implies, for $s \in \mathbb{R}$,

$$t^p F(s) \leq F(ts) \leq t^r F(s) \quad \text{if} \quad 0 \leq t \leq 1, \tag{2.1}$$

and

$$t^r F(s) \leq F(ts) \leq t^p F(s) \quad \text{if} \quad t > 1. \tag{2.2}$$

These inequalities will be used frequently in what follows.

For each $u \in S(a)$ and $t \in \mathbb{R}$, set

$$u^t(x) = e^{\frac{Nt}{2}} u(e^t x).$$

It is clear that $u^t \in S(a)$ if $u \in S(a)$ and $t \in \mathbb{R}$. The following lemma asserts that on the curve $t \mapsto u^t$ there exists a unique point belonging to $V(a)$, at which $J(u^t)$ achieves its maximum.

Lemma 2.1

(a) *For every $u \in S(a)$, there exists a unique $t(u) \in \mathbb{R}$ such that $u^{t(u)} \in V(a)$ and $J(u^{t(u)}) = \max_{t\in\mathbb{R}} J(u^t)$. The map*

$$S(a) \to V(a) \times \mathbb{R}, \quad u \mapsto \left(u^{t(u)}, t(u)\right)$$

is a homeomorphism with inverse

$$V(a) \times \mathbb{R} \to S(a), \quad (u,t) \mapsto u^{-t}.$$

(b) $I(u) = \left.\dfrac{d}{dt}\right|_{t=0} J(u^t)$.

Proof For $u \in S(a)$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} J(u^t) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u^t|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u^t))F(u^t)dx \\ &= \frac{e^{2t}}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2e^{(N+\alpha)t}}\int_{\mathbb{R}^N} \left(I_\alpha * F\left(e^{\frac{Nt}{2}}u\right)\right)F\left(e^{\frac{Nt}{2}}u\right)dx. \end{aligned} \tag{2.3}$$

Then

$$\begin{aligned} \frac{d}{dt}J(u^t) &= e^{2t}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{N+\alpha}{2e^{(N+\alpha)t}}\int_{\mathbb{R}^N} \left(I_\alpha * F\left(e^{\frac{Nt}{2}}u\right)\right)F\left(e^{\frac{Nt}{2}}u\right)dx \\ &\quad - \frac{N}{2e^{(N+\alpha)t}}\int_{\mathbb{R}^N} \left(I_\alpha * F\left(e^{\frac{Nt}{2}}u\right)\right)f\left(e^{\frac{Nt}{2}}u\right)e^{\frac{Nt}{2}}u dx \\ &= e^{2t}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{N}{2}\psi(t)\right), \end{aligned} \tag{2.4}$$

where

$$\psi(t) = \int_{\mathbb{R}^N} \left(I_\alpha * \frac{F(e^{\frac{Nt}{2}}u)}{(e^{\frac{Nt}{2}})^{\frac{N+\alpha+2}{N}}}\right)\frac{\widetilde{F}(e^{\frac{Nt}{2}}u)}{(e^{\frac{Nt}{2}})^{\frac{N+\alpha+2}{N}}}dx.$$

From (2.4) we obtain the result b).

For any $s \in \mathbb{R}$, $s \neq 0$, from (f_1) and (f_2) we see that both the functions $\frac{F(ts)}{t^r}$ and $\frac{\widetilde{F}(ts)}{t^{(N+\alpha+2)/N}}$ are nondecreasing in $t \in (0,\infty)$. Moreover, since $r > \frac{N+\alpha+2}{N}$ and

$$\frac{F(ts)}{t^{(N+\alpha+2)/N}} = \frac{F(ts)}{t^r}t^{r-\frac{N+\alpha+2}{N}},$$

we deduce that $\frac{F(ts)}{t^{(N+\alpha+2)/N}}$ is strictly increasing in $t \in (0,\infty)$. This implies $\psi(t)$ is strictly increasing in $t \in \mathbb{R}$ and there is at most one $t \in \mathbb{R}$ such that $\frac{d}{dt}J(u^t) = 0$. By (2.1) and (2.2), we have

$$\lim_{t\to 0}\frac{F(ts)}{t^{(N+\alpha+2)/N}}=0 \quad \text{and} \quad \lim_{t\to+\infty}\frac{F(ts)}{t^{(N+\alpha+2)/N}}=+\infty. \tag{2.5}$$

Since

$$\left(r-\frac{N+\alpha}{N}\right)F(s)\le \widetilde{F}(s)\le\left(p-\frac{N+\alpha}{N}\right)F(s)$$

and since $(N+\alpha+2)/N<r\le p$, we also have

$$\lim_{t\to 0}\frac{\widetilde{F}(ts)}{t^{(N+\alpha+2)/N}}=0 \quad \text{and} \quad \lim_{t\to+\infty}\frac{\widetilde{F}(ts)}{t^{(N+\alpha+2)/N}}=+\infty. \tag{2.6}$$

From (2.5) and (2.6), we deduce $\psi(t)\to 0$ as $t\to-\infty$ using the Lebesgue dominated convergence theorem, and we have $\psi(t)\to+\infty$ as $t\to\infty$ by the Fatou Lemma. As a consequence of (2.4), there exists exactly one $t=t(u)\in\mathbb{R}$ such that $\frac{d}{dt}J(u^t)|_{t=t(u)}=0$, and moreover, $\frac{d}{dt}J(u^t)>0$ for $t<t(u)$ and $\frac{d}{dt}J(u^t)<0$ for $t>t(u)$. Therefore, there exists a unique $t(u)\in\mathbb{R}$ such that $u^{t(u)}\in V(a)$ and $J(u^{t(u)})=\max_{t\in\mathbb{R}}J(u^t)$. Now assume that $\{u_n\}\subset S(a)$ and $u_n\to u$ in $H^1(\mathbb{R}^N)$, and let $t_n=t(u_n)$. Then

$$\frac{2}{N}\int_{\mathbb{R}^N}|\nabla u_n|^2dx=\int_{\mathbb{R}^N}\left(I_\alpha*\frac{F\left(e^{\frac{Nt_n}{2}}u_n\right)}{\left(e^{\frac{Nt_n}{2}}\right)^{\frac{N+\alpha+2}{N}}}\right)\frac{\widetilde{F}\left(e^{\frac{Nt_n}{2}}u_n\right)}{\left(e^{\frac{Nt_n}{2}}\right)^{\frac{N+\alpha+2}{N}}}dx.$$

Using the Lebesgue dominated convergence theorem and the Fatou Lemma again, we see that the sequence $\{t_n\}$ is bounded. Assume $t_n\to t^*\in\mathbb{R}$, passing to a subsequence if necessary. Then passing to the limit in the last equation yields

$$\frac{2}{N}\int_{\mathbb{R}^N}|\nabla u|^2dx=\int_{\mathbb{R}^N}\left(I_\alpha*\frac{F\left(e^{\frac{Nt^*}{2}}u\right)}{\left(e^{\frac{Nt^*}{2}}\right)^{\frac{N+\alpha+2}{N}}}\right)\frac{\widetilde{F}\left(e^{\frac{Nt^*}{2}}u\right)}{\left(e^{\frac{Nt^*}{2}}\right)^{\frac{N+\alpha+2}{N}}}dx,$$

which together with the uniqueness of $t(u)$ implies that $t(u)=t^*=\lim_{n\to\infty}t(u_n)$. Hence $t(u)$ is continuous in u, hence the map

$$S(a)\to V(a)\times\mathbb{R},\quad u\mapsto\left(u^{t(u)},t(u)\right)$$

is a homeomorphism because u^t is continuous in (u,t). This proves a). □

In the next lemma, we show that J has a mountain pass structure on $S(a)$ as in [12, 19], and we use the stretched functional method as in [12] to find a (PS) sequence $\{v_n\}$ of $J|_{S(a)}$ at the mountain pass level such that $I(v_n)\to 0$ as $n\to\infty$.

Lemma 2.2

(a) *Let* $D_k:=\{u\in S(a):\ \int_{\mathbb{R}^N}|\nabla u|^2dx\le k\}$. *There exist* $0<k_1<k_2$ *such that*

$$0<\sup_{u\in D_{k_1}}J(u)<\inf_{u\in\partial D_{k_2}}J(u) \quad \text{and} \quad J(u)>0 \quad \text{for} \quad u\in D_{k_2}.$$

(b) *Setting*

$$\Gamma(a) := \{\gamma \in C([0,1],\ S(a)):\ \gamma(0) \in D_{k_1}, J(\gamma(1)) < 0\},$$

there holds

$$m(a) = \inf_{\gamma \in \Gamma(a)} \sup_{t \in [0,1]} J(\gamma(t)) > 0.$$

Moreover, there exists a sequence $\{v_n\} \subset S(a)$ *such that, as* $n \to +\infty$,

$$J(v_n) \to m(a), \quad \left(J|_{S(a)}\right)'(v_n) \to 0, \quad I(v_n) \to 0.$$

Proof

(a) We first claim that J has a mountain pass geometry on $S(a)$. Let $u \in S(a)$. From (2.1) and (2.2), we have, for $s \in \mathbb{R}$,

$$F(s) \leq (F(-1) + F(1))(|s|^r + |s|^p),$$

and then

$$\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)dx \leq (F(-1) + F(1))^2 \int_{\mathbb{R}^N} (I_\alpha * (|s|^r + |s|^p))(|s|^r + |s|^p)dx.$$

Let $C > 0$ be a constant depending only on N, α, a, r, and p, which may change from line to line. Using the Hardy-Littlewood-Sobolev inequality, the Gagliardo-Nirenberg inequality and the Sobolev embedding inequality, we obtain for $u \in S(a)$

$$\begin{aligned}
&\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)dx \\
&\leq C\left[\int_{\mathbb{R}^N} (I_\alpha * |u|^r)|u|^r dx + 2\int_{\mathbb{R}^N} (I_\alpha * |u|^r)|u|^p dx + \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx\right] \\
&\leq C\left(\|u\|^{2r}_{L^{2Nr/(N+\alpha)}(\mathbb{R}^N)} + \|u\|^{2p}_{L^{2Np/(N+\alpha)}(\mathbb{R}^N)}\right) \\
&\leq C\left(\|\nabla u\|^{rN-N-\alpha}_{L^2(\mathbb{R}^N)} + \|\nabla u\|^{pN-N-\alpha}_{L^2(\mathbb{R}^N)}\right).
\end{aligned} \tag{2.7}$$

This implies that

$$J(u) \geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{C}{2}\|\nabla u\|^{rN-N-\alpha}_{L^2(\mathbb{R}^N)} - \frac{C}{2}\|\nabla u\|^{pN-N-\alpha}_{L^2(\mathbb{R}^N)}.$$

On the other hand, we have

$$J(u) \leq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

Since $pN - N - \alpha \geq rN - N - \alpha > 2$, there exist $0 < k_1 < k_2$ small enough such that

$$0 < \sup_{u \in D_{k_1}} J(u) < \inf_{u \in \partial D_{k_2}} J(u) \quad \text{and} \quad J(u) > 0 \quad \text{for} \quad u \in D_{k_2}.$$

(b) Clearly,

$$\lim_{t\to-\infty}\int_{\mathbb{R}^N}|\nabla u^t|^2dx=0 \quad \text{and} \quad \lim_{t\to+\infty}\int_{\mathbb{R}^N}|\nabla u^t|^2dx=+\infty.$$

Using (2.3), we rewrite $J(u^t)$ as

$$J(u^t)=\frac{e^{2t}}{2}\left(\int_{\mathbb{R}^N}|\nabla u|^2dx-\varphi(t)\right),$$

where

$$\varphi(t)=\int_{\mathbb{R}^N}\left(I_\alpha * \frac{F\left(e^{\frac{Nt}{2}}u\right)}{\left(e^{\frac{Nt}{2}}\right)^{\frac{N+\alpha+2}{N}}}\right)\frac{F\left(e^{\frac{Nt}{2}}u\right)}{\left(e^{\frac{Nt}{2}}\right)^{\frac{N+\alpha+2}{N}}}dx.$$

From (2.5), a similar discussion as in the proof of Lemma 2.1 shows that $\varphi(t)\to 0$ as $t\to-\infty$, and $\varphi(t)\to\infty$ as $t\to\infty$, hence

$$\lim_{t\to-\infty}J(u^t)=0 \quad \text{and} \quad \lim_{t\to+\infty}J(u^t)=-\infty. \tag{2.8}$$

Then for $u\in V(a)$, there exist $t_1<0$ and $t_2>0$ such that $u^{t_1}\in D_{k_1}$ and $J(u^{t_2})<0$. So, if we set $g(t)=u^{(1-t)t_1+tt_2}$ for $t\in[0,1]$, then $g\in\Gamma(a)$ and $\sup_{t\in[0,1]}J(g(t))=J(u)$. This implies

$$m(a)\geq\inf_{\gamma\in\Gamma(a)}\sup_{t\in[0,1]}J(\gamma(t))\geq\inf_{u\in\partial D_{k_2}}J(u)>0. \tag{2.9}$$

On the other hand, from Lemma 2.1 we see that $S(a)\backslash V(a)$ has precisely two components given by $S^{\pm}(a):=\{u\in S(a):\ \pm I(u)>0\}$. The result a) of this lemma implies that $D_{k_1}\subset S^+(a)$. By (f_1), if $J(u)<0$ then $I(u)<0$. This shows $J^0:=\{u\in S(a):\ J(u)<0\}\subset S^-(a)$. Hence any path in $\Gamma(a)$ must intersect $V(a)$. This property together with (2.9) implies

$$m(a)=\inf_{\gamma\in\Gamma(a)}\sup_{t\in[0,1]}J(\gamma(t))>0.$$

Now we recall the stretched functional introduced first in [12]:

$$\widetilde{J}:H^1(\mathbb{R}^N)\times\mathbb{R}\to\mathbb{R}, \quad (u,t)\mapsto J(u^t)$$

and define

$$\widetilde{\Gamma}(a):=\{g\in C([0,1],S(a)\times\mathbb{R}):\ g(0)\in D_{k_1}\times\{0\},\ g(1)\in J^0\times\{0\}\}.$$

Since for $\gamma\in\Gamma(a)$,

$$g(\cdot):=(\gamma(\cdot),0)\in\widetilde{\Gamma}(a) \quad \text{and} \quad \widetilde{J}(g(t))=J(\gamma(t)) \ \text{ for } \ t\in[0,1]$$

and for $g=(g_1,g_2)\in\widetilde{\Gamma}(a)$,

$$\gamma(\cdot):=g_1(\cdot)^{g_2(\cdot)}\in\Gamma(a) \quad \text{and} \quad J(\gamma(t))=\widetilde{J}(g(t)) \ \text{ for } \ t\in[0,1],$$

we have

$$\inf_{g\in\widetilde{\Gamma}(a)}\ \sup_{t\in[0,1]}\widetilde{J}(g(t))=m(a).$$

Then, using the Ekeland variational principle as in [12, Lemma 2.3], it follows that there exists a sequence $\{(v_n,t_n)\}\subset S(a)\times\mathbb{R}$ such that, as $n\to+\infty$,

$$t_n\to 0,\quad \widetilde{J}(v_n,t_n)\to m(a),\quad \left(\widetilde{J}|_{S(a)\times\mathbb{R}}\right)'(v_n,t_n)\to 0.$$

Without loss of generality, we may assume that $t_n=0$ because $\widetilde{J}(v_n,t_n)=\widetilde{J}(v_n^{t_n},0)$ and

$$\left(\widetilde{J}|_{S(a)\times\mathbb{R}}\right)'(v_n,t_n)[\varphi,s]=\left(\widetilde{J}|_{S(a)\times\mathbb{R}}\right)'(v_n^{t_n},0)[\varphi^{t_n},s]$$

for all $s\in\mathbb{R}$ and $\varphi\in H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N}v_n\varphi dx=0$. This implies that (see [12] for details), as $n\to+\infty$,

$$J(v_n)\to m(a),\quad \left(J|_{S(a)}\right)'(v_n)\to 0\quad\text{and}\quad \partial_t\widetilde{J}(v_n,0)\to 0.$$

Since $\partial_t\widetilde{J}(v_n,0)=I(v_n)$ the proof is complete.

□

Now we study the (PS) sequence $\{v_n\}$ of $J|_{S(a)}$ obtained in Lemma 2.2. With the help of the additional information $I(v_n)\to 0$, we will see that $\{v_n\}$ is bounded. We will use the concentration compactness principle to prove that, up to a translation and a subsequence, $v_n\to u_a$ weakly in $H^1(\mathbb{R}^N)$ for some $u_a\neq 0$. The fact that $I(v_n)\to 0$ will also be used to show that $\lim_{n\to\infty}\langle J'(v_n),v_n\rangle/a^2=\lambda_a$ for some $\lambda_a<0$ and that (λ_a,u_a) is a weak solution of (1.1).

Lemma 2.3 *If $\{v_n\}$ is the sequence obtained in Lemma 2.2, then there exists $u_a\in H^1(\mathbb{R}^N)\backslash\{0\}$ such that, up to a subsequence and a translation, $v_n\to u_a$ weakly in $H^1(\mathbb{R}^N)$. Moreover, there exists $\lambda_a<0$ such that (λ_a,u_a) is a weak solution of (1.1) and $I(u_a)=0$.*

Proof For $n\in\mathbb{N}$,

$$2J(v_n)-I(v_n)=\frac{N}{2}\int_{\mathbb{R}^N}(I_\alpha*F(v_n))\left[f(v_n)v_n-\frac{N+\alpha+2}{N}F(v_n)\right]dx. \tag{2.10}$$

Then by (f_1) there exist $C_1>0$ and $C_2>0$ such that

$$C_1\le\int_{\mathbb{R}^N}(I_\alpha*F(v_n))F(v_n)dx\le C_2. \tag{2.11}$$

Then we see that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$ from

$$\int_{\mathbb{R}^N}|\nabla v_n|^2dx=2J(v_n)+\int_{\mathbb{R}^N}(I_\alpha*F(v_n))F(v_n)dx.$$

Next we claim that

$$\varliminf_{n\to+\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|v_n|^2>0. \tag{2.12}$$

If this is false, we obtain $v_n\to 0$ in $L^q(\mathbb{R}^N)$ for $q\in(2,2^*)$ by Lions' vanishing lemma [23], and then, by the second inequality in (2.7), $\int_{\mathbb{R}^N}(I_\alpha * F(v_n))F(v_n)dx\to 0$ as $n\to+\infty$, contrary to (2.11). Hence (2.12) is true. Consequently, there is a sequence $\{y_n\}\subset\mathbb{R}^N$ and a $u_a\in H^1(\mathbb{R}^N)\backslash\{0\}$ such that, up to a subsequence, $v_n(\cdot-y_n)\to u_a$ weakly in $H^1(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. Replacing $v_n(\cdot-y_n)$ by v_n, we may assume $y_n=0$, and $v_n\to u_a$ weakly in $H^1(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. By (f_1), $\{F(v_n)\}$ is bounded in $L^{2N/(N+\alpha)}(\mathbb{R}^N)$ and $F(v_n)\to F(u_a)$ a.e. in $\mathbb{R}^N$. This in particular implies $F(v_n)\to F(u_a)$ weakly in $L^{2N/(N+\alpha)}(\mathbb{R}^N)$, and therefore $I_\alpha * F(v_n)\to I_\alpha * F(u_a)$ weakly in $L^{2N/(N-\alpha)}(\mathbb{R}^N)$ since $I_\alpha *: \ L^{2N/(N+\alpha)}(\mathbb{R}^N)\to L^{2N/(N-\alpha)}(\mathbb{R}^N)$ is a bounded linear operator.

Now assume $N\geq 3$. By (f_1) again, $\{f(v_n)\}$ is bounded in $L^{2N/(\alpha+2)}(\mathbb{R}^N)$ and $f(v_n)\to f(u_a)$ strongly in $L^{2N/(\alpha+2)}_{\text{loc}}(\mathbb{R}^N)$. Then, for any $\varphi\in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}(I_\alpha * F(v_n))f(v_n)\varphi\to\int_{\mathbb{R}^N}(I_\alpha * F(u_a))f(u_a)\varphi. \tag{2.13}$$

Since $\{(I_\alpha * F(v_n))f(v_n)\}$ is bounded in $L^{\frac{2N}{N+2}}(\mathbb{R}^N)$, which is seen by the Hölder inequality

$$\int_{\mathbb{R}^N}|(I_\alpha * F(v_n))f(v_n)|^{\frac{2N}{N+2}}\leq\left(\int_{\mathbb{R}^N}|I_\alpha * F(v_n)|^{\frac{2N}{N-\alpha}}\right)^{\frac{N-\alpha}{N+2}}\left(\int_{\mathbb{R}^N}|f(v_n)|^{\frac{2N}{\alpha+2}}\right)^{\frac{\alpha+2}{N+2}},$$

and since $C_0^\infty(\mathbb{R}^N)$ is dense in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, we deduce that

$$(I_\alpha * F(v_n))f(v_n)\to(I_\alpha * F(u_a))f(u_a)\qquad\text{weakly in } L^{\frac{2N}{N+2}}(\mathbb{R}^N). \tag{2.14}$$

Since $\left(J|_{S(a)}\right)'(v_n)\to 0$, we have (see [8, Lemma 3]), for $v\in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}\nabla v_n\cdot\nabla v dx-\mu_n\int_{\mathbb{R}^N}v_n v dx-\int_{\mathbb{R}^N}(I_\alpha * F(v_n))f(v_n)v dx=o(1)\|v\|, \tag{2.15}$$

where

$$\mu_n:=\frac{1}{a^2}\langle J'(v_n),v_n\rangle=\frac{1}{a^2}\left(\int_{\mathbb{R}^N}|\nabla v_n|^2dx-\int_{\mathbb{R}^N}(I_\alpha * F(v_n))f(v_n)v_n dx\right).$$

Using the definition of I in (1.5) and the fact that $\lim_{n\to\infty}I(v_n)=0$, μ_n can be expressed as

$$\begin{aligned}\mu_n&=\frac{1}{a^2}\left[I(v_n)-\frac{N+\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha * F(v_n))F(v_n)dx+\frac{N-2}{2}\int_{\mathbb{R}^N}(I_\alpha * F(v_n))f(v_n)v_n dx\right]\\&=o(1)+\frac{1}{2a^2}\int_{\mathbb{R}^N}(I_\alpha * F(v_n))[(N-2)f(v_n)v_n-(N+\alpha)F(v_n)]dx.\end{aligned}$$

By (f_1) and (2.11), for n large, μ_n is negative, bounded below and bounded away from 0. Therefore, there exists $\lambda_a<0$ such that, up to a subsequence, $\mu_n\to\lambda_a$. Moreover, it follows from (2.14) and (2.15) that, for any $v\in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \nabla u_a \cdot \nabla v dx - \lambda_a \int_{\mathbb{R}^N} u_a v dx - \int_{\mathbb{R}^N} (I_\alpha * F(u_a)) f(u_a) v dx = 0. \tag{2.16}$$

Then (λ_a, u_a) is a weak solution of (1.1). By [26, Theorem 3],

$$(N-2)\int_{\mathbb{R}^N} |\nabla u_a|^2 dx - N\lambda_a \int_{\mathbb{R}^N} |u_a|^2 dx - (N+\alpha)\int_{\mathbb{R}^N} (I_\alpha * F(u_a))F(u_a)dx = 0. \tag{2.17}$$

Letting $v = u_a$ in (2.16) and using (2.17) yields $I(u_a) = 0$.

If $N = 1, 2$, then by (f_1), $\{|f(v_n)|^s\}$ is bounded in $L^1(\mathbb{R}^N)$ for any $s \geq \frac{2N}{\alpha+2}$ and $f(v_n) \to f(u_a)$ strongly in $L^s_{\text{loc}}(\mathbb{R}^N)$ for any $s \geq 1$. This in particular implies that $\{f(v_n)\}$ is bounded in $L^{2N/(N+\alpha)}(\mathbb{R}^N)$ and $f(v_n) \to f(u_a)$ strongly in $L^{2N/(N+\alpha)}_{\text{loc}}(\mathbb{R}^N)$. Then, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, (2.13) is still valid. Since $\{(I_\alpha * F(v_n))f(v_n)\}$ is bounded in $L^2(\mathbb{R}^N)$ as seen from the inequality

$$\int_{\mathbb{R}^N} |(I_\alpha * F(v_n))f(v_n)|^2 \leq \left(\int_{\mathbb{R}^N} |I_\alpha * F(v_n)|^{\frac{2N}{N-\alpha}}\right)^{\frac{N-\alpha}{N}} \left(\int_{\mathbb{R}^N} |f(v_n)|^{\frac{2N}{\alpha}}\right)^{\frac{\alpha}{N}},$$

and since $C_0^\infty(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$, we deduce that

$$(I_\alpha * F(v_n))f(v_n) \to (I_\alpha * F(u_a))f(u_a) \qquad \text{weakly in } L^2(\mathbb{R}^N).$$

Then we use the same argument as above to find $\lambda_a < 0$ and see that (u_a, λ_a) satisfies (2.16). Since $N = 1, 2$, we have $u_a \in L^s(\mathbb{R}^N)$ for $s \geq 2$. Then $F(u_a) \in L^s(\mathbb{R}^N)$ for any $s \geq 1$. By the Hardy–Littlewood–Sobolev inequality, we deduce that

$$\int_{\mathbb{R}^N} |(I_\alpha * F(u_a))g| \leq C\|F(u_a)\|_{L^s(\mathbb{R}^N)}\|g\|_{L^t(\mathbb{R}^N)}$$

if $s > 1$, $t > 1$, $\frac{1}{s} + \frac{1}{t} = 1 + \frac{\alpha}{N}$, and $g \in L^t(\mathbb{R}^N)$. The range of $t > 1$ for which there exists $s > 1$ such that $\frac{1}{s} + \frac{1}{t} = 1 + \frac{\alpha}{N}$ is $1 < t < \frac{N}{\alpha}$. Then the range of numbers conjugate to such t is $(\frac{N}{N-\alpha}, +\infty)$. Therefore, $I_\alpha * F(u_a) \in L^t(\mathbb{R}^N)$ if $t > \frac{N}{N-\alpha}$. Then for any $\mu > 1$,

$$\int_{\mathbb{R}^N} |(I_\alpha * F(u_a))f(u_a)|^\mu \leq \|I_\alpha * F(u_a)\|^\mu_{L^{\mu N/(N-\alpha)}(\mathbb{R}^N)} \|f(u_a)\|_{L^{\mu N/\alpha}(\mathbb{R}^N)^\mu}.$$

Thus, by the Caldéron-Zygmund inequality [11, Theorem 9.9], $u_a \in W^{2,\mu}_{\text{loc}}(\mathbb{R}^N)$. Once we have this regularity of u_a, we can obtain (2.17) again in the spirit of the proof of [26, Theorem 3]. This combined with (2.16) yields $I(u_a) = 0$. □

Even though we have proved that (λ_a, u_a) is a solution of (1.1), we do not know whether $\|u_a\|_{L^2(\mathbb{R}^N)} = a$ and $J(u_a) = m(a)$ at this stage. This is not easy to see because the embedding $H^1(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$ is not compact for $q \in (2, 2^*)$. Here and in what follows, $2^* = 2N/(N-2)$ if $N \geq 3$ and $2^* = +\infty$ if $N = 1, 2$. In the proof of Theorem 1.1, we need the following lemma, in which the estimate is motivated by the paper [31].

Lemma 2.4 *For $a > 0$, the map $a \mapsto m(a)$ is strictly decreasing.*

Proof We fix $a_2 > a_1 > 0$. By the definition of $m(a)$, there exists $\{u_n\} \subset V(a_1)$ such that

$$m(a_1) \le J(u_n) \le m(a_1) + \frac{1}{n}.$$

Since $I(u_n) = 0$, we have

$$\begin{aligned} 2J(u_n) &= \int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n)dx \\ &= \frac{N}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) \left[f(u_n)u_n - \frac{N+\alpha+2}{N} F(u_n) \right] dx. \end{aligned} \tag{2.18}$$

It follows from (2.18) and (f_1) that

$$0 < \varliminf_{n\to+\infty} \int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n)dx \le \varlimsup_{n\to+\infty} \int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n)dx < +\infty \tag{2.19}$$

and

$$0 < \varliminf_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \le \varlimsup_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx < +\infty. \tag{2.20}$$

Setting $v_n(x) = \beta^{\frac{N-2}{2}} u_n(\beta x)$ with $\beta := a_1/a_2 < 1$, we obtain $v_n \in S(a_2)$. From Lemma 2.1 we deduce that there exists $t_n := t(v_n) \in \mathbb{R}$ such that $v_n^{t_n} \in V(a_2)$. Then we have

$$\begin{aligned} m(a_2) &\le J\left(v_n^{t_n}\right) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} \left|\nabla v_n^{t_n}\right|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} \left(I_\alpha * F\left(v_n^{t_n}\right)\right) F\left(v_n^{t_n}\right) dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} \left|\nabla u_n^{t_n}\right|^2 dx - \frac{1}{2(\beta e^{t_n})^{N+\alpha}} \int_{\mathbb{R}^N} \left(I_\alpha * F\left(\beta^{\frac{N-2}{2}} e^{\frac{Nt_n}{2}} u_n\right)\right) F\left(\beta^{\frac{N-2}{2}} e^{\frac{Nt_n}{2}} u_n\right) dx \\ &= J(u_n^{t_n}) + \frac{1}{2e^{(N+\alpha)t_n}} \int_{\mathbb{R}^N} \left(I_\alpha * F\left(e^{\frac{Nt_n}{2}} u_n\right)\right) F\left(e^{\frac{Nt_n}{2}} u_n\right) dx \\ &\quad - \frac{1}{2(\beta e^{t_n})^{N+\alpha}} \int_{\mathbb{R}^N} \left(I_\alpha * F\left(\beta^{\frac{N-2}{2}} e^{\frac{Nt_n}{2}} u_n\right)\right) F\left(\beta^{\frac{N-2}{2}} e^{\frac{Nt_n}{2}} u_n\right) dx \\ &\le J(u_n) + \frac{1}{2e^{(N+\alpha)t_n}} \int_{\mathbb{R}^N} \Phi\left(e^{\frac{Nt_n}{2}} u_n\right) dx \\ &\le m(a_1) + \frac{1}{n} + \frac{1}{2e^{(N+\alpha)t_n}} \int_{\mathbb{R}^N} \Phi\left(e^{\frac{Nt_n}{2}} u_n\right) dx, \end{aligned} \tag{2.21}$$

where

$$\Phi(u) = (I_\alpha * F(u))F(u) - \frac{1}{\beta^{N+\alpha}} \left(I_\alpha * F\left(\beta^{\frac{N-2}{2}} u\right)\right) F\left(\beta^{\frac{N-2}{2}} u\right).$$

We first consider the case $N \ge 3$. Since $\beta^{\frac{N-2}{2}} \le 1$, we have $F(\beta^{\frac{N-2}{2}} s) \ge \beta^{\frac{p(N-2)}{2}} F(s)$. This implies

$$\Phi(u) \le (1 - \beta^{(N-2)p-(N+\alpha)})(I_\alpha * F(u))F(u) \le 0, \tag{2.22}$$

because

$$\beta^{(N-2)p-(N+\alpha)} = \left(\frac{a_2}{a_1}\right)^{N+\alpha-p(N-2)} > 1.$$

Then we have

$$m(a_2) \le J(v_n^{t_n}) \le m(a_1) + \frac{1}{n} \tag{2.23}$$

which together with the fact that $v_n^{t_n} \in V(a_2)$, by estimates similar to (2.20), yields

$$0 < \varliminf_{n\to+\infty} \int_{\mathbb{R}^N} \left|\nabla v_n^{t_n}\right|^2 dx \le \varlimsup_{n\to+\infty} \int_{\mathbb{R}^N} \left|\nabla v_n^{t_n}\right|^2 dx < +\infty. \tag{2.24}$$

Observing that

$$\int_{\mathbb{R}^N} |\nabla v_n^{t_n}|^2 dx = e^{2t_n} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx = e^{2t_n} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx,$$

combining (2.20) and (2.24) implies the existence of $T > 0$ such that

$$-T < t_n < T. \tag{2.25}$$

Using (2.22) and (2.25) we can estimate the last term in (2.21):

$$\begin{aligned} \frac{1}{2e^{(N+\alpha)t_n}} \int_{\mathbb{R}^N} \Phi\left(e^{\frac{Nt_n}{2}} u_n\right) dx &\le -\frac{\beta^{(N-2)p-(N+\alpha)} - 1}{2e^{(N+\alpha)T}} \int_{\mathbb{R}^N} \left(I_\alpha * F\left(e^{\frac{Nt_n}{2}} u_n\right)\right) F(e^{\frac{Nt_n}{2}} u_n) dx \\ &\le -\frac{\beta^{(N-2)p-(N+\alpha)} - 1}{2e^{(N+Np+\alpha)T}} \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) F(u_n) dx, \end{aligned} \tag{2.26}$$

where the second inequality follows from

$$F\left(e^{\frac{Nt_n}{2}} u_n\right) \ge F\left(e^{-\frac{NT}{2}} u_n\right) \ge e^{-\frac{NTp}{2}} F(u_n).$$

In view of (2.19) and (2.21) we arrive at

$$m(a_2) \le m(a_1) + \frac{1}{n} - \delta \tag{2.27}$$

for n large and for some $\delta > 0$ independent of n. Then $m(a_2) < m(a_1)$ follows by letting $n \to +\infty$.

In the case $N = 1, 2$, we have $\beta^{\frac{N-2}{2}} \ge 1$ and $F(\beta^{\frac{N-2}{2}} s) \ge \beta^{\frac{r(N-2)}{2}} F(s)$, and we still have the estimates from (2.22) to (2.26) with p replaced by r. Therefore (2.27) is valid and the result follows. □

We are now ready to prove Theorem 1.1, using the previous lemmas. Again, the fact that $\lim_{n\to\infty} I(v_n) = 0 = I(u_a)$ plays an important role in the proof.

Proof of Theorem 1.1 Let $\{v_n\}$ be the sequence obtained in Lemma 2.2. By Lemma 2.3, there exist $\lambda_a < 0$ and $u_a \in H^1(\mathbb{R}^N)\backslash\{0\}$ such that (λ_a, u_a) is a weak solution of (1.1), $I(u_a) = 0$ and, up to a subsequence, $v_n \to u_a$ weakly in $H^1(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. Clearly we have

$$\int_{\mathbb{R}^N} |v_n|^2 dx = \int_{\mathbb{R}^N} |u_a|^2 dx + \int_{\mathbb{R}^N} |v_n - u_a|^2 dx + o(1).$$

Let $a_1 = \|u_a\|_{L^2(\mathbb{R}^N)}$ and $a_{2,n} = \|v_n - u_a\|_{L^2(\mathbb{R}^N)}$. Then

$$a_1 > 0 \quad \text{and} \quad a^2 = a_1^2 + a_{2,n}^2 + o(1).$$

Fatou's lemma implies that $I_\alpha * F(u_a) \le \liminf_{n\to\infty} I_\alpha * F(v_n)$. Since $I(u_a) = 0$, by (f_1) and Lemma 2.2 and using Fatou's lemma again, we have

$$\begin{aligned} J(u_a) =& J(u_a) - \frac{1}{2} I(u_a) \\ =& \frac{N}{4} \int_{\mathbb{R}^N} (I_\alpha * F(u_a)) \left[f(u_a)u_a - \frac{N+\alpha+2}{N} F(u_a) \right] dx \\ \le& \frac{N}{4} \varliminf_{n\to+\infty} \int_{\mathbb{R}^N} (I_\alpha * F(v_n)) \left[f(v_n)v_n - \frac{N+\alpha+2}{N} F(v_n) \right] dx \\ =& \varliminf_{n\to+\infty} (J(v_n) - \frac{1}{2} I(v_n)) = m(a). \end{aligned} \tag{2.28}$$

On the other hand, it follows from Lemma 2.4 that

$$J(u_a) \ge m(a_1) \ge m(a). \tag{2.29}$$

(2.28) together with (2.29) implies

$$J(u_a) = m(a_1) = m(a).$$

Now by Lemma 2.4 again, we see that $\|u_a\|_{L^2(\mathbb{R}^N)} = a_1 = a$. This completes the proof. □

The sequence $\{v_n\}$ converges strongly to u_a in $H^1(\mathbb{R}^N)$. Indeed, the above proof shows that

$$J(u_a) = \lim_{n\to+\infty} J(v_n) = m(a), \tag{2.30}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^N} (I_\alpha * F(u_a)) \left[f(u_a)u_a - \frac{N+\alpha+2}{N} F(u_a) \right] dx \\ &= \lim_{n\to+\infty} \int_{\mathbb{R}^N} (I_\alpha * F(v_n)) \left[f(v_n)v_n - \frac{N+\alpha+2}{N} F(v_n) \right] dx. \end{aligned} \tag{2.31}$$

From (2.31) and the decomposition

$$\begin{aligned} &\int_{\mathbb{R}^N} (I_\alpha * F(u)) \left[f(u)u - \frac{N+\alpha+2}{N} F(u) \right] dx \\ &= \int_{\mathbb{R}^N} (I_\alpha * F(u))[f(u)u - rF(u)]dx + \left(r - \frac{N+\alpha+2}{N} \right) \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)dx, \end{aligned}$$

using Fatou's lemma we see that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}(I_\alpha * F(v_n))F(v_n)dx = \int_{\mathbb{R}^N}(I_\alpha * F(u_a))F(u_a)dx.$$

Then it follows from (2.30) that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla v_n|^2dx = \int_{\mathbb{R}^N}|\nabla u_a|^2dx,$$

which together with the fact $\|v_n\|_{L^2(\mathbb{R}^N)} = \|u_a\|_{L^2(\mathbb{R}^N)}$ implies $v_n \to u_a$ strongly in $H^1(\mathbb{R}^N)$.

3 Proof of Theorem 1.3

In this section, we assume that $N \geq 2$, f is odd and $(f_1) - (f_2)$ hold. We adapt arguments from [1, 12] to the problem in question. Let $E := H^1_{\text{rad}}(\mathbb{R}^N)$ be the subspace of $H^1(\mathbb{R}^N)$ consisting of radially symmetric functions. Let $\|\cdot\|$ be the usual norm of $H^1(\mathbb{R}^N)$. We fix a strictly increasing sequence of finite-dimensional linear subspaces $V_k \subset E$ such that $\bigcup_k V_k$ is dense in E. Let $V_k^\perp$ be the orthogonal complement of V_k in E.

Lemma 3.1 *For any $k \in \mathbb{N}$, there exists $\rho_k > 0$ such that $b_k := \inf_{u\in B_k} J(u) \to +\infty$ as $k \to +\infty$ where*

$$B_k = \{u \in V_{k-1}^\perp \cap S(a) : \ \|\nabla u\|_{L^2(\mathbb{R}^N)} = \rho_k\}.$$

Proof For $q \in (2, 2^*)$, let

$$\mu_k(q) := \inf_{u\in V_{k-1}^\perp}\frac{\|u\|^2}{\|u\|^2_{L^q(\mathbb{R}^N)}}. \tag{3.1}$$

Then $\mu_k(q) \to +\infty$ as $k \to \infty$ (see [1, Lemma 2.1]). We see from (f_1) that $2 < 2Nr/(N+\alpha) \leq 2Np/(N+\alpha) < 2^*$ and, consequently,

$$\nu_k := \min\left\{\mu_k\left(\frac{2Nr}{N+\alpha}\right),\ \ \mu_k\left(\frac{2Np}{N+\alpha}\right)\right\} \to \infty \qquad \text{as } k \to \infty.$$

By (2.7) and (3.1), for $u \in V_{k-1}^\perp \cap S(a)$ with k sufficiently large,

$$\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)dx \leq C\left(\|u\|^{2r}_{L^{2Nr/(N+\alpha)}(\mathbb{R}^N)} + \|u\|^{2p}_{L^{2Np/(N+\alpha)}(\mathbb{R}^N)}\right) \leq \frac{C}{\nu_k^r}(\|u\|^{2r} + \|u\|^{2p}).$$

Since $\|u\|^2 \geq a^2$, we have $\|u\|^{2r} \leq a^{2r-2p}\|u\|^{2p}$, and it follows that

$$\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)dx \leq \frac{C}{\nu_k^r}\|u\|^{2p} \leq \frac{C}{\nu_k^r}\left(\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^p + 1\right),$$

where $C = C(f, N, \alpha, a) > 0$ is a constant depending only on f, N, α and a. Now we have

$$\begin{aligned} J(u) =& \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)dx \\ \geq & \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2 dx - \frac{C}{2v_k^r}\left(\left(\int_{\mathbb{R}^N}|\nabla u|^2 dx\right)^p + 1\right). \end{aligned}$$

Let

$$\rho_k = \left(\frac{v_k^r}{2C}\right)^{\frac{1}{2p-2}}.$$

Then, for $u \in B_k$,

$$J(u) \geq \frac{1}{2}\rho_k^2 - \frac{C\rho_k^{2p}}{2v_k^r} - \frac{C}{2v_k^r} = \frac{1}{4}\rho_k^2 - \frac{C}{2v_k^r}.$$

Since $v_k \to +\infty$ as $k \to \infty$, $\rho_k \to +\infty$ as $k \to \infty$ and the result follows. □

According to Lemma 3.1, there exists $k_0 \in \mathbb{N}$ such that $b_k > 1$ for $k \geq k_0$. For $k \geq k_0$, (2.8) and the compactness of $V_k \cap S(a)$ imply that there exists $t_k > 0$ large enough so that

$$\|\nabla u^{-t_k}\|_{L^2(\mathbb{R}^N)} < \rho_k < \|\nabla u^{t_k}\|_{L^2(\mathbb{R}^N)} \quad \text{and} \quad \max_{u \in V_k \cap S(a)}\{J(u^{-t_k}), J(u^{t_k})\} < 1. \tag{3.2}$$

Now we define

$$\Gamma_k = \{\gamma : [0,1] \times (S(a) \cap V_k) \to S(a) : \gamma \text{ is continuous, odd in } u, \\ \gamma(0,u) = u^{-t_k}, \text{ and } \gamma(1,u) = u^{t_k}\}.$$

A key role in the argument in [1] is played by the intersection property:

$$\gamma([0,1] \times (S(a) \cap V_k)) \cap B_k \neq \emptyset \quad \text{for every } \gamma \in \Gamma_k.$$

In [1, Lemma 2.3] this property was proved using the cohomological index theory for spaces with an action of the group $G = \{-1, 1\}$. Here we provide a new and more elementary proof of this property that does not require the cohomological index. We first show the following lemma using the Borsuk-Ulam theorem.

Lemma 3.2 *Let $L_1 \subsetneq L$ be finite dimensional normed vector spaces. Let $a > 0$, $S = \{u \in L : \|u\| = a\}$, $\alpha \in \mathbb{R}$, and $H = (H_1, H_2) : [0,1] \times S \to \mathbb{R} \times L_1$ be a continuous map such that $H_1(t,u)$ is even in u, $H_2(t,u)$ is odd in u, and*

$$H_1(0,u) < \alpha < H_1(1,u) \quad \text{for } u \in S. \tag{3.3}$$

Then there exists $(t,u) \in [0,1] \times S$ such that $H(t,u) = (\alpha, 0)$.

Proof Assume, by contradiction, that the conclusion is false. Let

$$K_1 = \{(t,u) \in [0,1] \times S : \ H_1(t,u) \leq \alpha, \ H_2(t,u) = 0\} \cup (\{0\} \times S)$$

and

$$K_2 = \{(t,u) \in [0,1] \times S : \ H_1(t,u) \geq \alpha, \ H_2(t,u) = 0\} \cup (\{1\} \times S).$$

Then K_1 and K_2 are closed subsets of $[0,1]\times S$, and hence compact. According to (3.3) and the assumption of contradiction, K_1 and K_2 are disjoint, and therefore $\delta := \frac{1}{4}\operatorname{dist}(K_1, K_2) > 0$. Set

$$N_\delta(K_i) = \{(t,u) \in [0,1]\times S : \ \operatorname{dist}((t,u),\ K_i) < \delta\}, \quad i = 1, 2.$$

Since $H_1(t,u)$ is even in u and $H_2(t,u)$ is odd in u, K_i and thus $N_\delta(K_i)$ are symmetric sets with respect to u. Moreover, $N_\delta(K_1) \cap N_\delta(K_2) = \emptyset$ and

$$H_2(t,u) \neq 0 \quad \text{if } (t,u) \in \partial^{[0,1]\times S} N_\delta(K_i), \tag{3.4}$$

where $\partial^{[0,1]\times S} N_\delta(K_i)$ is the boundary of $N_\delta(K_i)$ in $[0,1]\times S$. Denote $\Omega = \{u \in L : \|u\| < a\}$ and let $M = (M_1, M_2) : L \to (\{0\}\times\Omega) \cup ([0,1)\times S)$ be the homeomorphism induced by the stereographic projection with north pole $(1,0) \in \mathbb{R}\times L$. To be more precise,

$$M(u) = \begin{cases} (0,u) & \text{if } u \in \Omega, \\ (1 - a\|u\|^{-1},\ a\|u\|^{-1}u) & \text{if } u \in L\backslash\Omega. \end{cases}$$

Denote $\Omega_1 = M^{-1}((\{0\}\times\Omega) \cup N_\delta(K_1))$. Then Ω_1 is an open bounded symmetric neighborhood of 0 in L and $M(\partial\Omega_1) = \partial^{[0,1]\times S} N_\delta(K_1)$. Now we define $A : \partial\Omega_1 \to L_1$ as

$$A(u) = H_2 \circ M(u) = H_2(1 - a\|u\|^{-1},\ a\|u\|^{-1}u) \quad \text{for } u \in \partial\Omega_1.$$

Then A is odd and continuous and, by (3.4), $A(u) \neq 0$ for all $u \in \partial\Omega_1$. But this is in contradiction with the Borsuk–Ulam theorem. □

The following is [1, Lemma 2.3], for which we provide a new proof based on Lemma 3.2.

Lemma 3.3 *$\gamma([0,1]\times(S(a)\cap V_k)) \cap B_k \neq \emptyset$ for every $\gamma \in \Gamma_k$.*

Proof Set $L = V_k$ and $L_1 = V_{k-1}$ in which we use the $L^2(\mathbb{R}^N)$ norm. Choose $S = S(a)\cap V_k$ and $\alpha = \rho_k$. Let $P_{k-1} : E \to V_{n-1}$ be the orthogonal projection and define

$$h_k : S(a) \to \mathbb{R}\times V_{k-1}, \quad u \mapsto (\|\nabla u\|_{L^2(\mathbb{R}^N)},\ P_{k-1}u)$$

and

$$H = (H_1,\ H_2) = h_k \circ \gamma.$$

Then L, L_1, S, α, and H satisfy all the conditions of Lemma 3.2. By Lemma 3.2, there exists $(t,u) \in [0,1]\times(S(a)\cap V_k)$ such that $H(t,u) = (\alpha, 0)$. That is, $\gamma(t,u) \in B_k$. □

By Lemmas 3.1 and 3.3, we have

$$c_k := \inf_{\gamma\in\Gamma_k} \max_{t\in[0,1],\ u\in S(a)\cap V_k} J(\gamma(t,u)) \geq b_k \to +\infty.$$

We will show that c_k, $k \geq k_0$, is a critical value of $J|_{S(a)}$. For this we make use of the stretched functional

$$\widetilde{J} : \mathbf{E} := E \times \mathbb{R} \to \mathbb{R}, \quad \widetilde{J}(u,s) = J(u^s)$$

from the proof of Lemma 2.2, now constrained to the space E of radial functions. On $\mathbf{E}$ we consider the involution

$$\tau : \mathbf{E} \to \mathbf{E}, \quad \tau(u,s) := (-u,s),$$

which preserves the natural inner product on $\mathbf{E}$. Clearly $\widetilde{J}$ is invariant under τ because F is even in u. This implies that $\nabla\widetilde{J} : \mathbf{E} \to \mathbf{E}$ is τ-equivariant, i.e. $\nabla\widetilde{J} \circ \tau = \tau \circ \nabla\widetilde{J}$.

Now we define

$$\widetilde{c}_k := \inf_{g \in \widetilde{\Gamma}_k} \max_{t \in [0,1],\ u \in S(a) \cap V_k} \widetilde{J}(g(t,u))$$

where

$$\widetilde{\Gamma}_k := \{g : [0,1] \times (S(a) \cap V_k) \to S(a) \times \mathbb{R} : g \text{ is continuous and equivariant,}$$
$$g(0,u) = (u^{-t_k}, 0), \text{ and } g(1,u) = (u^{t_k}, 0)\}.$$

Here a map $g : [0,1] \times (S(a) \cap V_k) \to S(a) \times \mathbb{R}$ is said to be equivariant if $g(t,-u) = \tau g(t,u)$.

Lemma 3.4 $\widetilde{c}_k = c_k$.

Proof Observe that

$$\gamma \in \Gamma_k \implies g := (\gamma, 0) \in \widetilde{\Gamma}_k, \quad \widetilde{J}(g(t,u)) = J(\gamma(t,u)),$$

and

$$g \in \widetilde{\Gamma}_k \implies \gamma := g_1^{g_2} \in \Gamma_k, \quad J(\gamma(t,u)) = \widetilde{J}(g(t,u)).$$

Then the result follows. □

Recall that we fixed $k \geq k_0$ so that $b_k > 1$. Then Lemmas 3.3 and 3.4 yield

$$\widetilde{c}_k = c_k \geq b_k > 1.$$

To show that c_k is a critical value of $J|_{S(a)}$, we first prove the following result.

Lemma 3.5 *Let* $0 < \varepsilon < c_k - 1$ *and* $g \in \widetilde{\Gamma}_k$ *be such that*

$$\max_{t \in [0,1],\ u \in S(a) \cap V_k} \widetilde{J}(g(t,u)) \leq c_k + \varepsilon.$$

Then there exists $(v,s) \in S(a) \times \mathbb{R}$ *such that*:

(i) $\widetilde{J}(v,s) \in [c_k - \varepsilon, c_k + \varepsilon]$,
(ii) $\min_{t \in [0,1],\ u \in S(a) \cap V_k} \|(v,s) - g(t,u)\|_{\mathbf{E}} \leq \sqrt{\varepsilon}$,
(iii) $\|\left(\widetilde{J}|_{S(a) \times \mathbb{R}}\right)'(v,s)\| \leq 2\sqrt{\varepsilon}$.

Proof The proof is an equivariant version of the one of [12, Lemma 2.3]. On $\widetilde{\Gamma}_k$, we define the metric

$$d(g,h) = \max_{t\in[0,1],\ u\in S(a)\cap V_k} \|g(t,u) - h(t,u)\|_{\mathbf{E}}$$

and consider the continuous function

$$\Phi : \widetilde{\Gamma}_k \to \mathbb{R}, \quad \Phi(g) = \max_{t\in[0,1],\ u\in S(a)\cap V_k} \widetilde{J}(g(t,u)).$$

By Ekeland's variational principle there exists $h \in \widetilde{\Gamma}_k$ such that

(i) $\Phi(h) \leq \Phi(g)$,
(ii) $d(h,g) \leq \sqrt{\varepsilon}$,
(iii) $\Phi(h_1) > \Phi(h) - \sqrt{\varepsilon} d(h_1,h)$ for $h_1 \in \widetilde{\Gamma}_k$ with $h_1 \neq h$.

Assume, by contradiction, that the result is not true. For $(u,s) \in S(a) \times \mathbb{R}$ the tangent space is denoted by

$$T_{(u,s)} := T_{(u,s)}(S(a) \times \mathbb{R}) = T_u S(a) \times \mathbb{R} = \left\{(z_1, z_2) \in \mathbf{E} :\ \langle u, z_1\rangle_{L^2} = 0\right\}.$$

For $(t,u) \in [0,1] \times (S(a) \cap V_k)$ with $\widetilde{J}(h(t,u)) \geq c_k - \varepsilon$, there exists $z(t,u) \in T_{h(t,u)}$ with $\|z(t,u)\|_{\mathbf{E}} = 1$ and such that

$$\langle \nabla\widetilde{J}(h(t,u)), z(t,u)\rangle_{\mathbf{E}} < -2\sqrt{\varepsilon}. \tag{3.5}$$

Denote

$$S = \{(t,u) \in [0,1] \times (S(a) \cap V_k) : \quad \widetilde{J}(h(t,u)) \geq c_k - \varepsilon\}.$$

Then, using (3.5), we can construct a continuous vector field $U : S \to \mathbf{E}$ such that

(i) $\|U(t,u\|_{\mathbf{E}} = 1$,
(ii) $U(t,u) \in T_{h(t,u)}$,
(iii) $\langle \nabla\widetilde{J}(h(t,u)), U(t,u)\rangle_{\mathbf{E}} < -2\sqrt{\varepsilon}$.

Thus U is a normalized pseudo-gradient vector field for $\widetilde{J}$ along h. The vector field

$$V(t,u) := \frac{1}{2}\big(U(t,u) + \tau \circ U(t,-u)\big) \in T_{h(t,u)}$$

is equivariant, i.e. $V(t,-u) = \tau V(t,u)$. Moreover (iii) with V instead of U holds because $\nabla\widetilde{J} \circ h$ is equivariant. It also follows that $V(t,u) \neq 0$, hence we may pass to the normalized vector field

$$W : S \to \mathbf{E}, \quad W(t,u) = \frac{1}{\|V(t,u)\|_{\mathbf{E}}} V(t,u),$$

which is continuous and equivariant. Clearly (i)-(iii) from above hold with U replaced by W. Let $\Psi : [0,1] \times (S(a) \cap V_k) \to [0,1]$ be a continuous cut-off function satisfying

$$\Psi(t,u) = \begin{cases} 1, & \text{if } \widetilde{J}(h(t,u)) \geq c_k, \\ 0, & \text{if } \widetilde{J}(h(t,u)) \leq c_k - \varepsilon. \end{cases}$$

We may assume that $\Psi(t,-u)=\Psi(t,u)$ because $\widetilde{J}$ is even in u. If not we replace $\Psi(t,u)$ by $\frac{1}{2}\big(\Psi(t,u)+\Psi(t,-u)\big)$. Now we use W to deform $h\in\widetilde{\Gamma}$ as follows. For $r\in[0,1/2]$ we define $g_r=(g_{r,1},g_{r,2}):[0,1]\times(S(a)\cap V_k)\to S(a)\times\mathbb{R}$ for $(t,u)\in S$ by

$$g_{r,1}(t,u)=\sqrt{1-\frac{r^2\Psi^2(t,u)\|W_1(t,u)\|_{L^2}^2}{a^2}}h_1(t,u)+r\Psi(t,u)W_1(t,u),$$

and

$$g_{r,2}(t,u)=h_2(t,u)+r\Psi(t,u)W_2(t,u).$$

For $(t,u)\in[0,1]\times(S(a)\cap V_k)\backslash S$ we set $g_r(t,u)=h(t,u)$. Observe that g_r is continuous and equivariant: $g_r(t,-u)=\tau g_r(t,u)$. In addition, (3.2) implies

$$\max\{\widetilde{J}(h(0,u)),\widetilde{J}(h(1,u))\}=\max\{\widetilde{J}(u^{-t_k},0),\widetilde{J}(u^{t_k},0)\}=\max\{J(u^{-t_k}),J(u^{t_k})\}<1,$$

hence $c_k-\varepsilon>1$ yields

$$g_r(0,u)=h(0,u)=(u^{-t_k},0),\quad g_r(1,u)=h(1,u)=(u^{t_k},0).$$

Therefore, $g_r\in\widetilde{\Gamma}_k$ for $r\in[0,1/2]$.

The rest of the proof proceeds as the one of [12, Lemma 2.3], leading to a contradiction. □

From Lemma 3.5, it is possible to find a (PS) sequence $\{v_{k,n}\}\subset S(a)$ for $J|_{S(a)}$ with the additional property $\lim_{n\to\infty}I(v_{k,n})=0$ at the level c_k.

Lemma 3.6 *There exists a sequence* $\{v_{k,n}\}\subset S(a)$, *also denoted by* $\{v_n\}$ *for simplicity of notation, such that, as* $n\to+\infty$,

$$J(v_n)\to c_k,\quad \big(J|_{S(a)}\big)'(v_n)\to 0,\quad I(v_n)\to 0.$$

Proof The proof is essentially the same as the proof of [12, Lemma 2.4]. We only sketch it and refer to [12] for more details. For n large such that $\frac{1}{n}<c_k-1$, let $g_n\in\widetilde{\Gamma}_k$ be such that

$$\max_{t\in[0,1],\,u\in S(a)\cap V_k}\widetilde{J}(g_n(t,u))\le c_k+\frac{1}{n}.$$

We may assume that $g_n(t,u)=(g_{n,1}(t,u),0)$ with $g_{n,1}\in\Gamma_k$. By Lemma 3.5 there exists $(w_n,s_n)\in S(a)\times\mathbb{R}$ such that:

(i) $\widetilde{J}(w_n,s_n)\in[c_k-\frac{1}{n},c_k+\frac{1}{n}]$,
(ii) $\min_{t\in[0,1],\,u\in S(a)\cap V_k}\|(w_n,s_n)-g_n(t,u)\|_{\mathbf{E}}\le\frac{1}{\sqrt{n}}$,
(iii) $\|\big(\widetilde{J}|_{S(a)\times\mathbb{R}}\big)'(w_n,s_n)\|\le\frac{2}{\sqrt{n}}$.

Clearly (ii) implies

$$\lim_{n\to\infty} s_n = 0. \tag{3.6}$$

Set $v_n = w_n^{s_n}$. Then by (i)

$$\lim_{n\to\infty} J(v_n) = \lim_{n\to\infty} \widetilde{J}(w_n, s_n) = c_k.$$

Since $I(v_n) = \partial_s \widetilde{J}(w_n, s_n)$, by (iii) we have

$$\lim_{n\to\infty} I(v_n) = 0.$$

Moreover, since $\langle J'(v_n), \varphi\rangle_{E^*\times E} = \langle \partial_u \widetilde{J}(w_n, s_n), \varphi^{-s_n}\rangle_{E^*\times E}$ for any $\varphi \in T_{S(a)}v_n$, by (iii) and (3.6), for n sufficiently large,

$$\begin{aligned}\|\left(J|_{S(a)}\right)'(v_n)\| &= \sup_{\varphi\in T_{S(a)}v_n,\ \|\varphi\|_E=1} \langle J'(v_n), \varphi\rangle_{E^*\times E}\\ &= \sup_{\varphi\in T_{S(a)}v_n,\ \|\varphi\|_E=1} \langle \partial_u \widetilde{J}(w_n, s_n), \varphi^{-s_n}\rangle_{E^*\times E}\\ &\le \frac{2}{\sqrt{n}} \sup_{\varphi\in T_{S(a)}v_n,\ \|\varphi\|_E=1} \|\varphi^{-s_n}\|_E \le \frac{4}{\sqrt{n}}.\end{aligned}$$

This completes the proof. □

With (PS) sequences on hand, we are in a position to study their compactness. Since we are working in the space E consisting of radially symmetric functions so that E is imbedded compactly in $L^q(\mathbb{R}^N)$ for any $q \in (2, 2^*)$, it is easier to show the compactness of (PS) sequences in the present case compared with the argument in Sect. 2.

Lemma 3.7 *Let $\{v_{k,n}\}_{n=1}^\infty \subset S(a)$ be the sequence obtained in Lemma 3.6. Then up to a subsequence $\{v_{k,n}\}_{n=1}^\infty$ converges strongly in E to some $u_k \in S(a)$ as $n \to \infty$. Moreover, there exists $\lambda_k < 0$ such that (λ_k, u_k) is a solution of (1.2) and $J(u_k) = c_k$.*

Proof In view of the fact that $v_{k,n}$ satisfies (2.10) and (2.11), we see that $\{v_{k,n}\}_{n=1}^\infty$ is bounded in E. Therefore, we may assume that there exists $u_k \in E$ such that up to a subsequence $v_{k,n} \to u_k$ weakly in E. Since $N \ge 2$, E is imbedded compactly in $L^q(\mathbb{R}^N)$ for any $q \in (2, 2^*)$. Then $v_{k,n} \to u_k$ strongly in $L^q(\mathbb{R}^N)$ for any $q \in (2, 2^*)$.

Since

$$|F(s)| \le C(|s|^r + |s|^p) \quad \text{and} \quad 2 < \frac{2Nr}{N+\alpha} \le \frac{2Np}{N+\alpha} < 2^*,$$

we have $F(v_{k,n}) \to F(u_k)$ strongly in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ (see [29, Theorem A.4]). Then $I_\alpha * F(v_{k,n}) \to I_\alpha * F(u_k)$ strongly in $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$ as $I_\alpha * : L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) \to L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$ is a bounded linear operator. Since

$$2 < \frac{2N(r-1)}{\alpha+2} \le \frac{2N(p-1)}{\alpha+2} < 2^*,$$

we can choose a number q such that

$$\frac{2N}{\alpha+2}<q<\frac{2N}{\alpha} \tag{3.7}$$

and

$$2<q(r-1)\leq q(p-1)<2^*. \tag{3.8}$$

The condition

$$|f(s)|\leq C(|s|^{r-1}+|s|^{p-1})$$

together with (3.8) implies $f(v_{k,n})\to f(u_k)$ strongly in $L^q(\mathbb{R}^N)$ (see [29, Theorem A.4]). Let μ be the number defined by

$$\frac{N-\alpha}{2N}+\frac{1}{q}=\frac{1}{\mu}. \tag{3.9}$$

Then $(I_\alpha * F(v_{k,n}))f(v_{k,n})\to(I_\alpha * F(u_k))f(u_k)$ strongly in $L^\mu(\mathbb{R}^N)$. By (3.7) and (3.9), we have

$$\frac{2N}{N+2}<\mu<2. \tag{3.10}$$

Note that (see [8, Lemma 3]), for $v\in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}\nabla v_{k,n}\cdot\nabla v dx-\mu_{k,n}\int_{\mathbb{R}^N}v_{k,n}v dx-\int_{\mathbb{R}^N}(I_\alpha * F(v_{k,n}))f(v_{k,n})v dx=o(1)\|v\| \tag{3.11}$$

as $n\to\infty$, where

$$\mu_{k,n}:=\frac{1}{a^2}\langle J'(v_{k,n}),v_{k,n}\rangle.$$

Since we already proved the convergence of $(I_\alpha * F(v_{k,n}))f(v_{k,n})$ to $(I_\alpha * F(u_k))f(u_k)$ in $L^\mu(\mathbb{R}^N)$ and since μ is in the range of (3.10), we can argue in the same way as (2.15)–(2.16) to see that, for $v\in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}\nabla u_k\cdot\nabla v dx-\lambda_k\int_{\mathbb{R}^N}u_k v dx-\int_{\mathbb{R}^N}(I_\alpha * F(u_k))f(u_k)v dx=0, \tag{3.12}$$

where $\lambda_k=\lim_{n\to\infty}\mu_{k,n}<0$. Taking the difference between (3.11) and (3.12) with $v=v_{k,n}-u_k$, we see for $n\to\infty$:

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla(v_{k,n}-u_k)|^2dx-\lambda_k\int_{\mathbb{R}^N}(v_{k,n}-u_k)^2dx\\&=\int_{\mathbb{R}^N}\left[(I_\alpha * F(v_{k,n}))f(v_{k,n})-(I_\alpha * F(u_n))f(u_n)\right](v_{k,n}-u_n)dx+o(1)\to 0.\end{aligned}$$

Therefore, $\{v_{k,n}\}_{n=1}^\infty$ converges strongly in E to $u_k\in S(a)$ as $n\to\infty$ and (λ_k,u_k) is a solution of (1.1) with $J(u_k)=c_k$. □

Proof of Theorem 1.3 By Lemma 3.7, $(\lambda_k,\pm u_k)$ are radial solutions of (1.2) with $\lambda_k<0$ and $J(u_k)=c_k$. This proves the result. □

Acknowledgements Y. Liu is support by Beijing Advanced Innovation Center for Imaging Theory and Technology. Z. Liu is supported by National Natural Science Foundation of China (No. 11671272).

Note added in proof: After acceptance of our paper for publication we learned of the paper [18], which contains analogous results for a related Choquard equation involving the fractional Laplacian and a homogeneous nonlinearity.

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