SN Partial Differ. Equ. Appl. (2020) 1:29
https://doi.org/10.1007/s42985-020-00034-y

ORIGINAL PAPER

Nontrivial solutions to non-local problems with sublinear or superlinear nonlinearities

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Received: 22 March 2020 / Accepted: 31 July 2020 / Published online: 1 September 2020

Abstract
Existence of nontrivial and multiple solutions for two types of non-local problems with sublinear or superlinear nonlinearities are investigated by linking theorems and index theory in critical point theory. Some results in the literature are extended.

Keywords Non-local(Fractional) problems · Nontrivial(Multiple) solutions · Linking theorems · Index theory

Mathematics Subject Classification 35R11 · 35A15 · 47A10

1 Introduction

Fractional and non-local operators of elliptic type arise in a quite natural way in many different problems, such as the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering,minimal surfaces, materials science, water waves and so on. The investigations of the problems involved these non-local operators are interesting and important from both pure mathematical research aspects and real-world applications, eg see [1, 2] and references therein.

Recently, variational methods and critical point theory have been proved to be powerful in dealing with these non-local elliptic problems after the paper [3] establishing the framework for the solvability of the following problems

This article is part of the topical collection dedicated to Prof. Dajun Guo for his 85th birthday, edited by Yihong Du, Zhaoli Liu, Xingbin Pan, and Zhitao Zhang.

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$$\begin{cases} (-\Delta)^s u = g(x,u) & \text{in} \quad \Omega, \\ \quad u = 0 & \text{in} \quad \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded set with smooth boundary, $0<s<1$, $N > 2s$ and $(-\Delta)^s$ is the fractional Laplace operator, which (up to normalization factors) may be defined as

$$(-\Delta)^s u(x) = \int_{\mathbb{R}^N} \frac{(u(x+y) + u(x-y) - 2u(x))}{|y|^{N+2s}} dy, \qquad x \in \mathbb{R}^N.$$

They established the solvability of nontrivial solutions of the problems under the Ambrosetti- Rabinowitz superlinear condition for the nonlinearity $g(x, u)$: there exist $\mu > 2$ and $r > 0$ such that for a.e. $x \in \Omega,\ t \in \mathbb{R},\ |t| \geq r$, we have $0 < \mu G(x,t) \leq t g(x,t)$, where $G = \int_0^t g(x,\tau) d\tau$. When the nonlinearity $g(x, u)$ satisfies a linear growth condition, solvability of the problem is studied in [4]. When $g(x, u)$ is a lower order perturbation of the critical power, the classical Brezis-Nirenberg results are established in [5]. Some multiplicity results are also established either by Morse theory eg see [6, 7] or by fountain theorems eg see [8] where superlinear nonlinearities without Ambrosetti- Rabinowitz conditions are considered.

When $g(x, u)$ is a linear perturbation such as $g(x,u) = \lambda q(x) u + f(x,u)$, where $\lambda = 1$ or is a parameter related to the eigenvalues of certain eigenvalue problems and $q \in L^\infty(\Omega)$, several researches considered the so-called non-resonance or resonant problems with Landesman-Lazer conditions when $g(x, u)$ is bounded; for example, see [9, 10]. When the perturbation $g(x, u)$ is superlinear and satisfies the Ambrosetti- Rabinowitz conditions, we refer to [11] for some results concerning the solvability of nontrivial solutions of the problem. In this paper, we first consider the case where the perturbation $g(x, u)$ is sublinear and satisfies an extended form of Ahmad-Lazer-Paul type conditions; e.g. see [12–14] for some references. We remark that when $f(x, u)$ is bounded, the Ahmad-Lazer-Paul type is more general than the Landesman-Lazer condition used in [10]; see [13, 15] and the references therein. Hence our results generalize the corresponding ones in [9, 10]. The other case we consider is the superlinear perturbation $f(x, u)$, but we do not impose the standard Ambrosetti- Rabinowitz condition on it. This makes our results can be applied to more general nonlinearities. Moreover, not as in the literature we don't need that the function q is bounded which makes us to deal with a different eigenvalue problem.

2 Main results

As in [3], we consider a more general non-local operator $\mathcal{L}_K$ with $(-\Delta)^s$ as a special case, which is defined as follows:

$$\mathcal{L}_K u(x) = \int_{\mathbb{R}^N} \Big((u(x+y) + u(x-y) - 2u(x)) \Big) K(y) dy, \qquad x \in \mathbb{R}^N.$$

Here $K : \mathbb{R}^N \backslash \{0\} \to (0, +\infty)$ is a function such that

$$mK \in L^1(\mathbb{R}^N), \quad \text{where} \quad m(x) = \min\{|x|^2, 1\}; \tag{2}$$

$$\text{there exists } \theta > 0 \text{ such that } K(x) \geq \theta |x|^{-(N+2s)} \quad \text{for any} \quad x \in \mathbb{R}^N \backslash \{0\}; \tag{3}$$

$$K(x) = K(-x) \quad \text{for any } x \in \mathbb{R}^N \backslash \{0\}. \tag{4}$$

The function space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and $(g(x)-g(y))\sqrt{K(x-y)} \in L^2(\mathbb{R}^{2N}\backslash(\mathcal{C}\Omega \times \mathcal{C}\Omega), dxdy)$, where $\mathcal{C}\Omega = \mathbb{R}^{2N}\backslash\Omega$.

Instead of (1), we consider the more general problem

$$\begin{cases} -\mathcal{L}_K u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N\backslash\Omega. \end{cases} \tag{5}$$

By a solution of (5), we mean a weak one. That is a $u \in X_0$ such that

$$\int_{\mathbb{R}^{2N}} (u(x)-u(y))(\phi(x)-\phi(y))K(x-y)dxdy = \int_\Omega g(x,u(x))\phi(x)dx, \quad \forall \phi \in X_0,$$

where the Hillbert space X_0 denotes

$$X_0 := \{u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N\backslash\Omega\},$$

with the scalar product

$$\langle u, v\rangle = \int_{\mathbb{R}^{2N}} (u(x)-u(y))(v(x)-v(y))K(x-y)dxdy$$

and the norm $\|u\|^2 = \langle u, u\rangle$.

The main results of the paper are existence theorems (Theorems 1–3 and Theorem 5) and a multiplicity result (Theorem 4) for two types equations driven by general non-local operators including fractional operators as special cases with sublinear or superlinear nonlinearities.

To be precise, in the first part of the paper we study the following problem with sublinear nonlinearities

$$\begin{cases} -\mathcal{L}_K u + a(x)u - \lambda_k u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{6}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary, $s \in (0,1)$, $N > 2s$, $a(x) \in L^{\frac{N}{2s}}(\Omega)$, $g(x, u)$ is not necessarily bounded and λ_k is an eigenvalue of the problem

$$\begin{cases} -\mathcal{L}_K u + a(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N\backslash\Omega. \end{cases} \tag{7}$$

We notice that since $a(x) \in L^{\frac{N}{2s}}(\Omega)$ is not necessarily bounded, eigenvalue problem (7) does not seem to have been investigated in the literature. Hence we will first study problem (7). Particularly, we will prove that it has and only has a sequence of eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \ldots$ with a finite multiplicity for each eigenvalue. The eigenspace corresponding to λ_i is denoted by E_i. Suppose that $E_k = \text{span}\{\phi_1, \phi_2, \phi_3, \ldots, \phi_m\}$.

We impose the following assumptions, where $G(x,t) = \int_0^t g(x,\tau)d\tau$.

(g_1) $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $|g(x,t)| \le C|t|^\alpha + C$, for some $0 \le \alpha < 1$ and $C > 0$;

($G_\pm$) $\dfrac{\int_\Omega G\left(x, \sum_{i=1}^m \alpha_i\phi_i\right)dx}{\|a\|^{2\alpha}} \to \pm\infty$, as $\|a\| = \left(\sum_{i=1}^m \alpha_i^2\right)^{\frac{1}{2}} \to \infty$.

If $g(x, t)$ is odd for $t \in \mathbb{R}$, we can consider multiple solutions of (6) under some conditions. For example, we assume

(g_2) there exists $r > 0$, s.t. $G(x,t) > 0$, as $(x,t) \in \overline{\Omega} \times (0, r]$;

(g_3) $g(x,-t) = -g(x,t)$, as $(x,t) \in \overline{\Omega} \times \mathbb{R}$.
We have the following results.

Theorem 1 *Let $s \in (0,1)$, $N > 2s$, $\Omega \subset \mathbb{R}^N$ be an open bounded domain with smooth boundary and let $K : \mathbb{R}^N \backslash \{0\} \to (0,+\infty)$ be a function satisfying (2), (3) and (4). Suppose that condition pair (g_1) (G_+) or (g_1) (G_-) holds, then (6) has at least one solution $u \in X_0$.*

Theorem 2 *Let $s \in (0,1)$, $N > 2s$, $\Omega \subset \mathbb{R}^N$ be an open bounded domain with smooth boundary and let $K : \mathbb{R}^N \backslash \{0\} \to (0,+\infty)$ be a function satisfying (2), (3) and (4). Suppose that conditions (g_1) and (G_+) hold. If there exists $m \le k$ such that*

$$\limsup_{t \to 0} \frac{g(x,t)}{t} < \lambda_m - \lambda_k \tag{8}$$

and

$$\inf_{t \neq 0} \frac{g(x,t)}{t} \ge \lambda_{m-1} - \lambda_k \tag{9}$$

uniformly for almost everywhere $x \in \Omega$, then equation (6) has at least one nontrivial solution in X_0.

Theorem 3 *Let $s \in (0,1)$, $N > 2s$, $\Omega \subset \mathbb{R}^N$ be an open bounded domain with smooth boundary and let $K : \mathbb{R}^N \backslash \{0\} \to (0,+\infty)$ be a function satisfying (2), (3) and (4). Suppose that conditions (g_1) and (G_-) hold. If there exists $m \ge k$ such that*

$$\liminf_{t \to 0} \frac{g(x,t)}{t} > \lambda_m - \lambda_k$$

and

$$\sup_{t \neq 0} \frac{g(x,t)}{t} \le \lambda_{m+1} - \lambda_k$$

uniformly for almost everywhere $x \in \Omega$, then equation (6) has at least one nontrivial solution in X_0.

Theorem 4 *Let $s \in (0,1)$, $N > 2s$, $\Omega \subset \mathbb{R}^N$ be an open bounded domain with smooth boundary and let $K : \mathbb{R}^N \backslash \{0\} \to (0,+\infty)$ be a function satisfying (2), (3) and (4). Moreover, we assume that all the eigenfunctions of problem (7) belong to $L^\infty(\Omega)$. If conditions (G_-) and (g_1-g_3) hold, then problem (6) has at least m solutions in X_0, where m is the dimension of the eigenvalue λ_k.*

Remark 1 The regularity assumption in Theorem 4 on the eigenfunctions of (7) is not too strong; e.g. see [2] for some related discussion for fractional problems.

Remark 2 Conditions ($G_\pm$), sometimes called generalized Ahmad-Lazer-Paul conditions, are now typical and widely used in the literature for dealing with elliptic PDEs on bounded domains or periodic solutions for Hamitonian systems with **sublinear nonlinearities**. But it seems that they haven't appeared in non-local problems. As $\alpha = 0$, ($G_\pm$) are reduced to the classical Ahmad-Lazer-Paul conditions, which include the well-known Landesman-Lazer conditions as special cases in variational problems, for dealing with similar problems but with **bounded nonlinearities** (e.g. see [14, 15] or Theorem 4.12 in [16]).

Extensions of the classical Ahmad-Lazer-Paul conditions to the present forms ($G_\pm$) for investigating unbounded problems were considered by a number of authors in early 1990s; e.g. see some references in [17]. The extensions to the present forms ($G_\pm$), either for elliptic PDEs or for periodic solutions for Hamiltonian systems, were also independently obtained in the first author's Ph.D. thesis in 1992 under supervision of Professor Guo Dajun whom the paper dedicates to; e.g. see [12] and [18], where some further references are also available.

As in the literatures such as in [12] and the references therein, conditions ($G_\pm$) can be replaced by ones not involving the eigenfunctions of problem (7) (e.g. see [10]) if the following unique continuity property holds for the eigenfunctions of problem (7): all eigenfunctions corresponding to (7) have nodal set with zero Lebesgue measure, where the nodal set of a function ϕ in Ω is the level set $\{x \in \Omega, \phi(x) = 0\}$. Some information about the unique continuity property for non-local problems can be found in [10].

In the second part of the paper we study the following problem with superlinear nonlinearities

$$\begin{cases} -\mathcal{L}_K u + a(x)u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{10}$$

where K, Ω, $a(x) \in L^{\frac{N}{2s}}(\Omega)$, $s \in (0,1)$, $N > 2s$ are all as in the first part.

We impose the following assumptions, where $G(x,t) = \int_0^t g(x,\tau)d\tau$.

(a_0) $\lambda = 0$ is not an eigenvalue of problem (7): $-\mathcal{L}_K u + a(x)u = \lambda u,\ u \in X_0$.

(g_4) $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $G(x,t) \geq 0,\ g(x,t) = o(|t|)$ as $t \to 0$ uniformly in $\bar{\Omega}$.

(g_5) $\lim_{|t| \to +\infty} \frac{G(x,t)}{|t|^2} = +\infty$ uniformly in $x \in \bar{\Omega}$.

(g_6) Set $\widetilde{G}(x,t) = \frac{1}{2} g(x.t)t - G(x,t)$. Then $\widetilde{G}(x,t) > 0$ for $t \neq 0$ and there is $r_0, c_0 > 0,\ \sigma > \max\{1, \frac{N}{2s}\}$ s.t.

$$|g(x,t)|^\sigma \leq c_0 \widetilde{G}(x,t)|t|^\sigma, \text{ if } |t| \geq r_0.$$

Our main result reads as follows.

Theorem 5 *Let $s \in (0,1)$, $N > 2s$, $\Omega \subset \mathbb{R}^N$ be an open bounded domain with smooth boundary and let $K : \mathbb{R}^N \backslash \{0\} \to (0,+\infty)$ be a function satisfying (2), (3) and (4). Moreover, suppose that conditions (a_0), (g_4)–(g_6) hold, then (10) has at least one nontrivial solution in X_0.*

Remark 3 The conditions (g_4), (g_5),(g_6) are more general than the Ambrosetti- Rabinowitz condition and a simple computation can prove that the superlinear function

$$g(x,t) = |u|^2 \ln(1 + |u|) - \frac{1}{2}|u|^2 + |u| - \ln(1 + |u|)$$

satisfies (g_4) $-$ (g_6) but does not satisfy Ambrosetti-Rabinowitz condition [19].

3 Preliminaries

In order to investigate eigenvalue problem (7), we first prove the following lemma.

Lemma 1 *If* $s \in (0,1),\ N > 2s,\ \Omega \subset \mathbb{R}^{2N}$ *is an open domain*, $a(x) \in L^{\frac{N}{2s}}(\Omega)$ *and* $u_n \rightharpoonup u \in X_0$, *then*

$$\int_\Omega a(x)|u_n(x)|^2 dx \to \int_\Omega a(x)|u(x)|^2 dx.$$

Proof Since $u_n \rightharpoonup u$ in X_0, then by the imbedding theorem in [1, 3], u_n^2 is bounded in $L^{\frac{N}{N-2s}}(\Omega)$. So we may suppose

$$u_n^2 \rightharpoonup u^2 \quad \text{in } L^{\frac{N}{N-2s}}(\Omega).$$

We assume that, up to a subsequence,

$$u_n^2(x) \to u^2(x) \quad \text{a.e. } x \in \Omega.$$

Then the proof is given by Vitali theorem. □

Now we focus on the eigenvalue problem (7)

$$\begin{cases} -\mathcal{L}_K u + a(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Lemma 2 *Let* $s \in (0,1),\ N > 2s,\ \Omega \subset \mathbb{R}^{2N}$ *be an open bounded domain and* $a(x) \in L^{\frac{N}{2s}}(\Omega)$.

(a) *Define*

$$\lambda_1 := \inf_{u \in X_0, \|u\|_2 = 1} \frac{1}{2}\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 K(x-y)dxdy + \int_\Omega a(x)u(x)^2 dx.$$

Then λ_1 *is finite and is a simple eigenvalue of* (7) *with a non-negative eigenfunction* $e_1 \in X_0$.

(b) *The spectrum of problem* (7) *has and only has eigenvalues which can be listed* $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_n \le \ldots$ *and the corresponding eigenfunctions* $\{e_k\}_{k \in N}$ *form a base of Hilbert spaces* $L^2(\Omega)$ *and* X_0.

Proof Denote $\widetilde{\mathcal{J}}(u) = \frac{1}{2}\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 K(x-y)dxdy + \int_\Omega a(x)u(x)^2 dx$, where $u \in X_0$. By the Hölder and Sobolev inequalities, we have

$$\begin{aligned}\widetilde{\mathcal{J}}(u) &\ge \frac{1}{2}\|u\|^2 - \varepsilon\left(\int_\Omega |u|^{\frac{2N}{N-2s}}dx\right)^{\frac{N-2s}{N}} - C(\varepsilon)\left(\int_\Omega |a(x)|^{\frac{N}{2s}}dx\right)^{\frac{2s}{N}} \\ &\ge \frac{1}{2}\|u\|^2 - C\varepsilon\|u\|^2 - C(\varepsilon),\end{aligned}$$

for every $\varepsilon > 0$. Hence, by fixing ε small, $\widetilde{\mathcal{J}}(u)$ is coercive on X_0 and therefore is bounded below. Then λ_1 is a finite number. Let $u_n \in X_0, \|u_n\|_2 = 1$ be a minimizing sequence for $\widetilde{\mathcal{J}}$, that is, $\widetilde{\mathcal{J}}(u_n) \to \lambda_1$. It is clear that u_n is bounded in X_0. Up to a subsequence suppose that $u_n \rightharpoonup u$ in X_0. By the compact imbedding $X_0 \subset L^2(\Omega)$ ([3]), we have $u_n \to u$ in $L^2(\Omega)$. Hence $\|u\|_2 = 1$. By Lemma 1

$$\int_\Omega a(x)|u_n(x)|^2 dx \to \int_\Omega a(x)|u(x)|^2 dx.$$

Hence by the lower semi-continuity of the norm in X_0

$$\widetilde{\mathcal{J}}(u) \leq \liminf_{n\to+\infty} \widetilde{\mathcal{J}}(u_n) = \lambda_1.$$

Hence $\widetilde{\mathcal{J}}(u) = \lambda_1$, and λ_1 is achieved and is an eigenvalue of (7).

Since $u \in X_0$ implies that $|u| \in X_0$ and

$$||u_n(x)| - |u_n(y)||^2 \leq |u_n(x) - u_n(y)|^2$$

we have that $\{|u_n|\}$ is also a minimizing sequence if $\{u_n\}$ is a minimizing sequence for $\widetilde{\mathcal{J}}$ in X_0. Therefore, there exists a non-negative eigenfunction e_1 corresponding to the first eigenvalue λ_1. In fact we can prove that every eigenfunction e corresponding to the first eigenvalue λ_1 doesn't change sign. That is $e \geq 0$ or $e \leq 0$ a.e. in Ω. Obviously, $\widetilde{\mathcal{J}}(e) = \lambda_1 = \widetilde{\mathcal{J}}(e_1)$. But, if $x \in \{e > 0\}$ and $y \in \{e < 0\}$, we have that

$$||e(x)| - |e(y)|| < |e(x) - e(y)|.$$

This means that $\widetilde{\mathcal{J}}(|e|) < \widetilde{\mathcal{J}}(e)$, if both $\{e > 0\}$ and $\{e < 0\}$ have positive measure, which contradicts to $|e| \in X_0, \ \|e\|_2 = 1$. As the proof of (c) in Proposition 9 ([11]), we can get that λ_1 is simple.

For the proof of (b), we argue recursively. Assume that the claim holds for 1, ..., k and prove it for $k+1$. By the definition of λ_1, we have, just as Lemmas 2.14 and 2.15 in [20] for a similar problem,

$$\lambda_n = \inf\left\{\left(\|u\|^2 + \int_\Omega a(x)u(x)^2 dx\right) : \|u\|_2 = 1, (u, e_1) = \cdots = (u, e_{n-1}) = 0\right\}.$$

So we get a sequence of eigenvalues

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \ldots.$$

From standard theory of compact linear operators, we know that the spectrum of problem (7) consists of just the above eigenvalues and the corresponding eigenfunctions form a base for both X_0 and $L^2(\Omega)$. □

We can also characterize the eigenvalues by subspaces of X_0.

Proposition 1 *We have*

$$\lambda_n = \max_{V \in V_{n-1}} \inf_{u \in V^\perp, \|u\|_2 = 1} \left(\|u\|^2 + \int_\Omega a(x)u(x)^2 dx\right)$$

where V_{n-1} is the set of $n-1$ dimensional subspaces in X_0.

Proof Denote

$$\widetilde{\lambda}_n := \max_{V\in V_{n-1}} \inf_{u\in V^{\perp},\|u\|_2=1} \left(\|u\|^2 + \int_\Omega a(x)u(x)^2 dx\right).$$

By the variational definition of λ_n, $\widetilde{\lambda}_n \geq \lambda_n$. On the other hand, for any $V \in V_{n-1}$, there exists $u(u \neq 0)$, $u = \sum_{i=1}^{n} x_j e_j \in V^{\perp}$ where $\{e_i\}_{1\leq i\leq n}$ are the corresponding eigenfunctions of $\{\lambda_i\}_{1\leq i\leq n}$ such that

$$\|u\|^2 + \int_\Omega a(x)u^2(x)dx = \sum_{i=1}^{n} \lambda_j x_j^2 \int_\Omega e_j^2 dx \leq \lambda_n \int_\Omega u^2 dx. \tag{11}$$

Hence

$$\inf_{u\in V^{\perp},\|u\|_2=1} \left(\|u\|^2 + \int_\Omega a(x)u^2(x)dx\right) \leq \lambda_n.$$

So, $\widetilde{\lambda}_n \leq \lambda_n$. The proposition is proved. □

Lemma 3 *If the eigenvalues of (7) are listed as follows $\lambda_1 < \lambda_2 \leq ... \leq \lambda_k \leq 0 < \lambda_{k+1} \leq ...$ and*

$$Y = \{e_1, ..., e_k\} \text{ where } \{e_k\} \text{ are the corresponding eigenfunctions}$$
$$Z = \{u \in X_0, \langle u, v\rangle_{L^2} = 0, \forall v \in Y\},$$

then we have

$$\delta = \inf_{u\in Z,\|u\|=1} \|u\|^2 + \int_\Omega a(x)u^2(x)dx > 0.$$

Proof By the above variational characterization of λ_{k+1},

$$\|u\|^2 + \int_\Omega a(x)u^2(x)dx \geq \lambda_{k+1} \int_\Omega u^2, \text{ for any } u \in Z.$$

If the lemma were not true, there exist $u_n \in Z, n \in \mathbb{N}$

$$\|u_n\|^2 + \int_\Omega a(x)u_n^2(x)dx < \frac{1}{n}\|u_n\|^2.$$

Set $v_n = \frac{u_n}{\|u_n\|}$. Then

$$1 + \int_\Omega a(x)v_n^2(x)dx \leq \frac{1}{n}.$$

Without loss of generality, assume $v_n \rightharpoonup v$ in X_0.

By Lemma 1

$$\int_{\Omega} a(x)v_n^2(x)dx \to \int_{\Omega} a(x)v^2(x)dx.$$

Hence,

$$1 + \int_{\Omega} a(x)v^2(x)dx \leq 0, \tag{12}$$

which implies that $v \neq 0$. Since $v \in Z$,

$$\|v\|^2 + \int_{\Omega} a(x)v^2(x)dx \geq \lambda_{k+1} \int_{\Omega} v_n^2(x)dx. \tag{13}$$

We change (12) to

$$\|v\|^2 + \int_{\Omega} a(x)|v(x)|^2 dx \leq \|v\|^2 - 1. \tag{14}$$

By combining (13) and (14), we have

$$\|v\|^2 \geq 1 + \lambda_{k+1} \int_{\Omega} v^2(x)dx > 1.$$

But,

$$\|v\|^2 \leq \lim_{n \to +\infty} \|v_n\|^2 = 1.$$

The contraction ends the proof. □

The proofs of the existence of one (nontrivial) solution rely on the standard linking theorems (e.g. see [16, 20]) and the proof of Theorem 4 needs the following proposition from $\mathbb{Z}^2$-index theory (e.g. see [21, 22]).

Proposition 2 *Let $\mathcal{J}$ be an even and C^1-functional on a Banach space X, satisfying P-S condition and $f(\theta) = 0$. If there is $\rho > 0$ and a $m-$dimensional subspace X_1 s.t.*

$$\sup_{x \in X_1 \cap S(\theta, \rho)} \mathcal{J}(x) < 0;$$

and there is another j-dimensional subspace $X_2 (j < m)$ s.t.

$$\inf_{x \in X_2^{\perp}} \mathcal{J}(x) > -\infty$$

where $X_2^{\perp}$ is the direct compliment subspace of X_2, then $\mathcal{J}$ has at least $m - j$ pair critical points.

4 Proof of main results

Due to the variational nature of the problem, in order to find weak solutions for problem (6), in the following we will look for critical points of the functional $\mathcal{J}$ on X_0

$$\mathcal{J}(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_\Omega a(x)(u(x))^2 dx - \frac{1}{2}\int_\Omega \lambda_k (u(x))^2 dx - \int_\Omega G(x,u)dx,$$

where λ_k is an eigenvalue of (7). Here all eigenvalues of (7) are listed as $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \ldots$ and the corresponding eigenspaces are denoted by E_k $(k = 1, 2, \ldots)$.

Moreover, it is known that under the conditions of our theorems $\mathcal{J} \in C^1(X_0, \mathbb{R})$ and for any $u,\ \phi \in X_0$,

$$\langle \mathcal{J}'(u), \phi\rangle = \int_{\mathbb{R}^{2N}} (u(x)-u(y))(\phi(x)-\phi(y))K(x-y)dxdy + \int_\Omega a(x)u(x)\phi(x)dx \quad - \int_\Omega \lambda_k u(x)\phi(x)dx - \int_\Omega g(x,u(x))\phi(x)dx.$$

Let us write

$$u = \bar{u} + u^0 + \widetilde{u},\ u \in X_0,$$

where

$$\bar{u} \in \sum_{i<k} E_i, \quad u^0 \in E_k \text{ and } \widetilde{u} \in \overline{\sum_{i \geq k+1} E_i}.$$

Lemma 4 *Under condition pair* (g_1), (G_+) *or* (g_1), (G_-), *the functional* $\mathcal{J}$ *defined above satisfies P-S condition on* X_0.

Proof We only prove the case where (g_1) and (G_+) hold. The other case can be proved similarly.

Suppose that $(u_n) \in X_0$ satisfies

$$\mathcal{J}'(u_n) \to 0, \text{ as } n \to +\infty,$$

and

$$|\mathcal{J}(u_n)| \leq C.$$

We have, noticing a similar inequality in (11)

$$\begin{aligned}
&\langle \mathcal{J}'(u_n), -\bar{u}_n\rangle \\
&= -\int_{\mathbb{R}^{2N}} |\bar{u}_n(x) - \bar{u}_n(y)|^2 K(x-y)dxdy - \int_\Omega a(x)|\bar{u}_n(x)|^2 dx + \int_\Omega \lambda_k |\bar{u}_n(x)|^2 dx \\
&\quad + \int_\Omega g(x, u_n(x))\bar{u}_n(x)dx \\
&\geq (\lambda_k - \lambda_{k-1})\int_\Omega |\bar{u}_n(x)|^2 dx - \int_\Omega |\bar{u}_n(x)|\Big(C|\bar{u}(x) + u^0(x) + \widetilde{u}_n(x)|^\alpha + C\Big)dx \\
&\geq (\lambda_k - \lambda_{k-1})\int_\Omega |\bar{u}_n|^2 dx - C\int_\Omega |\bar{u}_n|dx - C\int_\Omega |\bar{u}_n|(|\bar{u}_n|^\alpha + |u_n^0|^\alpha + |\widetilde{u}_n|^\alpha)dx \\
&\geq (\lambda_k - \lambda_{k-1} - \varepsilon)\int_\Omega |\bar{u}_n|^2 dx - C\int_\Omega |\bar{u}_n||u_n^0|^\alpha dx - C\int_\Omega |\bar{u}_n||\widetilde{u}_n|^\alpha dx - C(\varepsilon),
\end{aligned}$$

where $C(\varepsilon) > 0$ is a universal constant dependent on the arbitrary $\varepsilon > 0$. Fixing $\varepsilon > 0$ sufficiently small and noticing that $\sum_{i \leq k-1} E_i$ is finite dimensional, we have

$$\|\bar{u}_n\|^2 \leq C\|\widetilde{u}_n\|_{2\alpha}^{2\alpha} + C\|u_n^0\|_{2\alpha}^{2\alpha} + C. \tag{15}$$

By a similar computation and noticing Lemma 3, we have

$$
\begin{aligned}
&\langle \mathcal{J}'(u_n), \widetilde{u}_n \rangle \\
&= \int_{\mathbb{R}^{2N}} |\widetilde{u}_n(x) - \widetilde{u}_n(y)|^2 K(x-y)dxdy + \int_\Omega a(x)|\widetilde{u}_n(x)|^2 - \int_\Omega \lambda_k |\widetilde{u}_n(x)|^2 \\
&\quad + \int_\Omega g(x, u_n(x))\widetilde{u}_n(x)dx \\
&\geq \int_{\mathbb{R}}^{2N} |\widetilde{u}_n(x) - \widetilde{u}_n(y)|^2 K(x-y)dxdy + \int_\Omega a(x)|\widetilde{u}_n(x)|^2 dx + \int_\Omega g(x, u_n(x))\widetilde{u}_n(x)dx \\
&\quad - \frac{\lambda_k}{\lambda_{k+1}} \Big[\int_{\mathbb{R}^{2N} |\widetilde{u}_n(x) - \widetilde{u}_n(y)|^2 K(x-y)dxdy + \int_\Omega a(x)|\widetilde{u}_n(x)|^2 dx} \Big] \\
&= \Big(1 - \frac{\lambda_k}{\lambda_{k+1}}\Big) \Big[\int_{\mathbb{R}}^{2N} |\widetilde{u}_n(x) - \widetilde{u}_n(y)|^2 K(x-y)dxdy + \int_\Omega a(x)|\widetilde{u}_n(x)|^2 dx \Big] \\
&\quad + \int_\Omega g(x, u_n(x))\widetilde{u}_n(x)dx \\
&\geq \Big(1 - \frac{\lambda_k}{\lambda_{k+1}}\Big) \Big[\int_{\mathbb{R}^{2N}} |\widetilde{u}_n(x) - \widetilde{u}_n(y)|^2 K(x-y)dxdy \\
&\quad + \int_\Omega a(x)|\widetilde{u}_n(x)|^2 \Big] - \int_\Omega (C|u_n|^\alpha + C)|\widetilde{u}_n(x)|dx \\
&\geq \delta \Big(1 - \frac{\lambda_k}{\lambda_{k+1}}\Big) \int_{\mathbb{R}^{2N}} |\widetilde{u}_n(x) - \widetilde{u}_n(y)|^2 K(x-y)dxdy \\
&\quad - C \int_\Omega |\widetilde{u}_n|(C|\widetilde{u}_n + \bar{u}_n + u_n^0|^\alpha + C)dx \\
&\geq \delta \Big(1 - \frac{\lambda_k}{\lambda_{k+1}}\Big) \|\widetilde{u}_n\|^2 - C \int_\Omega |\widetilde{u}_n|dx - C \int_\Omega |\widetilde{u}_n|(|\bar{u}_n|^\alpha + |u_n^0|^\alpha + |\widetilde{u}_n|^\alpha)dx \\
&\geq \delta \Big(1 - \frac{\lambda_k}{\lambda_{k+1}} - \varepsilon\Big) \|\widetilde{u}_n\|^2 - C(\varepsilon) \int_\Omega |\bar{u}_n|^{2\alpha} dx - C(\varepsilon) \int_\Omega |\bar{u}_n^0|^{2\alpha} dx - C(\varepsilon),
\end{aligned}
$$

where $C(\varepsilon) > 0$ is a universal constant dependent on the arbitrary $\varepsilon > 0$. Similarly, noticing that $\langle \mathcal{J}'(u_n), \widetilde{u}_n \rangle \leq o(1)\|\widetilde{u}_n\|$, and fixing $\varepsilon > 0$ sufficiently small, we have

$$
\|\widetilde{u}_n\|^2 \leq C\|\bar{u}_n\|_{2\alpha}^{2\alpha} + C\|u_n^0\|_{2\alpha}^{2\alpha} + C. \tag{16}
$$

By enlarging the term $\|\widetilde{u}_n\|_{2\alpha}^{2\alpha}$ in the right side of (15) to $C\|\widetilde{u}_n\|^{2\alpha}$ by embedding inequality and inserting the inequality (16) to (15) and then using Young inequality, we can get the following inequality

$$
\|\bar{u}_n\|^2 \leq C\|u_n^0\|_{2\alpha}^{2\alpha} + C. \tag{17}
$$

Similarly, we have

$$
\|\widetilde{u}_n\|^2 \leq C\|u_n^0\|_{2\alpha}^{2\alpha} + C. \tag{18}
$$

By

$$
\begin{aligned}
\Big|\int_\Omega \Big(G(x,u_n)-G(x,u_n^0)\Big)dx\Big| &= \Big|\int_\Omega dx\int_0^1 g(x,u_n^0+s(\widetilde{u}_n+\bar{u}_n))((\widetilde{u}_n+\bar{u}_n))ds\Big| \\
&\le \int_\Omega dx\int_0^1 (|\widetilde{u}_n|+|\bar{u}_n|)(C|u_n^0+s(\widetilde{u}_n+\bar{u}_n)|^\alpha+b)ds \\
&\le C\int_\Omega (|\widetilde{u}_n||u_n^0|^\alpha+|\widetilde{u}_n|^{1+\alpha}+|\widetilde{u}_n||\bar{u}_n|^\alpha+b|\widetilde{u}_n|)dx \\
&\quad+\int_\Omega (|\bar{u}_n||u_n^0|^\alpha+|\bar{u}_n||\widetilde{u}_n|^\alpha+|\bar{u}_n|^{1+\alpha}+b|\bar{u}_n|)dx,
\end{aligned}
$$

and estimating the above each term with Hölder and embedding inequalities and (17) and (18), we can get

$$
\Big|\int_\Omega \Big(G(x,u_n)-G(x,u_n^0)\Big)dx\Big| \le C\|u_n^0\|_{2\alpha}^{2\alpha}+C. \tag{19}
$$

By $|\mathcal{J}(u_n)|\le C$ and the inequality

$$
\int_{\mathbb{R}^{2N}} |\bar{u}_n(x)-\bar{u}_n(y)|^2K(x-y)dxdy+\int_\Omega a(x)|\bar{u}_n(x)|^2dx-\int_\Omega \lambda_k|\bar{u}_n(x)|^2dx\le 0,
$$

we have

$$
\begin{aligned}
-C\le{}& \frac{1}{2}\int_{\mathbb{R}^{2N}} |\widetilde{u}_n(x)-\widetilde{u}_n(y)|^2K(x-y)dxdy+\frac{1}{2}\int_\Omega a(x)|\widetilde{u}_n(x)|^2 \\
&-\int_\Omega \lambda_k|\widetilde{u}_n(x)|^2-\int_\Omega G(x,u_n)dx.
\end{aligned}
$$

Noticing

$$
\int_\Omega a(x)|\widetilde{u}_n(x)|^2dx\le \int_\Omega |a(x)||\widetilde{u}_n(x)|^2dx\le \|a\|_{\frac{N}{2s}}\|\widetilde{u}_n\|^2_{\frac{2N}{N-2s}}\le C\|\widetilde{u}_n\|^2
$$

and a similar inequality $-\int_\Omega \lambda_k|\widetilde{u}_n(x)|^2dx\le C\|\widetilde{u}_n\|^2$, we change the above inequality to

$$
-C\le C\|\widetilde{u}_n\|^2-\int_\Omega \Big[G(x,u_n)-G(x,{u_n}^0)\Big]dx-\int_\Omega G(x,{u_n}^0)dx.
$$

Moreover, by (18) and (19), we have

$$
-C\le C\|u_n^0\|_{2\alpha}^{2\alpha}+C-\int_\Omega G(x,{u_n}^0)dx. \tag{20}
$$

Hence, $\{u_n^0\}$ is bounded by (G_+). Therefore, by (17) and (18), $\{u_n\}$ is bounded in X_0. A standard argument ([3]) implies that $\mathcal{J}$ satisfies Palais-Smale condition on X_0. □

Proof of Theorem 2 Write $X_0=\sum_{i\le m-1}E_i\oplus\overline{\sum_{i\ge m}E_i}$. We claim that

(i) $\exists\ \rho,\ d>0$ s.t. $\mathcal{J}\ge d$ on $\Big\{u\in\sum_{i\ge m}E_i|\|u\|=\rho\Big\}$;

(ii) there are $e\in\overline{\sum_{i\ge m}E_i}$ with $\|e\|=1$, $R>\rho$ and $\epsilon<d$ s.t. if

$$
Q=\Big\{u\in\sum_{i<m}E_i|\|u\|\le R\Big\}\oplus\{te:0<t<R\},
$$

then $\mathcal{J} \leq \epsilon$ on $\partial\Omega$, where $\partial\Omega$ denotes the boundary of Q in $\sum_{i<m} E_i \oplus \mathbb{R}e$.
Step 1 We give the proof of claim (i).
By (8),$\forall\ \varepsilon > 0,\ \exists\ \delta$, as $|t| < \delta$,

$$\frac{g(x,t)}{t} < (\lambda_m - \varepsilon) - \lambda_k,$$

and then

$$G(x,t) \leq \frac{t^2}{2}(\lambda_m - \varepsilon - \lambda_k) + C(\varepsilon)|t|^q, \quad 2 < q < 2_s^*$$

for all real number t and $x \in \Omega$.
For $u \in \sum_{i \geq m} E_i$,

$$\begin{aligned}
\mathcal{J}(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_\Omega a(x)|u|^2 dx - \frac{1}{2}\lambda_k \int_\Omega |u|^2 dx - \int_\Omega G(x,u)dx \\
&\geq \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_\Omega a(x)|u|^2 dx - \frac{1}{2}\lambda_k \int_\Omega |u|^2 dx - C(\varepsilon)\int_\Omega |u|^q dx \\
&\quad - \frac{1}{2}(\lambda_m - \varepsilon - \lambda_k)\int_\Omega u^2 dx \\
&= \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_\Omega a(x)|u|^2 dx - \frac{1}{2}(\lambda_m - \varepsilon)\int_\Omega |u|^2 dx - C(\varepsilon)\int_\Omega |u|^q dx \\
&\geq \kappa(\varepsilon)\|u\|^2 - C(\varepsilon)\int_\Omega |u|^q dx \\
&\geq \kappa(\varepsilon)\|u\|^2 - C(\varepsilon)\|u\|^q
\end{aligned}$$

for some $\kappa(\varepsilon) > 0$, where the inequality in Lemma 3 is used. Hence, claim (i) holds.
Step 2 By (9), it is obvious that $\mathcal{J} \leq 0$ on $\sum_{i \leq m-1} E_i$. If we can prove that

$$\lim_{\|u\| \to \infty, u \in \sum_{i \leq m} E_i} \mathcal{J}(u) = -\infty, \tag{21}$$

then e can be taken as any element in E_m with $\|e\| = 1$, R any number sufficiently large and $\epsilon < d$ any number sufficiently small.

Suppose $m = k$. For $u \in \sum_{i \leq m} E_i,\ u = \bar{u} + u^0$, then

$$\begin{aligned}
\mathcal{J}(u) &= \frac{1}{2}\|\bar{u}\|^2 + \frac{1}{2}\int_\Omega a(x)|\bar{u}|^2 dx - \frac{1}{2}\lambda_k \int_\Omega |\bar{u}|^2 dx - \int_\Omega G(x,u)dx \\
&\leq \frac{1}{2}(\lambda_{k-1} - \lambda_k)\int_\Omega |\bar{u}|^2 dx - \int_\Omega \Big(G(x,u) - G(x,u^0)\Big)dx - \int_\Omega G(x,u^0)dx \\
&\leq \frac{1}{2}(\lambda_{k-1} - \lambda_k + 2\varepsilon)\int_\Omega |\bar{u}|^2 dx + C(\varepsilon)\int_\Omega |u^0|^{2\alpha} dx - \int_\Omega G(x,u^0)dx + C(\varepsilon).
\end{aligned}$$

Choosing $0 < \varepsilon < \frac{\lambda_k - \lambda_{k-1}}{2}$ and using the condition (G_+), we obtain (21).

If $m<k$, then $u=\bar{u}$. The proof of (21) is much easier. Hence, the theorem is proved by Rabinowitz's linking theorem, e.g. see Theorem 2,12 in [20]. □

Proof of Theorem 3 Under the conditions of the theorem, we can prove

(i) there are $p, d>0$ such that $\mathcal{J}<-d$ on $\left\{u \in \sum_{i \leq m} E_i \mid \|u\|=\rho\right\}$;
(ii) $\mathcal{J} \geq 0$ on $\sum_{i \geq m+1} E_i$;
(iii) $\mathcal{J} \to +\infty$ as $u \in \sum_{i \geq m} E_i$ and $\|u\| \to \infty$.

Then, for $I=-\mathcal{J}$, we use Theorem 5.29 in [16] and obtain a positive(nonzero) critical value for I. This completes the proof. □

Proof of Theorem 4 Consider the functional on X_0

$$\mathcal{J}(u)=\frac{1}{2}\|u\|^2+\frac{1}{2}\int_\Omega a(x)u^2(x)dx-\frac{1}{2}\int_\Omega \lambda_k u^2(x)dx-\int_\Omega G(x,u)dx.$$

Clearly, under our assumptions, $\mathcal{J}$ is even and C^1 on X_0, satisfying P-S condition.

We set $X_2=\sum_{i=1}^{k-1} E_i$ as the subspace X_2 in Proposition 2. Then $X_2^\perp=\overline{\sum_{i \geq k} E_i}$. For every $u \in X_0$, we write $u=\bar{u}+u^0+\widetilde{u}$. Write the decomposition as $X_0=X_2 \oplus X_2^\perp= X_2 \oplus E_k \oplus \overline{\sum_{i \geq k+1} E_i}$.

For $u \in X_2^\perp$, $u=u^0+\widetilde{u}$, by Lemma 3, we obtain that

$$\begin{aligned}\mathcal{J}(u)=&\frac{1}{2}\|\widetilde{u}\|^2+\frac{1}{2}\int_\Omega a(x)|\widetilde{u}(x)|^2dx-\frac{1}{2}\int_\Omega \lambda_k|\widetilde{u}(x)|^2dx-\int_\Omega G(x,u^0+\widetilde{u})dx\\ \geq&\kappa\|\widetilde{u}\|^2-\int_\Omega\left[G(x,u^0+\widetilde{u})-G(x,u^0)\right]dx-\int_\Omega G(x,u^0)dx.\end{aligned}$$

for some $\kappa>0$. By condition (g_1) and a similar argument of (19), we get the $\mathcal{J}$ is bounded below on $X_2^\perp$.

Set $X_1=\sum_{i=1}^{k} E_i \subset L^\infty(\Omega)$. Since X_1 is finite dimensional, there exists $\delta_2>0$ s.t. $|u|_\infty \leq r$, $\forall u \in X_1$ with $\|u\| \leq \delta_2$. Hence by (g_2),

$$\mathcal{J}(u) \leq -\int_\Omega G(x,u)dx<0, \quad \forall u \in X_1 \cap S_{\delta_2}.$$

Noting that $m-j=\dim X_1-\operatorname{codim} X_2^\perp=\dim E_k=m$, hence $\mathcal{J}$ has at least m pairs of critical points corresponding to negative critical values. This completes the proof. □

Proof of Theorem 5 We suppose that all the eigenvalues of (7) are listed as $\lambda_1<\cdots<\lambda_m<0<\lambda_{m+1}<\cdots<\lambda_n<\cdots$ and the corresponding eigenspaces are denoted by E_k $(k=1,2,\ldots)$. Set $X_0^-=\sum\limits_{i \leq m} E_i$, $X_0^+=\overline{\sum\limits_{i \geq m+1} E_i}$. So $X_0=X_0^- \oplus X_0^+$ and correspondingly $u=u^-+u^+$ for every $u \in X_0$.

$$
\begin{aligned}
\mathcal{J}(u) =& \frac{1}{2}\int_{\mathbb{R}^{2N}}\left[|(u^-(x)+u^+(x))-(u^-(y)+u^+(y))|^2K(x-y)\right]dxdy\\
&+\int_\Omega[\frac{1}{2}a(x)u^2-G(x,u)]dx\\
=&\frac{1}{2}\int_{\mathbb{R}^{2N}}|u^+(x)-u^+(y)|^2K(x-y)dxdy\\
&+\frac{1}{2}\int_{\mathbb{R}^{2N}}|u^-(x)-u^-(y)|^2K(x-y)dxdy\\
&+\frac{1}{2}\int_\Omega a(x)u^2dx-\int_\Omega G(x,u)dx.
\end{aligned}
$$

By (g_4) and (g_6), we have

$$|g(x,u)|\le C|u|^{\frac{\sigma+1}{\sigma-1}},\ |u|\ge r_0,\ \text{a.e. } x\in\Omega.$$

Since $\sigma>\frac{N}{2s}$, we have $\frac{\sigma+1}{\sigma-1}<2_s^*-1$. Hence $\mathcal{J}$ is differentiable on X_0. Combining (g_4) and the above inequality, we have

$$|g(x,u)|\le\varepsilon|u|+C_\varepsilon|u|^{p-1},\ \frac{2\sigma}{\sigma-1}\le p<2_s^*. \tag{22}$$

Step 1 We have the following result:

(i) $\exists\ r>0$, s.t. $m=\inf\mathcal{J}(S_r^+)>0$, where $S_r^+=\partial B_r\cap X_0^+$.

(ii) $\exists\ \widetilde{r}>0$, s.t. $\mathcal{J}(u)\le 0,\forall u\in\partial Q$, where $Q=\{u=u^-+se_{m+1}:u^-\in X_0^-,\ s\ge 0,\ \|u\|\le\widetilde{r}\}$, where e_{m+1} is a nontrivial eigenfunction corresponding to λ_{m+1}.

□

Proof The proof of (i) is standard by equality (22) and the embedding theorem ([3]).
$\forall u=u^-+se_{m+1}$, we have

$$
\begin{aligned}
\mathcal{J}(u)=&\frac{1}{2}\int_{\mathbb{R}^{2N}}|se_{m+1}(x)-se_{m+1}(y)|^2K(x-y)dxdy\\
&+\frac{1}{2}\int_{\mathbb{R}^{2N}}|u^-(x)-u^-(y)|^2K(x-y)dxdy\\
&+\frac{1}{2}\int_\Omega a(x)(u^-)^2dx+\frac{1}{2}\int_\Omega a(x)(se_{m+1})^2dx-\int_\Omega G(x,u^-+se_{m+1})dx.
\end{aligned}
$$

In order to prove (ii), we follow some arguments in [19] (p.72) and only give the outlines. We just need to prove

$$\lim_{u\in\text{span}\{X^-,e_{m+1}\},\|u\|\to\infty}J(u)=-\infty.$$

Otherwise, there is $M\in\mathbb{R}$ and a sequence $u_n\in\text{span}\{X^-,e_{m+1}\},\|u_n\|\to\infty$ such that $\mathcal{J}(u_n)\ge M$. We write $u_n=u_n^-+u_n^+$ and define $w_n=u_n/\|u_n\|$ with the property $\|w_n\|=1$. Noticing that the sequence lies in a finite dimensional space, without loss of generality,we assume that $u_n^-\to w^-,u_n^+\to w^+$. By using the inequality $\frac{M}{\|u_n\|^2}\le\frac{\mathcal{J}(u_n)}{\|u_n\|^2}$ and displaying the right term, with the help of the inequality such as (11), Lemma 3.5 and nonnegativity of

$G(x, u)$, we can get that $w^+ \neq 0$. Then the similar proof follows by the obvious existence of $\bar{\mu}$ and the choices of $\omega > \bar{\mu}$ and Ω being the one in our paper.

Noticing also that $\mathcal{J}|_{X_0^-} \leq 0$, we have that (ii) holds when $\widetilde{r} > 0$ is sufficiently large. This completes the proof of Step 1.

Step 2 $\mathcal{J}$ satisfies the Cerami condition at any level $c \in R$. That is, if $\{u_n\}$ is any sequence in X_0 such that

$$\mathcal{J}(u_n) \to c$$

and

$$(1 + \|u_n\|) \sup\{| < \mathcal{J}'(u_n), \phi > \ : \phi \in X_0, |\|\phi\| = 1\} \to 0,$$

then $\{u_n\}$ has a convergent subsequence in X_0 [22]. □

Proof It is clear that for n big enough, we have

$$C_0 \geq \mathcal{J}(u_n) - \frac{1}{2}\mathcal{J}'(u_n)u_n = \int_\Omega \widetilde{G}(x, u_n)dx. \tag{23}$$

We want to get that $\{u_n\}$ is bounded in X_0. Suppose, by contradiction, there is a subsequence of u_n, still denoted by u_n, such that $\|u_n\| \to +\infty$ as $n \to +\infty$.

Set

$$v_n = \frac{u_n}{\|u_n\|}.$$

Hence

$$\begin{aligned}
&\mathcal{J}'(u_n)(u_n^+ - u_n^-) \\
&= \int_{\mathbb{R}^{2N}} (u_n(x) - u_n(y))[(u_n^+(x) - u_n^-(x)) - (u_n^+(y) - u_n^-(y))]K(x-y)dxdy \\
&\quad + \int_\Omega a(x)u_n(u_n^+ - u_n^-)dx - \int_\Omega g(x, u_n)(u_n^+ - u_n^-)dx \\
&= \int_{\mathbb{R}^{2N}} |u_n^+(x) - u_n^+(y)|^2 K(x-y)dxdy + \int_\Omega a(x)(u_n^+)^2 dx \\
&\quad - \int_{\mathbb{R}^{2N}} |u_n^-(x) - u_n^-(y)|^2 K(x-y)dxdy \\
&\quad - \int_\Omega a(x)(u_n^-)^2 dx - \int_\Omega g(x, u_n)(u_n^+ - u_n^-)dx \\
&\geq \delta\|u_n^+\|^2 + \delta\|u_n^-\|^2 \\
&\quad - \int_\Omega g(x, u_n)(u_n^+ - u_n^-)dx \\
&\geq \|u_n\|^2\left(\delta - \int_\Omega \frac{g(x, u_n)(v_n^+ - v_n^-)}{\|u_n\|}dx\right),
\end{aligned}$$

where $\delta > 0$ comes from Lemma 3 and a similar inequality as (11),by noting the finite dimensionality of X_0^-. So,

$$\liminf_{n\to\infty}\int_{\Omega}\frac{g(x,u_n)(v_n^+-v_n^-)}{\|u_n\|}dx\geq\delta. \tag{24}$$

Set $g(r)=\inf\{\widetilde{G}(x,u):x\in\mathbb{R}^N,\ |u|\geq r\}$. Then $g(r)\to+\infty,$ as $r\to+\infty$; for the proof see p.45 in [19]. Moreover, by (g_6), $g(r)>0,\ \forall r>0$. By (23), we get

$$C_0\geq\int_{\Omega}\widetilde{G}(x,u_n)dx\geq\int_{\{x:|u_n(x)|\geq r\}}\widetilde{G}(x,u_n)dx\geq g(r)\cdot\text{meas}\{x:|u_n(x)|\geq r\}.$$

Hence

$$\text{meas}\{x:|u_n(x)|\geq r\}\leq\frac{C_0}{g(r)},\ \forall r>0.$$

For every $\varepsilon>0$, choose $a_\varepsilon>r_0$ s.t.

$$\begin{aligned}
&\text{meas}\{x:|u_n(x)|\geq a_\varepsilon\}<\varepsilon.\\
&\Big|\int_{\{x\in\Omega:|u_n(x)|\geq a_\varepsilon\}}\frac{g(x,u_n)(v_n^+-v_n^-)}{\|u_n\|}dx\Big|\\
&\quad\leq C_0\int_{\{x\in\Omega:|u_n(x)|\geq a_\varepsilon\}}\frac{\widetilde{G}(x,u_n)^{\frac{1}{\sigma}}|u_n||v_n^+-v_n^-|}{\|u_n\|}dx\ (\text{by } g_6)\\
&\quad\leq C_0\int_{\{x\in\Omega:|u_n(x)|\geq a_\varepsilon\}}\widetilde{G}(x,u_n)^{\frac{1}{\sigma}}|v_n||v_n^+-v_n^-|dx\\
&\quad\leq C_0\int_{\{x\in\Omega:|u_n(x)|\geq a_\varepsilon\}}\widetilde{G}(x,u_n)^{\frac{1}{\sigma}}|v_n^+|^2dx+C_0\int_{\{x:|u_n(x)|\geq a_\varepsilon\}}\widetilde{G}(x,u_n)^{\frac{1}{\sigma}}|v_n^-|^2dx\\
&\quad\leq C_0\int_{\{x\in\Omega:|u_n(x)|\geq a_\varepsilon\}}\widetilde{G}(x,u_n)dx(\|v_n^+\|_{2_s^*}^2+\|v_n^-\|_{2_s^*}^2)\cdot\text{m}\{x\in\Omega:|u_n(x)|\geq a_\varepsilon\}^{\frac{1}{r}}\leq C\varepsilon^{\frac{1}{r}},
\end{aligned}$$

where the last inequality comes from Höder inequality by choosing $p=\sigma,q=\frac{N}{N-2s},\frac{1}{r}=1-\frac{1}{p}-\frac{1}{q}$ and C is a constant independent of ε and n.

From condition (g_4), for the above $\varepsilon>0$, we can choose $0<\delta=\delta(\varepsilon)<a_\varepsilon$ s.t.

$$|g(x,u)|\leq\varepsilon|u|\ \text{for}\ |u|\leq\delta.$$

So, we derive

$$\begin{aligned}
&\Big|\int_{\{x\in\Omega:0\leq|u_n(x)|\leq\delta\}}\frac{g(x,u_n)(v_n^+-v_n^-)}{\|u_n\|}dx\Big|\\
&\quad\leq\int_{\{x\in\Omega:0\leq|u_n(x)|\leq\delta\}}\frac{\varepsilon|u_n||v_n^+-v_n^-|}{\|u_n\|}dx\\
&\quad\leq\varepsilon\int_{\Omega}[(v_n^+)^2+(v_n^-)^2]dx=\varepsilon[|v_n^+|_2^2+|v_n^-|_2^2]\leq C\varepsilon,
\end{aligned}$$

where C is a constant independent of ε and n.

We turn to estimate of final part of the integration.

It is clear that $\exists\gamma=\gamma(\varepsilon)$ s.t. $|g(x,u_n)|\leq\gamma,\ \forall x\in\{x\in\Omega:\delta<|u_n(x)|<a_\varepsilon\}$. Hence, for n large enough, we have

$$\Big|\int_{\{x\in\Omega:\delta<|u_n(x)|<a_\varepsilon\}}\frac{g(x,u_n)(v_n^+-v_n^-)}{\|u_n\|}dx\Big|\leq\frac{\gamma}{\|u_n\|}\int_\Omega(|v_n^+|+|v_n^-|)dx\leq\frac{C_\varepsilon}{\|u_n\|}<\varepsilon.$$

Combining the above arguments, for every $\varepsilon>0$ and we can choose N large enough such that for as $n>N$ we have

$$\Big|\int_\Omega\frac{g(x,u_n)(v_n^+-v_n^-)}{\|u_n\|}dx\Big|<C\varepsilon+C\varepsilon^{\frac{1}{r}},$$

where C is a constant independent of ε and n, which contradicts (24). So u_n is bounded in X_0. A simple argument implies that $\{u_n\}$ contains a convergent subsequence in X_0.

Hence, we have obtained both the compactness properties and the geometrical structure of the functional. Hence, by linking theorem, we complete the proof of Theorem 1.5. □

Acknowledgements The author would like to thank the referees for careful reading and pointing out some suggestions.

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