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ORIGINAL PAPER

Finite Difference Schemes for Time-Space Fractional Diffusion Equations in One- and Two-Dimensions

Yu Wang[1] **· Min Cai**[1]

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Abstract
In this paper, finite difference schemes for solving time-space fractional diffusion equations in one dimension and two dimensions are proposed. The temporal derivative is in the Caputo-Hadamard sense for both cases. The spatial derivative for the one-dimensional equation is of Riesz definition and the two-dimensional spatial derivative is given by the fractional Laplacian. The schemes are proved to be unconditionally stable and convergent. The numerical results are in line with the theoretical analysis.

Keywords Time-space fractional diffusion equation · Caputo-Hadamard derivative · Riesz derivative · Fractional Laplacian · Numerical analysis

Mathematics Subject Classification 35R11 · 65M06

1 Introduction

Fractional calculus has been applied to numerous fields such as fluid mechanics, physics, chemistry, epidemiology, and finance during the last decades [3, 15, 23, 24, 28], to characterize memory effects and/or nonlocality. Fractional models are more suitable than integer models for systems with memory and long-term interactions. The anomalous diffusion equation is a class of important fractional differential equations, which has been widely applied in modeling of random walk, unification of diffusion and wave propagation, etc. [1, 19]. In some scenarios, anomalous diffusion can be described by the time-space fractional diffusion equation.

In this paper, we first consider a numerical method for the following one-dimensional time-space fractional diffusion equation:

Yu Wang and Min Cai have contributed equally to this work.

✉ Min Cai
mcai@shu.edu.cn

[1] Department of Mathematics, Shanghai University, Shanghai 200444, China

$$\begin{cases} {}_{\mathrm{CH}}\mathrm{D}^{\alpha}_{\tilde{a},t}u(x,t) - {}_{\mathrm{RZ}}\mathrm{D}^{\beta}_{x}u(x,t) = f(x,t), & (x,t) \in \Omega \times (\tilde{a},T], \\ u(a,t) = u(b,t) = 0, & t \in (\tilde{a},T], \\ u(x,\tilde{a}) = u_0(x), & x \in \Omega \end{cases} \tag{1}$$

with $\alpha \in (0,1)$, $\beta \in (1,2)$, $\tilde{a} > 0$, $\Omega = [a,b] \subset \mathbb{R}$ being a bounded interval, and $u_0(x)$ being a given initial condition. Here ${}_{\mathrm{CH}}\mathrm{D}^{\alpha}_{\tilde{a},t}$ denotes the Caputo-Hadamard differentiation operator defined by

$${}_{\mathrm{CH}}\mathrm{D}^{\alpha}_{\tilde{a},t}\varphi(t) = \frac{1}{\Gamma(n-\alpha)}\int_{\tilde{a}}^{t}\left(\log\frac{t}{s}\right)^{n-\alpha-1}\delta^n\varphi(s)\frac{\mathrm{d}s}{s}, \quad 0 < \tilde{a} < t,$$

where

$$\delta^n\varphi(s) = \left(s\frac{\mathrm{d}}{\mathrm{d}s}\right)^n\varphi(s), \ n-1 < \alpha < n \in \mathbb{Z}^+.$$

The sufficient condition for the existence of the Caputo-Hadamard derivative ${}_{\mathrm{CH}}\mathrm{D}^{\alpha}_{\tilde{a},t}\varphi(t)$ is that $\varphi(t) \in AC^n_{\delta}[\tilde{a},T] = \left\{\varphi\colon \delta^{n-1}\varphi \in AC[\tilde{a},T]\right\}$ with $AC(\Omega)$ denoting the space of absolute continuous functions. The spatial derivative is the Riesz derivative of order β $(1 < \beta < 2)$ given by

$${}_{\mathrm{RZ}}\mathrm{D}^{\beta}_{x}\psi(x) = -\frac{1}{2\cos(\pi\beta/2)}\left[{}_{\mathrm{RL}}\mathrm{D}^{\beta}_{a,x}\psi(x) + {}_{\mathrm{RL}}\mathrm{D}^{\beta}_{x,b}\psi(x)\right],$$

where

$${}_{\mathrm{RL}}\mathrm{D}^{\beta}_{a,x}\psi(x) = \frac{1}{\Gamma(m-\beta)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\int_a^x (x-s)^{m-1-\beta}\psi(s)\mathrm{d}s, \ m-1 < \beta < m \in \mathbb{Z}^+,$$

and

$${}_{\mathrm{RL}}\mathrm{D}^{\beta}_{x,b}\psi(x) = \frac{(-1)^m}{\Gamma(m-\beta)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\int_x^b (s-x)^{m-1-\beta}\psi(s)\mathrm{d}s, \ m-1 < \beta < m \in \mathbb{Z}^+$$

are the left-hand side and right-hand side Riemann-Liouville fractional derivatives. Sufficient condition for the existence of ${}_{\mathrm{RZ}}\mathrm{D}^{\beta}_{x}\psi(x)$ is that $\psi(x) \in AC^{\lceil\beta\rceil}[a,b] = \{\psi\colon \psi^{(k)} \in AC[a,b], k = 0,1,\dots,\lfloor\beta\rfloor\}$.

We also consider the following two-dimensional time-space fractional diffusion equation:

$$\begin{cases} {}_{\mathrm{CH}}\mathrm{D}^{\alpha}_{\tilde{a},t}u(x,y,t) + (-\Delta)^{\frac{\beta}{2}}u(x,y,t) = f(x,y,t), & (x,y) \in \widetilde{\Omega}, t \in (\tilde{a},T], \\ u(x,y,t) = 0, & (x,y) \in \mathbb{R}^2\backslash\widetilde{\Omega}, t \in (\tilde{a},T], \\ u(x,y,\tilde{a}) = u_0(x,y), & (x,y) \in \widetilde{\Omega} \end{cases} \tag{2}$$

with $\alpha \in (0,1)$, $\beta \in (1,2)$, $\tilde{a} > 0$, $\widetilde{\Omega} = (-L,L)^2 \subset \mathbb{R}^2$ being a bounded domain, and $u_0(x,y)$ being a given initial condition. Here $(-\Delta)^{\frac{\beta}{2}}$ is the fractional Laplacian defined by the hyper-singular integral [16],

$$(-\Delta)^{\frac{\beta}{2}}\psi(\mathbf{x}) = c_\beta \mathrm{P.V.}\int_{\mathbb{R}^2}\frac{\psi(\mathbf{x})-\psi(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^{2+\alpha}}\mathrm{d}\mathbf{z}, \ c_\beta = \frac{2^\beta\Gamma(1+\beta/2)}{\pi|\Gamma(-\beta/2)|}, \tag{3}$$

where $\mathbf{x} = (x, y) \in \mathbb{R}^2$ and P.V. stands for the Cauchy principal value. One sufficient condition for the existence of the fractional Laplacian $(-\Delta)^{\frac{\beta}{2}}\psi(\mathbf{x})$ is that $\psi(\mathbf{x})$ belongs to the following Schwartz space:

$$\mathcal{S} = \left\{ \psi \in \mathrm{C}^\infty(\mathbb{R}^2)\colon \sup_{\mathbf{x}\in\mathbb{R}^2}(1 + |\mathbf{x}|)^N \sum_{k=0}^{N} |\mathrm{D}^k\psi(\mathbf{x})| < \infty, N = 0, 1, 2, \cdots \right\}.$$

There are some researches on numerical methods for time-space fractional diffusion equations. Liu et al. [20] proposed a first-order implicit finite difference scheme to solve the fractional diffusion equation with the temporal Caputo derivative and the spatial Riemann-Liouville derivative on a bounded domain in one spatial dimension. Cao and Li [4] derived two finite difference schemes for two kinds of time-space fractional diffusion equations by approximating Riemann-Liouville fractional derivatives with the second-order accuracy via the weighted and shifted Grünwald-Letnikov formula. Arshad et al. [2] constructed a numerical scheme for the time-space fractional diffusion equation with the second-order accuracy in both time and space directions, where the temporal and spatial fractional derivatives are in the senses of Caputo and Riesz, respectively. A finite difference scheme was developed for the time-space fractional diffusion equation with Dirichlet fractional boundary conditions in the work of Xie and Fang [29], where the fractional derivatives include the temporal Caputo derivative and the spatial Riemann-Liouville derivative.

In view of the aforementioned studies, some numerical schemes have been proposed for the time-space fractional diffusion equation. However, the temporal derivative is mainly given by the Caputo derivative which is adequate for characterizing algebraic decay. In this paper, the temporal derivative in considered time-space fractional diffusion equations is in the Caputo-Hadamard sense which is suitable in describing the ultra slow process [5, 12, 17]. For the numerical approximation to the Caputo-Hadamard derivative, Gohar et al. [13] introduced the L1 formula for the temporal Caputo-Hadamard derivative to deduce a semi-discrete difference scheme, and gave the stability and convergence analysis. Fan et al. [11] proposed three numerical formulae for the Caputo-Hadamard derivative of order α with the $(3-\alpha)$ order accuracy, including L1-2 and L2-1σ formulae for the case with $\alpha \in (0, 1)$, and H2N2 formula for the case with $\alpha \in (1, 2)$. Li et al. [18] proposed and numerically analyzed an LDG scheme for the Caputo-Hadamard fractional sub-diffusion equation. Ou et al. [22] investigated the numerical scheme for the Caputo-Hadamard fractional diffusion-wave equation using exponential type meshes. In the aforementioned two works, the spatial derivative is in the sense of classical Laplacian.

The Riesz derivative is in the form of a linear combination of a left Riemann-Liouville derivative and a right Riemann-Liouville derivative, which allows the modeling of flow regime impacts from either side of the domain [30, 31]. In the mean while, the fractional Laplacian was frequently adopted to take long-range interaction in higher dimensions into account [27]. Therefore, the spatial derivative is chosen as the Riesz derivative in one dimension and the fractional Laplacian in two dimensions in the present paper. Based on the numerical method of approximating the Riemann-Liouville derivative, several numerical approximations for evaluating the Riesz fractional derivative were proposed, such as the spline interpolation method [25], standard Grünwald-Letnikov formula and its modifications [21, 26]. In particular, a series of high order algorithms for Riesz derivatives were constructed by Ding et al. [6–9]. Since the linear combination of shifted Grünwald-Letnikov formulae with different displacements and appropriate weights can evaluate the Riemann-Liouville derivative with the higher order accuracy and the resulting finite difference schemes for time dependent problems are stable, we choose weighted and shifted Grünwald-Letnikov formula for the Riesz derivative. For the fractional Laplacian, obtaining numerical approximations is still difficult and hot. A recent work given by Hao et al. [14] proposed a fractional centered difference formula. It generates a symmetric block Toeplitz matrix with Toeplitz blocks which enables us to develop fast and efficient algorithms by fast Fourier transform. This novel approximation technique is adopted for solving the two-dimensional nonlinear Schrödinger equation with the fractional Laplacian [27], where the temporal derivative is still of integer order.

Numerical methods for partial differential equations with the temporal Caputo derivative and the spatial fractional derivative are rare. This situation and potential applications of ultra slow diffusion motivate us to study numerical algorithms for (1) and (2).

The remaining part of this paper is organized as follows. Numerical approximations adopted in this paper for evaluating the Caputo-Hadamard derivative, Riesz derivative, and fractional Laplacian are shown in Sect. 2, along with corresponding properties. Fully discrete schemes for the one-dimensional time-space fractional diffusion equation (1) and the two-dimensional equation (2) are derived in Sect. 3. Rigorous stability analysis and error estimates are discussed as well. Numerical simulations in Sect. 4 verify the feasibility of the proposed numerical schemes and the theoretical analysis.

2 Preliminaries

In this section, we introduce approximations of the Caputo-Hadamard derivative, Riesz derivative in one dimension, and integral fractional Laplacian in higher dimensions that are applied in constructing numerical schemes for (1) and (2). In the following discussion, the L1 formula for the Caputo-Hadamard derivative [13], the weighted and shifted Grünwald-Letnikov formula for the Riesz derivative [26], and the fractional centered difference formula for the fractional Laplacian [14] are adopted.

2.1 L1 Formula for Caputo-Hadamard Derivative

Let $t_k = \tilde{a} + k\tau$ with $k = 0, 1, \cdots, N\,(N \in \mathbb{Z}^+)$, where $\tau = (T - \tilde{a})/N$ is the time step. For $\varphi(t) \in C^2[\tilde{a}, T]$, its Caputo-Hadamard derivative of order $\alpha \in (0, 1)$ at $t = t_k$ can be evaluated by the following L1 approximation [13]:

$$ {}_{\mathrm{CH}}\mathrm{D}^{\alpha}_{\tilde{a},t}\varphi(t)|_{t=t_k} = \sum_{i=1}^{k} c_{i,k}^{(\alpha)} \left[\varphi(t_i) - \varphi(t_{i-1})\right] + R^k_{\mathrm{CH}}, \tag{4} $$

where

$$ c_{i,k}^{(\alpha)} = \frac{1}{\Gamma(2-\alpha)} \frac{1}{\log \frac{t_i}{t_{i-1}}} \left[\left(\log \frac{t_k}{t_{i-1}}\right)^{1-\alpha} - \left(\log \frac{t_k}{t_i}\right)^{1-\alpha} \right], \tag{5} $$

and

$$R_{\text{CH}}^k = \frac{1}{\Gamma(1-\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_k}{s}\right)^{-\alpha} \left(\delta\varphi(s) - \frac{\varphi(t_i) - \varphi(t_{i-1})}{\log \frac{t_i}{t_{i-1}}} \right) \frac{\mathrm{d}s}{s}. \tag{6}$$

Lemma 1 [13] *For* $0 < \alpha < 1$, *the coefficients* $c_{i,k}^{(\alpha)}$ $(1 \leqslant i \leqslant k,\ 1 \leqslant k \leqslant N)$ *given by* (5) *satisfy*

$$c_{k,k}^{(\alpha)} > c_{k-1,k}^{(\alpha)} > \cdots > c_{i,k}^{(\alpha)} > c_{i-1,k}^{(\alpha)} > \cdots > c_{1,k}^{(\alpha)} > 0.$$

Remark 1 Let $0 < \alpha < 1$ and the coefficients $c_{i,k}^{(\alpha)}$ $(1 \leqslant i \leqslant k,\ 1 \leqslant k \leqslant N)$ be defined by (5). There holds $\frac{1}{c_{1,k}^{(\alpha)}} < \frac{\Gamma(1-\alpha)}{\tilde{a}^\alpha}(k\tau)^\alpha$.

Proof According to the mean value theorem,

$$\begin{aligned} c_{1,k}^{(\alpha)} &= \frac{1}{\Gamma(2-\alpha)} \frac{1}{\log \frac{t_1}{t_0}} \left(\left(\log \frac{t_k}{t_0}\right)^{1-\alpha} - \left(\log \frac{t_k}{t_1}\right)^{1-\alpha} \right) \\ &= \frac{1}{\Gamma(2-\alpha)} \frac{1}{\log \frac{t_1}{t_0}} (1-\alpha)\xi^{-\alpha} \log \frac{t_1}{t_0} \\ &= \frac{1}{\Gamma(1-\alpha)} \xi^{-\alpha},\ \xi \in \left(\log \frac{t_k}{t_1}, \log \frac{t_k}{t_0}\right). \end{aligned}$$

As $t_0 = \tilde{a}$, we have

$$c_{1,k}^{(\alpha)} > \frac{1}{\Gamma(1-\alpha)} \left(\log \frac{t_k}{t_0}\right)^{-\alpha} = \frac{\left(\log\left(1 + \frac{k\tau}{\tilde{a}}\right)\right)^{-\alpha}}{\Gamma(1-\alpha)} > \frac{\left(\frac{k\tau}{\tilde{a}}\right)^{-\alpha}}{\Gamma(1-\alpha)}.$$

In other words,

$$\frac{1}{c_{1,k}^{(\alpha)}} < \frac{\Gamma(1-\alpha)}{\tilde{a}^\alpha}(k\tau)^\alpha.$$

The proof is thus completed.

Lemma 2 [13] *If* $0 < \alpha < 1$ *and* $\varphi(t) \in C^2[\tilde{a}, T]$, *then the local truncation error* R_{CH}^k $(1 \leqslant k \leqslant N)$ *in* (6) *has the following estimate*:

$$\begin{aligned} \left|R_{\text{CH}}^k\right| \leqslant& \left(\frac{1}{\Gamma(2-\alpha)} \left(\log \frac{t_k}{t_{k-1}}\right)^2 + \frac{1}{\Gamma(1-\alpha)} \max_{1\leqslant l\leqslant N} \left(\log \frac{t_l}{t_{l-1}}\right)^2 \right) \\ &\times \left(\log \frac{t_k}{t_{k-1}}\right)^{-\alpha} \max_{\tilde{a}\leqslant t\leqslant t_k} \left|\delta^2\varphi(t)\right|. \end{aligned}$$

Remark 2 [13] The local truncation error given by (6) is bounded in the following sense:

$$\left|R_{\text{CH}}^k\right| \leqslant C\tau^{2-\alpha} \tag{7}$$

with $C > 0$ being a constant independent of the temporal stepsize τ.

2.2 Weighted and Shifted Grünwald-Letnikov Formula for Riesz Derivative

Let $x_j = a + jh,\ j = 0, 1, \cdots, M,$ where $h = (b-a)/M$ is the spatial stepsize. Define grid function spaces $\mathcal{U}_h = \{w \mid w = (w_0, w_1, \cdots, w_M)\}$ and $\mathcal{U}_h^\circ = \{w \mid w \in \mathcal{U}_h, w_0 = w_M = 0\}$. For any grid functions $w = \{w_j\}$ and $v = \{v_j\}$ in $\mathcal{U}_h^\circ$, a discrete inner product and the associated norm are defined as

$$(w, v)_h = h \sum_{j=1}^{M-1} w_j v_j, \quad \|w\|_h^2 = (w, w)_h.$$

Let $\psi(x)$ and ${}_{\mathrm{RL}}\mathrm{D}_{a,x}^{\beta}\psi(x)$, ${}_{\mathrm{RL}}\mathrm{D}_{x,b}^{\beta}\psi(x)$ and its Fourier transform belong to $L^1(\mathbb{R})$. The Riesz derivative of $\psi(x)$ at $x = x_j$ $(1 \leqslant j \leqslant M-1)$ can be approximated by the following weighted and shifted Grünwald-Letnikov formula [26]:

$$\begin{aligned} {}_{\mathrm{RZ}}\mathrm{D}_x^{\beta}\psi(x_j) = -\frac{\Psi_\beta}{h^\beta}\Bigg[& v_1 \sum_{k=0}^{j+l_1} g_k^{(\beta)} \psi(x_{j-k+l_1}) + v_2 \sum_{k=0}^{j+l_2} g_k^{(\beta)} \psi(x_{j-k+l_2}) \\ & + v_1 \sum_{k=0}^{M-j+l_1} g_k^{(\beta)} \psi(x_{j+k-l_1}) + v_2 \sum_{k=0}^{M-j+l_2} g_k^{(\beta)} \psi(x_{j+k-l_2}) \Bigg] + \mathcal{O}(h^2), \end{aligned} \tag{8}$$

where $\Psi_\beta = \frac{1}{2\cos(\frac{\pi\beta}{2})}$, $g_k^{(\beta)} = (-1)^k \begin{pmatrix} \beta \\ k \end{pmatrix}$, and $v_1 = \frac{\beta - 2l_2}{2(l_1 - l_2)}$, $v_2 = \frac{2l_1 - \beta}{2(l_1 - l_2)}$, $l_1 \neq l_2$. In particular, the coefficients $g_k^{(\beta)}$ can be computed via the following recursive formula:

$$g_0^{(\beta)} = 1,\ g_k^{(\beta)} = \left(1 - \frac{\beta+1}{k}\right) g_{k-1}^{(\beta)},\ k = 1, 2, \cdots.$$

Lemma 3 [23, 26] *For* $1 < \beta < 2$, *the coefficients* $g_k^{(\beta)}$ $(k \geqslant 0)$ *in* (8) *satisfy*

$$\begin{cases} g_0^{(\beta)} = 1, g_1^{(\beta)} = -\beta, \\ 1 \geqslant g_2^{(\beta)} \geqslant g_3^{(\beta)} \geqslant \cdots \geqslant 0, \\ \sum\limits_{k=0}^{\infty} g_k^{(\beta)} = 0, \sum\limits_{k=0}^{m} g_k^{(\beta)} < 0,\ m \geqslant 1. \end{cases} \tag{9}$$

In the following discussion, we choose $(l_1, l_2) = (1, 0)$ in (8) and the resulting approximation reads

$${}_{\mathrm{RZ}}\mathrm{D}_x^{\beta}\psi(x_j) = -\frac{\Psi_\beta}{h^\beta}\left(\sum_{k=0}^{j+1} w_k^{(\beta)} \psi(x_{j-k+1}) + \sum_{k=0}^{M-j+1} w_k^{(\beta)} \psi(x_{j+k-1}) \right) + \mathcal{O}(h^2). \tag{10}$$

Here $w_0^{(\beta)} = \frac{\beta}{2} g_0^{(\beta)}$ and $w_k^{(\beta)} = \frac{\beta}{2} g_k^{(\beta)} + \frac{2-\beta}{2} g_{k-1}^{(\beta)}$ for $k \geqslant 1$. In view of Lemma 3, the coefficients $w_k^{(\beta)}$ in (10) satisfy

$$\begin{cases} w_0^{(\beta)} = \frac{\beta}{2}, w_1^{(\beta)} = 1 - \frac{\beta+\beta^2}{2} < 0, w_2^{(\beta)} = \frac{\beta(\beta^2+\beta-4)}{4}, \\ 1 \geqslant w_0^{(\beta)} \geqslant w_3^{(\beta)} \geqslant \cdots \geqslant 0, \\ \sum\limits_{k=0}^{\infty} w_k^{(\beta)} = 0, \sum\limits_{k=0}^{m} w_k^{(\beta)} < 0, m \geqslant 2. \end{cases} \tag{11}$$

As a matter of fact, (10) can be rewritten as

$$\begin{aligned} {}_{\mathrm{RZ}}\mathrm{D}_x^{\beta}\psi\left(x_j\right) &= -\frac{1}{h^{\beta}}\left(\sum_{k=0}^{j} r_{j-k}^{(\beta)}\psi\left(x_k\right) + \sum_{k=j+1}^{M} r_{k-j}^{(\beta)}\psi\left(x_k\right)\right) + \mathcal{O}(h^2) \\ &= -\frac{1}{h^{\beta}}\sum_{k=0}^{M} r_{j-k}^{(\beta)}\psi\left(x_k\right) + \mathcal{O}(h^2), \end{aligned} \tag{12}$$

where

$$\begin{cases} r_0^{(\beta)} = 2\Psi_{\beta} w_1^{(\beta)} = \frac{1}{\cos(\frac{\pi\beta}{2})}\left(1 - \frac{\beta+\beta^2}{2}\right) > 0, \\ r_1^{(\beta)} = \Psi_{\beta}(w_0^{(\beta)} + w_2^{(\beta)}) = \frac{1}{2\cos(\frac{\pi\beta}{2})}\frac{\beta(\beta^2+\beta-2)}{4} < 0, \\ r_k^{(\beta)} = \Psi_{\beta} w_{k+1}^{(\beta)} = \frac{1}{2\cos(\frac{\pi\beta}{2})} w_{k+1}^{(\beta)} < 0, \ k \geqslant 2, \\ r_k^{(\beta)} = r_{-k}^{(\beta)}, \ k \geqslant 1. \end{cases} \tag{13}$$

Lemma 4 *For* $1 < \beta < 2$, *the coefficients* $r_k^{(\beta)}$ $(k \geqslant 0)$ *satisfy*

$$\begin{cases} \sum\limits_{k=0}^{m} r_{j-k}^{(\beta)} > 0, \ \ m-1 \geqslant j \geqslant 1, m \geqslant 2, \\ \sum\limits_{k=1}^{m} r_k^{(\beta)} < 0, \ \ m \geqslant 1, \\ \sum\limits_{k=1-m}^{m-1} r_k^{(\beta)} > 0, \ \ m \geqslant 2. \end{cases}$$

Proof It follows from (11) that $\sum\limits_{k=0}^{m} w_k^{(\beta)} < 0$ with $m \geqslant 2$. Thus,

$$\sum_{k=0}^{m} r_{j-k}^{(\beta)} = \Psi_{\beta}\left(\sum_{k=0}^{j+1} w_k^{(\beta)} + \sum_{k=0}^{m-j+1} w_k^{(\beta)}\right) > 0, \ j+1 \geqslant 2, m-j+1 \geqslant 2.$$

In view of (13), $r_0^{(\beta)} > 0$ and $r_k^{(\beta)} < 0$ with $k \neq 0$. Therefore, when $m \geqslant 1$, $\sum\limits_{k=1}^{m} r_k^{(\beta)} < 0$. Furthermore, (11) and (13) yield

$$\sum_{k=1-m}^{m-1} r_k^{(\beta)} = 2\Psi_{\beta}\sum_{k=0}^{m} w_k^{(\beta)} > 0, \ m \geqslant 2.$$

All this completes the proof.

2.3 Fractional Centered Difference Formula for Fractional Laplacian

In the case with two dimensions, let $h = 2L/M$ with $M \in \mathbb{Z}^+$, $x_j = -L + jh$ with $0 \leqslant j \leqslant M$, and $y_k = -L + kh$ with $0 \leqslant k \leqslant M$. Denote $\widetilde{\Omega}_h = \{(x_j, y_k) \mid 0 \leqslant j, k \leqslant M\}$, $\Omega_h = \widetilde{\Omega}_h \cap \widetilde{\Omega}$, and $\partial\Omega_h = \widetilde{\Omega}_h \cap \partial\widetilde{\Omega}$.

For any grid function $w = \{w_{jk}\}, v = \{v_{jk}\}$ on $S_h^\circ = \{w \mid w = \{w_{jk}\}, w_{jk} = 0$ with $(x_j, y_k) \in \partial\Omega_h\}$, a discrete inner product and the associated norm are defined as

$$(w, v)_{h^2} = h^2 \sum_{j=1}^{M-1} \sum_{k=1}^{M-1} w_{jk} v_{jk}, \quad \|w\|_{L_h^2}^2 = (w, w)_{h^2}.$$

Set $L_h^2 = \left\{ w \mid w = \{w_{jk}\}, \|w\|_{L_h^2}^2 < +\infty \right\}$. For $w \in L_h^2$, we define the semi-discrete Fourier transform $\hat{w}\colon [-\frac{\pi}{h}, -\frac{\pi}{h}] \to \mathbb{C}$ as

$$\hat{w}(\eta_1, \eta_2) = h^2 \sum_{j=1}^{M-1} \sum_{k=1}^{M-1} w_{jk} \mathrm{e}^{-\mathbf{i}(\eta_1 jh + \eta_2 kh)},$$

and the inverse semi-discrete Fourier transform

$$w_{jk} = \frac{1}{4\pi^2} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \hat{w}(\eta_1, \eta_2) \mathrm{e}^{-\mathbf{i}(\eta_1 jh + \eta_2 kh)} \mathrm{d}\eta_1 \mathrm{d}\eta_2$$

with $\mathbf{i} \in \mathbb{C}$ being the imaginary unit. It follows from the Parseval's identity that the continuous definition of the inner product takes the form

$$(w, v)_{h^2} = \frac{1}{4\pi^2} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \hat{w}(\eta_1, \eta_2) \hat{v}(\eta_1, \eta_2) \mathrm{d}\eta_1 \mathrm{d}\eta_2$$

with the norm given by

$$\|w\|_{L_h^2}^2 = \frac{1}{4\pi^2} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \left|\hat{w}(\eta_1, \eta_2)\right|^2 \mathrm{d}\eta_1 \mathrm{d}\eta_2.$$

For an arbitrary positive constant s, define the fractional Sobolev semi-norm $|\cdot|_{H_h^s}$ as

$$|w|_{H_h^s}^2 = \frac{1}{4\pi^2} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \left(\eta_1^2 + \eta_2^2\right)^s \left|\hat{w}(\eta_1, \eta_2)\right|^2 \mathrm{d}\eta_1 \mathrm{d}\eta_2.$$

Based on the above the settings, the fractional Laplacian in two dimensions can be evaluated by the two-dimensional fractional centered difference formula

$$\left(-\Delta_h\right)^{\frac{\beta}{2}} \psi(x, y) = \frac{1}{h^\beta} \sum_{j,k \in \mathbb{Z}} a_{j,k}^{(\beta)} \psi(x + jh, y + kh) \tag{14}$$

with

$$a_{j,k}^{(\beta)} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left[4\sin^2\left(\frac{\eta_1}{2}\right) + 4\sin^2\left(\frac{\eta_2}{2}\right)\right]^{\frac{\beta}{2}} \mathrm{e}^{-\mathbf{i}(\eta_1 j + \eta_2 k)} \mathrm{d}\eta_1 \mathrm{d}\eta_2. \tag{15}$$

Here we introduce the way of calculating $a_{j,k}^{(\beta)}$ given by [14]. Take an integer number $K > M$ and stepsize $\delta = 2\pi/K$, we have

$$\begin{aligned} a_{j,k}^{(\beta)} &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left[4\sin^2\left(\frac{\eta_1}{2}\right)+4\sin^2\left(\frac{\eta_2}{2}\right)\right]^{\frac{\beta}{2}}\mathrm{e}^{-\mathbf{i}(\eta_1 j+\eta_2 k)}\mathrm{d}\eta_1\mathrm{d}\eta_2 \\ &\approx \frac{1}{K^2}\sum_{p=0}^{K-1}\sum_{q=0}^{K-1}\left[4\sin^2\left(\frac{\delta p}{2}\right)+4\sin^2\left(\frac{\delta q}{2}\right)\right]^{\frac{\beta}{2}}\mathrm{e}^{-\mathbf{i}(j\delta p+k\delta q)}. \end{aligned}$$

With the expression above, the coefficients $a_{j,k}^{(\beta)}(0 \leqslant j,k \leqslant K-1)$ can be computed efficiently by the built-in function “fft2” in Matlab, where the accuracy of approximation is $\mathcal{O}(K^{-\beta-2})$ [14]. Throughout the numerical examples in this paper, K is taken as 2^{12} to compute the coefficients $a_{j,k}^{(\beta)}$.

Before introducing properties of the fractional central finite difference formula, the following space should be introduced [10, 14, 27]:

$$\mathcal{B}^s(\mathbb{R}^2) = \left\{\psi \,\middle|\, \psi \in L^1(\mathbb{R}^2), \int_{\mathbb{R}^2}\left[1+|\eta|^2\right]^s|\hat{\psi}(\eta_1,\eta_2)|\mathrm{d}\eta_1\mathrm{d}\eta_2 < \infty\right\},$$

where $|\eta|^2 = \eta_1^2 + \eta_2^2$.

Lemma 5 [14] *Let $\psi(x,y) \in \mathcal{B}^{2+\beta}(\mathbb{R}^2)$. For the fractional centered difference operator in* (3), *it holds that*

$$(-\Delta)^{\frac{\beta}{2}}\psi(x,y) = \left(-\Delta_h\right)^{\frac{\beta}{2}}\psi(x,y) + R_L(x,y), \quad 0 < \beta \leqslant 2,$$

where the truncation error satisfies

$$\left|R_L(x,y)\right| \leqslant C_0 h^2 \int_{\mathbb{R}^2}(1+|\eta|)^{\beta+2}\left|\hat{\psi}(\eta_1,\eta_2)\right|\mathrm{d}\eta_1\mathrm{d}\eta_2 = Ch^2 \tag{16}$$

with C being a constant independent of h.

Lemma 6 (Fractional semi-norm equivalence) [14] *For $\psi \in H_h^{\frac{\beta}{2}}(\mathbb{R}^2)$, we have*

$$\left(\frac{2}{\pi}\right)^{\beta}|\psi|^2_{H_h^{\frac{\beta}{2}}(\mathbb{R}^2)} \leqslant \left(\left(-\Delta_h\right)^{\frac{\beta}{2}}\psi,\psi\right)_{h^2} \leqslant |\psi|^2_{H_h^{\frac{\beta}{2}}(\mathbb{R}^2)}.$$

3 Fully Discrete Schemes

In this section, we derive fully discrete schemes for the one-dimensional time-space fractional diffusion equation (1) and the two-dimensional time-space fractional diffusion equation (2), along with the corresponding stability analysis and error estimation.

3.1 Fully Discrete Scheme for (1)

Denote the exact solution $u(x, t)$ and the numerical solution $U(x, t)$ at the grid point (x_j, t_n) by u_j^k and U_j^n, respectively. Denote also $f_j^n = f(x_j, t_n)$. Substituting (4) and (12) into (1), omitting the high-order terms, we obtain the following implicit difference scheme:

$$\begin{cases} c_{1,1}^{(\alpha)} U_j^1 - c_{1,1}^{(\alpha)} U_j^0 + h^{-\beta} \sum_{k=0}^{M} r_{j-k}^{(\beta)} U_k^1 = f_j^1, 1 \leqslant j \leqslant M-1, \\ \left(c_{n,n}^{(\alpha)} U_j^n + \sum_{i=1}^{n-1} (c_{i,n}^{(\alpha)} - c_{i+1,n}^{(\alpha)}) U_j^i - c_{1,n}^{(\alpha)} U_j^0 \right) + h^{-\beta} \sum_{k=0}^{M} r_{j-k}^{(\beta)} U_k^n = f_j^n, \\ 2 \leqslant n \leqslant N, \ 1 \leqslant j \leqslant M-1, \\ U_j^0 = u_0(x_j), 1 \leqslant j \leqslant M-1, \\ U_0^n = U_M^n = 0, 1 \leqslant n \leqslant N. \end{cases} \tag{17}$$

In order to rigorously analyze the numerical stability and convergence of the fully discrete scheme in (17), we first derive the following lemma.

Lemma 7 *For any grid function* $v = (v_0, v_1, \cdots, v_M) \in \mathcal{U}_h^{\circ}$, *we have*

$$h^{-\beta} h \sum_{j=1}^{M-1} \left(\sum_{k=0}^{M} r_{j-k}^{(\beta)} v_k \right) v_j > 0,$$

where $r_k^{(\beta)}$ *is defined by* (13) *with* $\beta \in (1, 2)$.

Proof It follows from Lemma 4 and (13) that

$$\begin{aligned} A &\triangleq h^{-\beta} h \sum_{j=1}^{M-1} \left(\sum_{k=0}^{M} r_{j-k}^{(\beta)} v_k \right) v_j \\ &= h^{-\beta} \left[h \sum_{j=1}^{M-1} r_0^{(\beta)} v_j^2 + h \sum_{j=1}^{M-1} \left(\sum_{k=0,k\neq j}^{M} r_{j-k}^{(\beta)} v_k v_j \right) \right] \\ &\geqslant h^{-\beta} \left[h \sum_{j=1}^{M-1} r_0^{(\beta)} v_j^2 + \frac{h}{2} \sum_{j=1}^{M-1} \sum_{k=0,k\neq j}^{M} r_{j-k}^{(\beta)} (v_k^2 + v_j^2) \right] \\ &= h^{-\beta} \left(h \sum_{j=1}^{M-1} r_0^{(\beta)} v_j^2 + \frac{h}{2} \sum_{j=1}^{M-1} \sum_{k=0,k\neq j}^{M} r_{j-k}^{(\beta)} v_j^2 + \frac{h}{2} \sum_{j=1}^{M-1} \sum_{k=0,k\neq j}^{M} r_{j-k}^{(\beta)} v_k^2 \right) \\ &\triangleq h^{-\beta} (A_1 + A_2 + A_3). \end{aligned} \tag{18}$$

Note that

$$\begin{aligned} A_2 &= \frac{h}{2} \sum_{j=1}^{M-1} \sum_{k=0,k\neq j}^{M} r_{j-k}^{(\beta)} v_j^2 = \frac{h}{2} \sum_{j=1}^{M-1} v_j^2 \sum_{k=0,k\neq j}^{M} r_{j-k}^{(\beta)} \\ &= \frac{h}{2} \sum_{j=1}^{M-1} v_j^2 \sum_{k=j-M,k\neq 0}^{j} r_k^{(\beta)} \geqslant \frac{h}{2} \sum_{j=1}^{M-1} v_j^2 \sum_{|k|=1}^{M-1} r_k^{(\beta)}, \end{aligned} \tag{19}$$

and

$$
\begin{aligned}
A_3 &= \frac{h}{2}\sum_{j=1}^{M-1}\sum_{k=0,k\neq j}^{M} r_{j-k}^{(\beta)} v_k^2 \\
&= \frac{h}{2}\left(\sum_{j=1}^{M-1}\sum_{k=j-M}^{-1} r_k^{(\beta)} v_{j-k}^2 + \sum_{j=1}^{M-1}\sum_{k=1}^{j} r_k^{(\beta)} v_{j-k}^2\right) \\
&= \frac{h}{2}\left(\sum_{k=1-M}^{-1} r_k^{(\beta)} \sum_{j=1}^{k+M} v_{j-k}^2 + \sum_{k=1}^{M-1} r_k^{(\beta)} \sum_{j=k}^{M-1} v_{j-k}^2\right) \\
&\geqslant \frac{h}{2}\left(\sum_{k=1-M}^{-1} r_k^{(\beta)} + \sum_{k=1}^{M-1} r_k^{(\beta)}\right)\sum_{j=1}^{M-1} v_j^2 \\
&= \frac{h}{2}\sum_{j=1}^{M-1} v_j^2 \sum_{|k|=1}^{M-1} r_k^{(\beta)}.
\end{aligned}
\tag{20}
$$

Substituting (19) and (20) into (18) gives

$$
\begin{aligned}
A &\geqslant h^{-\beta}\left(h\sum_{j=1}^{M-1} r_0^{(\beta)} v_j^2 + h\sum_{j=1}^{M-1} v_j^2 \sum_{|k|=1}^{M-1} r_k^{(\beta)}\right) \\
&= h^{-\beta} h\left(r_0^{(\beta)} + \sum_{|k|=1}^{M-1} r_k^{(\beta)}\right)\sum_{j=1}^{M-1} v_j^2 \\
&= h^{1-\beta}\sum_{k=1-M}^{M-1} r_k^{(\beta)} \sum_{j=1}^{M-1} v_j^2 > 0.
\end{aligned}
\tag{21}
$$

The proof is thus completed.

Based on the above lemma, we are ready for the stability and convergence analysis of the fully discrete scheme in (17).

Theorem 1 *Let $\alpha \in (0,1)$ and $\beta \in (1,2)$, the fully discrete scheme in (17) for the time-space fractional diffusion equation (1) is unconditionally stable.*

Proof Let $\widetilde{U}_j^n$ be the approximation of U_j^n which is the exact solution to the fully discrete scheme in (17) at (x_j, t_n). The perturbation term $\xi_j^n = \widetilde{U}_j^n - U_j^n$ satisfies

$$
\begin{cases}
c_{1,1}^{(\alpha)}\xi_j^1 + h^{-\beta}\sum\limits_{k=0}^{M} r_{j-k}^{(\beta)}\xi_k^1 = c_{1,1}^{(\alpha)}\xi_j^0,\ 1 \leqslant j \leqslant M-1, \\
c_{n,n}^{(\alpha)}\xi_j^n + h^{-\beta}\sum\limits_{k=0}^{M} r_{j-k}^{(\beta)}\xi_k^n = \sum\limits_{i=1}^{n-1}(c_{i+1,n}^{(\alpha)} - c_{i,n}^{(\alpha)})\xi_j^i + c_{1,n}^{(\alpha)}\xi_j^0, \\
n \geqslant 2,\ 1 \leqslant j \leqslant M-1.
\end{cases}
\tag{22}
$$

We prove the unconditional stability by mathematical induction. When $n = 1$, multiplying both sides of the first equation in (22) by ξ_j^1 and summing j from 1 to $M-1$ yield that

$$c_{1,1}^{(\alpha)} h \sum_{j=1}^{M-1} (\xi_j^1)^2 + h^{-\beta} h \sum_{j=1}^{M-1} \left(\sum_{k=0}^{M} r_{j-k}^{(\beta)} \xi_k^1 \right) \xi_j^1 = c_{1,1}^{(\alpha)} h \sum_{j=1}^{M-1} \xi_j^0 \xi_j^1.$$

According to Lemma 7, we have

$$h^{-\beta} h \sum_{j=1}^{M-1} \left(\sum_{k=0}^{M} r_{j-k}^{(\beta)} \xi_k^1 \right) \xi_j^1 > 0,$$

which gives

$$c_{1,1}^{(\alpha)} \|\xi^1\|_h^2 \leqslant c_{1,1}^{(\alpha)} \|\xi^1\|_h \|\xi^0\|_h. \tag{23}$$

Note also that $c_{1,1}^{(\alpha)} > 0$. Therefore,

$$\|\xi^1\|_h \leqslant \|\xi^0\|_h.$$

Assume that $\|\xi^m\|_h \leqslant \|\xi^0\|_h$ holds for $1 \leqslant m \leqslant n-1$. When $m = n$, multiplying both sides of the second equation in (22) by ξ_j^n and summing j from 1 to M yield that

$$\begin{aligned} & c_{n,n}^{(\alpha)} h \sum_{j=1}^{M-1} (\xi_j^n)^2 + h^{-\beta} h \sum_{j=1}^{M-1} \left(\sum_{k=0}^{M} r_{j-k}^{(\beta)} \xi_k^n \right) \xi_j^n \\ & = h \sum_{j=1}^{M-1} \sum_{i=1}^{n-1} (c_{i+1,n}^{(\alpha)} - c_{i,n}^{(\alpha)}) \xi_j^i \xi_j^n + c_{1,n}^{(\alpha)} h \sum_{j=1}^{M-1} \xi_j^0 \xi_j^n. \end{aligned}$$

It follows from Lemma 7 that

$$h^{-\beta} h \sum_{j=1}^{M-1} \left(\sum_{k=0}^{M} r_{j-k}^{(\beta)} \xi_k^n \right) \xi_j^n > 0,$$

which gives

$$\begin{aligned} c_{n,n}^{(\alpha)} \|\xi^n\|_h^2 & \leqslant \sum_{i=1}^{n-1} (c_{i+1,n}^{(\alpha)} - c_{i,n}^{(\alpha)}) h \sum_{j=1}^{M-1} \xi_j^i \xi_j^n + c_{1,n}^{(\alpha)} h \sum_{j=1}^{M-1} \xi_j^0 \xi_j^n \\ & \leqslant \sum_{i=1}^{n-1} (c_{i+1,n}^{(\alpha)} - c_{i,n}^{(\alpha)}) \|\xi^i\|_h \|\xi^n\|_h + c_{1,n}^{(\alpha)} \|\xi^0\|_h \|\xi^n\|_h \\ & \leqslant \|\xi^n\|_h \left(\sum_{i=1}^{n-1} (c_{i+1,n}^{(\alpha)} - c_{i,n}^{(\alpha)}) + c_{1,n}^{(\alpha)} \right) \|\xi^0\|_h \\ & = c_{n,n}^{(\alpha)} \|\xi^n\|_h \|\xi^0\|_h, \end{aligned}$$

where Lemma 1 is used. As a result, we have

$$\|\xi^n\|_h \leqslant \|\xi^0\|_h.$$

The proof is thus completed.

Theorem 2 *Let $u(x, t)$ be the exact solution to (1) and $U(x, t)$ be the numerical solution given by scheme (17). Assume that $u(x, t)$, ${}_{\mathrm{RL}}\mathrm{D}_{a,x}^{\beta}u(x,t)$, ${}_{\mathrm{RL}}\mathrm{D}_{x,b}^{\beta}u(x,t)$ and its Fourier transform belong to $\mathrm{C}^2\big([\tilde{a},T], L^1(\mathbb{R})\big)$. Then for the numerical error $\varepsilon = u - U$, there holds*

$$\|\varepsilon^n\|_h \leqslant C\big(\tau^{2-\alpha} + h^2\big),\ 0 \leqslant n \leqslant N, \tag{24}$$

where $\varepsilon^n = (\varepsilon_0^n, \varepsilon_1^n, \cdots, \varepsilon_{M-1}^n, \varepsilon_M^n) \in \mathcal{U}_h^\circ$, $\alpha \in (0,1)$, and $C > 0$ is a constant.

Proof It follows from (1) and (17) that

$$\begin{cases} c_{1,1}^{(\alpha)}\varepsilon_j^1 + h^{-\beta}\sum\limits_{k=0}^{M} r_{j-k}^{(\beta)}\varepsilon_k^1 = c_{1,1}^{(\alpha)}\varepsilon_j^0 + R_j^1, 1 \leqslant j \leqslant M-1,\\ c_{n,n}^{(\alpha)}\varepsilon_j^n + h^{-\beta}\sum\limits_{k=0}^{M} r_{j-k}^{(\beta)}\varepsilon_k^n = \sum\limits_{i=1}^{n-1}(c_{i+1,n}^{(\alpha)} - c_{i,n}^{(\alpha)})\varepsilon_j^i + c_{1,n}^{(\alpha)}\varepsilon_j^0 + R_j^n,\\ 2 \leqslant n \leqslant N,\ 1 \leqslant j \leqslant M-1,\\ \varepsilon_j^0 = 0, 1 \leqslant j \leqslant M-1. \end{cases} \tag{25}$$

By Lemma 2 and (8), the truncation error R_j^n satisfies

$$\left|R_j^n\right| \leqslant \widetilde{C}\big(\tau^{2-\alpha} + h^2\big),\ j = 1, 2, \cdots, M-1,\ n = 1, 2, \cdots, N,$$

where $\widetilde{C}$ is a positive constant.

Now we give the convergence analysis by proving

$$\|\varepsilon^n\|_h \leqslant \frac{\Gamma(1-\alpha)}{\tilde{a}^\alpha} \max_{1\leqslant k\leqslant n}\big\{(k\tau)^\alpha \|R^k\|_h\big\},\quad 1 \leqslant n \leqslant N \tag{26}$$

via mathematical induction. The case with $n = 0$ is trivial as $\|\varepsilon^0\|_h = 0$.

For $n = 1$, multiplying both sides of the first equation in (25) by ε_j^1 and summing j from 1 to $M-1$ yield

$$c_{1,1}^{(\alpha)} h \sum_{j=1}^{M-1}(\varepsilon_j^1)^2 + h^{-\beta} h \sum_{j=1}^{M-1}\left(\sum_{k=0}^{M} r_{j-k}^{(\beta)}\varepsilon_k^1\right)\varepsilon_j^1 = c_{1,1}^{(\alpha)} h \sum_{j=1}^{M-1}\varepsilon_j^0\varepsilon_j^1 + h\sum_{j=1}^{M-1} R_j^1\varepsilon_j^1.$$

Based on Lemma 7 and Remark 1, we give the following estimates:

$$c_{1,1}^{(\alpha)}\|\varepsilon^1\|_h^2 \leqslant c_{1,1}^{(\alpha)}\|\varepsilon^0\|_h\|\varepsilon^1\|_h + \|R^1\|_h\|\varepsilon^1\|_h,$$

from which follows

$$\|\varepsilon^1\|_h \leqslant \|\varepsilon^0\|_h + \frac{1}{c_{1,1}^{(\alpha)}}\|R^1\|_h \leqslant \frac{\Gamma(1-\alpha)}{\tilde{a}^\alpha}\tau^\alpha\|R^1\|_h.$$

Assume that (26) holds for $1 \leqslant n \leqslant m$ with $1 \leqslant m \leqslant N$. When $n = m+1$, multiplying both sides of the second equation in (25) by ε_j^n and summing j from 1 to $M-1$ give

$$
\begin{aligned}
&c_{n,n}^{(\alpha)} h \sum_{j=1}^{M-1} (\varepsilon_j^n)^2 + h^{-\beta} h \sum_{j=1}^{M-1} \left(\sum_{k=0}^{M} r_{j-k}^{(\beta)} \varepsilon_k^n \right) \varepsilon_j^n \\
&= h \sum_{j=1}^{M-1} \sum_{i=1}^{n-1} (c_{i+1,n}^{(\alpha)} - c_{i,n}^{(\alpha)}) \varepsilon_j^i \varepsilon_j^n + c_{1,n}^{(\alpha)} h \sum_{j=1}^{M-1} \varepsilon_j^0 \varepsilon_j^n + h \sum_{j=1}^{M-1} R_j^n \varepsilon_j^n.
\end{aligned}
$$

It follows from Lemma 7 and Remark 1 that

$$
\begin{aligned}
c_{n,n}^{(\alpha)} \|\varepsilon^n\|_h^2 &\leqslant \sum_{i=1}^{n-1} (c_{i+1,n}^{(\alpha)} - c_{i,n}^{(\alpha)}) \|\varepsilon^i\|_h \|\varepsilon^n\|_h + c_{1,n}^{(\alpha)} \left(\|\varepsilon^0\|_h + \frac{\|R^n\|_h}{c_{1,n}^{(\alpha)}} \right) \|\varepsilon^n\|_h \\
&\leqslant \sum_{i=1}^{n-1} (c_{i+1,n}^{(\alpha)} - c_{i,n}^{(\alpha)}) \frac{\Gamma(1-\alpha)}{\tilde{a}^\alpha} \max_{1 \leqslant k \leqslant i} \left\{ (k\tau)^\alpha \|R^k\|_h \right\} \|\varepsilon^n\|_h \\
&\quad + c_{1,n}^{(\alpha)} \frac{\Gamma(1-\alpha)}{\tilde{a}^\alpha} (n\tau)^\alpha \|R^n\|_h \|\varepsilon^n\|_h \\
&\leqslant \left[\sum_{i=1}^{n-1} (c_{i+1,n}^{(\alpha)} - c_{i,n}^{(\alpha)}) + c_{1,n}^{(\alpha)} \right] \frac{\Gamma(1-\alpha)}{\tilde{a}^\alpha} \max_{1 \leqslant k \leqslant n} \left\{ (k\tau)^\alpha \|R^k\|_h \right\} \|\varepsilon^n\|_h \\
&= c_{n,n}^{(\alpha)} \frac{\Gamma(1-\alpha)}{\tilde{a}^\alpha} \max_{1 \leqslant k \leqslant n} \left\{ (k\tau)^\alpha \|R^k\|_h \right\} \|\varepsilon^n\|_h.
\end{aligned}
$$

Therefore,

$$
\|\varepsilon^n\|_h \leqslant \frac{\Gamma(1-\alpha)}{\tilde{a}^\alpha} \max_{1 \leqslant k \leqslant n} \left\{ (k\tau)^\alpha \|R^k\|_h \right\} \leqslant \frac{\Gamma(1-\alpha)}{\tilde{a}^\alpha} T^\alpha \widetilde{C} \left(\tau^{2-\alpha} + h^2 \right).
$$

Consequently, there exists a positive constant C, such that

$$
\|\varepsilon^n\|_h \leqslant C \left(\tau^{2-\alpha} + h^2 \right).
$$

This finishes the proof.

3.2 Fully Discrete Scheme for (2)

For the two-dimensional fractional diffusion equation in (2), we denote $u_{jk}^n = u(x_j, y_k, t_n)$ and $f_{jk}^n = f(x_j, y_k, t_n)$ with $x_j = -L + jh$, $y_k = -L + kh$, and $t_n = n\tau$. Here h and τ are the spatial stepsize and the temporal stepsize, respectively. Adopting the L1 approximation in (4) for the temporal Caputo-Hadamard derivative and the fractional centred difference formula in (14) for the fractional Laplacian, we obtain the following fully discrete scheme after omitting the high-order terms:

$$\begin{cases} c_{1,1}^{(\alpha)}U_{jk}^{1} - c_{1,1}^{(\alpha)}U_{jk}^{0} + \left(-\Delta_h\right)^{\frac{\beta}{2}}U_{jk}^{1} = f_{jk}^{1}, 1 \leqslant j,k \leqslant M-1, \\ c_{n,n}^{(\alpha)}U_{jk}^{n} + \sum\limits_{i=1}^{n-1}(c_{i,n}^{(\alpha)} - c_{i+1,n}^{(\alpha)})U_{jk}^{i} - c_{1,n}^{(\alpha)}U_{jk}^{0} + \left(-\Delta_h\right)^{\frac{\beta}{2}}U_{jk}^{n} = f_{jk}^{n}, \\ 2 \leqslant n \leqslant N,\ 1 \leqslant j,k \leqslant M-1, \\ U_{jk}^{0} = u_0(x_j, y_k), 1 \leqslant j,k \leqslant M-1, \\ U_{jk}^{n} = 0, (x_j, y_k) \in \partial\Omega_h, 0 \leqslant n \leqslant N. \end{cases} \tag{27}$$

Here $U_{jk}^{n} = U(x_j, y_k, t_n)$ with $U(x,\ y,\ t)$ being the approximation of $u(x,\ y,\ t)$. This finite different system can be written into the following matrix form:

$$\begin{cases} c_{1,1}^{(\alpha)}\mathbf{U}^{1} - c_{1,1}^{(\alpha)}\mathbf{U}^{0} + \frac{1}{h^\beta}\mathbf{A}\mathbf{U}^{1} = \mathbf{F}^{1}, n = 1, \\ c_{n,n}^{(\alpha)}\mathbf{U}^{n} + \sum\limits_{i=1}^{n-1}(c_{i,n}^{(\alpha)} - c_{i+1,n}^{(\alpha)})\mathbf{U}^{i} - c_{1,n}^{(\alpha)}\mathbf{U}^{0} + \frac{1}{h^\beta}\mathbf{A}\mathbf{U}^{n} = \mathbf{F}^{n}, n \geqslant 2, \end{cases} \tag{28}$$

where

$$\mathbf{F}^n = (f_{11}^n, \cdots, f_{(M-1)1}^n, f_{12}^n, \cdots, f_{(M-1)2}^n, \cdots, f_{1(M-1)}^n, \cdots, f_{(M-1)(M-1)}^n)^{\mathrm{T}}$$

and

$$\mathbf{U}^n = \left(\mathbf{U}_1^n, \mathbf{U}_2^n, \cdots, \mathbf{U}_{M-1}^n\right)^{\mathrm{T}}$$

with $\mathbf{U}_j^n = \left(U_{1j}^n, U_{2j}^n, \cdots, U_{(M-1)j}^n\right)^{\mathrm{T}}$. The coefficient matrix $\mathbf{A}$ is a real symmetric block Toeplitz matrix with Toeplitz blocks, say,

$$\mathbf{A} = \begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{M-3} & A_{M-2} \\ A_1 & A_0 & A_1 & \cdots & A_{M-4} & A_{M-3} \\ A_2 & A_1 & A_0 & \cdots & A_{M-5} & A_{M-4} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ A_{M-3} & A_{M-4} & A_{M-5} & \cdots & A_0 & A_1 \\ A_{M-2} & A_{M-3} & A_{M-4} & \cdots & A_1 & A_0 \end{pmatrix}$$

with

$$A_j = \begin{pmatrix} a_{0,j}^{(\beta)} & a_{1,j}^{(\beta)} & a_{2,j}^{(\beta)} & \cdots & a_{M-3,j}^{(\beta)} & a_{M-2,j}^{(\beta)} \\ a_{1,j}^{(\beta)} & a_{0,j}^{(\beta)} & a_{1,j}^{(\beta)} & \cdots & a_{M-4,j}^{(\beta)} & a_{M-3,j}^{(\beta)} \\ a_{2,j}^{(\beta)} & a_{1,j}^{(\beta)} & a_{0,j}^{(\beta)} & \cdots & a_{M-5,j}^{(\beta)} & a_{M-4,j}^{(\beta)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{M-3,j}^{(\beta)} & a_{M-4,j}^{(\beta)} & a_{M-5,M-5j}^{(\beta)} & \cdots & a_{0,j}^{(\beta)} & a_{1,j}^{(\beta)} \\ a_{M-2,j}^{(\beta)} & a_{M-3,j}^{(\beta)} & a_{M-4,j}^{(\beta)} & \cdots & a_{1,j}^{(\beta)} & a_{0,j}^{(\beta)} \end{pmatrix}.$$

It follows from Lemma 6 that the matrix $\mathbf{A}$ is a real symmetric positive definite matrix.

Now we show the unconditional stability of the fully discrete scheme (27).

Theorem 3 *Let* $1 < \beta < 2$ *and* $0 < \alpha < 1$. *The finite difference scheme* (27) *for the fractional diffusion equation* (2) *is unconditionally stable.*

Proof Let $\widetilde{U}_{jk}^n$ be the approximation of U_{jk}^n which is the exact solution to the fully discrete scheme in (27) at (x_j, y_k, t_n). It is evident that the perturbation term $\epsilon_{jk}^n = \widetilde{U}_{jk}^n - U_{jk}^n$ $(0 \leqslant j,k \leqslant M, 1 \leqslant n \leqslant N)$ satisfies

$$\begin{cases} c_{1,1}^{(\alpha)}\epsilon_{jk}^1 - c_{1,1}^{(\alpha)}\epsilon_{jk}^0 + \left(-\Delta_h\right)^{\frac{\beta}{2}}\epsilon_{jk}^1 = 0, 1 \leqslant j,k \leqslant M-1, \\ c_{n,n}^{(\alpha)}\epsilon_{jk}^n + \sum\limits_{i=1}^{n-1}(c_{i,n}^{(\alpha)} - c_{i+1,n}^{(\alpha)})\epsilon_{jk}^i - c_{1,n}^{(\alpha)}\epsilon_{jk}^0 + \left(-\Delta_h\right)^{\frac{\beta}{2}}\epsilon_{jk}^n = 0, \\ 2 \leqslant n \leqslant N,\ 1 \leqslant j,k \leqslant M-1. \end{cases} \tag{29}$$

Now we show that $\|\epsilon^n\|_{L_h^2} \leqslant \|\epsilon^0\|_{L_h^2}$ $(n \geqslant 0)$ via mathematical induction. The case with $n=0$ is trivial. For $n=1$, taking the inner product of the first equation of (29) with ϵ^1 gives

$$\left(c_{1,1}^{(\alpha)}\epsilon^1, \epsilon^1\right)_{h^2} - \left(c_{1,1}^{(\alpha)}\epsilon^0, \epsilon^1\right)_{h^2} + \left(\left(-\Delta_h\right)^{\frac{\beta}{2}}\epsilon^1, \epsilon^1\right)_{h^2} = 0. \tag{30}$$

Note that Lemma 6 gives $\left(\left(-\Delta_h\right)^{\frac{\beta}{2}}\epsilon^1, \epsilon^1\right)_{h^2} \geqslant 0$. By Cauchy inequality, we have

$$c_{1,1}^{(\alpha)}\|\epsilon^1\|_{L_h^2}^2 = \left(c_{1,1}^{(\alpha)}\epsilon^1, \epsilon^1\right)_{h^2} \leqslant \left(c_{1,1}^{(\alpha)}\epsilon^0, \epsilon^1\right)_{h^2} \leqslant c_{1,1}^{(\alpha)}\|\epsilon^0\|_{L_h^2}\|\epsilon^1\|_{L_h^2},$$

which yields that

$$\|\epsilon^1\|_{L_h^2} \leqslant \|\epsilon^0\|_{L_h^2}. \tag{31}$$

Assume that $\|\epsilon^m\|_{L_h^2}^2 \leqslant \|\epsilon^0\|_{L_h^2}$ holds for $m = 2, \cdots, n-1$. It follows from the second equation in (29) that

$$\left(c_{n,n}^{(\alpha)}\epsilon^n, \epsilon^n\right)_{h^2} + \sum_{i=1}^{n-1}\left((c_{i,n}^{(\alpha)} - c_{i+1,n}^{(\alpha)})\epsilon^i, \epsilon^n\right)_{h^2} - \left(c_{1,n}^{(\alpha)}\epsilon^0, \epsilon^n\right)_{h^2} + \left(\left(-\Delta_h\right)^{\frac{\beta}{2}}\epsilon^n, \epsilon^n\right)_{h^2} = 0. \tag{32}$$

Applying Lemma 6 and the Cauchy inequality, we have

$$\begin{aligned} c_{n,n}^{(\alpha)}\|\epsilon^n\|_{L_h^2}^2 &\leqslant \sum_{i=1}^{n-1}(c_{i+1,n}^{(\alpha)} - c_{i,n}^{(\alpha)})\left(\epsilon^i, \epsilon^n\right)_{h^2} + c_{1,n}^{(\alpha)}\left(\epsilon^0, \epsilon^n\right)_{h^2} \\ &\leqslant \sum_{i=1}^{n-1}(c_{i+1,n}^{(\alpha)} - c_{i,n}^{(\alpha)})\|\epsilon^i\|_{L_h^2}\|\epsilon^n\|_{L_h^2} + c_{1,n}^{(\alpha)}\|\epsilon^0\|_{L_h^2}\|\epsilon^n\|_{L_h^2} \\ &\leqslant \|\epsilon^n\|_{L_h^2}\left(\sum_{i=1}^{n-1}(c_{i+1,n}^{(\alpha)} - c_{i,n}^{(\alpha)})\|\epsilon^0\|_{L_h^2} + c_{1,n}^{(\alpha)}\|\epsilon^0\|_{L_h^2}\right) \\ &\leqslant c_{n,n}^{(\alpha)}\|\epsilon^n\|_{L_h^2}\|\epsilon^0\|_{L_h^2}, \end{aligned}$$

which yields

$$\|\epsilon^n\|_{L_h^2} \leqslant \|\epsilon^0\|_{L_h^2}.$$

The proof is thus finished.

Now we show the convergence of the fully discrete scheme in (27).

Theorem 4 *Assume that* $u(x,y,t) \in C^2([\tilde{a},T],\mathcal{B}^{2+\beta}(\widetilde{\Omega}))$ *is the solution of* (2). *Let* $U_{jk}^n\,(0 \leqslant j,k \leqslant M, 0 \leqslant n \leqslant N)$ *be the numerical solution given by* (27). *Then the numerical error* $e^n = \{e_{jk}^n\}_{j,k=0}^M \in S_h$ *with* $e_{jk}^n = U_{jk}^n - u_{jk}^n\,(0 \leqslant j,k \leqslant M, 0 \leqslant n \leqslant N)$ *satisfies*

$$\|e^n\|_{L_h^2} \leqslant C\big(\tau^{2-\alpha} + h^2\big).$$

Here C is a positive constant independent of h and τ.

Proof It follows from (2) and (27) that

$$\begin{cases} c_{1,1}^{(\alpha)} e_{jk}^1 - c_{1,1}^{(\alpha)} e_{jk}^0 + \big(-\Delta_h\big)^{\frac{\beta}{2}} e_{jk}^1 = R_{jk}^1, 1 \leqslant j \leqslant M-1, 1 \leqslant k \leqslant M-1, \\ c_{n,n}^{(\alpha)} e_{jk}^n + \sum\limits_{i=1}^{n-1} (c_{i,n}^{(\alpha)} - c_{i+1,n}^{(\alpha)}) e_{jk}^i - c_{1,n}^{(\alpha)} e_{jk}^0 + \big(-\Delta_h\big)^{\frac{\beta}{2}} e_{jk}^n = R_{jk}^n, \\ 2 \leqslant n \leqslant N,\ 1 \leqslant j \leqslant M-1, 1 \leqslant k \leqslant M-1, \\ e_{jk}^0 = 0, 1 \leqslant j \leqslant M-1, 1 \leqslant k \leqslant M-1, \\ e_{jk}^n = 0, (x_j, y_k) \in \partial\Omega_h, 0 \leqslant n \leqslant N. \end{cases} \tag{33}$$

By Lemmas 2 and 5, it is evident that for $1 \leqslant n \leqslant N$ and $1 \leqslant j,k \leqslant M-1$,

$$\left|R_{jk}^n\right| \leqslant \widetilde{C}(\tau^{2-\alpha} + h^2) \tag{34}$$

with $\widetilde{C}$ being a positive constant independent of h and τ. Now we prove

$$\|e^n\|_{L_h^2} \leqslant \frac{\Gamma(1-\alpha)}{\tilde{a}^\alpha} \max_{1 \leqslant \ell \leqslant n} \left\{ (\ell\tau)^\alpha \|R^\ell\|_{L_h^2} \right\},\ 0 \leqslant n \leqslant N \tag{35}$$

via mathematical induction. When $n = 0$, it is obvious that

$$\|e^0\|_{L_h^2} = 0. \tag{36}$$

When $n = 1$, taking the inner product of the first equation of (33) with e^1 gives

$$\left(c_{1,1}^{(\alpha)} e^1, e^1\right)_{h^2} - \left(c_{1,1}^{(\alpha)} e^0, e^1\right)_{h^2} + \left(\big(-\Delta_h\big)^{\frac{\beta}{2}} e^1, e^1\right)_{h^2} = \big(R^1, e^1\big)_{h^2}. \tag{37}$$

It follows from the above equation and Lemma 6 that

$$c_{1,1}^{(\alpha)} \|e^1\|_{L_h^2}^2 = \left(c_{1,1}^{(\alpha)} e^1, e^1\right)_{h^2} \leqslant c_{1,1}^{(\alpha)} \|e^0\|_{L_h^2} \|e^1\|_{L_h^2} + \|R^1\|_{L_h^2} \|e^1\|_{L_h^2},$$

where the Cauchy inequality is applied. As a result, (36) and Remark 1 yield that

$$\|e^1\|_{L_h^2} \leqslant \|e^0\|_{L_h^2} + \frac{1}{c_{1,1}^{(\alpha)}} \|R^1\|_{L_h^2} \leqslant \frac{\Gamma(1-\alpha)}{\tilde{a}^\alpha} (\tau)^\alpha \|R^1\|_{L_h^2}. \tag{38}$$

Assume that (35) holds for $2 \leqslant n \leqslant m-1$. For the case with $n = m$, taking the inner product of the second equation of (33) with e^n gives

$$\left(c_{n,n}^{(\alpha)} e^n, e^n\right)_{h^2} + \sum_{i=1}^{n-1} \left((c_{i,n}^{(\alpha)} - c_{i+1,n}^{(\alpha)}) e^i, e^n\right)_{h^2} - \left(c_{1,n}^{(\alpha)} e^0, e^n\right)_{h^2} + \left(\big(-\Delta_h\big)^{\frac{\beta}{2}} e^n, e^n\right)_{h^2} = (R^n, e^n)_{h^2}.$$

Combining (36), Lemma 6, and Remark 1 gives

$$
\begin{aligned}
c_{n,n}^{(\alpha)}\|e^n\|_{L_h^2}^2 &\leqslant \sum_{i=1}^{n-1}(c_{i+1,n}^{(\alpha)} - c_{i,n}^{(\alpha)})\big(e^i, e^n\big)_{h^2} + c_{1,n}^{(\alpha)}\big(e^0, e^n\big)_{h^2} + (R^n, e^n)_{h^2} \\
&\leqslant \sum_{i=1}^{n-1}(c_{i+1,n}^{(\alpha)} - c_{i,n}^{(\alpha)})\|e^i\|_{L_h^2}\|e^n\|_{L_h^2} + c_{1,n}^{(\alpha)}\left(\frac{1}{c_{1,n}^{(\alpha)}}\|R^n\|_{L_h^2}\|e^n\|_{L_h^2}\right),
\end{aligned}
$$

where the Cauchy inequality is applied. Therefore,

$$
\begin{aligned}
c_{n,n}^{(\alpha)}\|e^n\|_{L_h^2} &\leqslant \sum_{i=1}^{n-1}(c_{i+1,n}^{(\alpha)} - c_{i,n}^{(\alpha)})\frac{\Gamma(1-\alpha)}{\tilde{a}^\alpha}\max_{1\leqslant \ell \leqslant i}\left\{(\ell\tau)^\alpha\|R^\ell\|_{L_h^2}\right\} \\
&\quad + c_{1,n}^{(\alpha)}\frac{\Gamma(1-\alpha)}{\tilde{a}^\alpha}(n\tau)^\alpha\|R^n\|_{L_h^2} \\
&\leqslant \left[\sum_{i=1}^{n-1}(c_{i+1,n}^{(\alpha)} - c_{i,n}^{(\alpha)}) + c_{1,n}^{(\alpha)}\right]\frac{\Gamma(1-\alpha)}{\tilde{a}^\alpha}\max_{1\leqslant \ell \leqslant n}\left\{(\ell\tau)^\alpha\|R^\ell\|_{L_h^2}\right\} \\
&= c_{n,n}^{(\alpha)}\frac{\Gamma(1-\alpha)}{\tilde{a}^\alpha}\max_{1\leqslant \ell \leqslant n}\left\{(\ell\tau)^\alpha\|R^\ell\|_{L_h^2}\right\}.
\end{aligned}
$$

As a result,

$$
\|e^n\|_{L_h^2} \leqslant \frac{\Gamma(1-\alpha)}{\tilde{a}^\alpha}\max_{1\leqslant \ell \leqslant n}\left\{(\ell\tau)^\alpha\|R^\ell\|_{L_h^2}\right\} \leqslant \frac{\Gamma(1-\alpha)}{\tilde{a}^\alpha}T^\alpha\widetilde{C}(\tau^{2-\alpha} + h^2).
$$

Consequently, there exists a positive constant C, such that

$$
\|e^n\|_{L_h^2} \leqslant C\big(\tau^{2-\alpha} + h^2\big).
$$

This finishes the proof.

4 Numerical Experiments

In this section, we numerically demonstrate the aforementioned theoretical results on the convergence and numerical stability.

Example 1 Consider the following fractional diffusion equation in one dimension with $\alpha \in (0, 1)$ and $\beta \in (1, 2)$:

$$
{}_{\mathrm{CH}}\mathrm{D}_{1,t}^{\alpha}u(x,t) - {}_{\mathrm{RZ}}\mathrm{D}_{x}^{\beta}u(x,t) = f(x,t),\ t \in (1,2]
$$

on a finite domain $0 \leqslant x \leqslant 1$, with a given force term

$$
\begin{aligned}
f(x,t) = &\frac{6}{\Gamma(4-\alpha)}x^4(1-x)^4(\log t)^{3-\alpha} + \frac{(\log t)^{3-\alpha}}{2\cos(\pi\beta/2)} \\
&\times \sum_{l=1}^{4}\left(\frac{(-1)^l 4!(4+l)!}{l!(4-l)!\Gamma(5+l-\beta)}\big(x^{4+l-\beta} + (1-x)^{4+l-\beta}\big)\right).
\end{aligned}
$$

The initial condition and the boundary condition are given by $u(x,1)=0$ and $u(0,t)=u(1,t)=0$, respectively. Its exact solution is

$$u(x,t)=(\log t)^3 x^4 (1-x)^4.$$

The numerical error is defined as

$$\text{Error}=\sqrt{\left(h\sum_{j=1}^{M-1}\left|u_j^N-u(x_j,t_N)\right|^2\right)}.$$

Tables 1 and 2 show the numerical error and the convergence orders given by the fully discrete scheme (17). We can see that the scheme is stable and the numerical error is $\mathcal{O}\left(\tau^{2-\alpha}+h^2\right)$ indeed.

Example 2 Consider the following fractional diffusion equation in two dimensions with $\alpha\in(0,1)$ and $\beta\in(1,2)$:

Table 1 Spatial convergence of scheme (17) with $\tau=0.000\,05$ for Example 1

β	h	$\alpha=0.3$		$\alpha=0.6$		$\alpha=0.9$	
		Error	Order	Error	Order	Error	Order
1.3	$1/2^3$	5.181 2E–5	–	4.879 7E–5	–	4.475 6E–5	–
	$1/2^4$	1.303 6E–5	1.991	1.235 5E–5	1.982	1.144 6E–5	1.967
	$1/2^5$	3.318 2E–6	1.974	3.150 5E–6	1.972	2.927 2E–6	1.967
	$1/2^6$	8.410 6E–7	1.980	7.990 0E–7	1.979	7.441 0E–7	1.976
	$1/2^7$	2.120 1E–7	1.988	2.014 8E–7	1.988	1.889 4E–7	1.978
1.5	$1/2^3$	5.153 4E–5	–	4.952 2E–5	–	4.677 7E–5	–
	$1/2^4$	1.273 2E–5	2.017	1.227 7E–5	2.012	1.166 3E–5	2.004
	$1/2^5$	3.214 4E–6	1.986	3.102 5E–6	1.984	2.952 4E–6	1.982
	$1/2^6$	8.112 6E–7	1.986	7.832 8E–7	1.986	7.466 9E–7	1.983
	$1/2^7$	2.040 4E–7	1.991	1.970 5E–7	1.991	1.888 9E–7	1.983
1.7	$1/2^3$	4.722 9E–5	–	4.604 0E–5	–	4.439 6E–5	–
	$1/2^4$	1.143 8E–5	2.046	1.116 2E–5	2.044	1.078 7E–5	2.041
	$1/2^5$	2.860 3E–6	2.000	2.791 7E–6	1.999	2.699 8E–6	1.998
	$1/2^6$	7.179 9E–7	1.994	7.008 6E–7	1.994	6.786 8E–7	1.992
	$1/2^7$	1.800 5E–7	1.996	1.757 9E–7	1.995	1.710 5E–7	1.988
1.9	$1/2^3$	3.861 5E–5	–	3.802 9E–5	–	3.720 7E–5	–
	$1/2^4$	9.084 5E–6	2.088	8.939 7E–6	2.089	8.741 8E–6	2.090
	$1/2^5$	2.246 1E–6	2.016	2.209 1E–6	2.017	2.159 6E–6	2.017
	$1/2^6$	5.605 0E–7	2.003	5.512 1E–7	2.003	5.394 0E–7	2.001
	$1/2^7$	1.401 2E–7	2.000	1.378 1E–7	2.000	1.355 1E–7	1.993

Table 2 Temporal convergence of scheme (17) with $h = 0.000\,1$ for Example 1

β	τ	$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
		Error	Order	Error	Order	Error	Order
1.3	$1/2^3$	2.114 6E–6	–	1.002 4E–5	–	3.603 9E–5	–
	$1/2^4$	6.846 7E–7	1.627	3.883 4E–6	1.368	1.704 6E–5	1.080
	$1/2^5$	2.190 5E–7	1.644	1.491 0E–6	1.381	8.009 5E–6	1.090
	$1/2^6$	6.948 6E–8	1.656	5.695 5E–7	1.388	3.750 3E–6	1.095
	$1/2^7$	2.190 5E–8	1.665	2.169 1E–7	1.393	1.752 9E–6	1.097
1.5	$1/2^3$	1.708 2E–6	–	8.165 1E–6	–	2.988 7E–5	–
	$1/2^4$	5.524 8E–7	1.629	3.156 9E–6	1.371	1.413 0E–5	1.081
	$1/2^5$	1.766 2E–7	1.645	1.210 6E–6	1.383	6.636 1E–6	1.090
	$1/2^6$	5.599 9E–8	1.657	4.621 0E–7	1.389	3.106 3E–6	1.095
	$1/2^7$	1.764 8E–8	1.666	1.759 2E–7	1.393	1.451 6E–6	1.098
1.7	$1/2^3$	1.341 8E–6	–	6.448 3E–6	–	2.391 9E–5	–
	$1/2^4$	4.335 3E–7	1.630	2.488 1E–6	1.374	1.129 0E–5	1.083
	$1/2^5$	1.385 0E–7	1.646	9.530 2E–7	1.384	5.297 2E–6	1.092
	$1/2^6$	4.389 3E–8	1.658	3.635 3E–7	1.390	2.478 1E–6	1.096
	$1/2^7$	1.382 8E–8	1.666	1.383 3E–7	1.394	1.157 7E–6	1.098
1.9	$1/2^3$	1.026 2E–6	–	4.944 9E–6	–	1.848 8E–5	–
	$1/2^4$	3.312 4E–7	1.631	1.904 5E–6	1.377	8.706 7E–6	1.086
	$1/2^5$	1.057 6E–7	1.647	7.287 2E–7	1.386	4.079 7E–6	1.094
	$1/2^6$	3.350 1E–8	1.659	2.778 0E–7	1.391	1.907 2E–6	1.097
	$1/2^7$	1.055 2E–8	1.667	1.056 7E–7	2.486	8.906 2E–7	1.099

$$\begin{cases} {}_{\mathrm{CH}}\mathrm{D}^{\alpha}_{1,t}u(x,y,t) + (-\Delta)^{\frac{\beta}{2}}u(x,y,t) = f(x,y,t), & (x,y) \in \widetilde{\Omega}, t \in (1,2], \\ u(x,y,t) = 0, & (x,y) \in \mathbb{R}^2 \backslash \widetilde{\Omega}, t \in (1,2], \\ u(x,y,1) = 0, & (x,y) \in \mathbb{R}^2, \end{cases}$$

where $\widetilde{\Omega} = (-1,1)^2$ and the exact solution is set as $u(x,y,t) = (\log t)^3(1-x^2)^4(1-y^2)^4$. The source term f is not explicitly known and we use very fine stepsize to compute it. In this case we evaluate the source term by $f \approx f_h = {}_{\mathrm{CH}}\mathrm{D}^{\alpha}_{1,t}u(x,y,t) + \left(-\Delta_h\right)^{\frac{\beta}{2}}u(x,y,t)$ with $h = 2^{-8}$. The numerical error in the spatial direction is

$$E(h) = \sqrt{h^2 \sum_{j=0}^{M} \sum_{k=0}^{M} \left| u_{jk}^N(h,\tau) - u_{2j,2k}^N(h/2,\tau) \right|^2},$$

where τ is small enough. The numerical error in the temporal direction is

$$F(\tau) = \sqrt{h^2 \sum_{j=0}^{M} \sum_{k=0}^{M} \left| u_{jk}^N(h,\tau) - u_{jk}^{2N}(h,\tau/2) \right|^2},$$

where h is small enough.

Tables 3 and 4 display the errors and convergence orders for the finite difference scheme (27). We can observe that the numerical results are stable and of $(2-\alpha)$ order in time and of 2nd order in space, which verifies Theorems 3 and 4.

Table 3 Spatial convergence of scheme (27) with $\tau = 1/2^7$ for Example 2

β	h	$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
		$E(h)$	Order	$E(h)$	Order	$E(h)$	Order
1.3	$1/2^2$	–	–	–	–	–	–
	$1/2^3$	4.492 5E–3	–	4.054 9E–3	–	3.505 7E–3	–
	$1/2^4$	1.064 0E–3	2.078	9.643 4E–4	2.072	8.397 1E–4	2.062
	$1/2^5$	2.628 4E–4	2.017	2.384 6E–4	2.016	2.081 3E–4	2.012
	$1/2^6$	6.550 6E–5	2.004	5.945 5E–5	2.004	5.200 3E–5	2.001
1.5	$1/2^2$	–	–	–	–	–	–
	$1/2^3$	5.519 0E–3	–	5.103 3E–3	–	4.562 9E–3	–
	$1/2^4$	1.292 5E–3	2.094	1.199 3E–3	2.089	1.078 9E–3	2.080
	$1/2^5$	3.184 4E–4	2.021	2.956 7E–4	2.020	2.664 6E–4	2.018
	$1/2^6$	7.932 0E–5	2.005	7.366 6E–5	2.005	6.649 2E–5	2.003
1.7	$1/2^2$	–	–	–	–	–	–
	$1/2^3$	6.589 9E–3	–	6.212 8E–3	–	5.709 6E–3	–
	$1/2^4$	1.524 5E–3	2.112	1.441 3E–3	2.108	1.331 1E–3	2.101
	$1/2^5$	3.745 2E–4	2.025	3.541 8E–4	2.025	3.274 9E–4	2.023
	$1/2^6$	9.323 2E–5	2.006	8.817 7E–5	2.006	8.160 4E–5	2.005
1.9	$1/2^2$	–	–	–	–	–	–
	$1/2^3$	7.700 2E–3	–	7.370 1E–3	–	6.921 2E–3	–
	$1/2^4$	1.757 6E–3	2.131	1.685 9E–3	2.128	1.589 5E–3	2.122
	$1/2^5$	4.303 9E–4	2.030	4.128 5E–4	2.030	3.894 2E–4	2.029
	$1/2^6$	1.070 6E–4	2.007	1.026 9E–4	2.007	9.688 9E–5	2.007

Table 4 Temporal convergence of scheme (27) with $h = 1/2^6$ for Example 2

β	τ	$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
		$F(\tau)$	Order	$F(\tau)$	Order	$F(\tau)$	Order
1.3	$1/2^2$	–	–	–	–	–	–
	$1/2^3$	1.464 9E–3	–	5.174 6E–3	–	1.296 6E–2	–
	$1/2^4$	4.881 7E–4	1.585	2.065 8E–3	1.325	6.241 3E–3	1.055
	$1/2^5$	1.592 1E–4	1.616	8.068 6E–4	1.356	2.964 5E–3	1.074
	$1/2^6$	5.119 6E–5	1.637	3.112 8E–4	1.374	1.396 5E–3	1.086
1.5	$1/2^2$	–	–	–	–	–	–
	$1/2^3$	1.464 9E–3	–	5.174 6E–3	–	1.296 6E–2	–
	$1/2^4$	4.881 7E–4	1.585	2.065 8E–3	1.325	6.241 3E–3	1.055
	$1/2^5$	1.592 1E–4	1.616	8.068 6E–4	1.356	2.964 5E–3	1.074
	$1/2^6$	5.119 6E–5	1.637	3.112 8E–4	1.374	1.396 5E–3	1.086
1.7	$1/2^2$	–	–	–	–	–	–
	$1/2^3$	1.116 1E–3	–	4.012 3E–3	–	1.024 0E–2	–
	$1/2^4$	3.708 6E–4	1.590	1.595 6E–3	1.330	4.952 4E–3	1.048
	$1/2^5$	1.207 3E–4	1.619	6.213 0E–4	1.361	2.355 9E–3	1.072
	$1/2^6$	3.877 2E–5	1.639	2.392 2E–4	1.377	1.110 2E–3	1.085
1.9	$1/2^2$	–	–	–	–	–	–
	$1/2^3$	9.517 5E–4	–	3.448 3E–3	–	8.882 7E–3	–
	$1/2^4$	3.157 6E–4	1.592	1.367 6E–3	1.334	4.298 6E–3	1.047
	$1/2^5$	1.026 9E–4	1.621	5.315 3E–4	1.363	2.044 1E–3	1.072
	$1/2^6$	3.295 9E–5	1.640	2.044 3E–4	1.379	9.628 9E–4	1.086

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Compliance with Ethical Standards

Conflict of Interest On behalf of all authors, the corresponding author states that there are no conflicts of interests/competing interests.

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