SN Partial Differ. Equ. Appl. (2020) 1:27
https://doi.org/10.1007/s42985-020-00025-z

ORIGINAL PAPER

Strictly convex solutions for singular Monge–Ampère equations with nonlinear gradient terms: existence and boundary asymptotic behavior

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Received: 8 February 2020 / Accepted: 30 July 2020 / Published online: 1 September 2020

Abstract

A new existence criteria of strictly convex solutions is established for the singular Monge–Ampère equations

$$\begin{cases} det(D^2u) = b(x)f(-u) + g(|Du|) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} det(D^2u) = b(x)f(-u)(1 + g(|Du|)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Under b, f and g satisfying suitable conditions, we prove that the above boundary value problems admit a strictly convex solution, which turns out that this case is more difficult to handle than Monge–Ampère problems without gradient terms and needs some new ingredients in the arguments. Then we show the asymptotic behavior of strictly convex solutions under appropriate conditions. On the technical level, we adopt the sub-super-solution method and the Karamata regular variation theory.

Keywords Singular Monge–Ampère equations · Nonlinear gradient terms · Strictly convex solutions · Existence · Boundary asymptotic behavior

Mathematics Subject Classification 35J60 · 35J96

This article is part of the topical collectiondedicated to Prof. Dajun Guo for his 85th birthday, edited by YihongDu, Zhaoli Liu, Xingbin Pan, and Zhitao Zhang.

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1 Introduction

Let Ω be a strictly convex, bounded smooth domain in R^n with $n \geq 2$. We study the existence and boundary asymptotic behavior of strictly convex solutions to the singular Monge–Ampère equations

$$\begin{cases} det(D^2u) = b(x)f(-u) + g(|Du|) \;\; \text{in } \Omega, \\ u = 0 \;\; \text{on } \partial\Omega, \end{cases} \tag{$\Gamma+$}$$

and

$$\begin{cases} det(D^2u) = b(x)f(-u)(1 + g(|Du|)) \;\; \text{in } \Omega, \\ u = 0 \;\; \text{on } \partial\Omega, \end{cases} \tag{$\Gamma*$}$$

where $det(D^2u)$ is the Monge–Ampère operator, $b \in C^\infty(\Omega)$ is positive in Ω, $f \in C^\infty(0, +\infty)$ is positive and decreasing, and $g \in C^\infty(0, +\infty)$ is positive and nondecreasing.

The Monge–Ampère equation is a fully nonlinear equation arising in geometric problems, fluid mechanics and other applied subjects. For example, Monge–Ampère equation can describe Weingarten curvature, or reflector shape design (see [1]). In recent years, increasing attention has been paid to the study of the Monge–Ampère equation by different methods (see [2–10]).

At the same time, we notice that the boundary asymptotic behavior of solutions of singular elliptic problems has attracted the attention of Crandall, Rabinowitz and Tartar [11], Ghergu and Rădulescu [12], Lazer and McKenna [13], Zhang [14], Zhang and Li [15], Zhang and Feng [16, 17], Alsaedi, Mâagli, and Zeddini [18], Dumont, Dupaigne, Goubet, and Radulescu [19], Zhang and Bao [20], Huang, Li, and Wang [21], and Huang [22]. Especially, let us review several excellent results related to our problem of Monge–Ampère equations. In [23], Loewner and Nirenberg considered the existence of solution for the Monge–Ampère problem

$$\begin{cases} \det\, (D^2u) = u^{-(n+2)} \;\; \text{in } \Omega, \\ u = 0 \;\; \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

when $n = 2$. In [24], Cheng and Yau studied problem (1.1) in a more general case $n \geq 2$ and obtained the existence results of problem (1.1).

In [25], Lazer and McKenna presented a unique result for the Monge–Ampère problem

$$\begin{cases} \det\, (D^2u) = b(x)u^{-\gamma} \;\; \text{in } \Omega, \\ u = 0 \;\; \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\gamma > 1$ and $b \in C^\infty(\bar{\Omega})$ is positive. Applying regularity theory and sub-supersolution method, they got a unique solution u satisfying $u \in C^2(\Omega) \cap C(\bar{\Omega})$, and they proved that there exist two negative c_1 and c_2, such that u satisfies

$$c_1 d(x)^\beta \leq u(x) \leq c_2 d(x)^\beta \;\; \text{in } \Omega,$$

where $\beta = \frac{n+1}{n+\gamma}$ and $d(x) = dist(x, \partial\Omega)$.

Recently, Mohammed [26] established the existence and the global estimates of solutions of the Monge–Ampère problem:

$$\begin{cases} \det\,(D^2u) = b(x)f(-u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \in R^n (n \geq 2), f \in C^\infty(0, \infty)$ is positive and decreasing, and $b \in C^\infty(\Omega)$ is positive in Ω.

Very recently, Li and Ma [27] studied the existence and the boundary asymptotic behavior of solutions of problem (1.3) by using regular variation theory and sub-super-solution method. An overview of the asymptotic behaviour of solutions of elliptic problems can be found in Ghergu and Radulescu [28].

Moreover, it is well known that the Monge–Ampère operator is a fully nonlinear partial differential operator, and we notice that some fully nonlinear elliptic operators have attracted the attention of Dai [29], Jiang, Trudinger and Yang [30], Guan and Jiao [31], Jian, Wang and Zhao [32], Ji and Bao [33], Caffarelli, Li and Nirenberg [34], Amendola, Galise and Vitolo [35], Galise and Vitolo [36], Capuzzo-Dolcetta, Leoni and Vitolo [37], Bardi and Cesaroni [38], and Lazer and McKenna [25]. For the other latest related papers, see Zhang [39], and Feng and Zhang [40].

However, to the best of our knowledge, there is almost no paper on the existence and boundary asymptotic behavior of strictly convex solutions to singular Monge-Ampère equations with nonlinear gradient terms.

We suppose that f satisfies:

(f_1) $f \in C^\infty(0, +\infty), f(s) > 0, \lim_{s\to 0} f(s) = +\infty, f$ is decreasing on $(0, +\infty)$, and there exist positive d and c_1 such that $f(u) < \frac{c_1}{u^d}$;

(f_2) there exists $C_f > 0$ such that

$$\lim_{s\to 0^+} H'(s) \int_0^s \frac{d\tau}{H(\tau)} = -C_f,$$

where

$$H(\tau) = [(n+1)F(\tau)]^{\frac{1}{n+1}}, \quad F(\tau) = \int_\tau^a f(s)ds,$$

and a is a positive constant. Since what we will consider is that $f(s)$ as $s \to 0$, that is, the constant a is not a matter of cardinal significance.

Let φ satisfy

$$\int_0^{\varphi(t)} \frac{d\tau}{H(\tau)} = t. \tag{1.4}$$

Moreover we assume that b satisfies

(b_1) $b \in C^\infty(\bar{\Omega})$ is positive in Ω;

(b_2) there exist $\theta \in C^1(0,a)$, which is positive, monotone, and positive $\overline{b}, \underline{b}$ such that

$$\underline{b} = \lim_{d(x)\to 0} \inf_{x\in\Omega} \frac{b(x)}{\theta^{n+1}(d(x))} \leq \lim_{d(x)\to 0} \sup_{x\in\Omega} \frac{b(x)}{\theta^{n+1}(d(x))} = \overline{b}, \tag{1.5}$$

where $d(x) = dist(x, \partial\Omega)$, and we let $\Theta(t) = \int_0^t \theta(s)ds$, and there exists D_θ such that

$$\lim_{t\to 0^+} \left(\frac{\Theta(t)}{\theta(t)}\right)' = D_\theta.$$

Finally we assume that g satisfies

(g_1) $g \in C^\infty(0,\infty)$ is positive and nondecreasing on $(0,\infty)$,
(g_2) there exist constant c_g and $0 \leq q < n$ such that

$$g(x) \leq c_g x^q. \tag{1.6}$$

The first main result of the present paper is on the existence of strictly convex solutions to problem $\Gamma + (\Gamma*)$.

Theorem 1.1 *Let Ω be a smooth, bounded, strictly convex domain in R^n. If (f_1), (b_1), (g_1) and (g_2) hold, then problem $\Gamma + (\Gamma*)$ admits a unique strictly convex solution.*

The second main result of this paper is on the boundary asymptotic behavior of strictly convex solutions to problem $\Gamma + (\Gamma*)$.

For convenience's sake, we introduce the notations:

$$M_0 = \max_{x\in\Omega} \prod_{i=1}^{n-1} \kappa_i(x), \quad m_0 = \min_{x\in\Omega} \prod_{i=1}^{n-1} \kappa_i(x),$$

where $\kappa_1(\overline{x}), \ldots, \kappa_{n-1}(\overline{x})$ denote the principal curvatures of $\partial\Omega$ at the point $\overline{x}$.

Theorem 1.2 *Let Ω be a smooth, bounded, strictly convex domain in R^n. Suppose f satisfies (f_1), (f_2), b satisfies (b_1), (b_2), and g satisfies (g_1) and (g_2), if*

$$C_f > 1 - D_\theta, \tag{1.7}$$

then for any strictly convex solution $u(x)$ to $\Gamma + (\Gamma*)$, it holds

$$\lim_{d(x)\to 0} \sup_{x\in\Omega} \frac{u(x)}{-\varphi(\overline{\xi}\Theta(d(x)))} \leq 1 \leq \lim_{d(x)\to 0} \inf_{x\in\Omega} \frac{u(x)}{-\varphi(-\underline{\xi}\Theta(d(x)))},$$

where

$$\underline{\xi} = \left(\frac{\underline{b}}{M_0(1 - C_f^{-1}(1 - D_\theta))}\right)^{\frac{1}{n+1}}, \quad \overline{\xi} = \left(\frac{\overline{b}}{m_0(1 - C_f^{-1}(1 - D_\theta))}\right)^{\frac{1}{n+1}}.$$

2 Preliminaries

To consider the asymptotic behavior of strictly convex solution to problem $\Gamma + (\Gamma*)$, we will use Karamata regular variation theory which was introduced and established by Karamata in 1930, and it is a fundamental tool in stochastic processes (see [27, 39, 41–43]). In this part, we present some basic facts of Karamata regular variation theory, which was proved in [41, 43].

Definition 2.1 A positive measurable function f defined on $(0, a)$, for some constant $a > 0$, is called **regularly varying at zero** with index ρ, written $f \in RVZ_\rho$, if for each $\xi > 0$ and some $\rho \in R$,

$$\lim_{s \to 0^+} \frac{f(\xi s)}{f(s)} = \xi^\rho. \tag{2.1}$$

Clearly, if $f \in RVZ_\rho$, then $L(s) = \frac{f(s)}{s^\rho}$ is slowly varying at zero.

Definition 2.2 A positive measurable function f defined on (0, a), for some constant $a > 0$, is called **rapidly varying at zero**,

$$\begin{aligned} &\textit{if} \lim_{s \to 0^+} f(s) = \infty, \textit{and for each } \rho > 1, \lim_{s \to 0^+} f(s) s^\rho = \infty, \\ &\textit{or if} \lim_{s \to 0^+} f(s) = 0, \textit{and for each } \rho > 1, \lim_{s \to 0^+} f(s) s^{-\rho} = 0. \end{aligned}$$

Proposition 2.3 (Uniform convergence theorem) *If $f \in RVZ_\rho$, then (2.1) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$.*

Proposition 2.4 (Representation theorem) *A function L is slowly varying at zero if and only if it may be written in the form*

$$L(s) = \psi(s) exp\left(\int_s^{a_1} \frac{y(\tau)}{\tau} d\tau\right), \; s \in (0, a_1)$$

for some $0 < a_1 < a$, where the function ψ and y are measurable and for $s \to 0^+, \; y(s) \to 0$ and $\psi(s) \to c_0$, with $c_0 > 0$.

We say that

$$\hat{L}(s) = c_0 exp\left(\int_s^{a_1} \frac{y(\tau)}{\tau} d\tau\right)$$

*is **normalized slowly varying at zero** and*

$$f(s) = s^\rho \hat{L}(s), \; s \in (0, a_1)$$

*is **normalized regularly varying at zero** with index ρ and write $f \in NRVZ_\rho$.*

Proposition 2.5 *A function $f \in RVZ_\rho$ belongs to $NRVZ_\rho$ if and only if*

$$f \in C^1(0, a_1), \quad \text{for} \quad \text{some} \quad a_1 > 0 \quad \text{and} \quad \lim_{s \to 0^+} \frac{sf'(s)}{f(s)} = \rho.$$

Proposition 2.6 *If function f, g, L are slowly varying at zero, then*

(1) f^p *for every* $p \in R$, $c_1 f + c_2 g (c_1, c_2 \geq 0), f \circ g$(*if* $g(s) \to 0$ *as* $s \to 0^+$) *are also slowly varying at zero.*

(2) *For every* $\rho > 0$ *and* $s \to 0^+$,

$$s^\rho L(s) \to 0, \; s^{-\rho} L(s) \to \infty.$$

(3) *For* $\rho \in R$ *and* $s \to 0^+$, $\frac{\ln(L(s))}{\ln s} \to 0$ *and* $\frac{\ln(s^\rho L(s))}{\ln s} \to \rho$.

Proposition 2.7 *If* $f_1 \in RVZ_{\rho_1}, f_2 \in RVZ_{\rho_2}$, *then* $f_1 f_2 \in RVZ_{\rho_1+\rho_2}$ *and* $f_1 \circ f_2 \in RVZ_{\rho_1 \rho_2}$.

Proposition 2.8 (Asymptotic behavior) *If a function L is slowly varying at zero, then for* $a > 0$ *and* $t \to 0^+$,

(1) $\int_0^t s^\rho L(s) ds \cong (1+\rho)^{-1} t^{1+\rho} L(t)$ *for* $\rho > 1$;

(2) $\int_t^a s^\rho L(s) ds \cong (-1-\rho)^{-1} t^{1+\rho} L(t)$ *for* $\rho < -1$.

Lemma 2.1 (Lemma 2.9 of [27]) *Let* θ *and* Θ *be the functions given by* ($\mathbf{b}_2$). *Then*

(1) *If* θ *is non-decreasing, then* $0 \leq D_\theta \leq 1$; *and if* θ *is non-increasing, then* $D_\theta \geq 1$;

(2) $\lim_{t \to 0^+} \frac{\Theta(t)}{\theta(t)} = 0$ *and* $\lim_{t \to 0^+} \frac{\Theta(t)\theta'(t)}{\theta^2(t)} = 1 - D_\theta$;

(3) *If* $D_\theta > 0$, *then* $\theta \in NRVZ_{(1-D_\theta)/D_\theta}$ *and* $\Theta \in NRVZ_{D_\theta^{-1}}$;

(4) *If* $D_\theta = 0$, *then* θ *is rapidly varying to zero.*

(5) *If* $D_\theta = 1$, *then* θ *is normalized slowly varying at zero.*

Lemma 2.2 (Lemma 2.10 of [27]) *Let f satisfy* (f_1), (f_2). *We have*

(1) $C_f \leq 1$;

(2) *If* $0 < C_f < 1$, *then f satisfying* (f_2) *is equivalent to* $F \in NRVZ_{(n+1)C_f/(1-C_f)}$;

(3) *If* $C_f = 1$, *then F is rapidly varying to infinity at zero;*

(4) $\lim_{s \to 0^+} \frac{((n+1)F(s))^{n/(n+1)}}{f(s)\varphi^{-1}(s)} = C_f^{-1}$.

Lemma 2.3 (Lemma 2.11 of [27]) *Let f satisfy* (f_1), (f_2), *and* φ *is defined by* (1.4). *Then we have*

(1) $\varphi'(t) = [(n+1)F(\varphi)]^{1/(n+1)}$,*and* $\varphi''(t) = -[(n+1)F(\varphi(t))]^{(1-n)/(1+n)} f(\varphi(t))$;

(2) $\lim_{t \to 0^+} \frac{\varphi'(t)}{t\varphi''(t)} = -C_f^{-1}$;

(3) $\varphi \in NRVZ_{1-C_f}$;
(4) $\varphi' \in NRVZ_{-C_f}$;
(5) *If* (1.7) *holds, then* $\lim_{t\to 0^+} \frac{t}{\varphi(\xi\Theta(t))} = 0$ *for* $\xi \in [d_1, d_2]$ *with* $0 < d_1 < d_2$;
(6) *If* (1.7) *holds, then* $\lim_{t\to 0^+} \frac{\Theta(t)}{\theta(t)\varphi(\Theta(t))} = 0$.

Lemma 2.4 *If f satisfies* $(f_1), (f_2)$, φ *is defined by* (1.4), θ, Θ *are defined in* (b_2), *and* $0 \le q < n$, *then we have*

$$\lim_{t\to 0^+} \xi^q \frac{[(n+1)F(\varphi(\xi\Theta(t)))]^{q/(n+1)}}{\theta^{n+1-q}(t)f(\varphi(\xi\Theta(t)))} = 0$$

uniformly for $\xi \in [c_1, c_2]$ *with* $0 < c_1 < c_2$.

Proof By (1.4) and Lemmas 2.1–2.3, we get

$$\lim_{t\to 0^+} \frac{\Theta(t)}{\theta(t)} = 0 \text{ and } \lim_{t\to 0^+} \varphi(\Theta(t)) = 0.$$

When $0 < C_f < 1$, letting $s = \varphi(\Theta)$, we have

$$\lim_{t\to 0^+} \frac{\Theta(t)[(n+1)F(\varphi(\Theta(t)))]^{1/n+1}}{\varphi(\Theta(t))} = -\lim_{s\to 0} \frac{H(s)\int_0^s \frac{d\tau}{H(\tau)}}{s} = C_f - 1.$$

Then, by Lemma 2.3 (6), for $q \in [0, n)$, we have

$$\begin{aligned}
&\lim_{t\to 0^+} \xi^q \frac{[(n+1)F(\varphi(\xi\Theta(t)))]^{q/(n+1)}}{\theta^{n+1-q}(t)f(\varphi(\xi\Theta(t)))} \\
&= \xi^{1+q} \lim_{t\to 0^+} \frac{[(n+1)F(\varphi(\xi\Theta(t)))]^{n/(n+1)}}{\xi\Theta(t)f(\varphi(\xi\Theta(t)))} \cdot \lim_{t\to 0^+} \frac{\Theta(t)}{\theta(t)} \lim_{t\to 0^+} \left(\frac{\Theta(t)}{\theta(t)\varphi(\Theta(t))}\right)^{n-q} \\
&\cdot \lim_{t\to 0^+} [\frac{\Theta(t)[(n+1)F(\varphi(\Theta(t)))]^{1/n+1}}{\varphi(\Theta(t))}]^{q-n} \\
&= \xi^{1+q} C_f^{-1}(C_f - 1)^{-(n-q)} \lim_{t\to 0^+} \frac{\Theta(t)}{\theta(t)} \lim_{t\to 0^+} \left(\frac{\Theta(t)}{\theta(t)\varphi(\Theta(t))}\right)^{n-q} \\
&= 0
\end{aligned}$$

uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$.

If $C_f = 1$ and $D_\theta > 0$, by using Lemmas 2.1 and 2.3, then we find that $\theta \in NRVZ_{D_\theta^{-1}-1}$, $\Theta \in NRVZ_{D_\theta^{-1}}$ and $\varphi'(t) = [(n+1)F(\varphi(t))]^{1/(n+1)}$ belongs to $NRVZ_{-1}$. So Proposition 2.7 implies that $[(n+1)F(\varphi(\Theta(t)))]^{1/(n+1)}$ belongs to $NRVZ_{-D_\theta^{-1}}$ and $\theta(t)[(n+1)F(\varphi(\Theta(t)))]^{1/(n+1)}$ belongs $NRVZ_{-1}$. It follows by Proposition 2.6 that

$$\theta(t)[(n+1)F(\varphi(\Theta(t)))]^{1/(n+1)} \to \infty.$$

Thus, for $q \in [0, n)$, we have

$$\begin{aligned}
&\lim_{t\to 0^+}\xi^q\frac{[(n+1)F(\varphi(\xi\Theta(t)))]^{q/(n+1)}}{\theta^{n+1-q}(t)f(\varphi(\xi\Theta(t)))}\\
&=\xi^{1+q}\lim_{t\to 0^+}\frac{\Theta(t)}{\theta(t)}\lim_{t\to 0^+}\frac{[(n+1)F(\varphi(\xi\Theta(t)))]^{n/(n+1)}}{\xi\Theta(t)f(\varphi(\xi\Theta(t)))}\\
&\quad\lim_{t\to 0^+}\left[\theta(t)[(n+1)F(\varphi(\Theta(t)))]^{1/(n+1)}\right]^{(q-n)}\\
&=0.
\end{aligned}$$

□

3 Proofs of Theorem 1.1 and Theorem 1.2

Let $d(x)=\inf_{y\in\partial\Omega}|x-y|$. For any $\delta>0$, we define

$$\Omega_\delta=\{x\in\Omega:0<d(x)<\delta\}.$$

When Ω is C^∞-smooth, choose $\delta_1>0$ such that(see Lemmas 14.16 and 14.17 of [3]) $d\in C^\infty(\Omega_{\delta_1})$.

Let $\bar{x}\in\partial\Omega$ be the projection of the point $x\in\Omega_{\delta_1}$ to $\partial\Omega$, and $\kappa_i(\bar{x})(i=1,2\ldots,n-1)$ be the principal curvatures of $\partial\Omega$ at $\bar{x}$, then, according to a principal coordinate system at $\bar{x}$, we get, by Lemma 14.17 in [3],

$$\begin{aligned}
Dd(x)&=(0,0,\ldots,1),\\
D^2d(x)&=diag\left[\frac{-\kappa_1(\bar{x})}{1-d(x)\kappa_1(\bar{x})},\ldots,\frac{-\kappa_{n-1}(\bar{x})}{1-d(x)\kappa_{n-1}(\bar{x})},0\right].
\end{aligned}$$

We first collect some results for the convenience of later use and reference.

Lemma 3.1 (Lemma 2.1 of [44]) *Suppose that $\Omega\subset R^n$ is a bounded domain, and $u,v\in C^2(\Omega)$ are strictly convex. If*

(1) $\psi(x,z,p)\geq\phi(x,z,p)$, *forall* $(x,z,p)\in(\Omega\times R\times R^n)$;
(2) $det(D^2u)\geq\psi(x,u,Du)$ *and* $det(D^2v)\leq\phi(x,v,Dv)$ *in* Ω;
(3) $u\leq v$ *on* $\partial\Omega$;
(4) $\psi_z(x,z,p)>0$ *or* $\phi_z(x,z,p)>0$, *then* $u\leq v$ *in* Ω.

Lemma 3.2 (Theorem 7.1 of [2]) *The equation of the form*

$$\begin{cases} det(D^2u)=\varphi(x,u,Du),\ x\in\Omega,\\ u=\phi\in C^\infty,\ x\in\partial\Omega,\end{cases}$$

where $\varphi(x,u,p)$ is a positive C^∞ function for $x\in\overline{\Omega},u\leq\max\phi,p\in R^n$ and ϕ is a strictly convex function in all of Ω, admits a strictly convex solution $u\in C^\infty(\overline{\Omega})$, if there exists a subsolution $\underline{u}\in C^2(\overline{\Omega})$ which equals ϕ on $\partial\Omega$ and satisfies

$$det(\underline{u}_{ij})\geq\varphi(x,\underline{u},D\underline{u}),\forall x\in\Omega.$$

If $\varphi_u \geq 0$, *then this solution is unique.*

Lemma 3.3 (Proposition 2.1 of [45]) *Let* Ω *be an open subset of* R^n *with* $n \geq 2$. *If* $z \in C^2(\Omega)$ *and* $h \in C^2(R)$, *then it holds*

$$det(D^2 h(z(x))) = \Big\{ h'(z(x))^{n-1} h''(z(x)) (\nabla z(x))^T B(z) \nabla z(x) + + (h'(z(x)))^n \Big\} det(D^2(z(x))), \; x \in \Omega,$$

where A^T *denotes the transpose of matrix* A, *and* B *denotes the inverse of the matrix* (z_{ij}).

Moreover, when $z(x) = d(x)$, *we have*

$$detD^2(h(z(x))) = (-h'(z(x)))^{n-1} h''(z(x)) \prod_{i=1}^{n-1} \frac{\kappa_i(\bar{x})}{1 - z(x)\kappa_i(\bar{x})}.$$

Proof of Theorem 1.1 Set $z(x) = 1 - u_0(x)$, where $u_0 \in C^\infty(\overline{\Omega})$ is the unique strictly convex solution to problem

$$det(D^2 u_0) = 1 \; in \; \Omega, \; u_0 = 1 \; on \; \partial\Omega.$$

Then $z > 0$ in Ω and it is the unique strictly concave solution to problem

$$(-1)^n det(D^2 z) = 1 \; in \; \Omega, \; z = 0 \; on \; \partial\Omega.$$

Since $(z_{x_i x_j})$ is negative definite on $\overline{\Omega}$, its trace is negative, that is $\Delta z < 0$, and hence one can use the Hopf boundary lemma to obtain that $\|\nabla z\| > 0$ for $x \in \partial\Omega$. It follows that there exist constants b_1 and b_2 with $b_1 > 0, \; b_2 > 0$ such that

$$b_1 d(x) \leq z(x) \leq b_2 d(x) \; for \; x \in \Omega.$$

Let $0 < s < \min\left\{1, \frac{n+1}{n+d}\right\}$ and

$$c = \left[\frac{1}{M_1 s^n} (M c_1 |z|_{max}^{n+1-s(n+d)} + c_g s^{q-n} |z|_{max}^{(s-1)q+n+1-sn} |Dz|_{max}^q)\right]^{\frac{1}{n-q}} + 1,$$

where

$$M = \max_{x \in \overline{\Omega}} b(x).$$

Let $w = -c(z(x))^s$, where z is defined above.

Considering problem $(\Gamma+)$, we have from Lemma 3.3 that

$$\begin{aligned} det(D^2 w) &= (-c)^n det(D^2 z) s^n z^{n(s-1)} + (-c)^n (s-1) s^n z^{n(s-1)-1} (\nabla z)^T B(z) \nabla z \\ &= c^n s^n z^{n(s-1)-1} [z + (s-1)(\nabla z)^T B(z) \nabla z], \end{aligned}$$

where $B(z)$ denotes the inverse of the matrix $(z_{x_i x_j})$.

Let

$$\Delta_1 = z + (s-1)(\nabla z)^T B(z)\nabla z.$$

Then, in the light of the definition of z, we get $(z_{x_ix_j})$ is negative definite. It follows that there exist constants e_1 and $e_2 > 0$ such that

$$-e_1\|\nabla z\|^2 \le (\nabla z)^T B(z)\nabla z \le -e_2\|\nabla z\|^2,$$

and $\mathrm{trace}(-z_{x_ix_j}) = -\Delta z > 0$ on Ω, and $-z$ reaches its maximum on $\overline{\Omega}$ at each point of $\partial\Omega$, and then it follows from the maximum principle that there exist an open set U containing $\partial\Omega$ such that

$$\|\nabla z\| \ge e > 0.$$

Since that $0 < s < 1$, there exists $M_1 > 0$ such that

$$\Delta_1 \ge M_1.$$

Then we get

$$\begin{aligned} &det(D^2 w) \\ &\quad \ge c^n s^n z^{n(s-1)-1} M_1 \\ &\quad > \left[\frac{1}{M_1 s^n}(Mc_1|z|_{max}^{n+1-s(n+d)} + c_g s^{q-n}|z|_{max}^{(s-1)q+n+1-sn}|Dz|_{max}^q)\right] c^q s^n z^{n(s-1)-1} M_1 \\ &\quad \ge (Mc_1 z^{-sd} + c_g c^q s^q |z|^{(s-1)q}|Dz|^q) \\ &\quad > \frac{Mc_1}{(cz^s)^d} + c_g|csz^{s-1}Dz|^q \\ &\quad > b(x)f(-w) + g(|Dw|), \end{aligned}$$

where $|z|_{max} = \max_{x\in\overline{\Omega}}|z|$ and $|Dz|_{max} = \max_{x\in\overline{\Omega}}|Dz|$. By Lemma 3.2, problem $(\Gamma+)$ admits a strictly convex solution.

Similarly to the proof above, when it comes into problem $(\Gamma*)$, let

$$0 < s < \min\left\{\frac{n+1-q}{n+d-q}, \frac{n+1}{n+d}, 1\right\},$$

and

$$c = \left\{\frac{1}{M_1 s^n}\left(Mc_1|z|_{max}^{n+1-s(n+d)} + Mc_1 c_g s^{q-n}|z|_{max}^{(n+1-q)-s(n+d-q)}|Dz|_{max}^q\right)\right\}^{1/(n+d-q)} + 1.$$

Then we obtain

$$\begin{aligned}
&det(D^2 w)\\
&\quad \geq c^n s^n z^{n(s-1)-1} M_1\\
&\quad > \left[\frac{1}{M_1 s^n}\left(Mc_1 |z|_{max}^{n+1-s(n+d)} + Mc_1 c_g s^q |z|_{max}^{(n+1-q)-s(n+d-q)} |Dz|_{max}^q\right)\right] c^{-d+q} s^n z^{n(s-1)-1} M_1\\
&\quad \geq Mc_1 c^{-d} z^{-sd} + Mc_1 c^{q-d} c_g s^q z^{-sd} |z|^{(s-1)q} |Dz|^q\\
&\quad = \frac{Mc_1}{(cz^s)^d}(1 + c_g c^q |sz^{s-1} Dz|^q)\\
&\quad > b(x) f(-w)(1 + g(|Dw|)).
\end{aligned}$$

By Lemma 3.2, problem $(\Gamma*)$ admits a strictly convex solution.

The uniqueness of solution can be derived immediately by Lemma 3.1. The proof of Theorem 1.1 is completed. □

Proof of Theorem 1.2 For an arbitrary $\epsilon \in (0, \min\{1/2, \underline{b}\})$, let

$$\xi_{+\epsilon} = \left(\frac{(\overline{b}+\epsilon)(1+\epsilon)+\epsilon}{m_0(1 - C_f^{-1}(1 - D_\theta))}\right)^{1/(1+n)},$$

and

$$\xi_{-\epsilon} = \left(\frac{(\underline{b}-\epsilon)(1-\epsilon)-\epsilon}{M_0(1 - C_f^{-1}(1 - D_\theta))}\right)^{1/(1+n)},$$

where m_0, M_0 were given by (1.3), $\overline{b}$, $\underline{b}$ were given by (1.2). Using Lemmas 2.1–2.3, we see that

$$\begin{aligned}
&\lim_{d(x)\to 0} \frac{\Theta(d(x))}{\theta(d(x))} = 0;\\
&\lim_{d(x)\to 0} \frac{\Theta(d(x))\theta'(d(x))}{\theta^2(d(x))} = 1 - D_\theta;\\
&\lim_{d(x)\to 0} \frac{[(n+1)F(\varphi(\Theta d(x)))]^{n/(n+1)}}{\Theta(d(x)) f(\varphi(\Theta(d(x))))} = C_f^{-1};\\
&\lim_{d(x)\to 0} \prod_{i=1}^{n-1}(1 - d(x)\kappa_i(\overline{x})) = 1;\\
&m_0 \xi_{+\epsilon}^{n+1}(1 - C_f^{-1}(1 - D_\theta) - (\overline{b}+\epsilon)(1+\epsilon)) = \epsilon;\\
&M_0 \xi_{-\epsilon}^{n+1}(1 - C_f^{-1}(1 - D_\theta) - (\underline{b}-\epsilon)(1-\epsilon)) = -\epsilon;\\
&\lim_{t\to 0^+} \xi^q \frac{[(n+1)F(\varphi(\xi\Theta(t))]^{q/(n+1)}}{\theta^{n+1-q}(t) f(\varphi(\xi\Theta(t)))} = 0.
\end{aligned}$$

Since

$$\lim_{d(x)\to 0} \prod_{i=1}^{n-1}(1 - d(x)\kappa_i(\overline{x})) = 1,$$

we find ϵ, for any $x \in \Omega_{\delta_\epsilon}$, such that

$$1-\epsilon < \prod_{i=1}^{n-1}(1-d(x)\kappa_i(\overline{x})) < 1+\epsilon,$$

and it follows from (b_2) that for $x \in \Omega_{\delta_\epsilon}$,

$$(\underline{b}-\epsilon)\theta^{n+1}(d(x)) < b(x) < (\overline{b}+\epsilon)\theta^{n+1}(d(x)).$$

Letting

$$\overline{u}_\epsilon(x) = -\varphi(\xi_{-\epsilon}\Theta(d(x))) \quad \text{and} \quad \underline{u}_\epsilon(x) = -\varphi(\xi_{+\epsilon}\Theta(d(x))),$$

then we have

$$\begin{aligned}
&det(D^2\overline{u}_\epsilon) - b(x)f(-\overline{u}_\epsilon) - g(|D\bar{u}_\epsilon|)\\
&\quad \le det(D^2\overline{u}_\epsilon) - b(x)f(-\overline{u}_\epsilon)\\
&\quad \le det(D^2\overline{u}_\epsilon) - (\underline{b}-\epsilon)\theta^{n+1}(d(x))f(\varphi(\xi_{-\epsilon}\Theta(d(x))))\\
&\quad = (-1)^n[\xi_{-\epsilon}\varphi'(\xi_{-\epsilon}\Theta(d(x)))\theta(d(x))]^{n-1}\prod_{i=1}^{n-1}\frac{-\kappa_i(\overline{x})}{1-d(x)\kappa_i(\overline{x})}\\
&\quad \times[\xi_{-\epsilon}^2\varphi''(\xi_{-\epsilon}\Theta(d(x)))\theta^2(d(x)) + \xi_{-\epsilon}\varphi'(\xi_{-\epsilon}\Theta(d(x)))\theta'(d(x))]\\
&\qquad -(\underline{b}-\epsilon)\theta^{n+1}(d(x))f(\varphi(\xi_{-\epsilon}\Theta(d(x))))\\
&\quad \le (1-\epsilon)^{-1}\theta^{n+1}(d(x))f(\varphi(\xi_{-\epsilon}\Theta(d(x))))\\
&\quad \times\left[\xi_{-\epsilon}^{n+1}M_0\left(1-\frac{\Theta(d(x))\theta'(d(x))}{\theta^2(d(x))}\right)\frac{[(n+1)F(\varphi(\xi_{-\epsilon}\Theta(d(x))))]^{n/(n+1)}}{\xi_{-\epsilon}\Theta(d(x))f(\varphi(\xi_{-\epsilon}\Theta(d(x))))} - (\underline{b}-\epsilon)(1-\epsilon)\right]\\
&\quad \le 0,
\end{aligned}$$

which means $\overline{u}_\epsilon$ is a supersolution to problem $\Gamma+$ in Ω_{δ_ϵ}.

On the other hand, we can similarly show that $\underline{u}_\epsilon = -\varphi(\xi_{+\epsilon}\Theta(d(x)))$ is a subsolution to problem $(\Gamma+)$ in Ω_{δ_ϵ} as follow

$$\begin{aligned}
&det(D^2\underline{u}_\epsilon) - b(x)f(-\underline{u}_\epsilon) - g(|D\underline{u}_\epsilon|)\\
&\quad \ge det(D^2\underline{u}_\epsilon) - b(x)f(-\underline{u}_\epsilon) - c_g|D\underline{u}_\epsilon|^q\\
&\quad \ge det(D^2\underline{u}_\epsilon) - (\overline{b}+\epsilon)\theta^{n+1}(d(x))f(\varphi(\xi_{+\epsilon}\Theta(d(x)))) - c_g|D\underline{u}_\epsilon|^q\\
&\quad = (-1)^n[\xi_{+\epsilon}\varphi'(\xi_{+\epsilon}\Theta(d(x)))\theta(d(x))]^{n-1}\prod_{i=1}^{n-1}\frac{-\kappa_i(\overline{x})}{1-d(x)\kappa_i(\overline{x})}\\
&\quad \times[\xi_{+\epsilon}^2\varphi''(\xi_{+\epsilon}\Theta(d(x)))\theta^2(d(x)) + \xi_{+\epsilon}\varphi'(\xi_{+\epsilon}\Theta(d(x)))\theta'(d(x))]\\
&\quad -(\overline{b}+\epsilon)\theta^{n+1}(d(x))f(\varphi(\xi_{+\epsilon}\Theta(d(x)))) - c_g\xi_{+\epsilon}^q\theta^q(t)[(n+1)F(\varphi(\xi\Theta(t)))]^{q/(n+1)}\\
&\quad \ge (1+\epsilon)^{-1}\theta^{n+1}(d(x))f(\varphi(\xi_{+\epsilon}\Theta(d(x))))\\
&\quad \times\left[\xi_{+\epsilon}^{n+1}m_0\left(1-\frac{\Theta(d(x))\theta'(d(x))}{\theta^2(d(x))}\right)\frac{[(n+1)F(\varphi(\xi_{+\epsilon}\Theta(d(x))))]^{n/(n+1)}}{\xi_{+\epsilon}\Theta(d(x))f(\varphi(\xi_{+\epsilon}\Theta(d(x))))} - (\overline{b}+\epsilon)(1+\epsilon)\right.\\
&\quad \left. - c_g(1+\epsilon)\xi_{+\epsilon}^q\frac{[(n+1)F(\varphi(\xi\Theta(t)))]^{q/(n+1)}}{\theta^{n+1-q}(t)f(\varphi(\xi_{+\epsilon}\Theta(t)))}\right]\\
&\quad \ge 0.
\end{aligned}$$

Let $v = -d(x)$. Then we can choose a sufficiently large constant $M > 0$ such that

$$u + Mv \leq \overline{u}_\epsilon \text{ on } \Gamma := \{x \in \Omega : d(x) = \delta_\epsilon\}.$$

Since

$$u = v = \overline{u}_\epsilon = 0 \text{ on } \partial\Omega,$$

and

$$det(D^2(u + Mv)) \geq det(D^2 u) = b(x)f(-u) + g(|Du|) \geq b(x)f(-(u + Mv)) + g(|D(u + Mv)|),$$

we deduce from Lemma 3.2 that

$$u + Mv \leq \overline{u}_\epsilon \text{ in } \Omega_{\delta_\epsilon},$$

which implies that

$$\frac{u}{-\varphi(\xi_{-\epsilon}\Theta(d(x)))} \geq 1 - \frac{Md(x)}{-\varphi(\xi_{-\epsilon}\Theta(d(x)))} \text{ in } \Omega_{\delta_\epsilon}.$$

Letting $d(x) \to 0, \ \epsilon \to 0$, then we get by Lemma 2.3 (5) that

$$\lim_{d(x)\to 0} \inf_{x\in\Omega} \frac{u(x)}{-\varphi(\underline{\xi}\Theta(d(x)))} \geq 1.$$

Similarly, we derive

$$\lim_{d(x)\to 0} \sup_{x\in\Omega} \frac{u(x)}{-\varphi(\overline{\xi}\Theta(d(x)))} \leq 1.$$

The estimate of the solution to problem $\Gamma*$ is similar to the proof above.

The proof of Theorem 1.2 is finished. □

Acknowledgements This work is sponsored by the National Natural Science Foundation of China (11301178) and the Beijing Natural Science Foundation of China (1163007).

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