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ORIGINAL PAPER

Unconditionally Superconvergence Error Analysis of an Energy-Stable and Linearized Galerkin Finite Element Method for Nonlinear Wave Equations

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Abstract
In this paper, a linearized energy-stable scalar auxiliary variable (SAV) Galerkin scheme is investigated for a two-dimensional nonlinear wave equation and the unconditional superconvergence error estimates are obtained without any certain time-step restrictions. The key to the analysis is to derive the boundedness of the numerical solution in the H^1-norm, which is different from the temporal-spatial error splitting approach used in the previous literature. Meanwhile, numerical results are provided to confirm the theoretical findings.

Keywords Unconditionally superconvergence error estimate · Nonlinear wave equation · Linearized energy-stable scalar auxiliary variable (SAV) Galerkin scheme

Mathematics Subject Classification 65M15 · 65M60 · 65N15 · 65N30

1 Introduction

In this paper, we focus on the unconditional superconvergence error estimate of an energy-stable and linearized Galerkin finite element method (FEM) for the following two-dimensional nonlinear wave equations [7]:

$$u_{tt} - \Delta u + \lambda u + F'(u) = 0, \quad (\boldsymbol{x}, t) \in \Omega \times (0, T], \tag{1}$$

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \qquad u_t(\boldsymbol{x}, 0) = u_1(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \tag{2}$$

$$u(\boldsymbol{x}, t) = 0, \quad (\boldsymbol{x}, t) \in \partial\Omega \times (0, T], \tag{3}$$

where $\lambda \geqslant 0$ is a constant, $\Omega \subset \mathbb{R}^2$ is a rectangular domain with the boundary $\partial\Omega$, $u = u(\boldsymbol{x}, t)$ is the unknown function defined in $\Omega \times [0, T]$, u_0 and u_1 are sufficiently smooth

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functions, and $\boldsymbol{x} = (x, y)$ and $T > 0$ is a finite number. Moreover, $F \in C^2(\mathbb{R})$ is the nonlinear potential.

Nonlinear wave equations (1)–(3) are widely used to describe many of complicated natural phenomena in scientific fields [9, 37]. Numerous numerical methods and analyses for the nonlinear wave equations have been investigated extensively, e.g., see [1, 11, 31] for finite difference methods, [5–7, 10, 16, 26, 35, 36] for Galerkin FEMs. In particular, optimal error estimate was studied in [7] for the nonlinear wave equation with an energy-conserving and linearly implicit scalar auxiliary variable (SAV) Galerkin scheme with the help of the temporal-spatial error splitting technique. Optimal error estimates were derived using the standard Galerkin method for a linear second-order hyperbolic equation in [5]. An H^1-Galerkin mixed FEM was discussed and the corresponding error estimates were obtained for a class of second-order hyperbolic problems in [26].

As we all know, for nonlinear problems, some certain assumptions about the nonlinear term are indeed to obtain the corresponding error estimation. The most common assumption is that the nonlinear term is required to satisfy the Lipschitz continuity condition for differential problems [8, 13, 14, 18, 38]. However, as pointed out in [15, 27], the Lipschitz continuity assumption is not satisfied in most actual applications. For instance, the nonlinear terms appeared in phase field problems, nonlinear Schrödinger equations, and the viscous Burgers' equations. Moreover, certain time-step restrictions dependent on the spatial mesh size for high-dimensional nonlinear problems were required using a nonlinear/linearized scheme [3, 32]. In practical applications, if similar time-step restrictions are required, one may apply an unnecessarily small time-step and be extremely time-consuming in computations. A new approach called the time-splitting technique was proposed in [20, 21] to obtain error estimates for a nonlinear thermistor system and a nonlinear system from incompressible miscible flow in porous media without the time-step restrictions (i.e., unconditional convergence). Subsequently, the time-splitting technique was successfully applied to study the unconditional error estimates for different high-dimensional nonlinear problems, such as nonlinear thermistor equations [19], the Landau-Lifshitz equation [4, 12], nonlinear Schrödinger equations [23, 28, 30, 34], nonlinear parabolic problems [22, 24, 39], etc. The key of the analysis is to introduce an additional time-discrete system ((elliptic) time-discrete equations), which leads to the error estimation process being relatively complicated. Moreover, the L^∞-norm boundedness of the numerical solution is usually required in the time-splitting approach.

In this paper, the unconditional superconvergence error estimate is obtained for two-dimensional nonlinear wave equation based on an energy-stable and linearized Galerkin FEM. The difficulties come from the estimation of the nonlinear term without using the Lipschitz continuity assumption and the L^∞-boundedness of the numerical solution in the error analysis. We first established a priori boundedness of the numerical solution in H^1 -norm based on a rigorous analysis in terms of an energy inequality without introducing an additional time discrete system in the previous literature. Then, the unconditional superconvergence error estimates are obtained by treating the nonlinear term skillfully without using the L^∞-norm boundedness of the numerical solution required in the previous literature. Meanwhile, some numerical results are provided to confirm our theoretical findings.

The rest of the paper is organized as follows. In Sect. 2, some preliminaries and notations are introduced. In Sect. 3, an energy-stable and linearized SAV Galerkin finite-element scheme is proposed and the superconvergence error estimate of the scheme is presented. In Sect. 4, numerical results are provided to demonstrate the theoretical analysis.

2 Preliminaries and Notations

Let $W^{m,p}(\Omega)$ be the standard Sobolev space with norm $\|\cdot\|_{m,p}$ and semi-norm $|\cdot|_{m,p}$ [2]. $L^2(\Omega)$ is the space of square integrable functions defined in Ω, and its inner product and norm are denoted by $(\cdot,\cdot)$ and $\|\cdot\|_0$, respectively. For any Banach space X and $I=[0,T]$, let $L^p(I;X)$ be the space of all measurable functions $f\colon I\rightarrow X$ with the norm

$$\|f\|_{L^p(I;X)}=\begin{cases}\left(\int_0^T\|f\|_X^p\mathrm{d}t\right)^{\frac{1}{p}}, & 1\leqslant p<\infty,\\ \operatorname*{ess\,sup}_{t\in I}\|f\|_X, & p=\infty.\end{cases}$$

Moreover, let $\mathcal{T}_h=\{K\}$ be a conforming and shape regular simplicial triangulation of Ω, and $h=\max_{K\in\mathcal{T}_h}\{\mathrm{diam}\ K\}$ be the mesh size. Let V_h be the finite-dimensional subspace of $H_0^1(\Omega)$, which consists of continuous piecewise polynomials on $\mathcal{T}_h$. Then, for a given element $K\in\mathcal{T}_h$, we define the finite-element space V_h as

$$V_h=\{v_h\in C^0(\Omega):\ v_h|_K\in\mathrm{span}\{1,x,y,xy\},\ \forall K\in\mathcal{T}_h,\ v_h|_{\partial\Omega}=0\}. \tag{4}$$

Define the Ritz projection operator $R_h\colon H_0^1(\Omega)\rightarrow V_h$ by

$$(\nabla(u-R_hu),\nabla\chi)=0,\quad \forall\chi\in V_h. \tag{5}$$

Then, by the classical finite-element theory [33], there holds for $u\in H^2(\Omega)\cap H_0^1(\Omega)$

$$\|u-R_hu\|_0+h\|\nabla(u-R_hu)\|_0\leqslant Ch^2|u|_2. \tag{6}$$

Lemma 1 [25] *Suppose that $\mathcal{T}_h$ is a shape regular rectangular partition and $u\in H^3(\Omega)$, then there holds*

$$(\nabla(u-I_hu),\nabla v)\leqslant Ch^2\|u\|_3\|\nabla v\|_0,\quad \forall v\in V_h, \tag{7}$$

where $I_h\colon H^2(\Omega)\rightarrow V_h$ is the Lagrangian node interpolation operator.

With the help of Lemma 1, the following superclose error estimate between I_hu and R_hu in H^1-norm has been established in [29].

Lemma 2 *Suppose that $u\in H^3(\Omega)$, then we have*

$$\|\nabla(I_hu-R_hu)\|_0\leqslant Ch^2\|u\|_3. \tag{8}$$

Following the basic idea of [15, 27], we adopt the following assumption instead of Lipschitz continuity assumption in the error estimate:

$$|f(s)|\leqslant C(1+|s|^p),\quad p\geqslant 0, \tag{9}$$

$$|f'(s)|\leqslant C(1+|s|^p),\quad p\geqslant 0. \tag{10}$$

Here, we present the following Gronwall-typed inequality, which plays an important role in the error analysis.

Lemma 3 [17] *Let τ, B and a_k, b_k, c_k, γ_k, for integers $k > 0$, be nonnegative numbers, such that*

$$a_n + \tau \sum_{k=0}^{n} b_k \leqslant \tau \sum_{k=0}^{n} \gamma_k a_k + \tau \sum_{k=0}^{n} c_k + B \quad \text{for } n \geqslant 0.$$

Suppose that $\tau\gamma_k < 1$, for all k, and set $\sigma_k = (1 - \tau\gamma_k)^{-1}$. Then, there holds

$$a_n + \tau \sum_{k=0}^{n} b_k \leqslant \left(\tau \sum_{k=0}^{n} c_k + B \right) \exp\left(\tau \sum_{k=0}^{n} \gamma_k \sigma_k \right).$$

3 Superconvergence Error Estimates of the Energy-Stable and Linearized SAV Galerkin Scheme

Suppose

$$E_1(u) = \int_\Omega F(u)\mathrm{d}\boldsymbol{x} \geqslant -c_0$$

for some $c_0 > 0$, i.e., it is bounded from below, and let $C_0 > c_0$, such that

$$E_1(u) + C_0 > 0.$$

Then, we introduce the following SAV:

$$r(t) = \sqrt{E(u)}, \quad E(u) = \int_\Omega F(u)\mathrm{d}\boldsymbol{x} + C_0.$$

Therefore, we can rewrite (1)–(3) as

$$u_t = v, \tag{11}$$

$$v_t - \Delta u + \lambda u + \frac{r(t)}{\sqrt{E(u)}} f(u) = 0, \tag{12}$$

$$r_t = \frac{1}{2\sqrt{E(u)}} \int_\Omega f(u) u_t \mathrm{d}\boldsymbol{x}, \tag{13}$$

where $f(u) = F'(u)$.

The weak formulation of (1)–(3) is: for any $t \in [0, T]$, find $u \in H_0^1(\Omega)$, $v \in H_0^1(\Omega)$, and $r \in \mathbb{R}$, such that

$$(u_t, \chi_1) - (v, \chi_1) = 0, \quad \forall \chi_1 \in H_0^1(\Omega), \tag{14}$$

$$(v_t, \chi_2) + (\nabla u, \nabla \chi_2) + \lambda(u, \chi_2) + \frac{r(t)}{\sqrt{E(u)}} (f(u), \chi_2) = 0, \quad \forall \chi_2 \in H_0^1(\Omega), \tag{15}$$

$$r_t = \frac{1}{2\sqrt{E(u)}} \int_\Omega f(u) u_t \mathrm{d}\boldsymbol{x}. \tag{16}$$

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the time interval [0, T] with time-step size $\tau = T/N$ and $u^n = u(\cdot, t_n)$ for $0 \leqslant n \leqslant N$. For a smooth function ω on [0, T], we define

$$D_\tau \omega^n = \frac{\omega^n - \omega^{n-1}}{\tau}.$$

Based on the above notations, a linearized fully discrete SAV Galerkin scheme is to find $u_h^n \in V_h$, $v_h^n \in V_h$, $r_h^n \in \mathbb{R}$, for given $u_h^{n-1} \in V_h$, $v_h^{n-1} \in V_h$, $r_h^{n-1} \in \mathbb{R}$ and $n = 1, 2, \cdots, N$, such that

$$(D_\tau u_h^n, \chi_{1h}) - (v_h^n, \chi_{1h}) = 0, \quad \forall \chi_{1h} \in V_h, \tag{17}$$

$$(D_\tau v_h^n, \chi_{2h}) + (\nabla u_h^n, \nabla \chi_{2h}) + \lambda(u_h^n, \chi_{2h}) + \frac{r_h^n}{\sqrt{E(u_h^{n-1})}}(f(u_h^{n-1}), \chi_{2h}), \quad \forall \chi_{2h} \in V_h, \tag{18}$$

$$r_h^n - r_h^{n-1} = \frac{1}{2\sqrt{E(u_h^{n-1})}} \int_\Omega f(u_h^{n-1})(u_h^n - u_h^{n-1}) \mathrm{d}\boldsymbol{x}, \tag{19}$$

and the initial values are chosen as $(u_h^0, v_h^0, r_h^0) = (R_h u_0, R_h v_0, \sqrt{E(u_0)})$.

The scheme (17)–(19) is energy stable. In fact, taking $\chi_{1h} = v_h^n - v_h^{n-1}$ in (17), $\chi_{2h} = u_h^n - u_h^{n-1}$ in (18), and multiplying (19) by r_h^n, then one can derive

$$(v_h^n, v_h^n - v_h^{n-1}) + (\nabla u_h^n, \nabla(u_h^n - u_h^{n-1})) + \lambda(u_h^n, u_h^n - u_h^{n-1}) + (r_h^n - r_h^{n-1}) r_h^n = 0,$$

which shows that

$$\begin{aligned}&(\|v_h^n\|_0^2 - \|v_h^{n-1}\|_0^2 + \|v_h^n - v_h^{n-1}\|_0^2) + (\|\nabla u_h^n\|_0^2 - \|\nabla u_h^{n-1}\|_0^2 + \|\nabla(u_h^n - u_h^{n-1})\|_0^2)\\ &+ \lambda(\|u_h^n\|_0^2 - \|u_h^{n-1}\|_0^2 + \|u_h^n - u_h^{n-1}\|_0^2) + ((r_h^n)^2 - (r_h^{n-1})^2 + (r_h^n - r_h^{n-1})^2) = 0.\end{aligned} \tag{20}$$

Thus, we have

$$\|v_h^n\|_0^2 + \|\nabla u_h^n\|_0^2 + \lambda\|u_h^n\|_0^2 + (r_h^n)^2 \leqslant \|v_h^{n-1}\|_0^2 + \|\nabla u_h^{n-1}\|_0^2 + \lambda\|u_h^{n-1}\|_0^2 + (r_h^{n-1})^2. \tag{21}$$

Define the energy $\mathcal{E}^n$ by

$$\mathcal{E}^n = \sqrt{\|v_h^n\|_0^2 + \|\nabla u_h^n\|_0^2 + \lambda\|u_h^n\|_0^2 + (r_h^n)^2},$$

then we have

$$\mathcal{E}^n \leqslant \mathcal{E}^{n-1} \leqslant \cdots \leqslant \mathcal{E}^0, \tag{22}$$

which implies that the SAV Galerkin scheme (17)–(19) is energy stable.

Clearly, if $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega)$, one can check that the H^1-norm boundedness of the numerical solution, i.e.,

$$\|u_h^n\|_1 \leqslant C, \quad n = 0, 1, \cdots, N. \tag{23}$$

Then, we present the convergence and superclose error estimates in the following theorem.

Theorem 1 *Let* (u^n, v^n, r^n) *and* (u_h^n, v_h^n, r_h^n) *be the solutions of* (14)–(16) *and* (17)–(19), *respectively. Suppose that* $u \in L^\infty((0,T]; H^3)$, $u_t \in L^\infty((0,T]; H^2)$, $u_{tt} \in L^\infty((0,T]; H^2)$, $u_{ttt} \in L^\infty((0,T]; L^2)$, $v \in L^\infty((0,T]; H^2)$, *and* $v_t \in L^\infty((0,T]; H^2)$. *Then, we have for* $n = 1, 2, \cdots, N$,

$$\|v^n - v_h^n\|_0 + h\|\nabla(u^n - u_h^n)\|_0 + |r^n - r_h^n| \leqslant C(h^2 + \tau), \tag{24}$$

and the superclose error estimate

$$\|I_h u^n - u_h^n\|_1 \leqslant C(h^2 + \tau). \tag{25}$$

Proof For the convenience of error estimation, we denote

$$\begin{aligned} &u^n - u_h^n = u^n - R_h u^n + R_h u^n - u_h^n := \xi_u^n + \eta_u^n,\\ &v^n - v_h^n = v^n - R_h v^n + R_h v^n - v_h^n := \xi_v^n + \eta_v^n,\\ &r^n - r_h^n := e_r^n. \end{aligned}$$

At $t = t_n$, from (14)–(16), we have

$$(D_\tau u^n, \chi_{1h}) - (v^n, \chi_{1h}) = (D_\tau u^n - u_t^n, \chi_{1h}), \quad \forall \chi_{1h} \in H_0^1(\Omega), \tag{26}$$

$$\begin{aligned} &(D_\tau v^n, \chi_{2h}) + (\nabla u^n, \nabla \chi_{2h}) + \lambda(u^n, \chi_{2h}) + \frac{r^n}{\sqrt{E(u^{n-1})}}(f(u^{n-1}), \chi_{2h})\\ &= (D_\tau v^n - v_t^n, \chi_{2h}) + r^n\left(\frac{f(u^{n-1})}{\sqrt{E(u^{n-1})}} - \frac{f(u^n)}{\sqrt{E(u^n)}}, \chi_{2h}\right), \quad \forall \chi_{2h} \in H_0^1(\Omega), \end{aligned} \tag{27}$$

$$\begin{aligned} r^n - r^{n-1} = &\frac{1}{2\sqrt{E(u^{n-1})}}\int_\Omega f(u^{n-1})(u^n - u^{n-1})\mathrm{d}\boldsymbol{x} + \tau\left(\frac{r^n - r^{n-1}}{\tau} - r_t^n\right)\\ &- \frac{1}{2\sqrt{E(u^{n-1})}}\int_\Omega f(u^{n-1})(u^n - u^{n-1})\mathrm{d}\boldsymbol{x} + \frac{\tau}{2\sqrt{E(u^n)}}\int_\Omega f(u^n)u_t^n\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{28}$$

Then, from (26)–(28) and (17)–(19), we have the following error equations:

$$(D_\tau \eta_u^n, \chi_{1h}) - (\eta_v^n, \chi_{1h}) = -(D_\tau \xi_u^n, \chi_{1h}) + (\xi_v^n, \chi_{1h}) + (D_\tau u^n - u_t^n, \chi_{1h}), \quad \forall \chi_{1h} \in V_h, \tag{29}$$

$$
\begin{aligned}
&(D_\tau \eta_v^n, \chi_{2h}) + (\nabla \eta_u^n, \nabla \chi_{2h}) + \lambda(\eta_u^n, \chi_{2h}) + \left(r^n \frac{f(u^{n-1})}{\sqrt{E(u^{n-1})}} - r_h^n \frac{f(u_h^{n-1})}{\sqrt{E(u_h^{n-1})}}, \chi_{2h} \right) \\
&= -(D_\tau \xi_v^n, \chi_{2h}) - (\nabla \xi_u^n, \nabla \chi_{2h}) - \lambda(\xi_u^n, \chi_{2h}) + (D_\tau v^n - v_t^n, \chi_{2h}) \\
&\quad + r^n \left(\frac{f(u^{n-1})}{\sqrt{E(u^{n-1})}} - \frac{f(u^n)}{\sqrt{E(u^n)}}, \chi_{2h} \right), \quad \forall \chi_{2h} \in V_h,
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
e_r^n - e_r^{n-1} &= \frac{1}{2} \int_\Omega \left(\frac{f(u^{n-1})}{\sqrt{E(u^{n-1})}} - \frac{f(u_h^{n-1})}{\sqrt{E(u_h^{n-1})}} \right) (u^n - u^{n-1}) \mathrm{d}\boldsymbol{x} \\
&\quad + \frac{1}{2} \int_\Omega \frac{f(u_h^{n-1})}{E(u_h^{n-1})} ((u^n - u^{n-1}) - (u_h^n - u_h^{n-1})) \mathrm{d}\boldsymbol{x} + \frac{\tau}{2} \int_\Omega \left(\frac{f(u^n)}{\sqrt{E(u^n)}} - \frac{f(u^{n-1})}{\sqrt{E(u^{n-1})}} \right) u_t^n \mathrm{d}\boldsymbol{x} \\
&\quad + \frac{\tau}{2} \int_\Omega \frac{f(u^{n-1})}{\sqrt{E(u^{n-1})}} (u_t^n - D_\tau u^n) \mathrm{d}\boldsymbol{x} + \tau(D_\tau r^n - r_t^n).
\end{aligned}
\tag{31}
$$

Denote

$$
H(u) = \frac{f(u)}{\sqrt{E(u)}}.
$$

Then, taking $\chi_{1h} = D_\tau \eta_v^n$ in (29) and $\chi_{2h} = D_\tau \eta_u^n$ in (30), we have

$$
\begin{aligned}
&(\eta_v, D_\tau \eta_v^n) + (\nabla \eta_u^n, \nabla D_\tau \eta_u^n) + \lambda(\eta_u^n, D_\tau \eta_u^n) + (r^n H(u^{n-1}) - r_h^n H(u_h^{n-1}), D_\tau \eta_u^n) \\
&= (D_\tau \xi_u^n, D_\tau \eta_v^n) - (\xi_v^n, D_\tau \eta_v^n) - (D_\tau u^n - u_t^n, D_\tau \eta_v^n) \\
&\quad - (D_\tau \xi_v^n, D_\tau \eta_u^n) - (\nabla \xi_u^n, \nabla D_\tau \eta_u^n) - \lambda(\xi_u^n, D_\tau \eta_u^n) \\
&\quad + (D_\tau v^n - v_t^n, D_\tau \eta_u^n) + r^n (H(u^{n-1}) - H(u^n), D_\tau \eta_u^n).
\end{aligned}
\tag{32}
$$

Moreover, multiplying (31) by e_r^n, we derive

$$
\begin{aligned}
\frac{e_r^n - e_r^{n-1}}{\tau} \cdot e_r^n &= \frac{e_r^n}{2} (H(u^{n-1}) - H(u_h^{n-1}), D_\tau u^n) + \frac{e_r^n}{2} (H(u_h^{n-1}), D_\tau \xi_u^n) \\
&\quad + \frac{e_r^n}{2} (H(u_h^{n-1}), D_\tau \eta_u^n) + \frac{e_r^n}{2} (H(u^n) - H(u^{n-1}), u_t^n) \\
&\quad + \frac{e_r^n}{2} (H(u^{n-1}), u_t^n - D_\tau u^n) + \left(D_\tau r^n - r_t^n \right) \cdot e_r^n.
\end{aligned}
\tag{33}
$$

Note that

$$
r^n H(u^{n-1}) - r_h^n H(u_h^{n-1}) = r^n (H(u^{n-1}) - H(u_h^{n-1})) + e_r^n H(u_h^{n-1}),
$$

then substituting (33) into (32) yields

$$
\begin{aligned}
&(\eta_v^n, D_\tau \eta_v^n) + (\nabla \eta_u^n, \nabla D_\tau \eta_u^n) + \lambda(\eta_u^n, D_\tau \eta_u^n) + \frac{2}{\tau}(e_r^n - e_r^{n-1}) \cdot e_r^n \\
= &(D_\tau \xi_u^n, D_\tau \eta_v^n) - (\xi_v^n, D_\tau \eta_v^n) - (D_\tau u^n - u_t^n, D_\tau \eta_v^n) \\
&- (D_\tau \xi_v^n, D_\tau \eta_u^n) - (\nabla \xi_u^n, \nabla D_\tau \eta_u^n) - \lambda(\xi_u^n, D_\tau \eta_u^n) + (D_\tau v^n - v_t^n, D_\tau \eta_u^n) \\
&+ r^n(H(u^{n-1}) - H(u^n), D_\tau \eta_u^n) - r^n(H(u^{n-1}) - H(u_h^{n-1}), D_\tau \eta_u^n) \\
&+ e_r^n(H(u^n) - H(u^{n-1}), u_t^n) + e_r^n(H(u_h^{n-1}), D_\tau \xi_u^n) + e_r^n(H(u^{n-1}) - H(u_h^{n-1}), D_\tau u^n) \\
&+ e_r^n(H(u^{n-1}), u_t^n - D_\tau u^n) + 2(D_\tau r^n - r_t^n) \cdot e_r^n := \sum_{\ell=1}^{14} E_\ell.
\end{aligned}
\tag{34}
$$

One can check that the left-hand side of (34) is

$$
\begin{aligned}
&\frac{1}{2\tau}(\|\eta_v^n\|_0^2 - \|\eta_v^{n-1}\|_0^2 + \|\eta_v^n - \eta_v^{n-1}\|_0^2) + \frac{1}{2\tau}(\|\nabla \eta_u^n\|_0^2 - \|\nabla \eta_u^{n-1}\|_0^2 + \|\nabla(\eta_u^n - \eta_u^{n-1})\|_0^2) \\
&+ \frac{\lambda}{2\tau}(\|\eta_u^n\|_0^2 - \|\eta_u^{n-1}\|_0^2 + \|\eta_u^n - \eta_u^{n-1}\|_0^2) + \frac{1}{\tau}((e_r^n)^2 - (e_r^{n-1})^2 + (e_r^n - e_r^{n-1})^2).
\end{aligned}
\tag{35}
$$

Now, we start to estimate the terms on the right-hand side of (34). Using summation by parts, we have for $E_1 - E_2$ that

$$
\begin{aligned}
E_1 = (D_\tau \xi_u^n, D_\tau \eta_v^n) &= \frac{1}{\tau}[(D_\tau \xi_u^n, \eta_v^n) - (D_\tau \xi_u^{n-1}, \eta_v^{n-1})] - \frac{1}{\tau}(D_\tau \xi_u^n - D_\tau \xi_u^{n-1}, \eta_v^{n-1}) \\
&\leqslant \frac{1}{\tau}[(D_\tau \xi_u^n, \eta_v^n) - (D_\tau \xi_u^{n-1}, \eta_v^{n-1})] + Ch^2 \|u_{tt}\|_{L^\infty(H^2)} \|\eta_v^{n-1}\|_0,
\end{aligned}
\tag{36}
$$

and

$$
\begin{aligned}
E_2 = -(\xi_v^n, D_\tau \eta_v^n) &= -\frac{1}{\tau}[(\xi_v^n, \eta_v^n) - (\xi_v^{n-1}, \eta_v^{n-1})] + \frac{1}{\tau}(\xi_v^n - \xi_v^{n-1}, \eta_v^{n-1}) \\
&\leqslant -\frac{1}{\tau}[(\xi_v^n, \eta_v^n) - (\xi_v^{n-1}, \eta_v^{n-1})] + Ch^2 \|v_t\|_{L^\infty(H^2)} \|\eta_v^{n-1}\|_0.
\end{aligned}
\tag{37}
$$

In a similar way, we have

$$
\begin{aligned}
E_3 = -(D_\tau u^n - u_t^n, D_\tau \eta_v^n) &= -\frac{1}{\tau}[(D_\tau u^n - u_t^n, \eta_v^n) - (D_\tau u^{n-1} - u_t^{n-1}, \eta_v^{n-1})] \\
&\quad + \frac{1}{\tau}((D_\tau u^n - u_t^n) - (D_\tau u^{n-1} - u_t^{n-1}), \eta_v^{n-1}) \\
&\leqslant -\frac{1}{\tau}[(D_\tau u^n - u_t^n, \eta_v^n) - (D_\tau u^{n-1} - u_t^{n-1}, \eta_v^{n-1})] + C\tau \|\eta_v^{n-1}\|_0,
\end{aligned}
\tag{38}
$$

where we have used $(D_\tau u^n - u_t^n) - (D_\tau u^{n-1} - u_t^{n-1}) = O(\tau^2)$ by the Taylor expansion.

By the Cauchy-Schwarz inequality and the Ritz projection definition, we derive

$$
E_4 + E_5 + E_6 \leqslant Ch^2 \|D_\tau \eta_u^n\|_0. \tag{39}
$$

Applying the Taylor expansion gives that

$$
E_7 \leqslant C\tau \|D_\tau \eta_u^n\|_0. \tag{40}
$$

Note that

$$H(u^{n-1})-H(u^n)=\frac{f(u^{n-1})}{\sqrt{E(u^{n-1})}}-\frac{f(u^n)}{\sqrt{E(u^n)}}$$
$$=\frac{f(u^{n-1})}{\sqrt{E(u^{n-1})}}-\frac{f(u^n)}{\sqrt{E(u^{n-1})}}+\frac{f(u^n)}{\sqrt{E(u^{n-1})}}-\frac{f(u^n)}{\sqrt{E(u^n)}}$$
$$=\frac{f(u^{n-1})-f(u^n)}{\sqrt{E(u^{n-1})}}+f(u^n)\frac{E(u^n)-E(u^{n-1})}{\sqrt{E(u^{n-1})}\sqrt{E(u^n)}(\sqrt{E(u^{n-1})}+\sqrt{E(u^n)})}$$

and $E(s)>0$ for $s\in\mathbb{R}$, we have

$$\begin{aligned}E_8&=r^n(H(u^{n-1})-H(u^n),D_\tau\eta_u^n)=r^n\int_\Omega\frac{f(u^{n-1})-f(u^n)}{\sqrt{E(u^{n-1})}}D_\tau\eta_u^n\mathrm{d}\boldsymbol{x}\\&+r^n\int_\Omega f(u^n)\frac{E(u^n)-E(u^{n-1})}{\sqrt{E(u^{n-1})}\sqrt{E(u^n)}(\sqrt{E(u^{n-1})}+\sqrt{E(u^n)})}D_\tau\eta_u^n\mathrm{d}\boldsymbol{x}\\&\leqslant C\int_\Omega(f(u^{n-1})-f(u^n))D_\tau\eta_u^n\mathrm{d}\boldsymbol{x}+C\int_\Omega\Big(F(u^n)-F(u^{n-1})\Big)\mathrm{d}\boldsymbol{x}\cdot\int_\Omega f(u^n)D_\tau\eta_u^n\mathrm{d}\boldsymbol{x}\\&\leqslant C\tau\|D_\tau\eta_u^n\|_0.\end{aligned}\tag{41}$$

Similar to E_8, E_9 can be estimated as

$$\begin{aligned}E_9&\leqslant C\int_\Omega(f(u^{n-1})-f(u_h^{n-1}))D_\tau\eta_u^n\mathrm{d}\boldsymbol{x}\\&\quad+C\int_\Omega\big(F(u_h^{n-1})-F(u^{n-1})\big)\mathrm{d}\boldsymbol{x}\cdot\int_\Omega f(u_h^{n-1})D_\tau\eta_u^n\mathrm{d}\boldsymbol{x}\\&\leqslant C\int_\Omega(1+|u_h^{n-1}|^p)|u^{n-1}-u_h^{n-1}||D_\tau\eta_u^n|\mathrm{d}\boldsymbol{x}\\&\quad+C\int_\Omega f((1-\theta)u^{n-1}+\theta u_h^{n-1})|u_h^{n-1}-u^{n-1}|\mathrm{d}\boldsymbol{x}\int_\Omega f(u_h^{n-1})D_\tau\eta_u^n\mathrm{d}\boldsymbol{x}\\&\leqslant C(1+\|u_h^{n-1}\|_{0,4p}^p)\|u^{n-1}-u_h^{n-1}\|_{0,4}\|D_\tau\eta_u^n\|_0\\&\quad+C\int_\Omega(1+|u_h^{n-1}|^p)^2\mathrm{d}\boldsymbol{x}\|u^{n-1}-u_h^{n-1}\|_0\|D_\tau\eta_u^n\|_0\\&\leqslant C(1+\|\nabla u_h^{n-1}\|_0^p)(\|u^{n-1}-I_hu^{n-1}\|_{0,4}+\|\nabla(I_hu^{n-1}-R_hu^{n-1})\|_0\\&\quad+\|\nabla\eta_u^n\|_0)\|D_\tau\eta_u^n\|_0\\&\quad+C(1+\|\nabla u_h^{n-1}\|_0^{2p})(h^2+\|\eta_u^{n-1}\|_0)\|D_\tau\eta_u^n\|_0\\&\leqslant C(h^2+\|\nabla\eta_u^{n-1}\|_0+\|\eta_u^{n-1}\|_0)\|D_\tau\eta_u^n\|_0,\end{aligned}\tag{42}$$

where $0<\theta<1$ and we have used (9), (10), and (23) in the above estimate.

Thus, we obtain

$$E_4+E_5+E_6+E_7+E_8+E_9\leqslant C(h^2+\tau+\|\nabla\eta_u^{n-1}\|_0+\|\eta_u^{n-1}\|_0)\|D_\tau\eta_u^n\|_0.\tag{43}$$

On the other hand, taking $\chi_{1h}=D_\tau\eta_u^n$ in (29) results in

$$\begin{aligned}\|D_\tau\eta_u^n\|_0^2 &= (\eta_v^n, D_\tau\eta_u^n) - (D_\tau\xi_u, D_\tau\eta_u^n) + (\xi_v^n, D_\tau\eta_u^n) + (D_\tau u^n - u_t^n, D_\tau\eta_u^n)\\ &\leqslant \|\eta_v^n\|_0\|D_\tau\eta_u^n\|_0 + \|D_\tau\xi_u^n\|_0\|D_\tau\eta_u^n\|_0 + \|\xi_v^n\|_0\|D_\tau\eta_u^n\|_0 + \|D_\tau u^n - u_t^n\|_0\|D_\tau\eta_u^n\|_0\\ &\leqslant C(h^2 + \tau + \|\eta_v^n\|_0)\|D_\tau\eta_u^n\|_0,\end{aligned}$$

which shows that

$$\|D_\tau\eta_u^n\|_0 \leqslant C(h^2 + \tau + \|\eta_v^n\|_0). \tag{44}$$

Substituting (44) into (43) gives that

$$E_4 + E_5 + E_6 + E_7 + E_8 + E_9 \leqslant C(h^2 + \tau)^2 + C(\|\eta_u^{n-1}\|_0^2 + \|\nabla\eta_u^{n-1}\|_0^2 + \|\eta_v^n\|_0^2). \tag{45}$$

Similar to E_8, E_{10} can be bounded by

$$E_{10} = e_r^n(H(u^n) - H(u^{n-1}), u_t^n) \leqslant C\tau|e_r^n|. \tag{46}$$

Using (9) and (23), we have for E_{11} that

$$\begin{aligned}E_{11} &\leqslant C|e_r^n|\int_\Omega |f(u_h^{n-1})||D_\tau\xi_u^n|\mathrm{d}\boldsymbol{x} \leqslant C|e_r^n|\int_\Omega (1 + |u_h^{n-1}|^p)|D_\tau\xi_u^n|\mathrm{d}\boldsymbol{x}\\ &\leqslant C|e_r^n|(1 + \|u_h^{n-1}\|_{0,2p}^p)\|D_\tau\xi_u^n\|_0 \leqslant C(1 + \|\nabla u_h^{n-1}\|_0^p)|e_r^n|\|D_\tau\xi_u^n\|_0\\ &\leqslant Ch^2|e_r^n|.\end{aligned} \tag{47}$$

Using a process similar to E_9, we have

$$\begin{aligned}E_{12} &\leqslant C|e_r^n|\int_\Omega (f(u^{n-1}) - f(u_h^{n-1}))D_\tau u^n\mathrm{d}\boldsymbol{x}\\ &\quad + C|e_r^n|\int_\Omega \left(F(u_h^{n-1}) - F(u^{n-1})\right)\mathrm{d}\boldsymbol{x}\cdot\int_\Omega f(u_h^{n-1})D_\tau u^n\mathrm{d}\boldsymbol{x}\\ &\leqslant C(h^2 + \|\eta_u^{n-1}\|_0)|e_r^n|.\end{aligned} \tag{48}$$

With an application of (9) and the Taylor expansion, we obtain

$$E_{13} + E_{14} \leqslant C\tau|e_r^n|. \tag{49}$$

Thus, it follows that

$$E_{10} + E_{11} + E_{12} + E_{13} + E_{14} \leqslant C(h^2 + \tau + \|\eta_u^{n-1}\|_0)|e_r^n|. \tag{50}$$

Substituting (35), (36), (37), (38), (45), and (50) into (34) leads to

$$\begin{aligned}&\frac{1}{2\tau}(\|\eta_v^n\|_0^2 - \|\eta_v^{n-1}\|_0^2) + \frac{1}{2\tau}(\|\nabla\eta_u^n\|_0^2 - \|\nabla\eta_u^{n-1}\|_0^2) + \frac{\lambda}{2\tau}(\|\eta_u^n\|_0^2\\ &- \|\eta_u^{n-1}\|_0^2) + \frac{1}{\tau}(|e_r^n|^2 - |e_r^{n-1}|^2)\\ \leqslant\ &\tau^{-1}[(D_\tau\xi_u^n, \eta_v^n) - (D_\tau\xi_u^{n-1}, \eta_v^{n-1})] + \tau^{-1}[(\xi_v^n, \eta_v^n) - (\xi_v^{n-1}, \eta_v^{n-1})]\\ &+ \tau^{-1}[(D_\tau u^n - u_t^n, \eta_v^n) - (D_\tau u^{n-1} - u_t^{n-1}, \eta_v^{n-1})] + C(h^2 + \tau)^2 + C|e_r^n|^2\\ &+ C(\|\eta_u^{n-1}\|_0^2 + \|\nabla\eta_u^{n-1}\|_0^2 + \|\eta_v^{n-1}\|_0^2).\end{aligned} \tag{51}$$

Summing up the above inequality and using $\eta_u^0 = 0$, $\eta_v^0 = 0$, and $e_r^0 = 0$, we derive

$$\frac{1}{2\tau}\|\eta_v^n\|_0^2+\frac{1}{2\tau}\|\nabla\eta_u^n\|_0^2+\frac{\lambda}{2\tau}\|\eta_u^n\|_0^2+\frac{1}{\tau}|e_r^n|^2\leqslant\tau^{-1}(D_\tau\xi_u^n,\eta_v^n)+\tau^{-1}(\xi_v^n,\eta_v^n)$$
$$+\tau^{-1}(D_\tau u^n-u_t^n,\eta_v^n)+Cn(h^2+\tau)^2+C\sum_{k=1}^{n}(\|\eta_u^k\|_0^2+\|\nabla\eta_u^k\|_0^2+\|\eta_v^k\|_0^2+|e_r^k|^2). \tag{52}$$

Multiplying both sides of the above inequality by 2τ and using the Cauchy-Schwarz inequality for the first three terms appeared on the right-hand side of the above inequality yields that

$$\|\eta_v^n\|_0^2+\|\nabla\eta_u^n\|_0^2+\|\eta_u^n\|_0^2+|e_r^n|^2\leqslant C(h^2+\tau)^2+C\tau\sum_{k=1}^{n}(\|\eta_u^k\|_0^2+\|\nabla\eta_u^k\|_0^2+\|\eta_v^k\|_0^2+|e_r^k|^2). \tag{53}$$

Therefore, an application of the Gronwall inequality (see Lemma 3) gives that for the sufficiently small τ

$$\|\eta_v^n\|_0+\|\nabla\eta_u^n\|_0+\|\eta_u^n\|_0+|e_r^n|\leqslant C(h^2+\tau). \tag{54}$$

Then, the desired result (24) is obtained by the triangle inequality. Moreover, according to (54) and (8) and using the triangle inequality again, we have for $n=1,2,\cdots,N$

$$\|\nabla(I_hu^n-u_h^n)\|_0\leqslant\|\nabla(I_hu^n-R_hu^n)\|_0+\|\nabla(R_hu^n-u_h^n)\|_0\leqslant C(h^2+\tau), \tag{55}$$

which the desired result (25) can be derived using the Poincare inequality.

In what follows, based on the above superclose error estimate between u_h^n and I_hu^n in (55), we employ the interpolation post-processing approach to obtain the global superconvergence result in H^1-norm. To do this, we build a macroelement $\widetilde{K}$ consisting of four elements K_j, $j=1,2,3,4$ (see Fig. 1), and we adopt the local interpolation operator I_{2h}: $C(\widetilde{K})\rightarrow Q_{22}(\widetilde{K})$ as interpolation post-processing operator [25] with the following interpolation conditions:

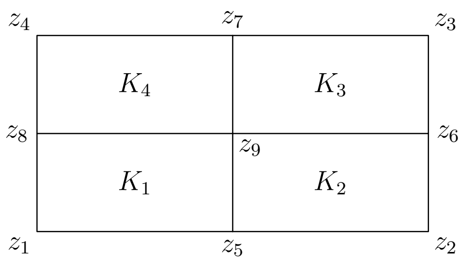
$$I_{2h}u(z_i)=u(z_i),\quad i=1,2,\cdots,9,$$

where z_i, $i=1,2,\cdots,9$ are the nine vertices of $\widetilde{K}$ and $Q_{22}(\widetilde{K})$ denotes the space of polynomials degree less than or equal to 2 in variables x and y on $\widetilde{K}$, respectively.

Moreover, the following properties for operator I_{2h} have been shown in [25]:

$$I_{2h}I_hu=I_{2h}u, \tag{56}$$

Fig. 1 The macroelement $\tilde{K}$



$$\|u - I_{2h}u\|_1 \leqslant Ch^2\|u\|_3, \quad \forall u \in H^3(\Omega), \tag{57}$$

$$\|I_{2h}v_h\|_1 \leqslant C\|v_h\|_1, \quad \forall v_h \in V_h. \tag{58}$$

Then, we have the following global superconvergent result.

Theorem 2 *Suppose that* $u \in L^\infty((0,T];H^3(\Omega))$ *together with the conditions of Theorem 1, we have for* $n = 1, 2, \cdots, N$

$$\|u^n - I_{2h}u_h^n\|_1 \leqslant C(h^2 + \tau). \tag{59}$$

Proof By the triangle inequality and the properties (56)–(58) and (55), we have

$$\begin{aligned}\|u^n - I_{2h}u_h^n\|_1 &\leqslant \|u^n - I_{2h}I_hu^n\|_1 + \|I_{2h}I_hu^n - I_{2h}u_h^n\|_1 \\ &\leqslant \|u^n - I_{2h}u^n\|_1 + \|I_{2h}(I_hu^n - u_h^n)\|_1 \\ &\leqslant Ch^2\|u^n\|_3 + C\|I_hu^n - u_h^n\|_1 \\ &\leqslant C(h^2 + \tau),\end{aligned} \tag{60}$$

which is the desired result and the proof is complete.

4 Numerical Results

In this section, we present some numerical results to verify the correctness of the theoretical findings.

Example 1 Consider the following Kelin-Gordon equation [7]:

$$u_{tt} - \Delta u + u^3 - u = g(\boldsymbol{x}, t), \quad (x, y) \in \Omega, \quad 0 < t \leqslant T.$$

Let the function $g(\boldsymbol{x}, t)$ and the initial and boundary conditions be chosen corresponding to the exact solution

$$u(x, y, t) = \exp(-t)x^2(1-x)^2y^2(1-y)^2.$$

We set the domain $\Omega = (0, 1) \times (0, 1)$ and the final time $T = 1.0$ in the computation.

A uniform partition with $m + 1$ nodes in both horizontal and vertical directions is made on the domain Ω. To confirm the error estimates in Theorems 1 and 2, choose $\tau = O(h^2)$. We present the numerical errors of $\|v^n - v_h^n\|_0$, $\|u^n - u_h^n\|_0$, $\|u^n - u_h^n\|_1$, $\|I_hu^n - u_h^n\|_1$, and $\|u^n - I_{2h}u_h^n\|_1$ at $t = 1.0$ in Table 1. Obviously, we can see that the numerical results agree well with the theoretical analysis, i.e., the convergence rate is $O(h^2)$, $O(h^2)$, $O(h)$, $O(h^2)$, and $O(h^2)$, respectively.

Example 2 Consider the following Kelin-Gordon equation:

$$u_{tt} - \Delta u + u^3 - u = 0, \quad (x, y) \in \Omega = (0, 1) \times (0, 1), \quad 0 < t \leqslant T = 100$$

with the initial conditions

Table 1 The numerical errors at $t = 1.0$

$m \times n$	4×4	8×8	16×16	32×32
$\|v^n - v_h^n\|_0$	7.982 5E−05	2.177 3E−05	5.422 2E−06	1.342 7E−06
Order	/	1.874 3	2.005 6	2.013 8
$\|u^n - u_h^n\|_0$	8.613 6E−05	2.443 8E−05	6.293 1E−06	1.584 4E−06
Order	/	1.817 5	1.957 3	1.989 8
$\|u^n - u_h^n\|_1$	1.198 8E−03	6.525 8E−04	3.324 7E−04	1.669 9E−04
Order	/	0.877 41	0.972 92	0.993 43
$\|I_h u^n - u_h^n\|_1$	1.117 3E−04	4.022 6E−05	1.104 7E−05	2.832 5E−06
Order	/	1.473 8	1.864 5	1.963 5
$\|u^n - I_{2h} u_h^n\|_1$	1.208 1E−03	3.303 5E−04	8.398 7E−05	2.107 5E−05
Order	/	1.870 6	1.975 8	1.994 6

$$u_0(x, y) = x^2(1-x)^2 y^2(1-y)^2, \qquad u_1(x, y) = -x^2(1-x)^2 y^2(1-y)^2.$$

The temporal direction is divided with time-step size 1, and the spatial direction is divided with stepsize $h = \frac{\sqrt{2}}{30}$. In Fig. 2, we present some values of the discrete energy for the backward Euler scheme at various time levels t_n. It can be seen that the numerical scheme preserves the nonincreasing property of the discrete energy, which is consistent with the theoretical analysis.

Example 3 Consider the following sine-Gordon equation [7]:

$$u_{tt} - \Delta u + \sin(u) = g(\boldsymbol{x}, t), \quad (x, y) \in \Omega, \quad 0 < t \leqslant T.$$

Let the function $g(\boldsymbol{x}, t)$ and the initial and boundary conditions be chosen corresponding to the exact solution

$$u(x, y, t) = \exp(-t)\sin(2\pi x)\sin(2\pi y).$$

We set the domain $\Omega = (0, 1) \times (0, 1)$ and the final time $T = 1.0$ in the computation.

A uniform partition with $m + 1$ nodes in both horizontal and vertical directions is made on the domain Ω. To confirm the error estimates in Theorems 1 and 2, choose $\tau = O(h^2)$. We present the numerical errors of $\|v^n - v_h^n\|_0$, $\|u^n - u_h^n\|_0$, $\|u^n - u_h^n\|_1$, $\|I_h u^n - u_h^n\|_1$, and $\|u^n - I_{2h} u_h^n\|_1$ at $t = 1.0$ in Table 2. Obviously, we can see that the numerical results agree well with the theoretical analysis, i.e., the convergence rate is $O(h^2)$, $O(h^2)$, $O(h)$, $O(h^2)$, and $O(h^2)$, respectively.

Example 4 Consider the following sine-Gordon equation:

$$u_{tt} - \Delta u + \sin(u) = 0, \quad (x, y) \in \Omega = (0, 1) \times (0, 1), \quad 0 < t \leqslant T = 100$$

with initial conditions

$$u_0(x, y) = v, \qquad u_1(x, y) = -\sin(2\pi x)\sin(2\pi y).$$

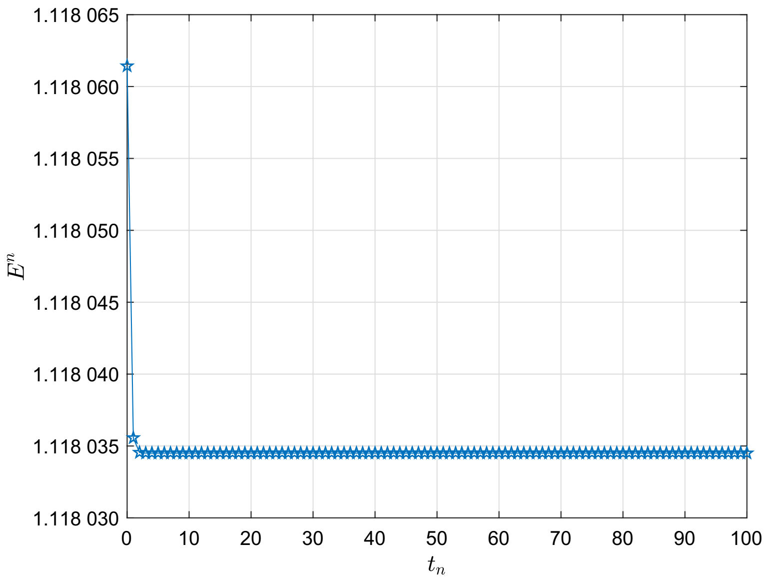


Fig. 2 The profile of the discrete energy for Example 2

The temporal direction is divided with time-step size 1, and the spatial direction is divided with stepsize $h = \frac{\sqrt{2}}{30}$. In Fig. 3, we present some values of the discrete energy for the backward Euler scheme at various time levels t_n. It can be seen that the numerical scheme preserves the nonincreasing property of the discrete energy, which is consistent with the theoretical analysis.

Example 5 Consider the following Kelin-Gordon equation [7]:

$$u_{tt} - \Delta u + u^3 - u = g(\boldsymbol{x}, t), \quad (x, y) \in \Omega, \quad 0 < t \leqslant T.$$

Let the function $g(\boldsymbol{x}, t)$ and the initial and boundary conditions be chosen corresponding to the exact solution

Table 2 The numerical errors at $t = 1.0$

$m \times n$	4×4	8×8	16×16	32×32
$\|v^n - v_h^n\|_0$	4.095 7E−02	8.908 6E−03	2.101 2E−03	5.195 2E−04
Order	/	2.200 8	2.084 0	2.016 0
$\|u^n - u_h^n\|_0$	4.439 8E−02	1.086 6E−02	2.671 3E−03	6.638 6E−04
Order	/	2.030 7	2.024 2	2.008 6
$\|u^n - u_h^n\|_1$	7.344 2E−01	3.690 6E−01	1.850 8E−01	9.261 6E−02
Order	/	0.992 74	0.995 74	0.998 80
$\|I_h u^n - u_h^n\|_1$	2.663 0E−01	8.328 7E−02	2.219 1E−02	5.644 6E−03
Order	/	1.676 9	1.908 1	1.975 0
$\|u^n - I_{2h} u_h^n\|_1$	5.635 4E−01	1.709 2E−01	4.359 7E−02	1.095 7E−02
Order	/	1.721 2	1.971 0	1.992 4

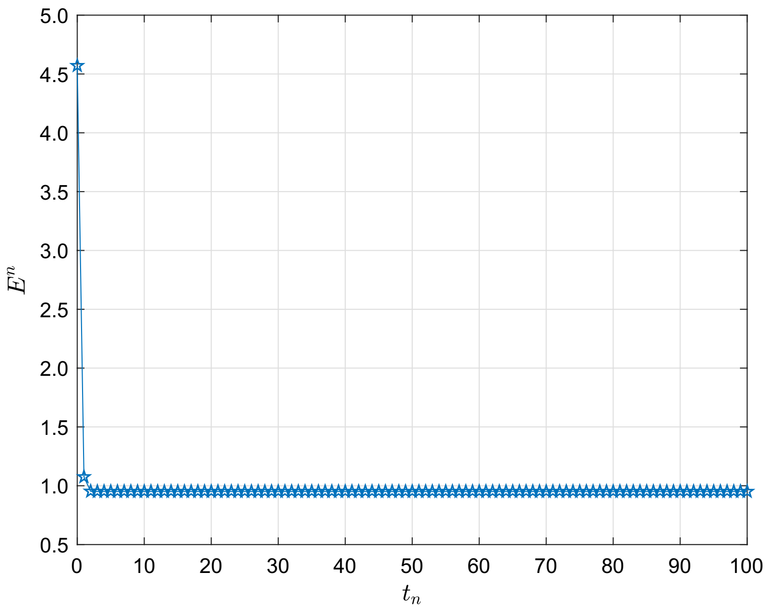


Fig. 3 The profile of the discrete energy for Example 4

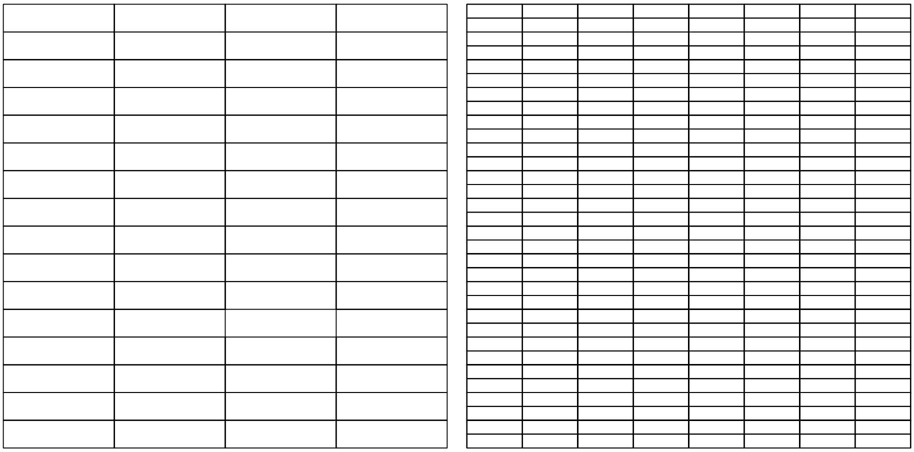
$$u(x, y, t) = \exp(-t)x^2(1-x)^2y^2(1-y)^2.$$

We set the domain $\Omega = (0, 1) \times (0, 1)$ and the final time $T = 1.0$ in the computation.

The domain Ω is divide into $m \times n$ rectangles with $m \times n = 4 \times 16$, 8×32, 16×64, 32×128, respectively (see Fig. 4 for the cases 4×16 and 8×32). We choose $\tau = 0.001$ and present the numerical errors of $\|v^n - v_h^n\|_0$, $\|u^n - u_h^n\|_0$, $\|u^n - u_h^n\|_1$, $\|I_h u^n - u_h^n\|_1$, and $\|u^n - I_{2h} u_h^n\|_1$ at $t = 1.0$ in Table 3. Obviously, we can see that the numerical results agree well with the theoretical analysis, i.e., the convergence rate is $O(h^2)$, $O(h^2)$, $O(h)$, $O(h^2)$, and $O(h^2)$, respectively. Moreover, for clarity, we present the graphics of the exact solution and numerical solution at $t = 1.0$ in Figs. 5–6 on mesh 32×128, which also shows that the numerical solution approximates the exact solution very well.

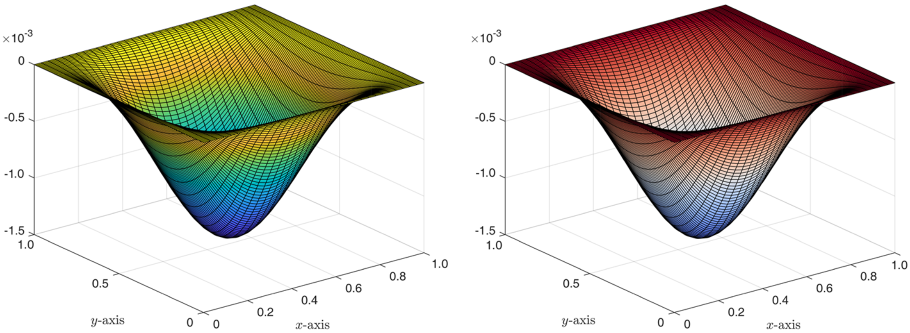
Table 3 The numerical errors at $t = 0.1$

$m \times n$	4×4	8×8	16×16	32×32
$\|v^n - v_h^n\|_0$	5.877 1E−05	1.621 4E−05	4.018 9E−06	1.002 4E−06
Order	/	1.857 9	2.012 3	2.003 3
$\|u^n - u_h^n\|_0$	5.892 6E−05	1.642 9E−05	4.238 6E−06	1.107 1E−06
Order	/	1.842 7	1.954 6	1.936 8
$\|u^n - u_h^n\|_1$	8.846 4E−04	4.768 5E−04	2.424 7E−04	1.217 4E−04
Order	/	0.891 55	0.975 71	0.994 06
$\|I_h u^n - u_h^n\|_1$	8.123 1E−05	2.555 7E−05	6.406 8E−06	1.356 4E−06
Order	/	1.668 3	1.996 0	2.239 8
$\|u^n - I_{2h} u_h^n\|_0$	8.626 9E−04	2.336 4E−04	5.931 3E−05	1.485 7E−05
Order	/	1.884 5	1.977 9	1.997 2



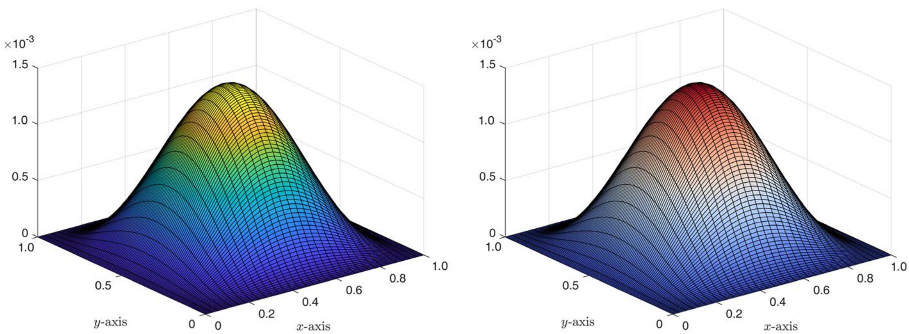
(a) $m \times n = 4 \times 16$ (b) $m \times n = 8 \times 32$

Fig. 4 The partition of Ω for Example 5



(a) Exact solution (b) Numerical solution

Fig. 5 The graphics of the solutions v and v_h at $t = 1.0$ on mesh 32×128



(a) Exact solution (b) Numerical solution

Fig. 6 The graphics of the solutions u and u_h at $t = 1.0$ on mesh 32×128

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Data Availability The data that support the findings of this study are available from the corresponding author upon reasonable request.

Compliance with Ethical Standards

Conflict of Interest The authors declare that they have no conflict of interest.

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