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ORIGINAL PAPER

A Note on Numerical Algorithm for the Time-Caputo and Space-Riesz Fractional Diffusion Equation

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Abstract

Recently, Zhang and Ding developed a novel finite difference scheme for the time-Caputo and space-Riesz fractional diffusion equation with the convergence order $\mathcal{O}(\tau^{2-\alpha}+h^2)$ in Zhang and Ding (Commun. Appl. Math. Comput. 2(1): 57–72, 2020). Unfortunately, they only gave the stability and convergence results for $\alpha \in (0,1)$ and $\beta \in \left[\frac{7}{8}+\frac{\sqrt[3]{621+48\sqrt{87}}}{24}+\frac{19}{8\sqrt[3]{621+48\sqrt{87}}}, 2\right]$. In this paper, using a new analysis method, we find that the original difference scheme is unconditionally stable and convergent with order $\mathcal{O}(\tau^{2-\alpha}+h^2)$ for all $\alpha \in (0,1)$ and $\beta \in (1,2]$. Finally, some numerical examples are given to verify the correctness of the results.

Keywords Caputo derivative · Riesz derivative · Time-Caputo and space-Riesz fractional diffusion equation

Mathematics Subject Classification 65M06 · 65M12

1 Introduction

In this paper, we consider the following time-Caputo and space-Riesz fractional diffusion equation:

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$$\begin{cases} {}_{\mathrm{C}}\mathrm{D}_{0,t}^{\alpha}u(x,t) = K\dfrac{\partial^{\beta}u(x,t)}{\partial|x|^{\beta}} + f(x,t),\ 0<x<L,\ 0<t\leqslant T,\\ u(x,0)=\varphi(x),\ 0\leqslant x\leqslant L,\\ u(0,t)=u(L,t)=0,\ 0<t\leqslant T, \end{cases} \tag{1}$$

where $K>0$ is the diffusion coefficient. ${}_{\mathrm{C}}\mathrm{D}_{0,t}^{\alpha}u(x,t)$ is the Caputo fractional derivative with respect to t, of order $0<\alpha<1$, defined by [4, 6, 7]

$$ {}_{\mathrm{C}}\mathrm{D}_{0,t}^{\alpha}u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,s)}{\partial s}\frac{1}{(t-s)^{\alpha}}\mathrm{d}s,\ 0<\alpha<1.$$

$\dfrac{\partial^{\beta}u(x,t)}{\partial|x|^{\beta}}$ is the Riesz fractional derivative with respect to x of order $\beta\in(1,2]$, which is defined as [6–8]

$$\frac{\partial^{\beta}u(x,t)}{\partial|x|^{\beta}} = -\frac{1}{2\cos\left(\frac{\pi\beta}{2}\right)}\left({}_{\mathrm{RL}}\mathrm{D}_{a,x}^{\beta} + {}_{\mathrm{RL}}\mathrm{D}_{x,b}^{\beta}\right)u(x,t),\ 1<\beta\leqslant 2,$$

where ${}_{\mathrm{RL}}\mathrm{D}_{a,x}^{\beta}$ denotes the left Riemann-Liouville fractional derivative

$$ {}_{\mathrm{RL}}\mathrm{D}_{a,x}^{\beta}u(x,t) = \frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial x^2}\int_a^x \frac{u(\xi,t)}{(x-\xi)^{\beta-1}}\mathrm{d}\xi, \quad 1<\beta<2,$$

and ${}_{\mathrm{RL}}\mathrm{D}_{x,b}^{\beta}$ is the right Riemann-Liouville fractional derivative

$$ {}_{\mathrm{RL}}\mathrm{D}_{x,b}^{\beta}u(x,t) = \frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial x^2}\int_x^b \frac{u(\xi,t)}{(\xi-x)^{\beta-1}}\mathrm{d}\xi, \quad 1<\beta\leqslant 2.$$

Recently, Zhang and Ding [9] developed a novel finite difference scheme with convergence order $\mathcal{O}(\tau^{2-\alpha}+h^2)$ for (1). However, it is a pity that they only proved that the stability and convergence in the case

$$\beta\in\left[\frac{7}{8}+\frac{\sqrt[3]{621+48\sqrt{87}}}{24}+\frac{19}{8\sqrt[3]{621+48\sqrt{87}}},2\right] \text{ and } \alpha\in(0,1)$$

based on the mathematical induction, while for case

$$\beta\in\left(1,\frac{7}{8}+\frac{\sqrt[3]{621+48\sqrt{87}}}{24}+\frac{19}{8\sqrt[3]{621+48\sqrt{87}}}\right) \text{ and } \alpha\in(0,1),$$

it seems difficult to prove the result using their method.

In this article, we will use another analysis method to reanalyze the difference scheme established in [9] and find that it is unconditionally stable and convergent with order $\mathcal{O}(\tau^{2-\alpha} + h^2)$ for all $\beta \in (1, 2]$ and $\alpha \in (0, 1)$.

The paper is organized as follows. In Sect. 2, we review the difference scheme established in [2] and its theoretical analysis. New stability and convergence analysis are introduced in Sect. 3. Numerical experiments are provided in Sect. 4 to prove the rationality of the theoretical analysis and the effectiveness of the algorithm.

2 Review the Algorithm in [9]

2.1 Establishment of the Algorithm

Define $t_n = n\tau, n = 0, 1, \cdots, N$, and $x_j = jh,\ j = 0, 1, \cdots, M$, where $\tau = T/N$ and $h = L/M$ are time and space mesh sizes, respectively.

For the numerical approximation of the Caputo fractional derivative ${}_{\mathrm{C}}\mathrm{D}^{\alpha}_{0,t}u(t)$ at $t = t_n$ $(n = 0, 1, \cdots, N)$, the authors used the following common $L1$ formula [2, 5]:

$$
\begin{aligned}
{}_{\mathrm{C}}\mathrm{D}^{\alpha}_{0,t}u(t)|_{t=t_n} &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{-\alpha} \frac{\partial u(x,s)}{\partial s} \mathrm{d}s \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{-\alpha} \left[\frac{u(x,t_{k+1}) - u(x,t_k)}{\tau} + \mathcal{O}(\tau) \right] \mathrm{d}s \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_{n-k-1} \big(u(x,t_{k+1}) - u(x,t_k) \big) + \mathcal{O}\big(\tau^{2-\alpha}\big),
\end{aligned} \tag{2}
$$

where

$$
b_k = (k+1)^{1-\alpha} - k^{1-\alpha},\ k = 0, 1, \cdots, n-1.
$$

At the same time, an effective second-order formula is used to numerically treat the spatial Riesz fractional derivative in [1], that is

$$
\frac{\partial^{\beta} u(x_j,t_n)}{\partial |x|^{\beta}} = -\frac{1}{2\cos\left(\frac{\pi}{2}\beta\right)} \left({}^{\mathrm{L}}\mathcal{A}^{\beta}_{2} + {}^{\mathrm{R}}\mathcal{A}^{\beta}_{2} \right) u(x_j,t_n) + \mathcal{O}\big(h^2\big), \tag{3}
$$

where

$$
{}^{\mathrm{L}}\mathcal{A}^{\beta}_{2} u(x_j,t_n) = \frac{1}{h^{\beta}} \sum_{\ell=0}^{j+1} \kappa^{(\beta)}_{2,\ell} u\big(x_j - (\ell-1)h, t_n\big)
$$

and

$$ {}^{\mathrm{R}}\mathcal{A}_2^{\beta}u(x_j,t_n)=\frac{1}{h^{\beta}}\sum_{\ell=0}^{M-j+1}\kappa_{2,\ell}^{(\beta)}u\big(x_j+(\ell-1)h,t_n\big). $$

Here, the coefficients

$$ \kappa_{2,\ell}^{(\beta)}=(-1)^{\ell}\left(\frac{3\beta-2}{2\beta}\right)^{\beta}\sum_{m=0}^{\ell}\left(\frac{\beta-2}{3\beta-2}\right)^{m}\binom{\beta}{m}\binom{\beta}{\ell-m},\quad \ell=0,1,\cdots, $$

which can also be calculated by the following recursive relations [1]:

$$ \begin{cases} \kappa_{2,0}^{(\beta)} &= \left(\dfrac{3\beta-2}{2\beta}\right)^{\beta},\\ \kappa_{2,1}^{(\beta)} &= \dfrac{4\beta(1-\beta)}{3\beta-2}\kappa_{2,0}^{(\beta)},\\ \kappa_{2,\ell}^{(\beta)} &= \dfrac{1}{\ell(3\beta-2)}\Big[4(1-\beta)(\beta-\ell+1)\kappa_{2,\ell-1}^{(\beta)}\\ &\qquad +(\beta-2)(2\beta-\ell+2)\kappa_{2,\ell-2}^{(\beta)}\Big],\quad \ell\geqslant 2. \end{cases} $$

Next, substituting (2) and (3) into (1), we obtain

$$ \begin{aligned} &\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1}b_{n-k-1}\big(u(x,t_{k+1})-u(x,t_k)\big)\\ &=-\frac{K}{2\cos\left(\frac{\pi}{2}\beta\right)}\left({}^{\mathrm{L}}\mathcal{A}_2^{\beta}+{}^{\mathrm{R}}\mathcal{A}_2^{\beta}\right)u(x_j,t_n)+f(x_j,t_n)+R_j^n, \end{aligned} \tag{4} $$

where there exists a constant C such that

$$ |R_j^n|\leqslant C\big(\tau^{2-\alpha}+h^2\big),\quad 1\leqslant j\leqslant M-1,\ 1\leqslant n\leqslant N. $$

Finally, omitting the high order terms R_j^n of (4). Replacing the function $u(x_j,t_n)$ with its numerical approximation value u_j^n, then we can obtain the following finite difference scheme [9]:

$$ \begin{cases} u_j^n+q\left[\displaystyle\sum_{\ell=0}^{j+1}\kappa_{2,\ell}^{(\beta)}u_{j-\ell+1}^n+\sum_{\ell=0}^{M-j+1}\kappa_{2,\ell}^{(\beta)}u_{j+\ell-1}^n\right]\\ =u_j^{n-1}-\displaystyle\sum_{k=1}^{n-1}b_k\left(u_j^{n-k}-u_j^{n-k-1}\right)+\tau^{\alpha}\Gamma(2-\alpha)f_j^n,\\ u_j^0=\varphi\big(x_j\big),\ 0\leqslant j\leqslant M,\\ u_0^n=u_M^n=0,\ 1\leqslant n\leqslant N, \end{cases} \tag{5} $$

where $q = \dfrac{\tau^\alpha \Gamma(2-\alpha) K}{2h^\beta \cos\left(\frac{\pi}{2}\beta\right)}$.

2.2 Theoretical Analysis of the Algorithm

Lemma 1 [1] *Under the condition*

$$\frac{7}{8} + \frac{\sqrt[3]{621+48\sqrt{87}}}{24} + \frac{19}{8\sqrt[3]{621+48\sqrt{87}}} \leqslant \beta < 2, \tag{6}$$

the coefficient $\kappa_{2,2}^{(\beta)}$ *satisfies*

$$\kappa_{2,2}^{(\beta)} \geqslant 0.$$

In [9], by using mathematical induction, the stability and convergence results of the proposed scheme are stated as follows:

Theorem 1 *Under the condition* (6) *and* $0 < \alpha < 1$, *the finite difference scheme* (5) *for time-Caputo and space-Riesz fractional diffusion* (1) *is unconditionally stable.*

Theorem 2 *Denote by* $u(x_j, t_n)$ $(j = 1, 2, \cdots, M-1;\ n = 1, 2, \cdots, N)$ *the exact solution of* (1) *at mesh point* (x_i, t_n), *and let* $\{U_j^n \mid 0 \leqslant j \leqslant M, 0 \leqslant n \leqslant N\}$ *be the solution of the finite difference scheme* (5). *Define*

$$\varepsilon_j^n = u(x_j, t_n) - U_j^n,\ \ j = 1, 2, \cdots, M;\ n = 1, 2, \cdots, N,$$

then there exists a positive constant C, *such that*

$$||\varepsilon^n||_\infty \leqslant C\left(\tau^{2-\alpha} + h^2\right),\ \ 0 \leqslant n \leqslant N,$$

under the condition (6) *and* $0 < \alpha < 1$.

Remark 1 From the conclusion of the above two theorems, we find that the method in [9] can only prove that the difference scheme (5) is stable and convergent under the condition (6). Below, we will use another method to prove that the difference scheme (5) is unconditionally stable and convergent for all $\alpha \in (0, 1)$ and $\beta \in (1, 2]$.

3 A New Theoretical Analysis Method for the Algorithm in [9]

Let

$$\mathcal{U}_h = \left\{u | u = (u_0, u_1, \cdots, u_M),\ u_0 = u_M = 0\right\}$$

be the space grid functions. For any $u, v \in \mathcal{U}_h$, define the following inner product

$$(u, v) = h \sum_{j=1}^{M-1} u_j v_j$$

and the corresponding norm

$$\|u\| = \sqrt{(u, u)}\ .$$

For convenience, denote the operator

$$\delta_x^\beta = -\frac{1}{2\cos\left(\frac{\pi}{2}\beta\right)}\left({}^{\mathrm{L}}\mathcal{A}_2^\beta + {}^{\mathrm{R}}\mathcal{A}_2^\beta\right), \tag{7}$$

then the numerical algorithm (5) can be rewritten as

$$\begin{cases} u_j^n - \sum\limits_{k=1}^{n-1}\left(b_{n-k-1} - b_{n-k}\right)u_j^k - b_{n-1}u_j^0 = \mu K \delta_x^\alpha u_j^k + \mu f_j^k, \\ 1 \leqslant j \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ u_j^0 = \varphi\left(x_j\right),\ 0 \leqslant j \leqslant M, \\ u_0^n = u_M^n = 0,\ 1 \leqslant n \leqslant N, \end{cases} \tag{8}$$

where $\mu = \tau^\alpha \Gamma(2-\alpha)$.

Next, we list several lemmas for the stability and convergence analysis.

Lemma 2 [3] *Let* $b_k = (k+1)^{1-\alpha} - k^{1-\alpha},\ k = 0, 1, 2, \cdots$ *and* $0 < \alpha < 1$. *Then there holds that*

(i) $1 = b_0 > b_1 > b_2 > \cdots > b_k \to 0,$ as $k \to +\infty$;

(ii) $\sum\limits_{k=0}^{n}(b_k - b_{k+1}) + b_{n+1} = (1 - b_1) + \sum\limits_{k=1}^{n-1}(b_k - b_{k+1}) + b_n = 1.$

Lemma 3 [1] *Let the operator* δ_x^β *be defined by* (7). *Then the following inequality*:

$$(\delta_x^\beta u, u) \leqslant 0$$

holds for all $\beta \in (1, 2]$.

Now, we give the stability result of the finite difference scheme (5).

Theorem 3 *The finite difference scheme* (5) *is unconditionally stable to the initial value* φ *and right term* f *for all* $\alpha \in (0,1)$ *and* $\beta \in (1,2]$.

Proof Taking the inner product of the first equation of (8) with u^n leads to

$$(u^n, u^n) - \sum_{k=1}^{n-1} \left(b_{n-k-1} - b_{n-k}\right)\left(u^k, u^n\right) - b_{n-1}\left(u^0, u^n\right) \\ = \mu K\left(\delta_x^\alpha u^n, u^n\right) + \mu(f^n, u^n). \tag{9}$$

For the second term of the left hand of (9), we obtain

$$-\sum_{k=1}^{n-1} \left(b_{n-k-1} - b_{n-k}\right)\left(u^k, u^n\right) \geqslant -\sum_{k=1}^{n-1} \left(b_{n-k-1} - b_{n-k}\right)\|u^k\| \cdot \|u^n\| \tag{10}$$

based on the Cauchy-Schwarz inequality and Lemma 2.

Similarly, for the third term of the left hand of (9), we also have

$$-b_{n-1}\left(u^0, u^n\right) \geqslant -b_{n-1}\|u^0\| \cdot \|u^n\|. \tag{11}$$

For the first term of the right hand of (9), it follows from Lemma 3 that

$$\mu K\left(\delta_x^\alpha u^n, u^n\right) \leqslant 0. \tag{12}$$

As to the second term of the right hand of (9), we easily know that

$$\mu(f^n, u^n) \leqslant \mu\|f^n\| \cdot \|u^n\|. \tag{13}$$

Substituting (10), (11), (12) and (13) into (9) yields

$$\|u^n\| \leqslant \sum_{k=1}^{n-1} \left(b_{n-k-1} - b_{n-k}\right)\|u^k\| + b_{n-1}\|u^0\| + \mu\|f^n\|. \tag{14}$$

Note that

$$\mu = \tau^\alpha \Gamma(2-\alpha) = (1-\alpha)\Gamma(1-\alpha)T^\alpha N^{-\alpha},$$

and

$$b_{n-1} = n^{1-\alpha} - (n-1)^{1-\alpha} = (1-\alpha)\zeta^{-\alpha},\ n = 1, 2, \cdots, N,\ \zeta \in (n-1, n),$$

then we obtain

$$\mu < T^\alpha \Gamma(1-\alpha) b_{n-1}, \quad n = 1, 2, \cdots, N.$$

Denote

$$U = \|u^0\| + T^\alpha \Gamma(1-\alpha) \max_{1 \leqslant \ell \leqslant N} \|f^\ell\|,$$

then (14) can be rewritten as

$$\|u^n\| \leqslant \sum_{k=1}^{n-1} \left(b_{n-k-1} - b_{n-k}\right) \|u^k\| + b_{n-1} U, \quad 1 \leqslant n \leqslant N.$$

Now, we will prove that

$$\|u^n\| \leqslant U, \quad 1 \leqslant n \leqslant N$$

by the mathematical induction.

First of all, it is obviously true for $n = 1$. Secondly, we assume that

$$\|u^k\| \leqslant U$$

is true for $k = 1, 2, \cdots, n-1$. Then we can get further

$$\|u^n\| \leqslant \sum_{k=1}^{n-1} \left(b_{n-k-1} - b_{n-k}\right) \|u^k\| + b_{n-1} U \leqslant U, \quad 1 \leqslant n \leqslant N,$$

that is

$$\|u^n\| \leqslant \|\varphi\| + T^\alpha \Gamma(1-\alpha) \max_{1 \leqslant \ell \leqslant N} \|f^\ell\|, \quad 1 \leqslant n \leqslant N. \tag{15}$$

This ends the proof.

Finally, we consider the convergence of the difference scheme (5).

Theorem 4 *Suppose that* $u(x_j, t_n)$ $(j = 1, 2, \cdots, M-1;\ n = 1, 2, \cdots, N)$ *and* $\{u_j^n \mid 0 \leqslant j \leqslant M, 0 \leqslant n \leqslant N\}$ *are the exact solution of* (1) *and finite difference scheme* (5), *respectively. Let*

$$\varepsilon_j^n = u(x_j, t_n) - u_j^n, \quad j = 1, 2, \cdots, M;\ n = 1, 2, \cdots, N.$$

Then it holds that

$$\|\varepsilon^n\| \leqslant C T^\alpha \Gamma(1-\alpha) \sqrt{L} \left(\tau^{2-\alpha} + h^2\right), \quad 1 \leqslant n \leqslant N$$

for all $\alpha \in (0, 1)$ *and* $\beta \in (1, 2]$.

Proof From (1) and (4), we obtain the following error equation:

$$\begin{cases} \varepsilon_j^n - \sum_{k=1}^{n-1} \left(b_{n-k-1} - b_{n-k}\right)\varepsilon_j^k - b_{n-1}\varepsilon_j^0 = \mu K \delta_x^\alpha \varepsilon_j^k + \mu R_j^k, \\ 1 \leqslant j \leqslant M-1,\ 1 \leqslant n \leqslant N, \\ \varepsilon_j^0 = 0,\ 0 \leqslant j \leqslant M, \\ \varepsilon_0^n = \varepsilon_M^n = 0,\ 1 \leqslant n \leqslant N. \end{cases} \tag{16}$$

For the first equation of (16), it follows from the inequality (15) that

$$\|\varepsilon^n\| \leqslant T^\alpha \Gamma(1-\alpha) \max_{1 \leqslant \ell \leqslant N} \|R^\ell\|, \quad 1 \leqslant n \leqslant N.$$

Due to

$$\|R^\ell\|^2 = \left(R^\ell, R^\ell\right) \leqslant C^2 L \left(\tau^{2-\alpha} + h^2\right)^2,$$

then we get

$$\|\varepsilon^n\| \leqslant C T^\alpha \Gamma(1-\alpha) \sqrt{L} \left(\tau^{2-\alpha} + h^2\right), \quad 1 \leqslant n \leqslant N.$$

This completes the proof.

4 Numerical Example

In this section, we give some numerical results to prove the rationality of our theoretical analysis.

Example 1 Consider the following equation:

$${}_{\mathrm{C}}\mathrm{D}_{0,t}^\alpha u(x,t) = \beta^4 \frac{\partial^\beta u(x,t)}{\partial |x|^\beta} + f(x,t),\ 0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant 1$$

with a given force term

$$\begin{aligned} f(x,t) = & \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\beta+2)} t^{\beta+1} x^4 (1-x)^4 \\ & + \frac{t^{\alpha+\beta+1}\beta^4}{\cos(\pi\beta/2)} \sum_{\ell=0}^{4} (-1)^\ell \frac{(4+\ell)!}{\ell!\,(4-\ell)!\,\Gamma(5+\ell-\beta)} \left[x^{4+\ell-\beta} + (1-x)^{4+\ell-\beta}\right]. \end{aligned}$$

Its exact solution is

$$u(x,t) = t^{\alpha+\beta+1} x^4 (1-x)^4.$$

By using the numerical algorithm (5), the maximum error, temporal convergence order and spatial convergence order were listed in Tables 1 and 2 for different values of α, β, τ and h, respectively. From these tables, we can conclude that the developed numerical algorithm is unconditionally stable and convergent with order $\mathcal{O}(\tau^{2-\alpha} + h^2)$.

Table 1 The maximum error and temporal convergence order with $\beta = 1.5$ and $h = \frac{1}{800}$

α	τ	The maximum error	The temporal convergence order
0.2	$\frac{1}{10}$	8.658 765 E–006	–
	$\frac{1}{12}$	6.413 437 E–006	1.646 4
	$\frac{1}{14}$	4.970 150 E–006	1.653 9
	$\frac{1}{16}$	3.982 490 E–006	1.659 1
	$\frac{1}{18}$	3.274 153 E–006	1.662 8
0.4	$\frac{1}{10}$	3.194 704 E–005	–
	$\frac{1}{12}$	2.433 552 E–005	1.492 7
	$\frac{1}{14}$	1.930 660 E–005	1.501 7
	$\frac{1}{16}$	1.578 412 E–005	1.508 6
	$\frac{1}{18}$	1.320 620 E–005	1.514 0
0.5	$\frac{1}{10}$	5.396 999 E–005	–
	$\frac{1}{12}$	4.174 471 E–005	1.408 8
	$\frac{1}{14}$	3.354 995 E–005	1.417 7
	$\frac{1}{16}$	2.773 867 E–005	1.424 4
	$\frac{1}{18}$	2.343955E–005	1.429 8
0.6	$\frac{1}{10}$	8.701 323 E–005	–
	$\frac{1}{12}$	6.836 268 E–005	1.323 1
	$\frac{1}{14}$	5.567 885 E–005	1.331 3
	$\frac{1}{16}$	4.657 145 E–005	1.337 6
	$\frac{1}{18}$	3.975 980 E–005	1.342 6
0.8	$\frac{1}{10}$	2.050 545 E–004	–
	$\frac{1}{12}$	1.661 756 E–004	1.153 1
	$\frac{1}{14}$	1.389 974 E–004	1.158 5
	$\frac{1}{16}$	1.190 086 E–004	1.162 7
	$\frac{1}{18}$	1.037 368 E–004	1.166 0

Table 2 The maximum error and spatial convergence order with $\alpha = 0.7$ and $\tau = \frac{1}{1\,000}$

β	h	The maximum error	The spatial convergence order
1.1	$\frac{1}{20}$	3.630 199 E–005	–
	$\frac{1}{25}$	2.342 366 E–005	1.963 4
	$\frac{1}{30}$	1.678 806 E–005	1.826 9
	$\frac{1}{35}$	1.239 086 E–005	1.970 2
	$\frac{1}{40}$	9.656 867 E–006	1.866 9
1.3	$\frac{1}{20}$	3.465 919 E–005	–
	$\frac{1}{25}$	2.224 634 E–005	1.987 0
	$\frac{1}{30}$	1.590 544 E–005	1.840 2
	$\frac{1}{35}$	1.171 728 E–005	1.982 5
	$\frac{1}{40}$	9.125 217 E–006	1.872 4
1.5	$\frac{1}{20}$	3.041 155 E–005	–
	$\frac{1}{25}$	1.942 731 E–005	2.044 8
	$\frac{1}{30}$	1.386 003 E–005	2.011 0
	$\frac{1}{35}$	1.019 512 E–005	2.055 4
	$\frac{1}{40}$	7.936 704 E–006	2.030 7
1.7	$\frac{1}{20}$	2.493 301 E–005	–
	$\frac{1}{25}$	1.585 946 E–005	2.027 5
	$\frac{1}{30}$	1.129 508 E–005	1.861 5
	$\frac{1}{35}$	8.301 045 E–006	1.998 0
	$\frac{1}{40}$	6.463 725 E–006	1.873 5
1.9	$\frac{1}{20}$	1.856 504 E–005	–
	$\frac{1}{25}$	1.177 327 E–005	2.041 1
	$\frac{1}{30}$	8.381 323 E–006	1.863 9
	$\frac{1}{35}$	6.163 894 E–006	1.993 5
	$\frac{1}{40}$	4.808 455 E–006	1.859 7

In addition, Figs. 1, 2, 3 and 4 compare the graphs of the exact and approximate solutions with different values of α, β, τ and h. From these figures, we can conclude that the developed numerical solutions are in excellent agreement with the exact solution.

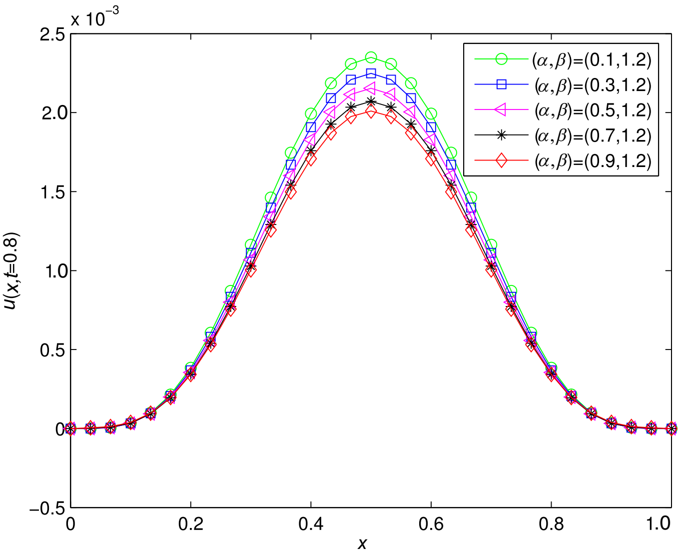

Fig. 1 The comparison of exact and numerical solutions for $\tau = \frac{1}{30}, h = \frac{1}{30}$ at $t = 0.8$ with different α

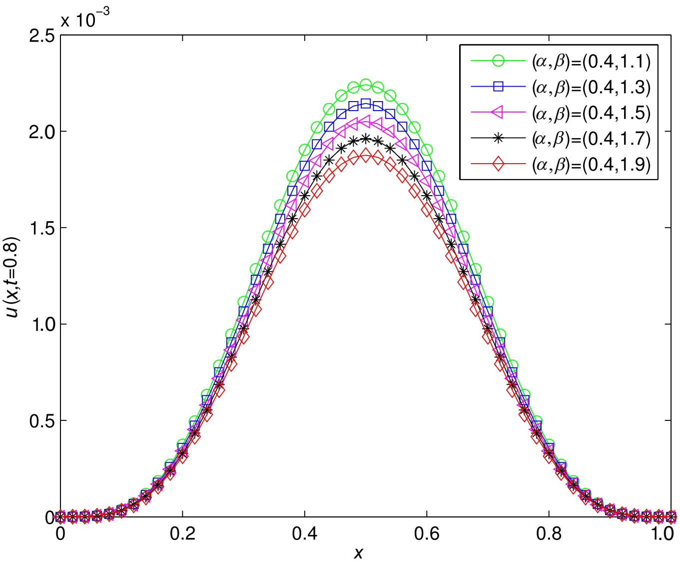

Fig. 2 The comparison of exact and numerical solutions for $\tau = \frac{1}{40}, h = \frac{1}{50}$ at $t = 0.8$ with different β

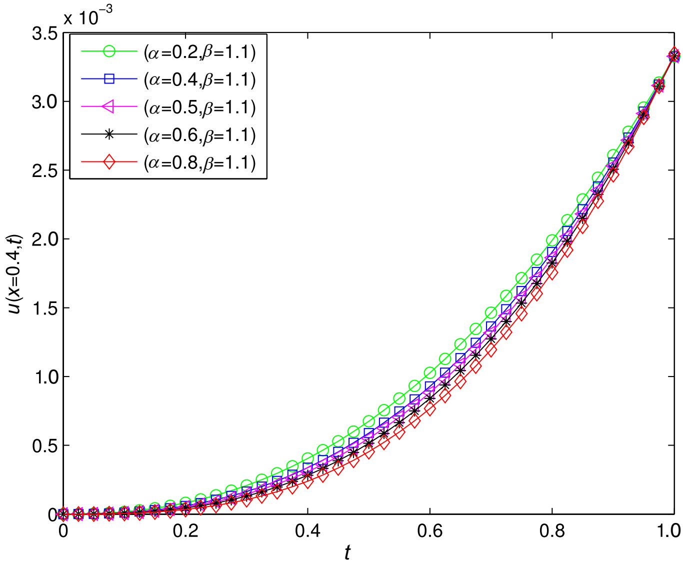

Fig. 3 The comparison of exact and numerical solutions for $\tau = \frac{1}{40}, h = \frac{1}{40}$ at $x = 0.4$ with different α

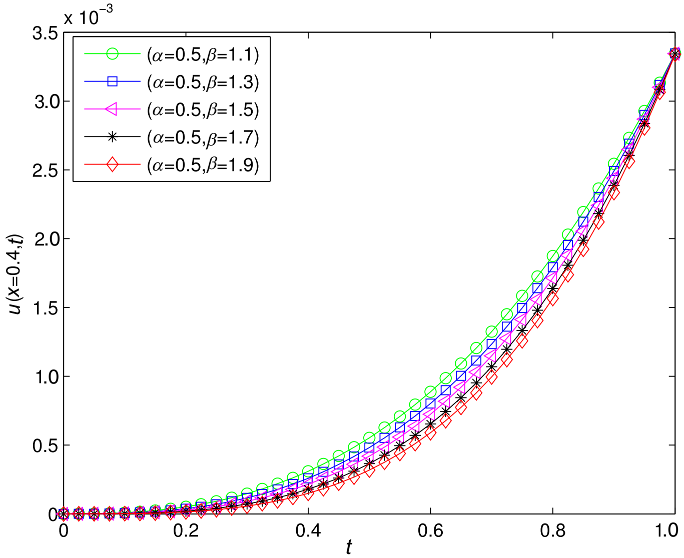

Fig. 4 The comparison of exact and numerical solutions for $\tau = \frac{1}{40}, h = \frac{1}{20}$ at $x = 0.4$ with different β

Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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