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ORIGINAL PAPER

Existence of positive solutions to Kirchhoff equations with vanishing potentials and general nonlinearity

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Abstract

We study the existence of positive solutions to the following Kirchhoff type equation with vanishing potential and general nonlinearity:

$$\begin{cases} -(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla v|^2)\Delta v + V(x)v = f(v), x \in \mathbb{R}^3, \\ v > 0, v \in H^1(\mathbb{R}^3), \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $a, b > 0$ are constants and the potential V can vanish, i.e., the zero set of V, $\mathcal{Z} := \{x \in \mathbb{R}^3 | V(x) = 0\}$ is non-empty. In our case, the method of Nehari manifold does not work any more. We first make a truncation of the nonlinearity and prove the existence of solutions for the equation with truncated nonlinearity, then by elliptic estimates, we prove that the solution of truncated equation is just the solution of our original problem for sufficiently small $\varepsilon > 0$.

Keywords Kirchhoff type problems · Vanishing potentials · Schrödinger equation · General nonlinearity

Mathematics Subject Classification 35J60 · 35J20 · 35B38

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1 Introduction

In this paper, we study the existence of positive solutions for Kirchhoff type equations with vanishing potentials and general nonlinearity:

$$\begin{cases} -(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla v|^2)\Delta v + V(x)v = f(v), x \in \mathbb{R}^3, \\ v > 0, v \in H^1(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $a, b > 0$ are constants. We assume that the potential $V(x)$ satisfies the following conditions:

(V1) $V \in C(\mathbb{R}^3, \mathbb{R})$, $V(x) \geq 0$, and $V_\infty := \liminf_{|x| \to \infty} V(x) > 0$.

(V2) The zero set of V, $\mathcal{Z} := \{x \in \mathbb{R}^3 | V(x) = 0\}$ is non-empty. Without loss of generality, we assume that $0 \in \mathcal{Z}$.

In recent years, the elliptic Kirchhoff type equations have been studied extensively by many researchers. The problem is related to the stationary analogue of the equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2\right)\Delta u = g(x, t) \tag{1.2}$$

proposed by Kirchhoff [20] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings, Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Several interesting existence and uniqueness results can be found in [3, 22–24, 26, 35–37] etc.

On the other hand, if $a = 1$, $b = 0$, $\mathbb{R}^3$ replaced by $\mathbb{R}^N, N \geq 1$ in (1.1), it reduces to the well-known Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = f(u) \text{in} \mathbb{R}^N, \tag{1.3}$$

which has been paid much attention after the celebrated work of Floer and Weinstein [16]. For vanishing potentials in (1.3), we first point out the papers [8, 9]. In [8], Byeon and Wang found many interesting results with vanishing potential V. For instance, they show that there exists a standing wave (solution) which is trapped in a neighborhood of isolated zero points of V and whose amplitude tends to 0 as $\varepsilon \to 0$; moreover, depending on the local behavior of the potential function V near the zero points, the limiting profiles of the standing wave solutions are shown to exhibit quite different characteristic features. In [9], they consider a more general nonlinearity and vanishing potential. This type of results for Schrödinger type equations have been studied extensively over the years. Before [8, 9] there had been some earlier works along this line such as [4, 5], in which equations with a large parameter were considered, but a simple scaling taken to the equations could convert them to a similar case as of (1.3). After [8, 9], there are also many papers closely related to the subject. In [2], the authors studied nonlinear Schrödinger equations with vanishing and decaying potentials. In [10], radially symmetric and vanishing potentials were considered. In [7, 11–13], multi-bump standing waves with critical frequency for nonlinear Schrödinger equations were studied. For more references about Schrödinger equations, we refer to [1, 6, 8, 9, 14, 25, 31–33] and references therein.

Recently, many authors studied the existence and concentration behavior of ground states for Kirchhoff type equations in $\mathbb{R}^3$. In [18], He and Zou studied (1.1) with

$\inf_{x\in\mathbb{R}^3} V(x) > 0$ and subcritical nonlinearity. In [30], Wang et al. treated (1.1) with $\inf_{x\in\mathbb{R}^3} V(x) > 0$ and critical growth. In [15], Figueiredo et al. obtained the existence and concentration of positive solutions for (1.1) with $\inf_{x\in\mathbb{R}^3} V(x) > 0$ and the almost optimal Berestycki–Lions type nonlinearity. In [27], Sun and Zhang investigated the uniqueness of positive ground state solutions for Kirchhoff type equations with constant coefficients and then studied the existence and concentration behaviour of Kirchhoff type problems in $\mathbb{R}^3$ with competing potentials. In [28], Sun and Zhang investigated the existence and asymptotic behaviour of the positive ground state solutions to Kirchhoff type equations with vanishing potentials. For more results, we refer to [19, 21, 29] etc.

In this paper, motivated by [9], we consider a more general nonlinearity than that in [28], i.e., we assume that $f : \mathbb{R} \to \mathbb{R}^+$ is continuous and satisfies

(f1) $\lim\limits_{t\to 0^+} \frac{f(t)}{t} = 0.$

(f2) There exists $4<\mu<6$ such that $\liminf\limits_{t\to 0^+} \frac{f(t)}{t^{\mu-1}} > 0.$

(f3) $0<\mu F(t)\leq tf(t)$ for $t > 0$ where μ is defined in (f2) and $F(t) = \int_0^t f(s)ds.$

Since we look for positive solutions of (1.1), we assume that $f(t) = 0$ for $t\leq 0$. We have the following result:

Theorem 1.1 *Suppose that* (V1), (V2), (f1)–(f3) hold. Then for sufficiently small $\varepsilon > 0$, *there exists a positive solution* v_ε *of* (1.1)*which satisfies*

$$\lim_{\varepsilon\to 0} \|v_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = 0.$$

Remark 1.2 Our assumption on f is more general than that in [27]. Actually, our nonlinearity here can be supercritical. Let $f(t) = t^{\mu-1} + t^{q-1}$ for $t\geq 0$ where $q > 6$. Then

$$\liminf_{t\to 0^+} \frac{f(t)}{t^{\mu-1}} = 1 > 0,$$

and

$$t^\mu + \frac{\mu}{q} t^q = \mu F(t) \leq tf(t) = t^\mu + t^q,$$

which implies that $f(t) = t^{\mu-1} + t^{q-1}$ satisfies (f1),(f2) and (f3).

The paper is organized as follows. In Sect. 2 we give some preliminary results and in Sect. 3 we give the proof of Theorem 1.1.

2 Preliminaries

First let $u(x) = v(\varepsilon x)$, then the Eq. (1.1) becomes the following equivalent equation

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + V(\varepsilon x)u = f(u)\text{in}\mathbb{R}^3, \\ u > 0, u \in H^1(\mathbb{R}^3). \end{cases} \tag{2.1}$$

Let $E_\varepsilon := \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\varepsilon x)u^2 < +\infty\}$ be the Hilbert subspace of $H^1(\mathbb{R}^3)$ with the norm

$$\|u\|_{E_\varepsilon} := \left(\int_{\mathbb{R}^3} a|\nabla u|^2 + V(\varepsilon x)u^2\right)^{1/2}.$$

For any set $A \subset \mathbb{R}^3$, we define

$$A_{\frac{1}{\varepsilon}} := \{x \in \mathbb{R}^3 | \varepsilon x \in A\}.$$

By (V1), we can choose $d > 0$ such that $\liminf_{|x|\to\infty} V(x) \geq 2d$. Then there exists $D > 0$ such that $V(\varepsilon x) \geq d$, for $|x| \geq \varepsilon^{-1}D$. Now we make a truncation of the nonlinearity f in (2.1). Define

$$\tilde{f}(t) = \begin{cases} f(t), & 0 < t \leq 1, \\ f(1)t^{\mu-1}, & t \geq 1, \\ 0, & t \leq 0, \end{cases}$$

where μ is defined in (f2). Since $\mu \int_0^1 f(s)ds \leq f(1)$, we know that for $t > 0$,

$$0 < \mu \int_0^t \tilde{f}(s)ds \leq \tilde{f}(t)t.$$

Let $\tilde{F}(t) := \int_0^t \tilde{f}(s)ds$. Now we define a function $g : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$g(x,t) = \begin{cases} \tilde{f}(t), & \text{for } |x| \leq D/\varepsilon, \\ min\{\tilde{f}(t), kt\}, & \text{for } |x| > D/\varepsilon, \end{cases}$$

where

$$0 < k < d - \frac{2d}{\mu}. \tag{2.2}$$

From above we know that for sufficiently small $t > 0$, $g(x,t) \equiv f(t)$.

Now we define the functional $I_\varepsilon : E_\varepsilon \to \mathbb{R}$ by

$$I_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^3} (a|\nabla u|^2 + V(\varepsilon x)u^2) + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - \int_{\mathbb{R}^3} G(x,u),$$

where $G(x,t) = \int_0^t g(x,s)ds$. Then we have that $I_\varepsilon \in C^1(E_\varepsilon, \mathbb{R})$ and we want to show that the critical point of I_ε on E_ε is just the solution of (2.1) for sufficiently small $\varepsilon > 0$, .

First we verify the mountain pass geometry of I_ε.

Lemma 2.1 *For any fixed* $\varepsilon > 0$, *there exists* $r_0 > 0$*such that*

$$\inf_{\|u\|_{E_\varepsilon} = r_0} I_\varepsilon(u) > 0.$$

Proof By (f1) and the definition of $\tilde{f}(t)$, we know

$$g(x,t) \leq \tilde{f}(t) \leq h|t| + C_h|t|^{\mu-1},$$

where $h > 0$ is arbitrary, $C_h > 0$ is a constant dependent on h. Thus

$$\begin{aligned} I_\varepsilon(u) &= \frac{1}{2}\|u\|_{E_\varepsilon}^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - \int_{\mathbb{R}^3} G(x,u) \\ &\geq \frac{1}{2}\|u\|_{E_\varepsilon}^2 - \frac{h}{2}\int_{\mathbb{R}^3} |u|^2 - \frac{C_h}{\mu}\int_{\mathbb{R}^3} |u|^\mu. \end{aligned}$$

As the embedding $E_\varepsilon \hookrightarrow L^p (2 \leq p \leq 6)$ is continuous, there exist constants $C_1, C_2 > 0$ such that

$$\int_{\mathbb{R}^3} u^2 \leq C_1\|u\|_{E_\varepsilon}^2, \int_{\mathbb{R}^3} |u|^\mu \leq C_2\|u\|_{E_\varepsilon}^\mu.$$

Choose $h > 0$ sufficiently small such that $\frac{h}{2}C_1 < \frac{1}{4}$, then

$$I_\varepsilon(u) \geq \frac{1}{2}\|u\|_{E_\varepsilon}^2 - \frac{h}{2}C_1\|u\|_{E_\varepsilon}^2 - \frac{C_h}{\mu}C_2\|u\|_{E_\varepsilon}^\mu \geq \frac{1}{4}\|u\|_{E_\varepsilon}^2 - \frac{C_h}{\mu}C_2\|u\|_{E_\varepsilon}^\mu.$$

Since $\mu > 2$, from above we know there exists $r_0 > 0$ such that

$$\inf_{\|u\|_{E_\varepsilon} = r_0} I_\varepsilon(u) > 0.$$

□

Lemma 2.2 *Let* $\phi \in C_0^\infty(M_{\frac{1}{\varepsilon}})$ *be fixed, where* $M := \{x \in \mathbb{R}^3 : |x| \leq D\}$, $\phi \geq 0$ *and* $\phi \not\equiv 0$. *Then for sufficiently large* $t > 0$, $I_\varepsilon(t\phi) < 0$.

Proof Notice that

$$\begin{aligned} I_\varepsilon(t\phi) &= \frac{1}{2}t^2\|\phi\|_{E_\varepsilon}^2 + \frac{b}{4}t^4\left(\int_{\mathbb{R}^3} |\nabla \phi|^2\right)^2 - \int_{M_{\frac{1}{\varepsilon}}} \tilde{F}(t\phi) \\ &\leq \frac{1}{2}t^2\|\phi\|_{E_\varepsilon}^2 + \frac{b}{4}t^4\left(\int_{\mathbb{R}^3} |\nabla \phi|^2\right)^2 - t^\mu \int_{M_{\frac{1}{\varepsilon}}} \phi^\mu + C, \end{aligned}$$

then by $4 < \mu < 6$ we know that for sufficiently large $t > 0$, $I_\varepsilon(t\phi) < 0$. □

Now we show that I_ε satisfies (PS) condition.

Lemma 2.3 *For each fixed* $\varepsilon > 0$, I_ε *satisfies* (*PS*) *condition.*

Proof Let $\{u_n\}_{n=1}^\infty \subset E_\varepsilon$ satisfy that $I_\varepsilon(u_n)$ is bounded and $\lim_{n\to\infty} I'_\varepsilon(u_n) = 0$. Then there exist $C_1, C_2 > 0$ such that for sufficiently large $n > 0$,

$$C_1 + C_2\|u_n\|_\varepsilon \geq I_\varepsilon(u_n) - \frac{1}{\mu} < I'_\varepsilon(u_n), u_n > \; = \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_\varepsilon^2 +$$
$$\left(\frac{1}{4} - \frac{1}{\mu}\right)\left(\int_{\mathbb{R}^3} |\nabla u_n|^2)^2 + \frac{1}{\mu}(\int_{\mathbb{R}^3} g(x, u_n)u_n - \mu G(x, u_n)\right)$$
$$\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_\varepsilon^2 + \frac{1}{\mu}\int_{\mathbb{R}^3 \setminus M_{\frac{1}{\varepsilon}}} (g(x, u_n)u_n - \mu G(x, u_n))$$
$$\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_\varepsilon^2 - \frac{k}{2}\int_{\mathbb{R}^3 \setminus M_{\frac{1}{\varepsilon}}} u_n^2 \geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_\varepsilon^2 - \frac{k}{2d}\int_{\mathbb{R}^3 \setminus M_{\frac{1}{\varepsilon}}} V(\varepsilon x)u_n^2$$
$$\geq \left(\frac{1}{2} - \frac{1}{\mu} - \frac{k}{2d}\right)\|u_n\|_\varepsilon^2.$$

By 2.2, we know that $\{u_n\}$ is bounded in E_ε. So if necessary to a subsequence, there exists $u_0 \in E_\varepsilon$ such that

$$u_n \rightharpoonup u_0, \text{ in } E_\varepsilon,$$
$$u_n \to u_0, \text{ in } L^\tau_{loc}(\mathbb{R}^3), 1 < \tau < 6,$$
$$u_n \to u_0, \text{ a.e. in } \mathbb{R}^3.$$

Now we show that $\{u_n\}$ converges to $\{u_0\}$ strongly in E_ε. It is sufficient to prove that for any $\delta > 0$, there exists $R > 0$ such that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3 \setminus B_R(0)} (|\nabla u_n|^2 + V(\varepsilon x)u_n^2) < \delta, \tag{2.3}$$

where $B_R(0)$ is a ball centered at zero and with radius R and let $M_{\frac{1}{\varepsilon}} \subset B_R(0)$.

Choose $\varphi_R \in C_0^\infty(\mathbb{R}^3)$ such that $\varphi_R \equiv 0$ in $B_R(0)$, $\varphi_R \equiv 1$ in $B^c_{2R}(0)$, $0 \leq \varphi_R \leq 1$, and $|\nabla \varphi_R| \leq \frac{2}{R}$. Thus for any $R > 0$, we have

$$\langle I'_\varepsilon(u_n), \varphi_R u_n\rangle \to 0. \tag{2.4}$$

Then for $R \geq D/\varepsilon$,

$$\int_{\mathbb{R}^3} (a|\nabla u_n|^2 + V(\varepsilon x)u_n^2)\varphi_R + a\int_{\mathbb{R}^3} u_n \nabla u_n \nabla \varphi_R +$$
$$b\int_{\mathbb{R}^3} |\nabla u_n|^2 \int_{\mathbb{R}^3} \nabla u_n \nabla(\varphi_R u_n) = \int_{\mathbb{R}^3} g(x, u_n)u_n\varphi_R + o(1).$$

It yields

$$\int_{\mathbb{R}^3 \setminus B_{2R}(0)} (a|\nabla u_n|^2 + V(\varepsilon x)u_n^2) \leq \int_{\mathbb{R}^3} (a|\nabla u_n|^2 + V(\varepsilon x)u_n^2)\varphi_R$$
$$\leq k\int_{\mathbb{R}^3} u_n^2\varphi_R - a\int_{\mathbb{R}^3} u_n\nabla u_n\nabla\varphi_R - b\int_{\mathbb{R}^3} |\nabla u_n|^2 \int_{\mathbb{R}^3} u_n\nabla u_n\nabla\varphi_R + o(1).$$

By (2.2), we have

$$k\int_{\mathbb{R}^3} u_n^2\varphi_R \leq \frac{k}{d}\int_{\mathbb{R}^3} V(\varepsilon x)u_n^2\varphi_R < \left(1 - \frac{2}{\mu}\right)\int_{\mathbb{R}^3} (a|\nabla u_n|^2 + V(\varepsilon x)u_n^2)\varphi_R.$$

Then by the boundedness of $\int_{\mathbb{R}^3} |\nabla u_n|^2$, we have constant $C > 0$ such that

$$\int_{\mathbb{R}^3 \setminus B_{2R}(0)} (|\nabla u_n|^2 + V(\varepsilon x) u_n^2) \leq \frac{C}{R} \|u_n\|_{L^2} \|\nabla u_n\|_{L^2} + o(1),$$

which implies (2.3). □

In this paper we will use Theorem 4.1 of [17]. We give the theorem here for readers' convenience:

Theorem 2.4 *Suppose* $y \in \mathbb{R}^n$, $a_{ij} \in L^\infty(B_1(y))$ *and* $c \in L^q(B_1(y))$ *for some* $q > n/2$ *satisfy the following assumptions*

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \, for\, any\, x \in B_1(y),\ \xi \in \mathbb{R}^n \, and \, |a_{ij}|_{L^\infty} + \|c\|_{L^q} \leq \Lambda$$

for some positive constants λ *and* Λ. *Suppose that* $u \in H^1(B_1(y))$ *is a subsolution in the following sense*

$$\int_{B_1(y)} (a_{ij} D_i u D_j \varphi + cu\varphi) \leq \int_{B_1} f\varphi \, for\, any\, \varphi \in H_0^1(B_1(y)) \, and \, \varphi \geq 0 \, in \, B_1(y).$$

If $f \in L^q(B_1(y))$, *then* $u^+ \in L^\infty_{loc}(B_1(y))$. *Moreover, there holds for any* $\theta \in (0,1)$ *and any* $p > 0$

$$\sup_{B_\theta(y)} u^+ \leq C\left\{ \frac{1}{(1-\theta)^{n/p}} \|u^+\|_{L^p(B_1(y))} + \|f\|_{L^q(B_1(y))} \right\}$$

where $C = C(n, \lambda, \Lambda, p, q)$ *is a positive constant.*

3 Proof of Theorem 1.1

From Lemma 2.1, Lemma 2.2 and Lemma 2.3, we know that I_ε satisfies mountain pass geometry and (PS) condition. Now we define

$$c_\varepsilon = \inf_{\eta \in \Gamma_\varepsilon} \max_{0 \leq t \leq 1} I_\varepsilon(\eta(t)),$$

where

$$\Gamma_\varepsilon := \{\eta \in C([0,1], E_\varepsilon) : \eta(0) = 0,\, I_\varepsilon(\eta(1)) < 0\}.$$

Then by the general minimax theorem(Theorem 2.8, [34]), we can get a sequence $\{u_n\} \subset E_\varepsilon$ such that

$$I_\varepsilon(u_n) \to c_\varepsilon,\ I'_\varepsilon(u_n) \to 0 \ as\, n \to \infty.$$

Since I_ε satisfies (PS) condition, thus there exists $u_\varepsilon \in E_\varepsilon$ such that $u_n \to u_\varepsilon$ in E_ε. Then u_ε is the critical point of I_ε and $I_\varepsilon(u_\varepsilon) = c_\varepsilon$. Furthermore, u_ε satisfies

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2\right)\Delta u_\varepsilon + V(\varepsilon x) u_\varepsilon = g(x, u_\varepsilon). \tag{3.1}$$

By the elliptic regularity theory and maximum principle, we know that u_ε is continuous and $u_\varepsilon > 0$. Now we show that $\|u_\varepsilon\|_{L^\infty} \to 0$ as $\varepsilon \to 0$, which implies that for sufficiently small $\varepsilon > 0$, u_ε is just the positive solution of (2.1) since $g(x,t) \equiv f(t)$ for sufficiently small $t > 0$. Then we equivalently prove Theorem 1.1.

Now we estimate the limit of c_ε as $\varepsilon \to 0^+$:

Lemma 3.1 $\limsup_{\varepsilon\to 0^+} c_\varepsilon = 0.$

Proof By (f2) and (f3), there exists $C > 0$ such that $\tilde{F}(t) \geq Ct^\mu$. For any $R > 0$, consider

$$\begin{cases} -\left(a + b\displaystyle\int_{B_R} |\nabla w|^2\right)\Delta w = C\mu w^{\mu-1}, \\ w > 0 \text{ in } B_R(0), \quad w = 0 \text{ on } \partial B_R(0), \end{cases} \tag{3.2}$$

where $B_R(0) \subset \mathbb{R}^3$ is a ball centered at zero and with radius R. Let w_R be the least energy solution of (3.2). We can take w_R as a function defined on $\mathbb{R}^3$, with $w_R = 0$ outside the ball $B_R(0)$. Then for $t > 0$ and sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned} I_\varepsilon(tw_R) &= \frac{1}{2}t^2\int_{\mathbb{R}^3}(a|\nabla w_R|^2 + V(\varepsilon x)w_R^2) + \frac{b}{4}t^4\left(\int_{\mathbb{R}^3}|\nabla w_R|^2\right)^2 - \int_{\mathbb{R}^3}\tilde{F}(tw_R) \\ &\leq \frac{t^2}{2}\|w_R\|_\varepsilon^2 + \frac{b}{4}t^4\left(\int_{\mathbb{R}^3}|\nabla w_R|^2\right)^2 - Ct^\mu\int_{\mathbb{R}^3} w_R^\mu. \end{aligned}$$

Since $4 < \mu < 6$, there exists $t_0 > 0$ such that $I_\varepsilon(tw_R) < 0$ for $t > t_0$. Then

$$\begin{aligned} &\limsup_{\varepsilon\to 0^+} c_\varepsilon \leq \limsup_{\varepsilon\to 0^+} \max_{t\in(0,\infty)} I_\varepsilon(tw_R) \\ &\leq \limsup_{\varepsilon\to 0^+}\left\{\max_{t\in(0,\infty)}\left[\frac{t^2}{2}\int_{\mathbb{R}^3}(a|\nabla w_R|^2 + V(\varepsilon x)w_R^2) + \frac{b}{4}t^4\left(\int_{\mathbb{R}^3}|\nabla w_R|^2\right)^2\right.\right. \\ &\left.\left.-Ct^\mu\int_{\mathbb{R}^3} w_R^\mu\right]\right\} \leq \limsup_{\varepsilon\to 0^+}\left[\frac{t_0^2}{2}\max_{x\in B_R(0)} V(\varepsilon x)\int_{\mathbb{R}^3} w_R^2\right] + \max_{t\in(0,\infty)}\left[\frac{t^2}{2}\int_{\mathbb{R}^3} a|\nabla w_R|^2\right. \\ &\left.+\frac{b}{4}t^4\left(\int_{\mathbb{R}^3}|\nabla w_R|^2\right)^2 - Ct^\mu\int_{\mathbb{R}^3} w_R^\mu\right] = 0 + I_R(w_R), \end{aligned}$$

where I_R is the energy functional of (3.2). Thus in order to prove Lemma 3.1, we only need to show that $g(R) := I_R(w_R) \to 0$ as $R \to \infty$.

Claim: $g(R) = I_R(w_R) \to 0$ as $R \to \infty$.

For any $m > 0$, let c_R^m be the least energy level of the energy functional associated to the equation

$$\begin{cases} -\left(a + b\displaystyle\int_{B_R} |\nabla w|^2\right)\Delta w + mw = C\mu w^{\mu-1}, \\ w > 0 \text{ in } B_R(0), \quad w = 0 \text{ on } \partial B_R(0). \end{cases} \tag{3.3}$$

Then $g(R) \leq c_R^m$, and as the proof of Lemma 2.3 in [27], we can prove $c_R^m \to c^m$ as $R \to \infty$, where c^m is the ground energy level of of the energy functional associated to the equation

$$-(a+b\int_{\mathbb{R}^3}|\nabla u|^2)\Delta u+mu=C\mu u^{\mu-1}\quad\text{in }\mathbb{R}^3. \tag{3.4}$$

By the Lemma 3.6 of [27] (Claim 3.6.1 in the proof of it), $c^m\to 0$ as $m\to 0^+$. Then from the above arguments we can get $g(R)=I_R(w_R)\to 0$ as $R\to\infty$. □

Remark 3.2 As a result of condition (V2)(the zero set of potential V is not empty), we can estimate in Lemma 3.1 that the energy level $\limsup_{\varepsilon\to 0^+}c_\varepsilon=0$, this is very different to the case $\inf_{x\in\mathbb{R}^3}V(x)>0$(see Lemma 2.3 in [29] for instance) and this result can be used in the subsequent arguments to prove $\|u_\varepsilon\|_{L^\infty}\to 0$ as $\varepsilon\to 0^+$ which in the end yields u_ε is the solution of the original problem for sufficiently small $\varepsilon>0$.

Now we estimate the limit of $\|u_\varepsilon\|_\varepsilon$ and $\|u_\varepsilon\|_{L^\mu}$ as $\varepsilon\to 0^+$.

Lemma 3.3 $\|u_\varepsilon\|_\varepsilon\to 0,\ \|u_\varepsilon\|_{L^\mu}\to 0$ *as* $\varepsilon\to 0^+$.

Proof We first prove $\|u_\varepsilon\|_\varepsilon\to 0$ as $\varepsilon\to 0^+$. Since

$$c_\varepsilon=\frac{1}{2}\|u_\varepsilon\|_\varepsilon^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2\right)^2-\int_{M_{\frac{1}{\varepsilon}}}\tilde{F}(u_\varepsilon)-\int_{\mathbb{R}^3\setminus M_{\frac{1}{\varepsilon}}}G(x,u_\varepsilon),$$

and

$$0=\|u_\varepsilon\|_\varepsilon^2+b\left(\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2\right)^2-\int_{M_{\frac{1}{\varepsilon}}}\tilde{f}(u_\varepsilon)u_\varepsilon-\int_{\mathbb{R}^3\setminus M_{\frac{1}{\varepsilon}}}g(x,u_\varepsilon)u_\varepsilon,$$

we get

$$\begin{aligned}c_\varepsilon=&\left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_\varepsilon\|_\varepsilon^2+b\left(\frac{1}{4}-\frac{1}{\mu}\right)\left(\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2\right)^2\\&+\int_{M_{\frac{1}{\varepsilon}}}\left(\frac{1}{\mu}\tilde{f}(u_\varepsilon)\right)u_\varepsilon-\tilde{F}(u_\varepsilon))+\int_{\mathbb{R}^3\setminus M_{\frac{1}{\varepsilon}}}\left(\frac{1}{\mu}g(x,u_\varepsilon)\right)u_\varepsilon-G(x,u_\varepsilon))\\&\geq\left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_\varepsilon\|_\varepsilon^2-\frac{k}{2}\int_{\mathbb{R}^3\setminus M_{\frac{1}{\varepsilon}}}u_\varepsilon^2\geq\left(\frac{1}{2}-\frac{1}{\mu}-\frac{k}{2d}\right)\|u_\varepsilon\|_\varepsilon^2.\end{aligned}$$

Thus by (2.2) and Lemma 3.1, we have $\|u_\varepsilon\|_\varepsilon\to 0$ as $\varepsilon\to 0^+$.

Now we show $\|u_\varepsilon\|_{L^\mu}\to 0$ as $\varepsilon\to 0^+$.

Since

$$c_\varepsilon=\frac{1}{2}\|u_\varepsilon\|_\varepsilon^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2\right)^2-\int_{\mathbb{R}^3}G(x,u_\varepsilon),$$

we know

$$\int_{\mathbb{R}^3}G(x,u_\varepsilon)=\frac{1}{2}\|u_\varepsilon\|_\varepsilon^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2\right)^2-c_\varepsilon\to 0,\ as\ \varepsilon\to 0^+.$$

Then

$$\int_{M_{\frac{1}{\varepsilon}}} \tilde{F}(u_\varepsilon) \to 0, as\, \varepsilon \to 0^+.$$

By (f2) and (f3), there exists $C > 0$ such that $\tilde{F}(t) \geq Ct^\mu$, thus

$$C \int_{M_{\frac{1}{\varepsilon}}} |u_\varepsilon|^\mu \leq \int_{M_{\frac{1}{\varepsilon}}} \tilde{F}(u_\varepsilon) \to 0,\ as\, \varepsilon \to 0^+. \tag{3.5}$$

On $\mathbb{R}^3 \backslash M_{\frac{1}{\varepsilon}}$, we know

$$\int_{\mathbb{R}^3 \backslash M_{\frac{1}{\varepsilon}}} (|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2) \to 0\ as\, \varepsilon \to 0^+,$$

thus

$$\|u_\varepsilon\|_{H^1(\mathbb{R}^3 \backslash M_{\frac{1}{\varepsilon}})} \to 0\ as\, \varepsilon \to 0^+.$$

By the Sobolev embedding theorem, we know

$$|u_\varepsilon\|_{L^\mu(\mathbb{R}^3 \backslash M_{\frac{1}{\varepsilon}})} \to 0\ as\, \varepsilon \to 0^+. \tag{3.6}$$

Thus by (3.5) and (3.6), we have

$$\|u_\varepsilon\|_{L^\mu} \to 0\ as\, \varepsilon \to 0^+.$$

□

Now we apply Theorem 2.4 to estimate $\|u_\varepsilon\|_{L^\infty}$ as $\varepsilon \to 0^+$. We know u_ε satisfies

$$-(a + b\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2)\Delta u_\varepsilon + V(\varepsilon x) u_\varepsilon = g(x, u_\varepsilon).$$

From the above arguments we know that there exist $C_1, C_2 > 0$ such that

$$C_1 \leq a + b\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \leq C_2,$$

for sufficiently small $\varepsilon > 0$. Since $g(x, u_\varepsilon) \leq \tilde{f}(u_\varepsilon)$, we have

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2\right)\Delta u_\varepsilon + V(\varepsilon x) u_\varepsilon - \tilde{f}(u_\varepsilon) \leq 0.$$

It yields

$$\left(a + b\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2)\Delta u_\varepsilon + \tilde{f}(u_\varepsilon)\right) \geq V(\varepsilon x) u_\varepsilon \geq 0.$$

By the definition of $\tilde{f}(t)$, there exist $C_3, C_4 > 0$ such that

$$\tilde{f}(u_\varepsilon) \leq C_3 u_\varepsilon + C_4 u_\varepsilon^{\mu-1}.$$

Thus from above arguments, there exist $C, C' > 0$ such that

$$\Delta u_\varepsilon + C u_\varepsilon + C' u_\varepsilon^{\mu-1} \geq 0.$$

It implies that

$$\int_{B_1(0)} \nabla u_\varepsilon \nabla \varphi + c u_\varepsilon \varphi \leq 0 \text{ for any } \varphi \in H_0^1(B_1(0)),\ \varphi \geq 0 \text{ in } B_1(0),$$

where $c = -C - C' u_\varepsilon^{\mu-2}$, $B_1(0)$ is any ball in $\mathbb{R}^3$ with radius 1. We have that $c(x) \in L^q(B_1(0))$, $q = \frac{\mu}{\mu-2}$, and

$$q - \frac{3}{2} = \frac{\mu}{\mu-2} - \frac{3}{2} = \frac{6-\mu}{2(\mu-2)} > 0.$$

Thus $q > \frac{3}{2}$ and there exists $\Lambda > 0$ such that $\|c\|_{L^q(B_1(0))} < \Lambda$. Now we can apply Theorem 2.4 and get that $u_\varepsilon \in L^\infty_{loc}(B_1(0))$ and

$$\|u_\varepsilon\|_{L^\infty} \leq \bar{C} \|u_\varepsilon\|_{L^\mu},$$

where $\bar{C}$ is a constant only dependent on Λ and independent of $\varepsilon > 0$. Now by Lemma 3.3 we know $\|u_\varepsilon\|_{L^\infty} \to 0$ as $\varepsilon \to 0^+$. Thus we have shown that for sufficiently small $\varepsilon > 0$, u_ε is the solution of (2.1) and equivalently proved Theorem 1.1.

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