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ORIGINAL PAPER

Existence of Boundary Value Problems for Impulsive Fractional Differential Equations with a Parameter

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Abstract
We investigate a class of boundary value problems for nonlinear impulsive fractional differential equations with a parameter. By the deduction of Altman's theorem and Krasnoselskii's fixed point theorem, the existence of this problem is proved. Examples are given to illustrate the effectiveness of our results.

Keywords Fractional differential equations · Boundary value problems · Impulsive · Existence

Mathematics Subject Classification 34A08 · 34A37 · 34A12

1 Introduction

Fractional differential equations are essential tools of describing memory and hereditary properties of materials, and they are widely applied in physics, control theory, and engineering. Recently, they also attract many researchers and have a rapid development [5, 6, 9, 10, 14]. Boundary value problems of fractional differential equations have been studied by many scholars [1, 8, 13, 15, 16, 18] during the past years.

The theory of impulsive differential equations has rapid development over the years, see for the monographs by Bainov et al. [3], Lakshmikantham et al. [7] and references

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therein. The most prominent feature of impulsive fractional differential equations is taking the influence of the condition of system of sudden and abrupt phenomenon into full consideration, which also plays a very important role in modern science, and has been extensively used in population ecological dynamics system, infectious disease dynamics, and so on. In recent years, some researchers have studied impulsive fractional differential equations, see for example [11, 17]. However, the theory for fractional differential equations has not yet been sufficiently developed. For example, there are few papers considering the boundary value problems for impulsive fractional differential equations of order $2 < \alpha \leqslant 3$ with a parameter, the form of this equation is more complicated and the property of this equation is worth to study.

In this paper, we consider the existence of the solution for a class of boundary value problems of nonlinear impulsive fractional differential equations

$$\begin{cases} D^\alpha u(t) = \lambda f\big(t, u(t)\big),\ t \in J', \\ \Delta u(t_k) = Q_k\big(u(t_k)\big), k = 1, 2, \cdots, m, \\ \Delta u'(t_k) = R_k\big(u(t_k)\big), k = 1, 2, \cdots, m, \\ \Delta u''(t_k) = S_k\big(u(t_k)\big), k = 1, 2, \cdots, m, \\ u(0) + u'(0) = 0, u(1) + u'(1) = 0, u''(0) + u''(1) = 0, \end{cases} \tag{1}$$

where D^α is the Caputo derivative, $2 < \alpha \leqslant 3$, $J = [0, 1]$, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $Q_k, R_k, S_k \in C(\mathbb{R}, \mathbb{R})$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = 1$, $J' = J \backslash \{t_1, \cdots, t_m\}$, $J_0 = [0, t_1]$, $J_1 = (t_1, t_2]$, $\cdots$, $J_m = (t_m, 1]$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$, $\Delta u''(t_k) = u''(t_k^+) - u''(t_k^-)$.

The paper is organized as follows. In Sect. 2, we introduce some necessary notions, basic definitions, and Lemmas. In Sect. 3, using the deduction of Altman's theorem and Krasnoselskii's fixed point theorem, the existence of solutions is proved. In Sect. 4, two examples are given to demonstrate the applications of our results.

2 Preliminaries

We firstly introduce the following space:

$PC(J, \mathbb{R}) = \{u : J \to \mathbb{R} \mid u \in C(J_k), u(t_k^+), u(t_k^-) \text{ exist}, u(t_k^-) = u(t_k), k = 0, 1, \cdots, m\}$ with the norm $\|u\| = \sup\limits_{t \in J} |u(t)|$. Clearly, $PC(J, \mathbb{R})$ is a Banach space.

Definition 1 ([10]) The Caputo derivative of order α for a function u is given by

$$D_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} u^{(n)}(s) \mathrm{d}s, n-1 < \alpha < n.$$

Definition 2 ([3]) The set F is said to be quasi-equicontinuous in J, if for any $\varepsilon > 0$, there exists a $\delta > 0$, such that if $x \in F, k \in \mathbb{Z}$, $t', t'' \in (t_{k-1}, t_k] \cap J$ and $|t' - t''| < \delta$, then $|x(t') - x(t'')| < \varepsilon$.

Lemma 1 ([3]) *The set $F \subset PC(J, \mathbb{R})$ is relatively compact if and only if*

(i) *F is bounded*;
(ii) *F is quasi-equicontinuous in J*.

Lemma 2 ([12]) *Let D be a Banach space. Assume that Ω is an open bounded subset of D, $\theta \in D$, let $T : \overline{\Omega} \to D$ be a completely continuous operator, such that*

$$\|Tu\| \leqslant \|u\|, \forall u \in \partial\Omega.$$

Then, T has a fixed point in $\overline{\Omega}$.

Lemma 3 ([4]) *Let D be a convex closed and non-empty subset of Banach space $X \times Y$. Let F, G be the operators, such that*

(i) *$Fx + Gy \in D$, whenever $x, y \in D$*;
(ii) *F is compact and continuous and G is a contraction.*

Then, there exists a $z \in D$, such that $z = Fz + Gz$, where $z = (u, v) \in X \times Y$.

Lemma 4 ([2]) *Let $u(t) \in C[0, 1]$. Then, the solution of the fractional differential equation*

$$D^\alpha u(t) = 0$$

is $u(t) = C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1} t^{n-1}, C_i \in \mathbb{R}, i = 0, 1, 2, \cdots, n, n = [\alpha] + 1$.

Lemma 5 *For a given $f \in C(J \times \mathbb{R}, \mathbb{R})$, a function u is a solution of the following impulsive boundary value problem*:

$$\begin{cases} D^\alpha u(t) = f(t, u(t)), 2 < \alpha \leqslant 3, \quad t \in J', \\ \Delta u(t_k) = Q_k\big(u(t_k)\big), k = 1, 2, \cdots, m, \\ \Delta u'(t_k) = R_k\big(u(t_k)\big), k = 1, 2, \cdots, m, \\ \Delta u''(t_k) = S_k\big(u(t_k)\big), k = 1, 2, \cdots, m, \\ u(0) + u'(0) = 0, u(1) + u'(1) = 0, u''(0) + u''(1) = 0, \end{cases} \tag{2}$$

if and only if u is a solution of the impulsive fractional integral equation

$$\begin{aligned}
u(t) = {} & \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f\big(s, u(s)\big) \mathrm{d}s \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} f\big(s, u(s)\big) \mathrm{d}s \\
& + \sum_{i=1}^{k-1} \frac{t_k - t_i}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f\big(s, u(s)\big) \mathrm{d}s \\
& + \sum_{i=1}^{k-1} \frac{(t_k - t_i)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s \\
& + \sum_{i=1}^{k} \frac{t - t_k}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f\big(s, u(s)\big) \mathrm{d}s \\
& + \sum_{i=1}^{k-1} \frac{(t_k - t_i)(t - t_k)}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s \\
& + \sum_{i=1}^{k} \frac{(t - t_k)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s \\
& + \sum_{i=1}^{k} Q_i\big(u(t_i)\big) + \sum_{i=1}^{k-1} (t_k - t_i) R_i\big(u(t_i)\big) \\
& + \sum_{i=1}^{k-1} \frac{(t_k - t_i)^2}{2} S_i\big(u(t_i)\big) + \sum_{i=1}^{k} (t - t_k) R_i\big(u(t_i)\big) \\
& + \sum_{i=1}^{k-1} (t - t_k)(t_k - t_i) S_i\big(u(t_i)\big) \\
& + \sum_{i=1}^{k} \frac{(t - t_k)^2}{2} S_i\big(u(t_i)\big) - a_1 - a_2 t - a_3 t^2, \quad t \in J_k, k = 0, 1, 2, \cdots, m,
\end{aligned} \tag{3}$$

where

$$
\begin{aligned}
a_1 = -\Bigg(& \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} f\big(s, u(s)\big) \mathrm{d}s \\
& + \sum_{i=1}^{m-1} \frac{t_m - t_i}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f\big(s, u(s)\big) \mathrm{d}s \\
& + \sum_{i=1}^{m} \frac{1 - t_m}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f\big(s, u(s)\big) \mathrm{d}s \\
& + \sum_{i=1}^{m-1} \frac{(4 - t_m - t_i)(t_m - t_i)}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s \\
& + \sum_{i=1}^{m} \frac{t_m^2 - 4t_m + 3}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s \\
& + \sum_{i=1}^{m} Q_i\big(u(t_i)\big) + \sum_{i=1}^{m-1} (t_m - t_i) R_i\big(u(t_i)\big) \\
& + \sum_{i=1}^{m} \frac{2t_m^2 - 8t_m + 3}{4} S_i\big(u(t_i)\big) + \sum_{i=1}^{m} (2 - t_m) R_i\big(u(t_i)\big) \\
& + \sum_{i=1}^{m-1} \frac{(t_m - t_i)(4 - t_m - t_i)}{2} S_i\big(u(t_i)\big) \\
& + \sum_{i=1}^{m+1} \frac{1}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f\big(s, u(s)\big) \mathrm{d}s \\
& - \frac{3}{4} \sum_{i=1}^{m+1} \frac{1}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s \Bigg),
\end{aligned} \tag{4}
$$

$$
a_2 = -a_1, \tag{5}
$$

$$
\begin{aligned}
a_3 = & \frac{1}{4} \sum_{i=1}^{m+1} \frac{1}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s \\
& + \frac{1}{4} \sum_{i=1}^{m} S_i\big(u(t_i)\big).
\end{aligned} \tag{6}
$$

Proof Let u be a solution of (2). Then, by Lemma 4, we have

$$
\begin{aligned}
u(t) &= I^\alpha f\big(t, u(t)\big) - a_1 - a_2 t - a_3 t^2 \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f\big(s, u(s)\big) \mathrm{d}s - a_1 - a_2 t - a_3 t^2, t \in J_0.
\end{aligned}
$$

Therefore

$$u'(t) = \frac{1}{\Gamma(\alpha - 1)} \int_0^t (t-s)^{\alpha-2} f\big(s, u(s)\big) \mathrm{d}s - a_2 - 2a_3 t, t \in J_0,$$

$$u''(t) = \frac{1}{\Gamma(\alpha - 2)} \int_0^t (t-s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s - 2a_3, t \in J_0.$$

If $t \in J_1$, we have

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} f\big(s, u(s)\big) \mathrm{d}s - b_1 - b_2(t - t_1) - b_3(t - t_1)^2,$$

and

$$u'(t) = \frac{1}{\Gamma(\alpha - 1)} \int_{t_1}^t (t-s)^{\alpha-2} f\big(s, u(s)\big) \mathrm{d}s - b_2 - 2b_3(t - t_1),$$

$$u''(t) = \frac{1}{\Gamma(\alpha - 2)} \int_{t_1}^t (t-s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s - 2b_3,$$

$$\begin{aligned} u(t_1^-) &= \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f\big(s, u(s)\big) \mathrm{d}s - a_1 - a_2 t_1 - a_3 t_1^2, u({t_1}^+) = -b_1, \\ u'(t_1^-) &= \frac{1}{\Gamma(\alpha - 1)} \int_0^{t_1} (t_1-s)^{\alpha-2} f\big(s, u(s)\big) \mathrm{d}s - a_2 - 2a_3 t_1, u'({t_1}^+) = -b_2, \\ u''(t_1^-) &= \frac{1}{\Gamma(\alpha - 2)} \int_0^{t_1} (t_1-s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s - 2a_3, u''({t_1}^+) = -2b_3. \end{aligned}$$

While

$$\begin{aligned} \Delta u(t_1) &= u(t_1^+) - u(t_1^-) = Q_1\big(u(t_1)\big), \\ \Delta u'(t_1) &= u'(t_1^+) - u'(t_1^-) = R_1\big(u(t_1)\big), \\ \Delta u''(t) &= u'(t_1^+) - u'(t_1^-) = S_1\big(u(t_1)\big). \end{aligned}$$

We have

$$\begin{aligned} -b_1 &= \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f\big(s, u(s)\big) \mathrm{d}s - a_1 - a_2 t_1 - a_3 t_1^2 + Q_1\big(u(t_1)\big), \\ -b_2 &= \frac{1}{\Gamma(\alpha - 1)} \int_0^{t_1} (t_1-s)^{\alpha-2} f\big(s, u(s)\big) \mathrm{d}s - a_2 - 2a_3 t_1 + R_1\big(u(t_1)\big), \\ -2b_3 &= \frac{1}{\Gamma(\alpha - 2)} \int_0^{t_1} (t_1-s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s - 2a_3 + S_1\big(u(t_1)\big). \end{aligned}$$

Consequently, for $t \in J_1$

$$
\begin{aligned}
u(t) =& \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} f\big(s, u(s)\big) \mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f\big(s, u(s)\big) \mathrm{d}s \\
&+ \frac{(t-t_1)}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1-s)^{\alpha-2} f\big(s, u(s)\big) \mathrm{d}s + \frac{(t-t_1)^2}{2\Gamma(\alpha-2)} \int_0^{t_1} (t_1-s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s \\
&+ Q_1\big(u(t_1)\big) + (t-t_1) R_1\big(u(t_1)\big) + \frac{1}{2}(t-t_1)^2 S_1\big(u(t_1)\big) - a_1 - a_2 t - a_3 t^2.
\end{aligned}
$$

We can also get

$$
\begin{aligned}
u(t) =& \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f\big(s, u(s)\big) \mathrm{d}s \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} f\big(s, u(s)\big) \mathrm{d}s \\
&+ \sum_{i=1}^{k-1} \frac{t_k - t_i}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f\big(s, u(s)\big) \mathrm{d}s \\
&+ \sum_{i=1}^{k-1} \frac{(t_k - t_i)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s \\
&+ \sum_{i=1}^{k} \frac{t - t_k}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f\big(s, u(s)\big) \mathrm{d}s \\
&+ \sum_{i=1}^{k-1} \frac{(t_k - t_i)(t - t_k)}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s \\
&+ \sum_{i=1}^{k} \frac{(t - t_k)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s \\
&+ \sum_{i=1}^{k} Q_i\big(u(t_i)\big) + \sum_{i=1}^{k-1} (t_k - t_i) R_i\big(u(t_i)\big) \\
&+ \sum_{i=1}^{k-1} \frac{(t_k - t_i)^2}{2} S_i\big(u(t_i)\big) + \sum_{i=1}^{k} (t - t_k) R_i\big(u(t_i)\big) \\
&+ \sum_{i=1}^{k-1} (t - t_k)(t_k - t_i) S_i\big(u(t_i)\big) \\
&+ \sum_{i=1}^{k} \frac{(t - t_k)^2}{2} S_i\big(u(t_i)\big) - a_1 - a_2 t - a_3 t^2, \ t \in J_k, k = 0, 1, \cdots, m.
\end{aligned}
$$

On the condition that $u(0) + u'(0) = 0$, we have

$$a_1 = -a_2. \tag{7}$$

By $u(1) + u'(1) = 0$, we have

$$\begin{aligned} a_2 + 3a_3 =& \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} f\big(s, u(s)\big) \mathrm{d}s \\ &+ \sum_{i=1}^{m-1} \frac{t_m - t_i}{\Gamma(\alpha - 1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f\big(s, u(s)\big) \mathrm{d}s \\ &+ \sum_{i=1}^{m} \frac{1 - t_m}{\Gamma(\alpha - 1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f\big(s, u(s)\big) \mathrm{d}s \\ &+ \sum_{i=1}^{m-1} \frac{(4 - t_m - t_i)(t_m - t_i)}{2\Gamma(\alpha - 2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s \\ &+ \sum_{i=1}^{m} \frac{t_m^2 - 4t_m + 3}{2\Gamma(\alpha - 2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s \\ &+ \sum_{i=1}^{m} Q_i\big(u(t_i)\big) + \sum_{i=1}^{m-1} (t_m - t_i) R_i\big(u(t_i)\big) \\ &+ \sum_{i=1}^{m} \frac{t_m^2 - 4t_m + 3}{2} S_i\big(u(t_i)\big) \\ &+ \sum_{i=1}^{m} (2 - t_m) R_i\big(u(t_i)\big) + \sum_{i=1}^{m-1} \frac{(t_m - t_i)(4 - t_m - t_i)}{2} S_i\big(u(t_i)\big) \\ &+ \sum_{i=1}^{m+1} \frac{1}{\Gamma(\alpha - 1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f\big(s, u(s)\big) \mathrm{d}s. \end{aligned} \tag{8}$$

Combining (7) and (8) with the condition $u''(1) + u''(0) = 0$, we conclude that

$$
\begin{aligned}
a_1 = &-\Bigg(\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}f\big(s,u(s)\big)\mathrm{d}s\\
&+\sum_{i=1}^{m-1}\frac{t_m-t_i}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f\big(s,u(s)\big)\mathrm{d}s+\sum_{i=1}^{m}\frac{1-t_m}{\Gamma(\alpha-1)}(t_i-s)^{\alpha-2}f\big(s,u(s)\big)\mathrm{d}s\\
&+\sum_{i=1}^{m-1}\frac{(4-t_m-t_i)(t_m-t_i)}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)\mathrm{d}s\\
&+\sum_{i=1}^{m}\frac{t_m^2-4t_m+3}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)\mathrm{d}s\\
&+\sum_{i=1}^{m}Q_i\big(u(t_i)\big)+\sum_{i=1}^{m-1}(t_m-t_i)R_i\big(u(t_i)\big)\\
&+\sum_{i=1}^{m}\frac{2t_m^2-8t_m+3}{4}S_i\big(u(t_i)\big)+\sum_{i=1}^{m}(2-t_m)R_i\big(u(t_i)\big)\\
&+\sum_{i=1}^{m-1}\frac{(t_m-t_i)(4-t_m-t_i)}{2}S_i\big(u(t_i)\big)+\sum_{i=1}^{m+1}\frac{1}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f\big(s,u(s)\big)\mathrm{d}s\\
&-\frac{3}{4}\sum_{i=1}^{m+1}\frac{1}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)\mathrm{d}s\Bigg),\\
a_3 = &\frac{1}{4}\sum_{i=1}^{m+1}\frac{1}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)\mathrm{d}s+\frac{1}{4}\sum_{i=1}^{m}S_i\big(u(t_i)\big).
\end{aligned}
$$

Conversely, if we assume that u is the solution of (4), through directly calculation, it shows that the solution given by (3) satisfies (2). This completes the proof.

Assume the following holds.

(H_1) There exist constants K, L, M, N, such that $|f\big(t,u(t)\big)| \leqslant K$, $|Q_k| \leqslant L$, $|R_k| \leqslant M$, $|S_k| \leqslant N$.

(H_2) $\limsup\limits_{u\to 0}\frac{\max_{t\in[0,1]} f(t,u)}{u} < \infty$, $\lim\limits_{u\to 0}\frac{Q_k(u)}{u} < \infty$, $\lim\limits_{u\to 0}\frac{R_k(u)}{u} < \infty$, $\lim\limits_{u\to 0}\frac{S_k(u)}{u} < \infty$.

(H_3) There exist constants $L_1, L_2, L_3, L_4 > 0$, $u, v \in \mathbb{R}$, such that

$$
\begin{aligned}
|f(t,u)-f(t,v)| \leqslant L_1|u-v|,\ |Q_k(u)-Q_k(v)| \leqslant L_2|u-v|,\\
|R_k(u)-R_k(v)| \leqslant L_3|u-v|,\ |S_k(u)-S_k(v)| \leqslant L_4|u-v|.
\end{aligned}
$$

We define an open bounded set $\Omega = \{u \in PC(J,\mathbb{R}) |\ \|u\| < r\}$, and then, $\overline{\Omega} = \{u \in PC(J,\mathbb{R}) |\ \|u\| \leqslant r\}$.

3 Main Results

In this section, we will prove the existence of (1) by Lemmas 2 and 3.

First of all, we define an operator $T:\overline{\Omega}\to PC(J,\mathbb{R})$ as

$$\begin{aligned} Tu(t)=&\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f\big(s,u(s)\big)\mathrm{d}s\\ &+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}f\big(s,u(s)\big)\mathrm{d}s\\ &+\sum_{i=1}^{k-1}\frac{t_k-t_i}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f\big(s,u(s)\big)\mathrm{d}s\\ &+\sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)\mathrm{d}s\\ &+\sum_{i=1}^{k}\frac{t-t_k}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}f\big(s,u(s)\big)\mathrm{d}s\\ &+\sum_{i=1}^{k-1}\frac{(t_k-t_i)(t-t_k)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)\mathrm{d}s\\ &+\sum_{i=1}^{k}\frac{(t-t_k)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}f\big(s,u(s)\big)\mathrm{d}s\\ &+\sum_{i=1}^{k}Q_i\big(u(t_i)\big)+\sum_{i=1}^{k-1}(t_k-t_i)R_i\big(u(t_i)\big)\\ &+\sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2}S_i\big(u(t_i)\big)+\sum_{i=1}^{k}(t-t_k)R_i\big(u(t_i)\big)+\sum_{i=1}^{k-1}(t-t_k)(t_k-t_i)S_i\big(u(t_i)\big)\\ &+\sum_{i=1}^{k}\frac{(t-t_k)^2}{2}S_i\big(u(t_i)\big)-a_1-a_2t-a_3t^2,\quad t\in J_k, k=0,1,2,\cdots,m, \end{aligned} \tag{9}$$

where a_1, a_2, and a_3 are defined as (4)–(6).

Lemma 6 $T:\overline{\Omega}\to PC(J,\mathbb{R})$ *is completely continuous with the assumption* (H_1).

Proof First, we note that T is continuous on account of the continuity of f, Q_k, R_k, S_k. Then, under the assumption (H_1), for any $u\in\overline{\Omega}$, $t\in J_k, k=0,1,\cdots,m$, we can obtain

$$
\begin{aligned}
|a_1| \leqslant{} & \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |f(s, u(s))| \mathrm{d}s \\
& + \sum_{i=1}^{m-1} \frac{t_m - t_i}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} |f(s, u(s))| \mathrm{d}s + \sum_{i=1}^{m} \frac{1 - t_m}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} |f(s, u(s))| \mathrm{d}s \\
& + \sum_{i=1}^{m-1} \frac{(4 - t_m - t_i)(t_m - t_i)}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} |f(s, u(s))| \mathrm{d}s + \sum_{i=1}^{m} \frac{t_m^2 - 4t_m + 3}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} |f(s, u(s))| \mathrm{d}s \\
& + \sum_{i=1}^{m} |Q_i(u(t_i))| + \sum_{i=1}^{m-1} (t_m - t_i)|R_i(u(t_i))| + \sum_{i=1}^{m} \frac{|2t_m^2 - 8t_m + 3|}{4} |S_i(u(t_i))| \\
& + \sum_{i=1}^{m-1} \frac{(t_m - t_i)(4 - t_m - t_i)}{2} |S_i(u(t_i))| + \sum_{i=1}^{m} (2 - t_m)|R_i(u(t_i))| \\
& + \sum_{i=1}^{m+1} \frac{1}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} |f(s, u(s))| \mathrm{d}s + \frac{3}{4} \sum_{i=1}^{m+1} \frac{1}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} |f(s, u(s))| \mathrm{d}s \\
\leqslant{} & \frac{K}{\Gamma(\alpha)} \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \mathrm{d}s + \sum_{i=1}^{m-1} \frac{K}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} \mathrm{d}s \\
& + \sum_{i=1}^{m} \frac{K}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} \mathrm{d}s + \sum_{i=1}^{m-1} \frac{2K}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} \mathrm{d}s \\
& + \sum_{i=1}^{m} \frac{3K}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} \mathrm{d}s + mL + (m-1)M + \frac{7m-1}{4} N \\
& + \sum_{i=1}^{m+1} \frac{K}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} \mathrm{d}s + \frac{3}{4} \sum_{i=1}^{m+1} \frac{K}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} \mathrm{d}s \leqslant \frac{(m+1)K}{\Gamma(\alpha+1)} \\
& + \frac{3mK}{\Gamma(\alpha)} + \frac{(17m-5)K}{4\Gamma(\alpha-1)} + mL + (m-1)M + \left(\frac{7m}{4} - 1\right)N, \\
|a_3| ={} & \frac{1}{4} \sum_{i=1}^{m+1} \frac{1}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} |f(s, u(s))| \mathrm{d}s + \frac{1}{4} \sum_{i=1}^{m} |S_i(u(t_i))| \leqslant \frac{(m+1)K}{4\Gamma(\alpha-1)} + \frac{1}{4} mN.
\end{aligned}
$$

$$
\begin{aligned}
|(Tu(t))| \leqslant{} & \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t_i - s)^{\alpha-1} |f(s, u(s))| \mathrm{d}s + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |f(s, u(s))| \mathrm{d}s \\
& + \sum_{i=1}^{k-1} \frac{t_k - t_i}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} |f(s, u(s))| \mathrm{d}s + \sum_{i=1}^{k-1} \frac{(t_k - t_i^2)}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} |f(s, u(s))| \mathrm{d}s \\
& + \sum_{i=1}^{k} \frac{t - t_k}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} |f(s, u(s))| \mathrm{d}s + \sum_{i=1}^{k-1} \frac{(t_k - t_i)(t - t_k)}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} |f(s, u(s))| \mathrm{d}s \\
& + \sum_{i=1}^{k} \frac{(t - t_k)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} |f(s, u(s))| \mathrm{d}s + \sum_{i=1}^{k} |Q_i(u(t_i))| + \sum_{i=1}^{k-1} (t_k - t_i)|R_i(u(t_i))| \\
& + \sum_{i=1}^{k-1} \frac{(t_k - t_i)^2}{2} |S_i(u(t_i))| + \sum_{i=1}^{k} (t - t_k)|R_i(u(t_i))| \\
& + \sum_{i=1}^{k-1} (t - t_k)(t_k - t_i)|S_i(u(t_i))| + \sum_{i=1}^{k} \frac{(t - t_k)^2}{2} |S_i(u(t_i))| + |a_1| + |a_2| + |a_3| \\
\leqslant{} & \frac{3(m+1)K}{\Gamma(\alpha+1)} + \frac{(8m-1)K}{\Gamma(\alpha)} + \frac{(43m-15)K}{4\Gamma(\alpha-1)} + 4mL + (4m-3)M + \frac{22m-13}{4} N := H.
\end{aligned}
$$

Therefore, $T\overline{\Omega}$ is bounded.

We can also obtain that for any $u \in \overline{\Omega},\ t \in J_k, k = 0, 1, 2, \cdots, m$

$$
\begin{aligned}
|\big((Tu)'(t)\big)| \leqslant{}& \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t_i - s)^{\alpha-1} |f\big(s, u(s)\big)| \mathrm{d}s \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |f\big(s, u(s)\big)| \mathrm{d}s \\
&+ \sum_{i=1}^{k-1} \frac{(t_k - t_i)}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} |f\big(s, u(s)\big)| \mathrm{d}s \\
&+ \sum_{i=1}^{k} \frac{(t_k - t_i)}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} |f\big(s, u(s)\big)| \mathrm{d}s \\
&+ \sum_{i=1}^{k} |R_i\big(u(t_i)\big)| + \sum_{i=1}^{k-1} (t_k - t_i) |S_i\big(u(t_i)\big)| \\
&+ \sum_{i=1}^{k} (t - t_k) |S_i\big(u(t_i)\big)| + |a_2| + 2|a_3| \\
\leqslant{}& \frac{(4m+1)K}{\Gamma(\alpha)} + \frac{(m+1)K}{\Gamma(\alpha+1)} \\
&+ \frac{(20m-5)K}{2\Gamma(\alpha-1)} + mL + (2m-1)M + \frac{(17m-8)N}{4} := \overline{H}.
\end{aligned}
$$

For any $u \in \overline{\Omega}$, $t', t'' \in J_k, t' \leqslant t''$, there exists at least one point $\xi_k \in (t', t''), k = 0, 1, 2, \cdots, m$

$$
|(Tu)(t'') - (Tu)(t')| \leqslant |(Tu)'(\xi_k)|(t'' - t') \leqslant \overline{H}(t'' - t')
$$

when $|t'' - t'| \to 0$, $\|(Tu)(t'') - (Tu)(t')\| \to 0$. Therefore, $T\Omega$ is quasi-equicontinuous.

Thus, by Lemma 1, $T : \overline{\Omega} \to PC(J, \mathbb{R})$ is completely continuous.

This completes the proof.

In view of the solution of existence for a class of boundary value problem of impulsive fractional differential equation with a parameter, define T_λ as (9), where we use λf to take place of f. By Lemma 5, problem (1) has a solution if and only if $u = T_\lambda u$ has a fixed point.

We denote

$$
p = \frac{1 - 4ml_2 - (4m-3)l_3 - \frac{(22m-13)}{4} l_4}{\frac{3(m+1)l_1}{\Gamma(\alpha+1)} + \frac{(8m-1)l_1}{\Gamma(\alpha)} + \frac{(43m-15)l_1}{4\Gamma(\alpha-1)}},
$$
$$
p' = \frac{1 - mL_2 - (3m-1)L_3 - 2(m-1)L_4}{\frac{(m+1)L_1}{\Gamma(\alpha+1)} + \frac{(3m+2)L_1}{\Gamma(\alpha)} + \frac{7mL_1}{2\Gamma(\alpha-1)}}.
$$

Theorem 1 *Assume that* (H_1) – (H_2) *and* $|\lambda| \leqslant p$ *hold. Then the problem* (1) *has at least one solution.*

Proof From (H_2), there exist positive constants l_1, l_2, l_3, l_4, such that $|f(t, u)| \leqslant l_1|u|$, $|Q_k(u)| \leqslant l_2|u|$, $|R_k(u)| \leqslant l_3|u|$, $|S_k(u)| \leqslant l_4|u|$. When $u \in \partial\Omega$, i.e., $\|u\| = r$, we have

$$
\begin{aligned}
\|T_\lambda u\| \leqslant{} & \frac{2|\lambda|}{\Gamma(\alpha)} \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |f\big(s, u(s)\big)| \mathrm{d}s \\
& + \sum_{i=1}^{m-1} \frac{2|\lambda|(t_m - t_i)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} |f\big(s, u(s)\big)| \mathrm{d}s \\
& + \sum_{i=1}^{m-1} \frac{|\lambda|(t_m - t_i)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} |f\big(s, u(s)\big))| \mathrm{d}s \\
& + \sum_{i=1}^{m} \frac{|\lambda| 2(1 - t_m)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} |f\big(s, u(s)\big)| \mathrm{d}s \\
& + \sum_{i=1}^{m-1} \frac{|\lambda|(t_m - t_i)(3 - 2t_m - t_i)}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} |f\big(s, u(s)\big)| \mathrm{d}s \\
& + \sum_{i=1}^{m} \frac{|\lambda|(t_m^2 - 3t_m + 2)}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} |f\big(s, u(s)\big)| \mathrm{d}s \\
& + 2\sum_{i=1}^{m} |Q_i\big(u(t_i)\big)| + 2\sum_{i=1}^{m-1} (t_m - t_i)|R_i\big(u(t_i)\big)| \\
& + \sum_{i=1}^{m-1} \frac{(t_m - t_i)^2}{2} |S_i\big(u(t_i)\big)| + \sum_{i=1}^{m} (3 - 2t_m)|R_i\big(u(t_i)\big)| \\
& + \sum_{i=1}^{m-1} (1 - t_m)(t_m - t_i)|S_i\big(u(t_i)\big)| + \sum_{i=1}^{m} \frac{|2t_m^2 - 6t_m + 3|}{2} |S_i\big(u(t_i)\big)| \\
& + \sum_{i=1}^{m} \frac{(t_m - t_i)(4 - t_m - t_i)}{2} |S_i\big(u(t_i)\big)| + \sum_{i=1}^{m+1} \frac{1}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} |f\big(s, u(s)\big)| \mathrm{d}s \\
& + \sum_{i=1}^{m+1} \frac{1}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} |f\big(s, u(s)\big)| \mathrm{d}s \\
\leqslant{} & \Bigg(\frac{2l_1|\lambda|}{\Gamma(\alpha)} \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \mathrm{d}s + \sum_{i=1}^{m-1} \frac{2l_1|\lambda|(t_m - t_i)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} \mathrm{d}s \\
& + \sum_{i=1}^{m-1} \frac{(t_m - t_i)^2|\lambda| l_1}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} \mathrm{d}s + \sum_{i=1}^{m} \frac{2(1 - t_m) l_1|\lambda|}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} \mathrm{d}s \\
& + \sum_{i=1}^{m-1} \frac{(t_m - t_i)(3 - 2t_m - t_i)|\lambda| l_1}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} \mathrm{d}s \\
& + \sum_{i=1}^{m} \frac{(t_m^2 - 3t_m + 2)|\lambda| l_1}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} \mathrm{d}s \\
& + 2ml_2 + 2\sum_{i=1}^{m-1} (t_m - t_i) l_3 + \sum_{i=1}^{m-1} \frac{(t_m - t_i)^2}{2} l_4 + \sum_{i=1}^{m} (3 - 2t_m) l_3 \\
& + \sum_{i=1}^{m-1} (1 - t_m)(t_m - t_i) l_4 + \sum_{i=1}^{m} \frac{|2t_m^2 - 6t_m + 3|}{2} l_4 + \sum_{i=1}^{m} \frac{(t_m - t_i)(4 - t_m - t_i)}{2} l_4 \\
& + \sum_{i=1}^{m+1} \frac{|\lambda| l_1}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} \mathrm{d}s + \sum_{i=1}^{m+1} \frac{|\lambda| l_1}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} \mathrm{d}s \Bigg) \|u\| \\
\leqslant{} & \Bigg(\bigg(\frac{3(m+1)l_1}{\Gamma(\alpha+1)} + \frac{(8m-1)l_1}{\Gamma(\alpha)} + \frac{(43m-15)l_1}{4\Gamma(\alpha-1)} \bigg) |\lambda| \\
& + 4ml_2 + (4m-3)l_3 + \frac{(22m-13)}{4} l_4 \Bigg) \|u\|,
\end{aligned}
$$

so we have $\|T_\lambda u\| \leqslant \|u\|$, for any $u \in \partial\Omega$. T_λ is completely continuous.

By Lemma 2, the operator T_λ has at least one fixed point in $\overline{\Omega}$, which also implies that problem (1) has at least one solution in $\overline{\Omega}$.

This completes the proof.

Now, we split the operator T_λ into two parts as $T_\lambda = F_\lambda + G_\lambda$, where

$$\begin{aligned}
F_\lambda u(t) =& \frac{\lambda}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f\big(s, u(s)\big)\mathrm{d}s \\
&+ \frac{\lambda}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} f\big(s, u(s)\big)\mathrm{d}s \\
&+ \sum_{i=1}^{k-1} \frac{\lambda(t_k - t_i)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f\big(s, u(s)\big)\mathrm{d}s \\
&+ \sum_{i=1}^{k-1} \frac{\lambda(t_k - t_i)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big)\mathrm{d}s \\
&+ \sum_{i=1}^{k} \frac{\lambda(t - t_k)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f\big(s, u(s)\big)\mathrm{d}s \\
&+ \sum_{i=1}^{k-1} \frac{\lambda(t_k - t_i)(t - t_k)}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big)\mathrm{d}s \\
&+ \sum_{i=1}^{k} \frac{\lambda(t - t_k)^2}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big)\mathrm{d}s \\
&+ \sum_{i=1}^{k} Q_i\big(u(t_i)\big) + \sum_{i=1}^{k-1} (t_k - t_i) R_i\big(u(t_i)\big) \\
&+ \sum_{i=1}^{k-1} \frac{(t_k - t_i)^2}{2} S_i\big(u(t_i)\big) + \sum_{i=1}^{k} (t - t_k) R_i\big(u(t_i)\big) + \sum_{i=1}^{k-1} (t - t_k)(t_k - t_i) S_i\big(u(t_i)\big) \\
&+ \sum_{i=1}^{k} \frac{(t - t_k)^2}{2} S_i\big(u(t_i)\big), t \in J_k, k = 0, 1, 2, \cdots, m.
\end{aligned}$$

Obviously, F_λ is completely continuous. And

$$
\begin{aligned}
G_\lambda u(t) =& \Bigg(\frac{\lambda}{\Gamma(\alpha)} \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} f\big(s, u(s)\big) \mathrm{d}s \\
&+ \sum_{i=1}^{m-1} \frac{\lambda(t_m - t_i)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f\big(s, u(s)\big) \mathrm{d}s \\
&+ \sum_{i=1}^{m} \frac{\lambda(1 - t_m)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f\big(s, u(s)\big) \mathrm{d}s \\
&+ \sum_{i=1}^{m-1} \frac{\lambda(4 - t_m - t_i)(t_m - t_i)}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s \\
&+ \sum_{i=1}^{m} \frac{\lambda(t_m^2 - 4t_m + 3)}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s \\
&+ \sum_{i=1}^{m} Q_i\big(u(t_i)\big) + \sum_{i=1}^{m-1} (t_m - t_i) R_i\big(u(t_i)\big) \\
&+ \sum_{i=1}^{m} \frac{(2t_m^2 - 8t_m + 3)}{4} S_i\big(u(t_i)\big) + \sum_{i=1}^{m} (2 - t_m) R_i\big(u(t_i)\big) \\
&+ \sum_{i=1}^{m-1} \frac{(t_m - t_i)(4 - t_m - t_i)}{2} S_i\big(u(t_i)\big) \\
&+ \sum_{i=1}^{m+1} \frac{\lambda}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} f\big(s, u(s)\big) \mathrm{d}s \\
&- \frac{3}{4} \sum_{i=1}^{m+1} \frac{\lambda}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s \Bigg)(1 - t) \\
&- \Bigg(\frac{1}{4} \sum_{i=1}^{m+1} \frac{\lambda}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} f\big(s, u(s)\big) \mathrm{d}s \\
&+ \frac{1}{4} \sum_{i=1}^{m} S_i\big(u(t_i)\big) \Bigg) t^2.
\end{aligned}
$$

Theorem 2 *Under the assumption of* $(\mathrm{H}_1)-(\mathrm{H}_3)$, *and* $|\lambda| \leqslant \min(p, p')$, *problem* (1) *has at least one solution.*

Proof Step 1 From (H_2), during similarly calculation as before, we have

$$
\begin{aligned}
|F_\lambda(u(t))+G_\lambda(v(t))| \leqslant & \frac{|\lambda|}{\Gamma(\alpha)}\int_{t_k}^{t}(t_i-s)^{\alpha-1}|f\big(s,u(s)\big)|\mathrm{d}s+\frac{|\lambda|}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|f\big(s,u(s)\big)|\mathrm{d}s\\
&+|\lambda|\sum_{i=1}^{k-1}\frac{t_k-t_i}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}|f\big(s,u(s)\big)|\mathrm{d}s\\
&+|\lambda|\sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}|f\big(s,u(s)\big))|\mathrm{d}s\\
&+|\lambda|\sum_{i=1}^{k}\frac{t-t_k}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}|f\big(s,u(s)\big)|\mathrm{d}s\\
&+|\lambda|\sum_{i=1}^{k-1}\frac{(t_k-t_i)(1-t_k)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}|f\big(s,u(s)\big)|\mathrm{d}s\\
&+|\lambda|\sum_{i=1}^{k}\frac{(t-t_k)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}|f\big(s,u(s)\big)|\mathrm{d}s+\sum_{i=1}^{k}|Q_i\big(u(t_i)\big)|+\sum_{i=1}^{k-1}(t_k-t_i)|R_i\big(u(t_i)\big)|\\
&+\sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2}|S_i\big(u(t_i)\big)|+\sum_{i=1}^{k}(t-t_k)|R_i\big(u(t_i)\big)|\\
&+\sum_{i=1}^{k-1}(t-t_k)(t_k-t_i)|S_i\big(u(t_i)\big)|+\sum_{i=1}^{k}\frac{(t-t_k)^2}{2}|S_i\big(u(t_i)\big)|\\
&+|\lambda|\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|\big(f(s,v(s)\big)|\mathrm{d}s+|\lambda|\sum_{i=1}^{m-1}\frac{t_m-t_i}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}|\big(f(s,v(s)\big)|\mathrm{d}s\\
&+|\lambda|\sum_{i=1}^{m}\frac{1-t_m}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}|\big(f(s,v(s)\big))|\mathrm{d}s\\
&+|\lambda|\sum_{i=1}^{m-1}\frac{(4-t_m-t_i)(t_m-t_i)}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}|\big(f(s,v(s)\big)|\mathrm{d}s\\
&+|\lambda|\sum_{i=1}^{m}\frac{t_m^2-4t_m+3}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}|\big(f(s,v(s)\big)|\mathrm{d}s+\sum_{i=1}^{m}|Q_i\big(v(t_i)\big)|+\sum_{i=1}^{m-1}(t_m-t_i)|R_i\big(v(t_i)\big)|\\
&+\sum_{i=1}^{m-1}(2+t_m^2-4t_m)|S_i\big(v(t_i)\big)|+\sum_{i=1}^{m}(2-t_m)|R_i\big(v(t_i)\big)|\\
&+\sum_{i=1}^{m-1}\frac{(t_m-t_i)(4-t_m-t_i)}{2}|S_i\big(v(t_i)\big)|+|\lambda|\sum_{i=1}^{m+1}\frac{1}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}|\big(f(s,v(s)\big)|\mathrm{d}s\\
&+|\lambda|\sum_{i=1}^{m+1}\frac{1}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}|\big(f(s,v(s)\big)|\mathrm{d}s\\
\leqslant&\bigg(\frac{(m+1)|\lambda|l_1}{\Gamma(\alpha+1)}+\frac{(2m-1)|\lambda|l_1}{\Gamma(\alpha)}\frac{(4m-3)|\lambda|l_1}{2\Gamma(\alpha-1)}+ml_2+(2m-1)l_3+\frac{4m-3}{2}l_4\bigg)\|u\|\\
&+\bigg(\frac{(m+1)|\lambda|l_1}{\Gamma(\alpha+1)}+\frac{3m|\lambda|l_1}{\Gamma(\alpha)}+\frac{7m|\lambda|l_1}{2\Gamma(\alpha-1)}+ml_2+2ml_3+3(m-1)l_4\bigg)\|v\|\\
\leqslant&\bigg(\frac{2(m+1)|\lambda|l_1}{\Gamma(\alpha+1)}+\frac{(5m-1)|\lambda|l_1}{\Gamma(\alpha)}+\frac{(11m-3)|\lambda|l_1}{2\Gamma(\alpha-1)}\\
&+2ml_2+(4m-1)l_3+\frac{10m-9}{2}l_4\bigg)r<r.
\end{aligned}
$$

Thus, $|F_\lambda(u(t))+G_\lambda(v(t))|\in\overline{\Omega}$.

Step 2 From (H_3), for any $u,v\in\overline{\Omega}$, $t\in J_k$, $k=0,1,\cdots,m$, we can get

$$
\begin{aligned}
|G_\lambda u(t) - G_\lambda v(t)| \leqslant & |\lambda| \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |\big(f(s,u(s)) - f(s,v(s))\big)| \mathrm{d}s \\
& + |\lambda| \sum_{i=1}^{m-1} \frac{t_m - t_i}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} |\big(f(s,u(s)) - f(s,v(s))\big)| \mathrm{d}s \\
& + |\lambda| \sum_{i=1}^{m} \frac{1 - t_m}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} |\big(f(s,u(s)) - f(s,v(s))\big)| \mathrm{d}s \\
& + |\lambda| \sum_{i=1}^{m-1} \frac{(4 - t_m - t_i)(t_m - t_i)}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} |\big(f(s,u(s)) - f(s,v(s))\big)| \mathrm{d}s \\
& + |\lambda| \sum_{i=1}^{m} \frac{t_m^2 - 4t_m + 3}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} |\big(f(s,u(s)) - f(s,v(s))\big)| \mathrm{d}s \\
& + \sum_{i=1}^{m} |Q_i\big(u(t_i)\big) - Q_i\big(v(t_i)\big)| + \sum_{i=1}^{m-1} (t_m - t_i) |R_i\big(u(t_i)\big) - R_i\big(v(t_i)\big)| \\
& + \sum_{i=1}^{m-1} (2 + t_m^2 - 4t_m) |S_i\big(u(t_i)\big) - S_i\big(v(t_i)\big)| + \sum_{i=1}^{m} (2 - t_m) |R_i\big(u(t_i)\big) - R_i\big(v(t_i)\big)| \\
& + \sum_{i=1}^{m-1} \frac{(t_m - t_i)(4 - t_m - t_i)}{2} |S_i\big(u(t_i)\big) - S_i\big(v(t_i)\big)| \\
& + |\lambda| \sum_{i=1}^{m+1} \frac{1}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} |\big(f(s,u(s)) - f(s,v(s))\big)| \mathrm{d}s \\
& + |\lambda| \sum_{i=1}^{m+1} \frac{1}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} |\big(f(s,u(s)) - f(s,v(s))\big)| \mathrm{d}s \\
\leqslant & \Bigg(|\lambda| \frac{L_1}{\Gamma(\alpha)} \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \mathrm{d}s \\
& + |\lambda| \sum_{i=1}^{m-1} \frac{(t_m - t_i)L_1}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} \mathrm{d}s \\
& + |\lambda| \sum_{i=1}^{m} \frac{(1 - t_m)L_1}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} \mathrm{d}s \\
& + |\lambda| \sum_{i=1}^{m-1} \frac{(4 - t_m - t_i)(t_m - t_i)L_1}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} \mathrm{d}s \\
& + |\lambda| \sum_{i=1}^{m} \frac{(t_m^2 - 4t_m + 3)L_1}{2\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} \mathrm{d}s + \sum_{i=1}^{m} L_2 + \sum_{i=1}^{m-1} (t_m - t_i) L_3 \\
& + \sum_{i=1}^{m-1} (2 + t_m^2 - 4t_m) L_4 + \sum_{i=1}^{m} (2 - t_m) L_3 + \sum_{i=1}^{m-1} \frac{(t_m - t_i)(4 - t_m - t_i)}{2} L_4 \\
& + |\lambda| \sum_{i=1}^{m+1} \frac{L_1}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} \mathrm{d}s + |\lambda| \sum_{i=1}^{m+1} \frac{L_1}{\Gamma(\alpha-2)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-3} \mathrm{d}s \Bigg) \|u - v\| \\
\leqslant & \Bigg(\frac{(m+1)|\lambda| L_1}{\Gamma(\alpha+1)} + \frac{(3m+2)|\lambda| L_1}{\Gamma(\alpha)} \\
& + \frac{7m|\lambda| L_1}{2\Gamma(\alpha-1)} + mL_2 + (2m-1)L_3 + 3(m-1)L_4 \Bigg) \|u - v\|.
\end{aligned}
$$

Therefore, G_λ is a contraction.

This completes the proof. When $|\lambda| \leqslant \min(p, p')$, by Lemma 3, problem (1) has at least one solution in $\overline{\Omega}$.

4 Examples

Example 1 Consider the following boundary value problems of impulsive fractional differential equations:

$$\begin{cases} D^{\alpha}u(t) = \lambda(\frac{\sin u^3}{1+u^2}), 2 < \alpha \leqslant 3, t \neq \frac{1}{3}, \\ \Delta u(\frac{1}{3}) = e^{u^2} - 1, \\ \Delta u'(\frac{1}{3}) = \ln(1+u^2) - u, \\ \Delta u''(\frac{1}{3}) = (1+u^2)^{\frac{1}{4}} - 1, \\ u(0) + u'(0) = 0, u(1) + u'(1) = 0, u''(0) + u''(1) = 0. \end{cases} \tag{10}$$

Let

$$l_1 = l_2 = l_3 = l_4 = \frac{1}{20},$$
$$|\lambda| \leqslant \frac{51}{4(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} + \frac{1}{4(\Gamma(\alpha-1))})},$$

which satisfy

(i) $|f(t,u)| \leqslant 1, |Q(u)| \leqslant 1, |R(u)| \leqslant 1, |S(u)| \leqslant 1,$

(ii) $\limsup\limits_{u\to 0} \frac{\max f\left(t, u(t)\right)}{u} = 0, \lim\limits_{u\to 0} \frac{Q(u)}{u} = 0, \lim\limits_{u\to 0} \frac{R(u)}{u} = -1, \lim\limits_{u\to 0} \frac{S(u)}{u} = 0.$

All the assumptions of Theorem 1 are satisfied. Thus, problem (10) has at least one solution.

Example 2 Consider the following boundary value problems of impulsive fractional differential equations:

$$\begin{cases} D^{\alpha}u(t) = \lambda(\frac{\ln(1+u^2)}{u}),\ 2 < \alpha \leqslant 3, t \neq \frac{1}{4}, \\ \Delta u(\frac{1}{4}) = \frac{u^3}{3+u^2}, \\ \Delta u'(\frac{1}{4}) = \frac{u^3}{4+u^2}, \\ \Delta u''(\frac{1}{4}) = \frac{u^3}{5+u^2}, \\ u(0) + u'(0) = 0, u(1) + u'(1) = 0, u''(0) + u''(1) = 0. \end{cases} \tag{11}$$

Let

$$
\begin{aligned}
&l_1 = l_2 = l_3 = l_4 = \frac{1}{30},\\
&p = \frac{91}{4(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha-1)})},\\
&p' = \frac{193}{140(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha-1)})},\\
&|\lambda| \leqslant \min(p, p'),
\end{aligned}
$$

which satisfy

(i) $|f(t,u)| \leqslant 1, |Q(u)| \leqslant 1, |R(u)| \leqslant 1, |S(u)| \leqslant 1,$

(ii) $\limsup\limits_{u\to 0} \dfrac{\max f\big(t, u(t)\big)}{u} = 0, \lim\limits_{u\to 0} \dfrac{Q(u)}{u} = 0, \lim\limits_{u\to 0} \dfrac{R(u)}{u} = 0, \lim\limits_{u\to 0} \dfrac{S(u)}{u} = 0,$

(iii) $|f(t,u) - f(t,v)| \leqslant \dfrac{1}{20}|u-v|, |Q_k(u) - Q_k(v)| \leqslant \dfrac{1}{40}|u-v|,$

$|R_k(u) - R_k(v)| \leqslant \dfrac{1}{100}|u-v|, |S_k(u) - S_k(v)| \leqslant \dfrac{1}{60}|u-v|.$

All the assumptions of Theorem 2 are satisfied. Thus, the problem (11) has at least one solution.

5 Conclusions

In this paper, we investigate a class of boundary value problems for nonlinear impulsive fractional differential equations with a parameter. We give sufficient conditions to obtain the existence of this problem for the first time by deduction of Altman's theorem and Krasnoselskii's fixed point theorem. Our results enrich the study for impulsive fractional differential equations of order $2 < \alpha \leqslant 3$ with a parameter. We will explore the eigenvalue problems for this equation in the future.

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Compliance with Ethical Standards

Conflict of Interest The authors have no any conflict of interest.

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