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ORIGINAL PAPER

Nonuniform Dependence on the Initial Data for Solutions of Conservation Laws

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Dedicated to the memory of Ching-Shan Chou.

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Abstract
In this paper, we study systems of conservation laws in one space dimension. We prove that for classical solutions in Sobolev spaces H^s, with $s > 3/2$, the data-to-solution map is not uniformly continuous. Our results apply to all nonlinear scalar conservation laws and to nonlinear hyperbolic systems of two equations.

Keywords Conservation laws · Data-to-solution map · Nonuniform dependence

Mathematics Subject Classification 35L65 · 35L03 · 35L45 · 26B05 · 34A12

1 Introduction

In 1975, Kato [11] studied the following system of symmetric hyperbolic equations:

$$\begin{cases} a_0(t,x,u)u_t + \sum_{j=1}^{m} a_j(t,x,u)u_{x_j} = f(t,x,u), \\ u(0,x) = u_0(x) \in H^s(\mathbb{R}^n) \end{cases}$$

and showed the existence, uniqueness, and continuous dependence of solutions on the initial data using a semigroup approach. This approach for proving the well-posedness for partial differential equations (PDEs) has been shown to be applicable to a wide variety of Cauchy problems. However, the limitation of using a general approach for a particular

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problem is that one may not find the optimal results for a particular PDE. For example, the Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + \frac{1}{2} u u_x = 0$$

is well-posed in the Sobolev space $H^s(\mathbb{R})$ for $s > -3/4$ (Colliander et al. [4]). This result required the advent of Bourgain spaces [2]; indeed Kato's approach was useful only in showing the well-posedness for $s > 3/2$.

One of the deficiencies of the semigroup approach concerns the smoothness of the data-to-solution map. The semigroup approach yields only that the map $H^s \ni u_0 \mapsto u \in C([0,T]; H^s)$ is continuous, even if the map is actually Lipschitz or C^1, as in the case of the KdV equation. This is an artifact of the generality of the approach. In an appendix, Kato [11] showed that for Burgers' equation

$$u_t + u u_x = 0,$$

the data-to-solution map is not Hölder continuous for any Hölder exponent. Kato's method is enlightening, and we now review his argument briefly.

Classical (H^s, for $s \in \mathbb{Z}$ with $s \geqslant 2$) solutions to Burgers' equation satisfy the implicit formula

$$u(x,t) = u_0(x - tu).$$

For each $\epsilon > 0$ and $\alpha > s - 1/2$, Kato considered the two pieces of initial data

$$u_0(x) = (\epsilon + \max\{x, 0\}^\alpha)\phi(x), \quad v_0(x) = \max\{x, 0\}^\alpha \phi(x),$$

where $\phi(x)$ is a smooth bump function equal to 1 for $|\, x \,| \leqslant 2$. Using the implicit formula, the difference between the solutions u and v can be estimated for $0 < x - tu < 1$ by

$$u(x,t) - v(x,t) = u_0(x - tu) - v_0(x - tv) = \epsilon + (x - tu)^\alpha - (x - tv)^\alpha \,.$$

Then, one takes s derivatives and uses the implicit formula for the solution again to find

$$\partial_x^s u(x,t) - \partial_x^s v(x,t) = C(\alpha, s)\left[(x - \epsilon t)^{\alpha - s} - (x)^{\alpha - s}\right] + \cdots,$$

where the dots indicate lower order terms. After taking the L^2 norm of this difference, the contribution of the leading term yields

$$\|u(t) - v(t)\|_{H^s} \geqslant C(\alpha, s, t)\epsilon^{\alpha - s + 1/2}.$$

Since $\left\|u_0 - v_0\right\|_{H^s} = \epsilon \|\phi\|_{H^s}$ and $\alpha > s - 1/2$ was arbitrary, this exponent can be made arbitrarily small. Hence, for every $\delta \in (0, 1]$, there exist initial data $u_0, v_0 \in H^s$ such that for small $t > 0$,

$$\|u(t) - v(t)\|_{H^s} \geqslant c t^\delta \left\|u_0 - v_0\right\|_{H^s}^\delta.$$

This leaves the following open questions.

(i) Is the lack of the Hölder continuity in the data-to-solution map a general property of nonlinear equations?
(ii) Is the data-to-solution map even uniformly continuous?

In 2005, Koch and Tzvetkov [12] studied the Cauchy problem for the Benjamin-Ono (BO) equation

$$u_t + Hu_{xx} + uu_x = 0,$$

where H is the Hilbert transform. It was known that the BO equation is well-posed in H^s, $s \geqslant 1$ [14]. Using high-frequency initial data, Molinet et al. [13] showed that the data-to-solution map is not C^2. Koch and Tzvetkov showed that the data-to-solution map is not uniformly continuous. The interesting innovation they made was that they introduced a pair of low-high frequency approximate solutions:

$$u_{\text{ap}}^{\omega,n}(t,x) = \left[\frac{\omega}{n} + \frac{1}{n^{s+\delta/2+1/2}} \cos\left(nx + \omega t - n^2 t\right)\right] \phi\left(\frac{x}{n^{1+\delta}}\right),$$

where ϕ is a cutoff function to ensure $u_{\text{ap}}^{\omega,n} \in H^s(\mathbb{R})$ and the pair is given by $\omega = 1$ and $\omega = -1$. They then took advantage of the nonlinear interaction in $u_t + uu_x$. Himonas et al. [6], and Himonas and Kenig [5] used the same method, low-high frequency approximate solutions of the form

$$u_{\text{ap}}^{\omega,n}(t,x) = \frac{\omega}{n} + n^{-s} \cos(nx - \omega t),$$

to prove non-uniform dependence for the Camassa-Holm (CH) equation in H^s, $s > 3/2$. In the non-periodic case, they introduced suitable cutoff functions similar to those of Koch and Tzvetkov. Then, in 2010, Himonas and Misiolek [7] discovered that the low-high frequency functions

$$u^{\omega,n}(t,x,y) = \left(\frac{\omega}{n} + n^{-s} \cos(ny - \omega t), \frac{\omega}{n} + n^{-s} \cos(nx - \omega t)\right)$$

are *exact* solutions to the incompressible Euler equations on $\mathbb{T}^2$ (the periodic case). This enabled them to prove directly that the data-to-solution map is not uniformly continuous. This result extends to the non-periodic case using cutoff functions, and to higher dimensions.

After these advances, the answers to the above questions are clear. There is a general class of equations for which the Cauchy problem is well-posed, and yet the data-to-solution map is not smoother than continuous. Several authors have since applied the method of low-high frequency approximate solutions to other special equations, first in Sobolev spaces, and then in Besov spaces (see [8, 9, 15, 16] and the references there). However, a theorem which can be applied to a general class of equations is elusive. Indeed, each of these results appears to be tailored carefully to both the function space and the PDE in question. The precise nonlinearity appears to be important in the construction of the particular approximate solutions considered. An approach applicable to a variety of equations seemed out of reach. This is the question we are answering in part in this paper.

We consider the data-to-solution map for nonlinear hyperbolic conservation laws in one space dimension. We prove for scalar equations and for systems of two equations that the data-to-solution map is not uniformly continuous in Sobolev spaces $H^s \ni u_0 \mapsto u \in C([0,T]; H^s)$. Our first result is for periodic solutions ($x \in \mathbb{T}$).

Theorem 1 *The data-to-solution map for the Cauchy problem* $u_t + f'(u)u_x = 0,\ f \in C^\infty$, $u(x,0) = u_0(x)$ *on* $\mathbb{T}$, *with* $u_0 \in H^s$, $s > 3/2$, *is not uniformly continuous in* H^s.

The same result holds for data on the real line.

Theorem 2 *The data-to-solution map for the Cauchy problem* $u_t + f'(u)u_x = 0,\ f \in C^\infty$, $u(x,0) = u_0(x)$ *on* $\mathbb{R}$, *with* $u_0 \in H^s$, $s > 3/2$, *is not uniformly continuous in* H^s.

Our method applies to nonlinear systems of two conservation laws, using Riemann invariants, but cannot (in general) be extended to systems of three or more equations; however, our results hold for the equations of compressible gas dynamics.

Unlike the proof for the BO equation and those which follow it, our method does not require the construction of approximate solutions tailored to the particular nonlinear PDE. Rather, our approach is to use the low-high frequency initial data, and the implicit formula for classical solutions to conservation laws along the lines of Kato's approach. Our approach does not directly extend to nonlocal perturbations of a conservation law, such as the BO or CH equation.

This result has implications for the numerical approximation of solutions to these PDEs. The accuracy, stability, and convergence for particular numerical methods have been proved for many classes of equations, including hyperbolic conservation laws and for weak solutions (for example, using discontinuous Galerkin methods [3, 10]). However, special care needs to be taken for nonlinear problems. Indeed, our result shows that even for exact solutions, one cannot find an estimate in Sobolev or C^k spaces on the difference between two solutions u, v with slightly different initial data u_0, v_0. Our result does not preclude the possibility that estimates can be found using weaker topologies. On the other hand, these results show that uniform dependence in H^s does not hold for conservation laws in more than one space dimension.

The paper is organized as follows. In Sect. 2, we prove the data-to-solution map is not uniformly continuous for classical solutions of scalar conservation laws under the H^s norm. Then in Sect. 3, we extend the results to systems of two equations.

2 Classical Solutions to Scalar Conservation Laws

We consider the Cauchy problem

$$\begin{cases} u_t + f'(u)u_x = 0, \\ u(x,0) = u_0(x), \end{cases}$$

where $x \in \mathbb{T}$ or $\mathbb{R}$ and $u_0 \in H^s$, $s > 3/2$; we assume $f \in C^\infty$ in a neighborhood of 0. Before we begin, we make some inessential normalizations. First, if we are interested in data near a value $\bar{u}$, it is translated to the origin by $u \mapsto u - \bar{u}$. We may also assume that $f'(0) = 0$ by making a change of coordinates $x \mapsto x - f'(0)t$. Then, since f is smooth, there is a positive integer m such that $f'(u) = u^m + \mathcal{O}(u^{m+1})$. In the case of Burgers' equation, $m = 1$ and $f'(u) = u$. We first show all of the details in the periodic case; then we show the changes necessary to deal with the non-periodic case.

By the method of characteristics, one can show that the solution satisfies the implicit relation

$$u(x,t) = u_0\big(x - tf'(u(x,t))\big).$$

From the ODE existence and uniqueness theorem, if the initial data are C^1 and f is C^2, there exists a unique solution to the above implicit formula. Moreover, the map $u_0 \mapsto u$ is continuous. We will show that this map is not uniformly continuous. We begin by recalling a size and lifespan estimate for classical solutions, proved using induction.

Lemma 1 *If* $u_0 \in C^k$ *and* $f \in C^{k+1}$, *then* $u \in C^k$ *and there exists a* T, $0 < T < \frac{1}{-2\tau}$, *where* $\tau = \min_{x,y}\{u_0'(x)f''(u_0(y))\} < 0$ *and a* $C = C(f, k)$, *such that for* $0 < t < T$,

$$\|u(t)\|_{C^k} \leqslant C\|u_0\|_{C^k}.$$

Proof From the implicit relation, we see that $\|u\|_{L^\infty} \leqslant \|u_0\|_{L^\infty}$. Now differentiate the implicit formula to find

$$u_x(x,t) = u_0'\big(x - tf'(u(x,t))\big)\big[1 - tf''(u(x,t))u_x(x,t)\big].$$

Solve for u_x to obtain

$$u_x(x,t) = \frac{u_0'\big(x - tf'(u(x,t))\big)}{1 + tu_0'(x - tf'(u(x,t)))f''(u(x,t))}.$$

We see that the denominator is bounded below by 1/2 for $t < \frac{1}{-2\tau}$ since $f''(\cdot)$ is bounded and u is bounded by the initial data. Continue differentiating the implicit formula for u to obtain

$$\partial_x^n u(x,t) = u_0'\big(x - tf'(u(x,t))\big)\big[1 - tf''(u(x,t))\partial_x^n u(x,t)\big] + R, 1 < n \leqslant k,$$

where R is composed of derivatives of u_0, f, and u, where the maximum number of derivatives of u_0 is n, the maximum number of derivatives of f is $n + 1$ and the maximum number of derivatives of u is $n - 1$. By assumption, these terms are bounded, and after solving for $\partial_x^n u(x,t)$, we find

$$\partial_x^n u(x,t) = \frac{u_0'\big(x - tf'(u(x,t))\big) + R}{1 + tu_0'(x - tf'(u(x,t)))f''(u(x,t))}.$$

From here we see that the result follows by induction on n.

Remark 1 Notice that the time interval of the existence depends upon the C^1 norm of u_0 and the C^2 norm of f, not the C^k norm of u_0, regardless of the space in which the initial data and solution are measured.

Remark 2 The result also holds if the C^k norm is replaced with the H^s norm, for $s > 3/2$.

We take the initial data $u_0^\omega(x) = v^\omega(x,0)$, where

$$v^\omega(x,t) = \frac{\omega}{\lambda^{1/m}} + \frac{1}{\lambda^s}\cos(\lambda x - \omega t)$$

with $\omega = 1$ or 0, and we denote by u^ω the corresponding solution to the conservation law. Note that $\|v^\omega\|_{H^k} \approx \lambda^{k-s} + \lambda^{-1/m}$.

Proof of Theorem 1 We will show the following three properties; these imply that the data-to-solution map is not uniformly continuous.

(i) For $\omega = 0, 1$, and small $t > 0$, $\lim_{\lambda\to\infty} \|u^\omega(t) - v^\omega(t)\|_{H^s} = 0$.

(ii) $\lim_{\lambda\to\infty} \left\| u_0^1 - u_0^0 \right\|_{H^s} = 0$.

(iii) $\liminf_{\lambda\to\infty} \left\| v^1 - v^0 \right\|_{H^s} \geqslant ct$ for $t > 0$.

Proof of (i). We compute $u^\omega - v^\omega$ and obtain

$$u^\omega - v^\omega = \frac{1}{\lambda^s}\left[\cos\left(\lambda x - \lambda t f'(u^\omega)\right) - \cos\left(\lambda x - \omega t\right)\right].$$

Differentiating, we find for integers $k \leqslant s$,

$$\begin{aligned}
\partial_x(u^\omega - v^\omega) =& \frac{-1}{\lambda^s}\left[\lambda \sin\left(\lambda x - \lambda t f'(u^\omega)\right)\left[1 - tf''(u^\omega)u_x^\omega\right] - \lambda \sin\left(\lambda x - \omega t\right)\right],\\
\partial_x^2(u^\omega - v^\omega) =& \frac{-1}{\lambda^s}\Big[\lambda^2 \cos\left(\lambda x - \lambda t f'(u^\omega)\right)\left[1 - tf''(u^\omega)u_x\right]^2\\
&-\lambda \sin\left(\lambda x - \lambda t f'(u^\omega)\right)\left[tf'''(u^\omega)\left(u_x^\omega\right)^2 + tf''(u^\omega)u_{xx}^\omega\right]\\
&-\lambda^2 \cos\left(\lambda x - \omega t\right)\Big],\\
&\cdots\\
|\partial_x^k(u^\omega - v^\omega)| =& \frac{1}{\lambda^s}\,|\lambda^k SC\left(\lambda x - \lambda t f'(u^\omega)\right)\left[1 - tf''(u^\omega)u_x\right]^k\\
&- \lambda^k SC(\lambda x - \omega t) + \mathcal{O}\left(\lambda^{k-1}\right)|\\
=& \frac{1}{\lambda^s}\,|\lambda^k SC\left(\lambda x - t\lambda f'(u^\omega)\right) - \lambda^k SC(\lambda x - \omega t) + \mathcal{O}\left(\lambda^{k-1}\right)|.
\end{aligned}$$

In the last equation, we introduced the notation $\mathcal{O}(\lambda^{k-1})$. We say a function $g(\lambda)$ is $\mathcal{O}(\lambda^{k-1})$ if there exists a $C \in \mathbb{R}$ such that $\lim_{\lambda\to\infty} |g(\lambda)|/\lambda^{k-1} \leqslant C$. We also used the facts that $f \in C^\infty$ in a neighborhood of 0 and for every $k \in \mathbb{N}$, $u \in C^k \cap H^k$ with $\|u\|_{C^k} \approx \lambda^{k-s}$ and $\|u\|_{H^k} \approx \lambda^{k-s}$ for $t \in [0, T]$. We use the notation $SC(x)$ to denote either $\sin(x)$ or $\cos(x)$. Using the identities

$$SC(a) - SC(b) = \pm 2 SC\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right),$$

we conclude

$$|\partial_x^k(u^\omega - v^\omega)| \leqslant \frac{2}{\lambda^{s-k}}\sin\left(\frac{\omega t - \lambda t f'(u^\omega)}{2}\right) + \mathcal{O}\left(\lambda^{-1}\right). \tag{1}$$

Now we use the assumption that $f'(u) = u^m + \mathcal{O}(u^{m+1})$ to find that the argument of the first term in (1) becomes (up to a factor of 2)

$$\omega t-\lambda t\left(\left[u^{\omega}(\lambda x-\lambda u^{\omega}t)\right]^{m}+\cdots\right)\approx t\left(\omega-\lambda\left(\frac{\omega}{\lambda^{1/m}}+\frac{\cos(\lambda x-\cdots)}{\lambda^{s}}\right)^{m}\right)$$
$$=-t\sum_{j=1}^{m}\binom{m}{j}\frac{\omega^{m-j}[\cos(\lambda x-\cdots)]^{j}}{\lambda^{j(s-1/m)}}.$$

Therefore, for large λ, and using $\sin(x)\approx x$ for $x\approx 0$, (1) becomes

$$|\,\partial_x^k(u^{\omega}-v^{\omega})\,|\lesssim\mathcal{O}\big(\lambda^{k-s}\big)\mathcal{O}\big(\lambda^{-s+1/m}\big)+\mathcal{O}\big(\lambda^{k-s}\big)\mathcal{O}\big(\lambda^{1-ms}\big)+\mathcal{O}\big(\lambda^{-1}\big).$$

Computing the $\dot{H}^k$ norm of the difference we see that

$$\left\|\partial_x^k(u^{\omega}-v^{\omega})\right\|_{L^2}\leqslant\sqrt{2\pi}\left\|\partial_x^k(u^{\omega}-v^{\omega})\right\|_{L^\infty}\lesssim\lambda^{\alpha},\tag{2}$$

where

$$\alpha=\max\{k-2s+1/m,1+k-(m+1)s,-1\}<0$$

for $k\leqslant s$. In particular, for integer values of s,

$$\lim_{\lambda\to\infty}\left\|\partial_x^s(u^{\omega}-v^{\omega})\right\|_{L^2}\to 0.$$

In the case $s\notin\mathbb{Z}$, we write $s=\lfloor s\rfloor+\beta$, $\beta\in(0,1)$, and $s>3/2$. Then in the case $s\in(3/2,2)$, from estimate (2), we have

$$\|u^{\omega}-v^{\omega}\|_{H^1}\lesssim\lambda^{-2\beta}.$$

Using the solution size estimate found in Lemma 1, we obtain for $k\geqslant s$,

$$\|u^{\omega}-v^{\omega}\|_{H^k}\lesssim\left\|u_0^{\omega}\right\|_{H^k}+\left\|v_0^{\omega}\right\|_{H^k}\approx\lambda^{k-s}.$$

Now we use real interpolation of Sobolev spaces [1] to find

$$\|u^{\omega}-v^{\omega}\|_{H^s}\lesssim\|u^{\omega}-v^{\omega}\|_{H^1}^{1-\beta}\|u^{\omega}-v^{\omega}\|_{H^2}^{\beta}\approx\lambda^{-2\beta+2\beta^2+2\beta-\beta s}=\lambda^{\beta(2\beta-s)}=\lambda^{\beta(\beta-1)}.$$

Since $\beta\in(1/2,1)$, the exponent is negative and the difference tends to zero as $\lambda\to\infty$.

If $2\leqslant\lfloor s\rfloor<s<\lfloor s\rfloor+1$, then by interpolation,

$$\|u^{\omega}-v^{\omega}\|_{H^s}\lesssim\|u^{\omega}-v^{\omega}\|_{H^{\lfloor s\rfloor}}^{1-\beta}\|u^{\omega}-v^{\omega}\|_{H^{\lfloor s\rfloor+1}}^{\beta}\approx\lambda^{1+\lfloor s\rfloor+\beta s-2s}=\lambda^{(1-\beta)(1-s)},$$

which again tends to zero as $\lambda\to\infty$.

Proof of (ii). The fact that $\lim_{\lambda\to\infty}\left\|u_0^1-u_0^0\right\|_{H^s}=0$ follows immediately from $u_0^1-u_0^0=\frac{1}{\lambda^{1/m}}$.

Proof of (iii). Using the difference of cosines, we compute

$$\left\|v^1-v^0\right\|_{H^s}\geqslant\left\|\frac{1}{\lambda^s}\sin\left(\frac{t}{2}\right)\sin\left(\lambda x-\frac{t}{2}\right)\right\|_{H^s}-\left\|\frac{1}{\lambda^{1/m}}\right\|_{H^s}.$$

Using $\|\sin(\lambda x-a)\|_{H^s}\approx\lambda^s$, we have

$$\left\|v^1-v^0\right\|_{H^s}\geqslant c\sin\left(\frac{t}{2}\right)-\frac{1}{\lambda^{1/m}},$$

and the theorem now follows from taking the limit $\lambda\to\infty$.

One may notice that our computations did not rely on the Sobolev norm. Indeed, one may replace the Sobolev space with the classical space $C^k(\mathbb{T})$ in the above computations and obtain the following result.

Corollary 1 *For all integers $k \geqslant 1$, the data-to-solution map is not uniformly continuous from $C^k(\mathbb{T})$ to $C([0,T];\ C^k(\mathbb{T}))$.*

2.1 The Non-periodic Case

The case when $x \in \mathbb{R}$ is similar, though some technical adjustments must be made to the approximating functions $v^{\omega}(x,t)$. We let $0 \leqslant \varphi(x)$ be a smooth function with support in $[-2,2]$, and equal to 1 when $x \in [-1,1]$. Since we see that the degree m of the nonlinearity in f does not matter, we will assume for the sake of simplicity that $m=1$, and that we have $f'(u) = u + \mathcal{O}(u^2)$. For a large integer λ and $\omega = 1, 0$, we set

$$v^{\omega}(x,t) = \frac{\omega}{\lambda}\varphi\Big(\frac{x}{\lambda^q}\Big) + \frac{1}{\lambda^{s+q/2}}\varphi\Big(\frac{x}{\lambda^q}\Big)\cos\Big(\lambda x - \omega t \varphi\Big(\frac{x}{\lambda^q}\Big)\Big)$$

with $q \in (0,1)$ being a small number, and we take $u_0^{\omega}(x) = v^{\omega}(x,0)$. The H^k norm of the solution is bounded by

$$\|u^{\omega}(t)\|_{H^k} \leqslant \frac{\omega\|\varphi\|_{H^k}}{\lambda^{1-q/2}} + \lambda^{k-s}\|\varphi\|_{H^k},$$

where we used that for any $s \geqslant 0$, $\|\varphi(\cdot\lambda^{-q})\|_{H^k} \approx \lambda^{q/2}\|\varphi\|_{H^k}$, and for any constant $a \in \mathbb{R}$,

$$\left\|\varphi\Big(\frac{\cdot}{\lambda^q}\Big)\cos(\lambda\cdot - a)\right\|_{H^k} \approx \lambda^{k+q/2}\|\varphi\|_{H^k}.$$

Proof of Theorem 2 Proceeding as we did in the periodic case, we will show the following three properties.

(i) For $\omega = 0, 1$, and small $t > 0$, $\lim\limits_{\lambda\to\infty}\|u^{\omega}(t) - v^{\omega}(t)\|_{H^s} = 0$.

(ii) $\lim\limits_{\lambda\to\infty}\left\|u_0^1 - u_0^0\right\|_{H^s} = 0$.

(iii) $\liminf\limits_{\lambda\to\infty}\left\|v^1 - v^0\right\|_{H^s} \geqslant ct$ for $t > 0$.

Proof of (i). We compute $u^{\omega} - v^{\omega}$ and obtain

$$
\begin{aligned}
u^\omega - v^\omega = \frac{\omega}{\lambda}&\left[\varphi\left(\lambda^{-q}x - t\lambda^{-q}f'(u^\omega)\right) - \varphi(\lambda^{-q}x)\right] \\
&+ \frac{1}{\lambda^{s+q/2}}\left[\varphi\left(\lambda^{-q}x - t\lambda^{-q}f'(u^\omega)\right)\cos\left(\lambda x - t\lambda f'(u^\omega)\right)\right. \\
&\left.-\varphi(\lambda^{-q}x)\cos(\lambda x - \omega t\varphi(\lambda^{-q}x))\right].
\end{aligned}
$$

Differentiating, we find

$$
\begin{aligned}
\partial_x(u^\omega - v^\omega) =&\frac{\omega}{\lambda^{1+q}}\left[\varphi'\left(\lambda^{-q}x - t\lambda^{-q}f'(u^\omega)\right)\left(1 - tf''(u^\omega)u^\omega_x\right) - \varphi'(\lambda^{-q}x)\right] \\
&+ \frac{1}{\lambda^{s+3q/2}}\left[\varphi'\left(\lambda^{-q}x - t\lambda^{-q}f'(u^\omega)\right)\left(1 - tf''(u^\omega)u^\omega_x\right)\cos\left(\lambda x - t\lambda f'(u^\omega)\right)\right. \\
&\left.-\varphi'(\lambda^{-q}x)\cos(\lambda x - \omega t\varphi(\lambda^{-q}x))\right] \\
&+ \frac{1}{\lambda^{s-1+q/2}}\left[\varphi\left(\lambda^{-q}x - t\lambda^{-q}f'(u^\omega)\right)\sin\left(\lambda x - t\lambda f'(u^\omega)\right)\left(1 - tf''(u^\omega)u^\omega_x\right)\right. \\
&\left.-\ \varphi(\lambda^{-q}x)\sin(\lambda x - \omega t\varphi(\lambda^{-q}x))\left(1 - \lambda^{-q-1}\omega t\varphi'(\lambda^{-q}x)\right)\right].
\end{aligned}
$$

We rearrange the terms as follows:

$$
\partial_x(u^\omega - v^\omega) =\frac{\omega}{\lambda^{1+q}}\left[\varphi'\left(\lambda^{-q}x - t\lambda^{-q}f'(u^\omega)\right) - \varphi'(\lambda^{-q}x)\right] \tag{3}
$$

$$
- \frac{\omega tf''(u^\omega)u^\omega_x}{\lambda^{1+q}}\left[\varphi'\left(\lambda^{-q}x - t\lambda^{-q}f'(u^\omega)\right)\right] \tag{4}
$$

$$
\begin{aligned}
&+ \frac{1}{\lambda^{s+3q/2}}\left[\varphi'\left(\lambda^{-q}x - t\lambda^{-q}f'(u^\omega)\right)\cos\left(\lambda x - t\lambda f'(u^\omega)\right)\right. \\
&\left.-\varphi'(\lambda^{-q}x)\cos(\lambda x - \omega t\varphi(\lambda^{-q}x))\right] \\
&+ \frac{\omega t}{\lambda^{s+3q/2}}\left[\varphi\left(\lambda^{-q}x\right)\sin\left(\lambda x - \omega t\varphi\left(\lambda^{-q}x\right)\right)\varphi'\left(\lambda^{-q}x\right)\right]
\end{aligned} \tag{5}
$$

$$
- \frac{tf''(u^\omega)u^\omega_x}{\lambda^{s+3q/2}}\left[\varphi'\left(\lambda^{-q}x - t\lambda^{-q}f'(u^\omega)\right)\cos\left(\lambda x - t\lambda f'(u^\omega)\right)\right] \tag{6}
$$

$$
\begin{aligned}
&+ \frac{1}{\lambda^{s-1+q/2}}\left[\varphi\left(\lambda^{-q}x - t\lambda^{-q}f'(u^\omega)\right)\sin\left(\lambda x - t\lambda f'(u^\omega)\right)\right. \\
&\left.-\varphi(\lambda^{-q}x)\sin(\lambda x - \omega t\varphi(\lambda^{-q}x))\right]
\end{aligned} \tag{7}
$$

$$
- \frac{tf''(u^\omega)u^\omega_x}{\lambda^{s-1+q/2}}\left[\varphi\left(\lambda^{-q}x - t\lambda^{-q}f'(u^\omega)\right)\sin\left(\lambda x - t\lambda f'(u^\omega)\right)\right]. \tag{8}
$$

Since the derivatives of φ are bounded, and since the L^∞ norm of the solution u^ω is bounded, we obtain estimates of each of the six lines in this equation as

$$\begin{aligned}\left\|\partial_x(u^\omega - v^\omega)\right\|_{L^2} \leqslant & \frac{\omega C_\phi}{\lambda^{1+q/2}} + \frac{\omega t C_{\phi,f}}{\lambda^{1+q}}\|u^\omega\|_{H^1} + \frac{C_\phi(1+\omega t)}{\lambda^{s+q}} \\ & + \frac{tC_{\phi,f}}{\lambda^{s+3q/2}}\|u^\omega\|_{H^1} + \frac{C_\phi}{\lambda^{s-1}} + \frac{tC_{\phi,f}}{\lambda^{s-1+q/2}}\|u^\omega\|_{H^1}.\end{aligned}$$

Therefore, we can recognize that after differentiating k times, the terms with slowest decay in λ are the terms which contributed to the fifth term, $\frac{C_\phi}{\lambda^{s-1}}$. In other words, from line (7) above,

$$\begin{aligned}\partial_x^k(u^\omega - v^\omega) = & \frac{1}{\lambda^{s-k+q/2}}\left[\varphi\big(\lambda^{-q}x - t\lambda^{-q}f'(u^\omega)\big)SC\big(\lambda x - t\lambda f'(u^\omega)\big)\right. \\ & \left.-\varphi(\lambda^{-q}x)SC(\lambda x - \omega t\varphi(\lambda^{-q}x))\right] \\ & + \lambda^{-\alpha}P\big(\varphi', \cdots, \phi^{[k]}, \partial_x u^\omega, \cdots, \partial_x^k u^\omega\big),\end{aligned} \tag{9}$$

where $\alpha > 0$ and the L^2 norm of P is bounded independent of λ. We add and subtract

$$\varphi\big(\lambda^{-q}x - t\lambda^{-q}f'(u^\omega)\big)SC(\lambda x - \omega t\varphi(\lambda^{-q}x))$$

inside the brackets to obtain for $\partial_x^k(u^\omega - v^\omega)$:

$$\begin{aligned}&\frac{1}{\lambda^{s-k+q/2}}\left[\varphi\Big(\frac{x}{\lambda^q} - \frac{t}{\lambda^q}f'(u^\omega)\Big)\Big(SC\big(\lambda x - t\lambda f'(u^\omega)\big) - SC\Big(\lambda x - \omega t\varphi\Big(\frac{x}{\lambda^q}\Big)\Big)\Big)\right. \\ &+ \left.\Big(\varphi\Big(\frac{x}{\lambda^q} - \frac{t}{\lambda^q}f'(u^\omega)\Big) - \varphi\Big(\frac{x}{\lambda^q}\Big)\Big)SC\Big(\lambda x - \omega t\varphi\Big(\frac{x}{\lambda^q}\Big)\Big)\right] \\ &+ \lambda^{-\alpha}P\big(\varphi', \cdots, \phi^{[k]}, \partial_x u^\omega, \cdots, \partial_x^k u^\omega\big).\end{aligned} \tag{10}$$

The term on the second line of (10) is bounded, since it is a shift:

$$\left\|\varphi\big(\lambda^{-q}x - t\lambda^{-q}f'(u^\omega)\big) - \varphi\big(\lambda^{-q}x\big)\right\|_{L^2} \lesssim \lambda^{-1-q/2}.$$

For the first line of expression (10), we use the difference of cosines or sines to find

$$SC\big(\lambda x - \lambda f'(u^\omega)t\big) - SC\Big(\lambda x - \omega t\varphi\Big(\frac{x}{\lambda^q}\Big)\Big) \lesssim \sin\Big(\frac{1}{2}\Big(\omega t\varphi\Big(\frac{x}{\lambda^q}\Big) - \lambda f'(u^\omega)t\Big)\Big).$$

Using the implicit solution formula, the $\omega t\varphi$ cancels and we have the bound

$$\left\|SC\big(\lambda x - \lambda f'(u^\omega)t\big) - SC\Big(\lambda x - \omega t\varphi\Big(\frac{x}{\lambda^q}\Big)\Big)\right\|_\infty \lesssim \frac{\omega t^2}{2\lambda^q} + \cdots,$$

where the dots represent a function of u^ω. From here, one can see that for large λ, the L^2 norm of the first term of expression (10) also tends to zero at the rate λ^{k-s-q}.

The above argument is sufficient to conclude that for $\omega = 0, 1$, and small $t > 0$, $\lim\limits_{\lambda\to\infty}\|u^\omega(t) - v^\omega(t)\|_{H^s} = 0$ when $s = k$. When s is not an integer, the interpolation argument presented in the periodic case can be augmented without any adjustments. This proves the first property in the non-periodic case.

Proof of (ii). The fact that $\lim\limits_{\lambda\to\infty}\left\|u_0^1 - u_0^0\right\|_{H^s} = 0$ follows immediately from $u_0^1 - u_0^0 = \varphi(x/\lambda^q)/\lambda$.

Proof of (iii). Using the difference of cosines, we compute

$$\left\|v^1 - v^0\right\|_{H^s} \geqslant \left\| \frac{1}{\lambda^{s+q/2}} \varphi\left(\frac{x}{\lambda^q}\right) \sin\left(\frac{1}{2} t\varphi\left(\frac{x}{\lambda^q}\right)\right) \sin\left(\lambda x - \frac{1}{2} t\varphi\left(\frac{x}{\lambda^q}\right)\right) \right\|_{H^s} - \left\| \frac{1}{\lambda} \varphi\left(\frac{x}{\lambda^q}\right)\right\|_{H^s}.$$

Using the estimate $\left\|\varphi\left(\frac{x}{\lambda^q}\right)\sin(\lambda x)\right\|_{H^s} \approx \lambda^{s+q/2}$, we have

$$\liminf_{\lambda\to\infty} \left\|v^1 - v^0\right\|_{H^s} \geqslant C \sin\left(\frac{1}{2}t\right).$$

This completes the proof of Theorem 2.

3 Systems of Conservation Laws

All strictly hyperbolic genuinely nonlinear systems of two conservation laws in one space variable possess Riemann invariants, which satisfy transport equations. Classical solutions of the Riemann invariant equations correspond to classical solutions of the original conservation laws. Since smooth initial data yield the existence of a smooth solution, locally in time, we may work completely with Riemann invariants, and write the system as

$$\begin{cases} z_t + \nu(z, w) z_x = 0, \\ w_t + \mu(z, w) w_x = 0 \end{cases} \tag{11}$$

with $\mu \neq \nu$. This is paired with the initial data $z(x, 0) = z_0$ and $w(x, 0) = w_0$. We take the following steps.

(i) Translate the state variables $(z, w) \mapsto (z - z_0, w - w_0)$, so that $(0, 0)$ is a physical state.
(ii) Scale the independent variables $(x, t) \mapsto (x + ct, kt)$, so that $\nu(0, 0) = 0$ and $\nu_z(0, 0) = 1$. This can be done by the assumption of the genuine nonlinearity in the ν characteristic. Since the two characteristics are interchangeable in this development, we need assume only that there is one genuinely nonlinear family[1].
(iii) Take the initial data $w_0 = 0$ and $z_0 = \frac{\omega}{\lambda} + \frac{1}{\lambda}\cos(\lambda x)$.
(iv) It is easy to see that one of the solutions is $w(x, t) = 0$.
(v) The first equation is now $z_t + (z + g(z)) z_x = 0$ with $| g(z) | \leqslant \kappa z^2$.
(vi) The arguments from the previous two sections now show that the data-to-solution map $z_0 \mapsto z$ is not uniformly continuous in H^s.

When there are more than two equations, typically Riemann invariants do not exist, as the defining conditions lead to an overdetermined system. An example of a physical system lacking Riemann invariants is the system of three equations for compressible, ideal gas dynamics:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \left(\rho u^2 + p\right)_x = 0, \\ (\rho E)_t + (\rho u E + u p)_x = 0 \end{cases} \tag{12}$$

with $E = u^2/2 + p/(\gamma - 1)\rho$, with the density, velocity, and pressure, (ρ, u, p), the state variables and γ the ratio of specific heats. As is well known, classical solutions also satisfy conservation of entropy,

[1] There are situations where this fails, for example the system $z_t + w z_x = 0$, $w_t + z w_x = 0$, for which one conservation law system is the isentropic gas dynamics system with $p(\rho) = P_0 - \rho^{-3}/3$.

$$(\rho S)_t + (\rho u S)_x = 0 \quad \text{with} \quad S = \log p - \gamma \log \rho,$$

and also an entropy transport equation, $S_t + uS_x = 0$, which has $S =$ const. as a solution. It is also straightforward to show that if one assumes the isentropic gas dynamics relation, $p = A\rho^\gamma$, then any solution of the first two equations in (12) satisfies the third. Thus, we may consider solutions of the first two equations of (12). The Riemann invariants for this system, with $p = \rho^\gamma/\gamma$, are $z, w = u \mp 2\rho^{(\gamma+1)/2}/(\gamma + 1)$. If we set $u = -2\rho^{(\gamma+1)/2}/(\gamma + 1)$, so $w = 0$, then the scalar equation $z_t + F(z)_x = 0$ displays non-uniform dependence on the initial data in a neighborhood of any constant density ρ_0.

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Compliance with Ethical Standards

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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